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Problem set ⑤

Problem ①

$Y \backslash X$	0	1	
0	1/4	1/4	$P(Y=0) = 1/2$
1	0	1/2	$P(Y=1) = 1/2$
	$P(X=0) = 1/4$	$P(X=1) = 3/4$	

⚠ $P(X|Y) = \dfrac{P(X,Y)}{P(Y)}$

$$\left.\begin{array}{l} P(X=0 \mid Y=0) = 1/2 \\ P(X=1 \mid Y=0) = 1/2 \end{array}\right\} = 1 \qquad \left.\begin{array}{l} P(Y=0 \mid X=0) = 1 \\ P(Y=1 \mid X=0) = 0 \end{array}\right\} 1$$

$$\left.\begin{array}{l} P(X=0 \mid Y=1) = 0 \\ P(X=1 \mid Y=1) = 1 \end{array}\right\} = 1 \qquad \left.\begin{array}{l} P(Y=0 \mid X=1) = 1/3 \\ P(Y=1 \mid X=1) = 2/3 \end{array}\right\} = 1$$

ⓐ $H(X)$

$$H(X) = \sum_{x \in \{0,1\}} P_x \cdot \log_2\left(\frac{1}{P_x}\right)$$

$$\cong 0.811$$

ⓑ $H(Y)$

$$H(Y) = \sum_{y \in \{0,1\}} P_y \cdot \log_2\left(\frac{1}{P_y}\right)$$

$$= 1$$

ⓒ $H(X|Y)$

$$H(X|Y) = \sum_{x \in \{0,1\}} \sum_{y \in \{0,1\}} P_{X,Y}(x,y) \cdot \log_2\left(1/P_{X|Y}(x|y)\right)$$

$$= 1/2$$

ⓓ $H(Y|X)$

$$H(Y|X) = \sum_{Y \in \{0,1\}} \sum_{X \in \{0,1\}} P_{Y,X}(x,y) \cdot \log_2\left(1/P_{Y|X}(y|x)\right)$$

$$\cong 0{,}689$$

(e) $H(X,Y) = H(X) + H(Y|X)$

$= 0{,}811 + 0{,}689$

$= 1{,}5$

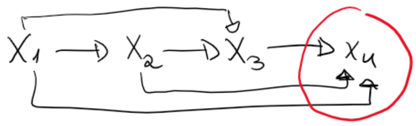
$$H(X,Y) = H(Y,X)$$

(f) $-H(X) \rightarrow 0{,}689$

(g) $-H(Y) \rightarrow 0{,}5$

problem (2)

(a)

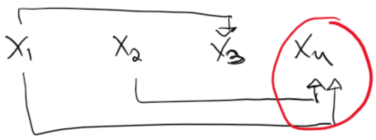


(i) $p(x_1,x_2,x_3,x_4) = p(x_1)\cdot p(x_2|x_1)\cdot p(x_3|x_1,x_2)\cdot p(x_4|x_1,x_2,x_3)$

(ii) $H(X_1,X_2,X_3,X_4) = H(X_1) + H(X_2|X_1) + H(X_3|X_1,X_2) + \underbrace{H(X_4|X_1,X_2,X_3)}$

X_4 dépend de X_1, de X_2 et de X_3

(b)



(i) $P(x_1,x_2,x_3,x_4) = p(x_1)\cdot p(x_2)\cdot p(x_3|x_1)\cdot p(x_4|x_1,x_2)$

(ii) $H(X_1,X_2,X_3,X_4) = H(X_1) + H(X_2) + H(X_3|X_1) + H(X_4|X_1,X_2)$

X_4 dépend de X_1 et X_2

problem (3)

X et Y two random variables
show that

$$H(X|Y) \leq H(X)$$

On a :

$$H(X|Y) = \underbrace{H(X,Y)}_{H(X,Y) = H(X) + H(Y)} - H(Y) = H(X) + H(Y) - H(Y) = H(X)$$

si X et Y indépendants.

$\downarrow$

entropie conditionnel max :

$\rightsquigarrow H(X|Y) \leq H(X)$

problem 4

$X_1, X_2, \dots, X_n$ random variables
show that

$$H(X_1, \dots, X_n) \leqslant \sum_{i=1}^{n} H(X_i)$$

on utilise la chain rule pour l'entropie

$$H(X_1, \dots, X_n) = \sum_{i=1}^{n} H(X_i \mid X_1, \dots, X_{i-1})$$

par problem 3

$$\sum_{i=1}^{n} H(X_i \mid X_1, \dots, X_{i-1}) \leqslant \sum_{i=1}^{n} H(X_i)$$

on conclut avec:

$$H(X_1, \dots, X_n) \leqslant \sum_{i=1}^{n} H(X_i)$$