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Problem Set 4

Problem ①

Let X and Y be two random variables that follow a Bernoulli distribution.

$Z = X \oplus Y$

ⓐ Show that $H(Z|X) = H(Y|X)$

$$H(Z|X) = -\sum\sum p(z,x)\log_2 p(z|x)$$
$$= -\sum\sum p(x\oplus y, x)\log_2 p(x \oplus y|x) \quad \text{since } Z = X\oplus Y$$
$$= -\sum\sum 0\oplus p(y,x)\log_2 0\oplus p(y|x)$$
$$= -\sum\sum p(y,x)\log_2 (y|x) = H(Y|X)$$

ⓑ Show that if X and Y are independent, $H(Z) \geq H(X) \wedge H(Z) \geq H(Y)$

$H(Z) \geq H(Z|X)$

$\rightarrow H(Z) \geq \underbrace{H(Z|X) = H(Y|X)}_{1.a} \overset{\text{independent}}{=} H(Y)$

- we can do the same thing to find $H(Z) \geq H(X)$

ⓒ when is $H(X,Y) = H(X) + H(Y)$ true

$$H(X,Y) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x,y)$$
$$= -\sum_{x\in X}\sum_{y\in Y} p(x)p(y)\log_2 p(x)p(y)$$
$$= -\sum_{x\in X} p(x)\log_2 p(x) - \sum_{y\in Y} p(y)\log_2 p(y)$$
$$= H(X) + H(Y)$$

problem 2

- X is a random variable
- two distributions p(x) and q(x)

	p(x)	q(x)
a	1/2	1/3
b	1/4	1/3
c	1/4	1/3

- calculate $H(p)$, $H(q)$, $D(p\|q)$, $D(q\|p)$

$$H(p) = -\sum_{x\in X} p(x)\log_2 p(x) = 3/2$$

$$H(q) = -\sum_{x\in X} q(x)\log_2 q(x) \cong 1{,}58$$

$$D(p\|q) = \sum_{x\in X} p(x)\log_2 \frac{p(x)}{q(x)} \simeq 0{,}0849$$

$$D(q\|p) = \sum_{x\in X} q(x)\log_2 \frac{q(x)}{p(x)} \simeq 0{,}0817$$

- Show that $D(p\|q) \neq D(q\|p)$

$$\left.\begin{aligned} D(p\|q) &= \sum_{x\in X} p(x)\log_2 \frac{p(x)}{q(x)} \\ D(q\|p) &= \sum_{x\in X} q(x)\log_2 \frac{q(x)}{p(x)} \end{aligned}\right\} p(x) \neq q(x)$$

$\rightsquigarrow D(p\|q) \neq D(q\|p)$

problem 3

p(x) and q(x) are two distributions of X

Show that $D(p\|q) \geq 0$

$A = \{x \mid p(x) > 0\}$

$$-D(p\|q) = -\sum_{x\in X} p(x)\log_2 \frac{p(x)}{q(x)}$$

$$= \sum_{x\in A} p(x)\log_2 \frac{q(x)}{p(x)}$$

jensen $$\leq \log \sum_{x\in A} p(x)\cdot\frac{q(x)}{p(x)}$$

$$= \log \sum_{x\in A} q(x)$$

$$\leq \log \sum_{x\in X} q(x) = \log 1 = 0$$

problem 4

show that
$$\sum_{i=1}^{N} a_i \log\left(\frac{a_i}{b_i}\right) \geq \left(\sum_{i=1}^{N} a_i\right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

<u>Jensen</u> $\quad \sum \alpha_i f(t_i) \geq f\left(\sum \alpha_i t_i\right)$

$$\alpha_i \geq 0, \quad \sum_i \alpha_i = 1 \quad \leadsto \quad \alpha_i = \frac{b_i}{\sum_{j=1}^{n} b_j}, \quad t_i = \frac{a_i}{b_i}$$

$$\sum \frac{a_i}{\sum b_j} \log \frac{a_i}{b_j} \geq \sum \frac{a_i}{\sum b_j} \log \sum \frac{a_i}{\sum b_j}$$

$$\Leftrightarrow \sum_{i=1}^{n} a_i \log\left(\frac{a_i}{b_j}\right) \geq \left(\sum_{i=1}^{n} a_i\right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$