Lista 2 - 218733 - Joao pedro de Moraes Bonucci

1. Prove que a equação $cos(\frac{x+3}{x+5}) + \frac{x}{5}$ possui pelo menos possui pelo menos uma raiz real. Encontre um intervalo de comprimento finito que contenha uma raiz desta equação.

Utilizando o teorema de bolzano, que diz que se f eh uma funcao continua em $[a, b]$ e se $f(a) \cdot f(b) < 0$ (ou seja f(a) e f(b) tem sinais opostos), entao existe x em $[a, b]$ tal que $f(x) = 0$ podemos provar que existe ao menos uma raiz real para a equacao dada.

Primeiramente, precisamos provar a que a equacao em questao eh continua, pelo menos em um trecho de interesse.

Pela definicao de funcao continua, sendo um intervalo $A \subset \real$, podemos definir uma funcao $f : A- > R$. Seja $c \in A$, entao $f(x)$ eh continua em c se e somente se $\forall E > 0$ $\exists \delta > 0$ tal que $|x - c| < \delta => |f(x) - f(c)| < E$.

Para x = -5 temos $cos(\frac{-5+3}{-5+5}) + \frac{-5}{5}$ como o denominador da fracao eh 0, temos uma descontinuidade

Para x > -5 temos que
$|x - c| < \delta => |cos(\frac{x+3}{x+5}) + \frac{x}{5} - cos(\frac{c+3}{c+5}) + \frac{c}{5}| < \epsilon => |cos(\frac{x+3}{x+5}) - cos(\frac{c+3}{c+5})| + |\frac{x-c}{5}|$

como $|cos(\frac{x+3}{x+5}) - cos(\frac{c+3}{c+5})| \leq 2$podemos reescrever a expressao acima para

$|x - c| < \delta => |2 + \frac{x-c}{5}| < \epsilon => |x - c| < 5 \cdot \epsilon - 10$

assima achamos a prova para $\delta = 5 \cdot \epsilon - 10$

Como a funcao eh continua para esse intervalo e a funcao troca de sinal, podemos garantir que exite ao menos uma raiz.

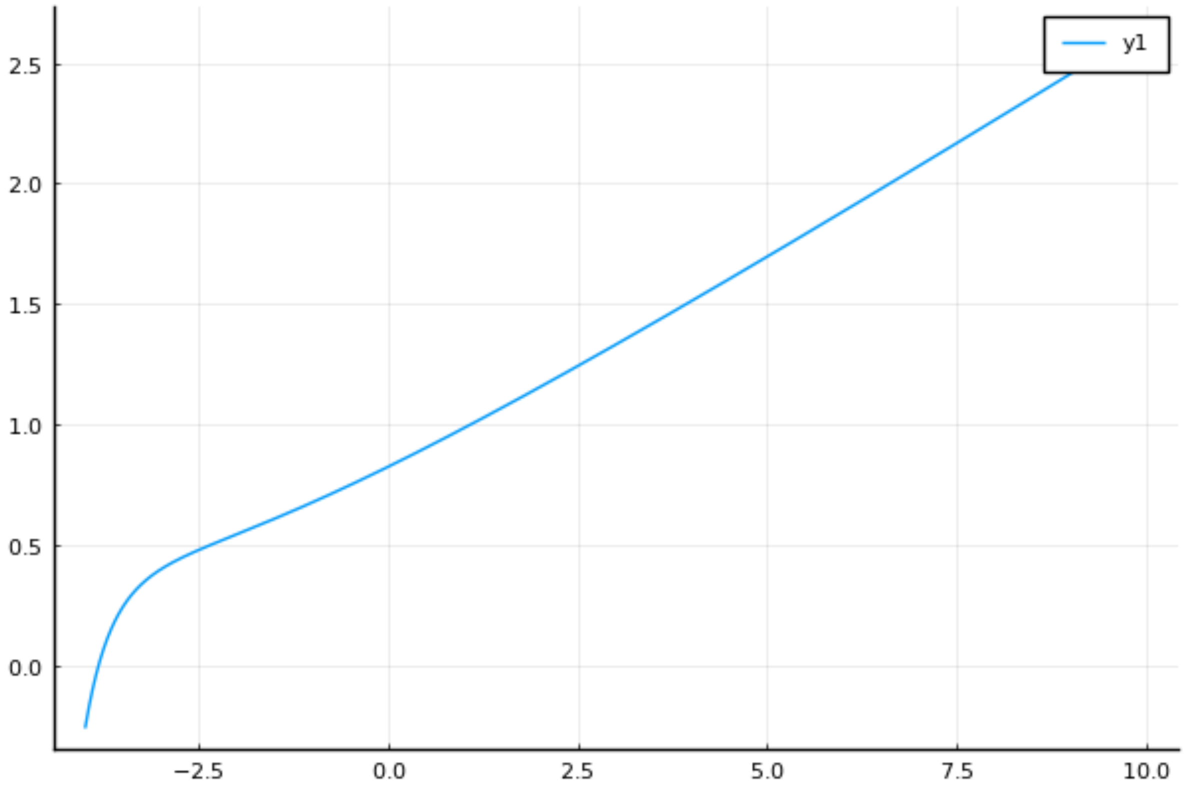
sabendo que para x = -4 a funcao eh negativa e para x = 0 a funcao eh positiva, podemos determinar este como um intervalo finito que contem ao menos uma raiz da funcao pelo teorema de bolzano

In [9]:
```
using Plots
using LaTeXStrings
pyplot()
```

Out[9]: `Plots.PyPlotBackend()`

In [10]:
```
function expr(x)
    return cos((x+3)/(x+5))+x/5
end
x = LinRange{Float64}(-4, 10, 2000)
y = expr.(x)
plot(x, y)
```

Out[10]:



2. Quantas raízes a função $f(x) = x^{\frac{-1}{2}} - e^{\frac{x}{5}}$ possui?

Se possível, exiba um intervalo que contem uma raiz.

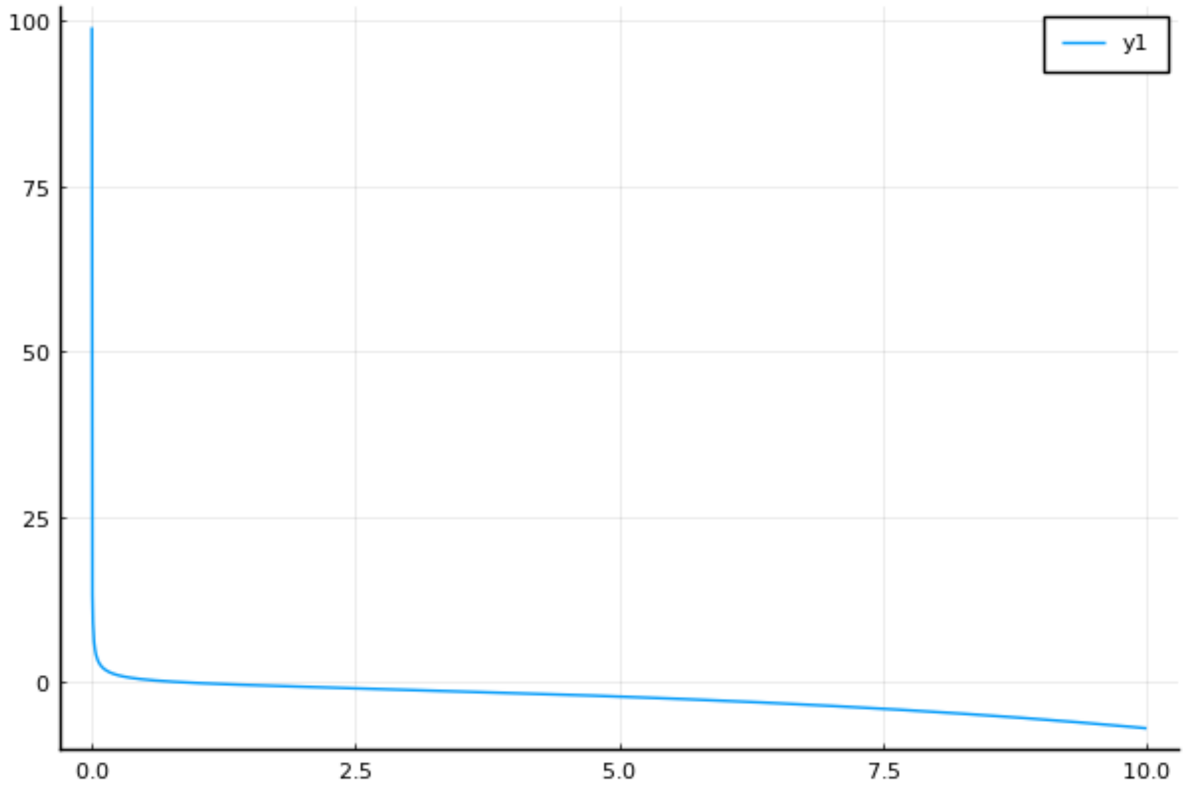
Obs: A função não tem domínio em toda a reta real.

a funcao contem uma unica raiz devido seu comportamento decrescente monotono

In [11]:

```
function expr(x)
    return x^(-1/2) - 2.71828182845904523536028747135266249775724709369995
end
x = LinRange{Float64}(0.0001, 10, 2000)
y = expr.(x)
plot(x, y)
```

Out[11]:



3. Dado $\epsilon = 10^{-4}$ e a função $f(x) = \frac{x+1}{x^2} - \frac{x^3}{10}$

Identifique um intervalo que contenha um zero de f . Partindo desse intervalo aproxime um zero usando:

- **O método da bissecção parando com** $|x_{k+1} - x_k| < \epsilon$
- **O método da bissecção parando com** $|f(x_k) < \epsilon|$

Quantas iterações foram necessárias em cada caso?

Qual foi a aproximação encontrada?

Qual tinha o menor valor de $|f(x)|$**?**

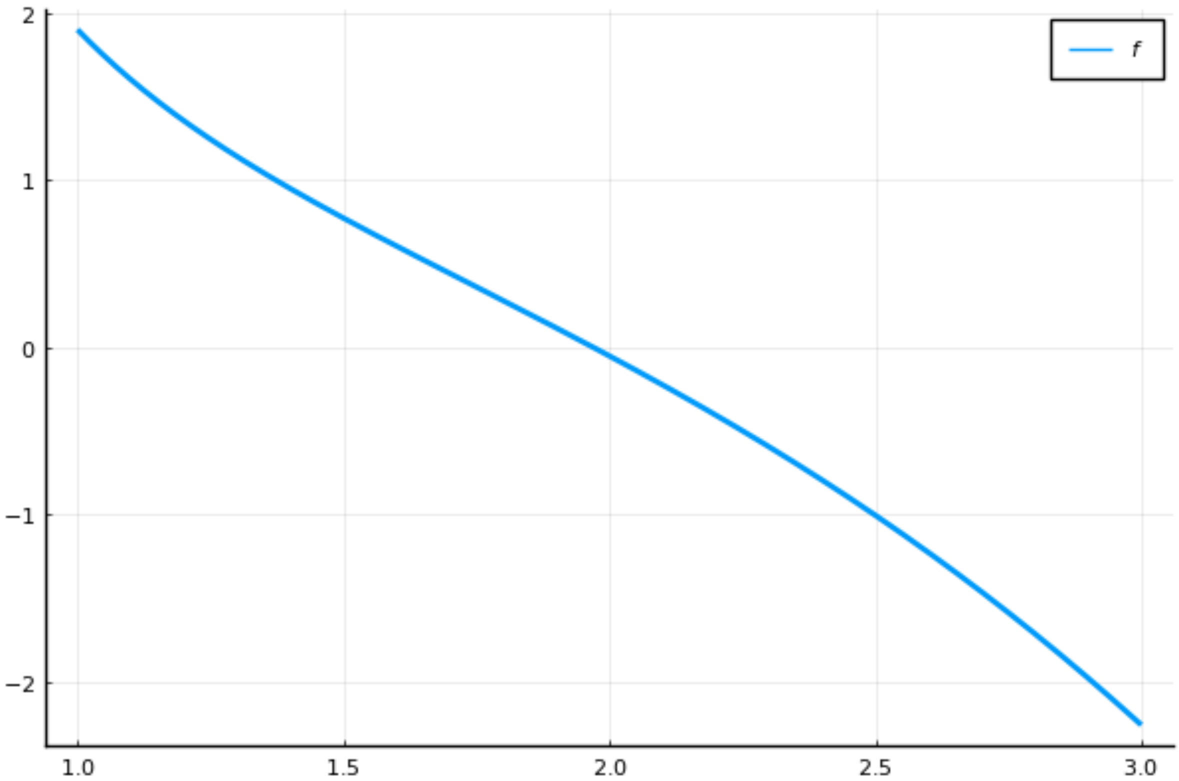
- Dominio: $D = \reals \setminus 0$
- Raizes: $\frac{x+1}{x^2} - \frac{x^3}{10} = 0 \rightarrow 10 \cdot (x+1) - x^5 = 0 \rightarrow x^5 - 10x - 10 = 0 \rightarrow x \approx 1.97044492657$
- Interseccao com eixo y: $\lim_{x \to 0} \frac{x+1}{x^2} - \frac{x^3}{10} = \infty$
- pontos criticos: $f'(x) = -\frac{3x^5+10x+20}{10x^3} = 0 \rightarrow x \approx -1.212807637953$
- minimo local: $f''(x) = \frac{6}{x^4} + \frac{2}{x^3} - \frac{3x}{5} \rightarrow f''(-1.212807637953) \approx 2.379774200263 > 0$ o ponto eh minimo local com convavidade para cima
- crescente: $\infty < x \leq -1.212807637953...$
- derescente: $0 < x < \infty$
- assintota: x = 0
- tende ao infinito negativo para x positivo
- tende para infinito positivo para x negativo

Com a analise acima podemos ver que a funcao apresenta uma raiz entre 0 e ∞, no entanto podemos ver que se $x^3 > \frac{10x+10}{x^2}$ a funcao passa a ser negativa. Para x = 3 temos que $3^3 = 27 > \frac{10 \cdot 3 - 10}{3^2} = \frac{40}{9}$ e para x = 1 temos que $1^3 = 1 < \frac{10 \cdot 1 + 10}{1^2} = 20$, logo analisarei o intervalo entre 1 e 3 pois pelo teorema de bolzano, garantimos que existe uma raiz contida nele

In [12]:

```
# Define a função
f(x) = (x+1)/x^2 - x^3/10
#f(x) = (10 + 10*x - x^5)/(10*x^2)
# Define o intervalo
x = LinRange{BigFloat}(1, 3, 10000)
# Desenha o gráfico e o eixo x.
plot(x, f.(x), label=L"f", lw=2)
```

Out[12]:



In [13]:

```
# Método da bisseccao
function bisseccao(f, a, b, usa_medio=true, epsilon=1.0e-5)
        iter = 0

        medio = (a + b)/2.0
        anterior = abs(b * 10)
        while true
                medio = (a + b)/2.0
                println("$iter: $medio")
                if f(medio)*f(a) > 0.0
                        a = medio
                else
                        b = medio
                end
                iter += 1
                if abs(medio - anterior) < epsilon
                        break
                end
                anterior = medio
        end
        println("$iter: $medio")
        return medio
end

println("com a segunda condicao:")
raiz = bisseccao(f, 1.0, 3.0, false)
@show f(raiz)
```

```
com a segunda condicao:
0: 2.0
1: 1.5
2: 1.75
3: 1.875
4: 1.9375
5: 1.96875
6: 1.984375
7: 1.9765625
8: 1.97265625
9: 1.970703125
10: 1.9697265625
11: 1.97021484375
12: 1.970458984375
13: 1.9703369140625
14: 1.97039794921875
15: 1.970428466796875
16: 1.9704437255859375
17: 1.9704513549804688
18: 1.9704513549804688
f(raiz) = -1.0823969794904187e-5
```

Out[13]: -1.0823969794904187e-5

In [14]:

```julia
# Método da bisseccao
function bisseccao(f, a, b, usa_medio=true, epsilon=1.0e-5)
        iter = 0

        medio = (a + b)/2.0
        while true
                medio = (a + b)/2.0
                println("$iter: $medio")
                if f(medio)*f(a) > 0.0
                        a = medio
                else
                        b = medio
                end
                iter += 1
                if abs(f(medio)) < abs(epsilon)
                        break
                end
        end
        println("$iter: $medio")
        return medio
end

println("coma a segunda condicao:")
raiz = bisseccao(f, 1.0, 3.0)
@show f(raiz)
```

```
coma a segunda condicao:
0: 2.0
1: 1.5
2: 1.75
3: 1.875
4: 1.9375
5: 1.96875
6: 1.984375
7: 1.9765625
8: 1.97265625
9: 1.970703125
10: 1.9697265625
11: 1.97021484375
12: 1.970458984375
13: 1.9703369140625
14: 1.97039794921875
15: 1.970428466796875
16: 1.9704437255859375
17: 1.9704437255859375
f(raiz) = 2.0221921093188655e-6
```

Out[14]: 2.0221921093188655e-6

4. Deduza um método para computo da raiz cúbica de um número x a partir do método de Newton, de forma análoga ao que fizemos com a raiz quadrada nas notas de aula. Aplique 4 passos seu método para calcular a raiz cúbica de 10 partindo do número 3. Quantas casas decimais corretas a aproximação obtida possui?

$$f(x) = x^3 - a = 0$$

$$f'(x) = 3x^2$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{{x_k}^3 - a}{3{x_k}^2}$$

Com essa implementacao e utilizando Float64 foi possivel atingir 7 casa de precisão com 4 iterações

In [15]:

```
a = 10
xk = 3
prec = 1.0e-15
iters = 0
println(iters, ": ", xk)
while iters < 4 && abs((xk*xk*xk - a)/a) > prec
xk = Float64(1/3*(2*xk + a/xk^2))
iters += 1
println(iters, ": ", xk)
end
a = BigFloat(a^(1/3))
println("\nValor 'exato': ", a)
@show xk*xk*xk
```

```
0: 3
1: 2.3703703703703702
2: 2.173508632330247
3: 2.154601586556419
4: 2.154434702959439

Valor 'exato': 2.154434690031883814498314677621237933635711669921875
xk * xk * xk = 10.000000180013187
```

Out[15]:

```
10.000000180013187
```

5. Escreva explicitamente qual o próximo iterado de Newton, x_{k+1} , quando usando para resolver a equação $x^3 = 0$ partindo de um valor $0 < x^k \leq 1$

Qual razão entre a distância de x_{k+1} à raiz da equação quando comparada a distância de x_k a essa raiz? Considerando o teorema de convergência visto em sala de aula o resultado é esperado? Há alguma contradição com o que foi provado? Justifique.

funcao estudada: $f(x) = x^3$ primeira derivada: $f'(x) = 3x^2$ segunda derivada: $f''(x) = 6x$ terceira derivada: $f^{(3)}(x) = 6$ quarta derivada: $f^{(4)}(x) = 0$

Pelo teorema de taylor existe x e y tal que:

$$f(y) = f(x) + f'(x)(y - x) + \frac{f''(x)}{2}(y - x^2) + \frac{f^{(3)}(x)}{6}(y - x^3)$$

que para a funcao estuda pode ser reescrito como:

$$f(y) = f(x) + 3x^2(y - x) + \frac{6x}{2}(y - x^2) + \frac{6}{6}(y - x^3)$$

e simplificada para:

$$f(y) = f(x) + 3x^2(y-x) + 3x(y-x^2) + (y-x^3)$$

Pelo metodo de Newton sabemos que dado um x_k qualquer o proximo ponto iterado sera dado por

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{{x_k}^3}{3{x_k}^2}$$

$$x_{k+1} = \frac{1}{3}(2x_k)$$

partindo de $x_k = 1$ temos que $x_{k+1} = \frac{2}{3}$ partindo de $x_k = 1 \cdot 10^{-200}$ temos que $x_{k+1} = \frac{2 \cdot 10^{-200}}{3}$

ou seja, a cada iteracao a distancia da aproximacao e da raiz real reduz em $\frac{1}{3}$

Utilizando as mesmas variaveis do teorema de convergencia estudado em aula temos que:

$$x_k = 1$$

$$x_{k+1} = \frac{2}{3}$$

$$x^* = 0$$

Logos deve existir M tal que

$$\frac{|x_{k+1} - x^*|}{|xk - x^*|^2} = \frac{2}{3} \leq M$$

Logo o resultado eh esperado pelo teorema de convergência

In [16]:

```
a = 0
xk = 1
prec = 1.0e-18
iters = 0
println(iters, ": ", xk)
while iters < 10000 && abs((xk*xk*xk - a)/a) > prec
xk = Float64(1/3*(2*xk))
iters += 1
println(iters, ": ", xk)
end

@show xk
```

```
0: 1
1: 0.6666666666666666
2: 0.4444444444444444
3: 0.2962962962962963
4: 0.19753086419753085
5: 0.1316872427983539
6: 0.08779149519890259
7: 0.058527663465935055
8: 0.03901844231062337
9: 0.026012294873748912
```

```
10: 0.017341529915832606
11: 0.011561019943888404
12: 0.007707346629258935
13: 0.005138231086172623
14: 0.0034254873907817486
15: 0.0022836582605211654
16: 0.0015224388403474436
17: 0.0010149592268982957
18: 0.0006766394845988638
19: 0.00045109298973257584
20: 0.0003007286598217172
21: 0.00020048577321447815
22: 0.00013365718214298543
23: 8.910478809532361e-5
24: 5.940319206354907e-5
25: 3.960212804236605e-5
26: 2.64014186949107e-5
27: 1.760094579660713e-5
28: 1.1733963864404753e-5
29: 7.822642576269835e-6
30: 5.215095050846556e-6
31: 3.476730033897704e-6
32: 2.317820022598469e-6
33: 1.5452133483989794e-6
34: 1.0301422322659861e-6
35: 6.86761488177324e-7
36: 4.5784099211821597e-7
37: 3.0522732807881063e-7
38: 2.0348488538587375e-7
39: 1.3565659025724915e-7
40: 9.043772683816609e-8
41: 6.029181789211072e-8
42: 4.0194545261407147e-8
43: 2.6796363507604763e-8
44: 1.7864242338403173e-8
45: 1.1909494892268781e-8
46: 7.93966326151252e-9
47: 5.293108841008347e-9
48: 3.528739227338898e-9
49: 2.352492818225932e-9
50: 1.5683285454839546e-9
51: 1.0455523636559697e-9
52: 6.970349091039797e-10
53: 4.6468993940265314e-10
54: 3.097932929351021e-10
55: 2.0652886195673472e-10
56: 1.3768590797115648e-10
57: 9.179060531410432e-11
58: 6.119373687606954e-11
59: 4.079582458404636e-11
60: 2.7197216389364237e-11
61: 1.813147759290949e-11
62: 1.2087651728606328e-11
63: 8.058434485737552e-12
64: 5.372289657158368e-12
65: 3.5815264381055786e-12
66: 2.3876842920703856e-12
67: 1.5917895280469236e-12
68: 1.061193018697949e-12
69: 7.074620124652993e-13
70: 4.716413416435328e-13
71: 3.1442756109568856e-13
72: 2.0961837406379236e-13
73: 1.397455827091949e-13
```

```
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83: 2.4234022031569625e-15
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85: 1.0770676458475387e-15
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87: 4.786967314877949e-16
88: 3.1913115432519656e-16
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91: 9.455737905931748e-17
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97: 8.301333689706881e-18
98: 5.534222459804587e-18
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104: 4.858576645095933e-19
105: 3.2390510967306217e-19
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108: 9.597188434757396e-20
109: 6.398125623171597e-20
110: 4.265417082114398e-20
111: 2.843611388076265e-20
112: 1.8957409253841763e-20
113: 1.2638272835894508e-20
114: 8.425515223929672e-21
115: 5.617010149286448e-21
116: 3.744673432857631e-21
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125: 9.740754954138628e-23
126: 6.493836636092418e-23
127: 4.3292244240616116e-23
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129: 1.9240997440273827e-23
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137: 7.507537486221761e-25
```

```
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144: 4.3939862745147916e-26
145: 2.929324183009861e-26
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147: 1.3019218591154935e-26
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191: 2.3257847596903563e-34
192: 1.5505231731269042e-34
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200: 6.049899898193679e-36
201: 4.033266598795786e-36
```

```
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209: 1.5737178010847746e-37
210: 1.049145200723183e-37
211: 6.994301338154553e-38
212: 4.6628675587697017e-38
213: 3.108578372513134e-38
214: 2.0723855816754226e-38
215: 1.381590387783615e-38
216: 9.210602585224099e-39
217: 6.140401723482732e-39
218: 4.093601148988488e-39
219: 2.729067432658992e-39
220: 1.819378288439328e-39
221: 1.2129188589595519e-39
222: 8.086125726397012e-40
223: 5.390750484264675e-40
224: 3.5938336561764497e-40
225: 2.395889104117633e-40
226: 1.5972594027450887e-40
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232: 1.4022579118749745e-41
233: 9.348386079166497e-42
234: 6.232257386110997e-42
235: 4.154838257407331e-42
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