Problem Set 4

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Problem 1

$\rho = 0,1$

U =

0.3599	0.5938	-0.6087	0.3839
-0.6087	-0.3839	-0.3599	0.5938
0.6087	-0.3839	0.3599	0.5938
-0.3599	0.5938	0.6087	0.3839

D =

0.8468	0	0	0
0	0.9299	0	0
0	0	1.0522	0
0	0	0	1.1711

K_XX =

1.0000	0.1000	0.0100	0.0010
0.1000	1.0000	0.1000	0.0100
0.0100	0.1000	1.0000	0.1000
0.0010	0.0100	0.1000	1.0000

$\rho = 0,5$

U =

0.3162	0.5573	-0.6325	0.4352
-0.6325	-0.4352	-0.3162	0.5573
0.6325	-0.4352	0.3162	0.5573
-0.3162	0.5573	0.6325	0.4352

D =

0.3750	0	0	0
0	0.5394	0	0
0	0	1.0000	0
0	0	0	2.0856

K_XX =

1.0000	0.5000	0.2500	0.1250
0.5000	1.0000	0.5000	0.2500
0.2500	0.5000	1.0000	0.5000
0.1250	0.2500	0.5000	1.0000

$\rho = 0,95$

U =

0.2747	-0.5062	0.6516	0.4937
-0.6516	0.4937	0.2747	0.5062
0.6516	0.4937	-0.2747	0.5062
-0.2747	-0.5062	-0.6516	0.4937

D =

0.0300	0	0	0
0	0.0506	0	0
0	0	0.1627	0
0	0	0	3.7568

K_XX =

1.0000	0.9500	0.9025	0.8574
0.9500	1.0000	0.9500	0.9025
0.9025	0.9500	1.0000	0.9500
0.8574	0.9025	0.9500	1.0000

- en suivant le sens de la flèche on voit que les valeurs de D sont décroissantes.

Problem 2

1) SVD is $A: \mathbb{R}^n \to \mathbb{R}^m$

we can write A as: $A = USV^T$

EVD is $A: \mathbb{R}^n \to \mathbb{R}^n$,
Let e be eigenvector of A,
$\rightsquigarrow A \cdot e = \lambda \cdot e \to$ (λ is the corresponding eigenvalue)

$\rightsquigarrow r = \text{rank}(A)$

$\rightsquigarrow$ we can make a diagonal matrix $r \times r$, of the non-zero eigenvalues, let it be Λ

and E the eigenvectors matrix $n \times r$

$\rightsquigarrow AE = E\Lambda$
$\Leftrightarrow A = E\Lambda E^{-1}$, similar to SVD.

if A is Symetric and defined Positive (SPD) then:

$$A = E\Lambda E^{-1} = USV^T$$

$U = V^T$, since A is Symetric

Given the non square matrix $A = USV^T$

then:
$$A^T A = VS^2V^T$$
$$AA^T = US^2U^T$$

$\rightsquigarrow$ the SVD of original A can be used to compute their SVD. and since they are SPD
$\Rightarrow \Lambda = S^2$

2) La décomposition en valeurs singulières:
$\rightsquigarrow$ s'applique aux matrices rectangulaires.

La décomposition en valeur propres
$\rightsquigarrow$ s'applique unique aux matrices carrés.

problem 3

.1)

$$P_X(x) = \frac{P_Z(T^{-1}(x))}{|\det J_T(x)|} = \frac{P_Z(A^{-1}x)}{|\det J_T(z)|} = \frac{P_Z(A^{-1}x)}{|\det A|}$$

$$= \frac{1}{|\det A|} \frac{1}{\sqrt{(2\pi)^N |\det K_{zz}|}} \exp\left[-\frac{1}{2} \underbrace{(A^{-1}x - \bar{z})^T}_{\begin{subarray}{l} = (A^{-1}x - \bar{z})^T \\ = (x - A\bar{z})^T A^{-1T} \end{subarray}} K_{zz}^{-1} \underbrace{(A^{-1}x - \bar{z})}_{= A^{-1}(x - A\bar{z})}\right]$$

$$= (x - A\bar{z})^T (A K_{zz} A^T)^{-1} (x - A\bar{z})$$

$$A^{-1T} K_{zz}^{-1} A^{-1} = (A K_{zz} A^T) = K_{xx}^{-1}$$

$$|\det K_{xx}| = |\det A K_{zz} A^T| = |\det A| |\det K_{zz}| |\det A^T|$$
$$= |\det A|^2 \cdot |\det K_{zz}|$$

$$\bar{x} = A\bar{z}$$

$$\boxed{P_X(x) = \frac{1}{(2\pi)^{N/2} |\det(K_{xx})|^{1/2}} \exp\left[-1/2 \, (x - \bar{x})^T K_{xx}^{-1} (x - \bar{x})\right]}$$

.2) a liniar transformation changes the covariation Matrix, when both old variances are multiplied by something other than 1. if we only add to the old covariances, then the new (after transformation) will not change.

Problem 4 see matlab file

Sur les histogrammes matlab, on voit que la distribution de W

we have mean: $E[W] = N\theta$
and variance: $Var[W] = N\theta(1-\theta)$

the weak law of large numbers is verified:

$$\frac{\text{Mean}}{\sqrt{\text{variance}}} = \frac{E[W]}{\sqrt{Var[W]}} = \frac{N\theta}{\sqrt{N\theta(1-\theta)}} = \sqrt{\frac{N\theta}{(1-\theta)}}$$

In the matlab plots we see that the distribution of W peaks at $\theta \cdot N$, which means if N grows, the mean grows too.