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# Problem Set 7 22/04/2020

## Problem ①

Prove that $H(X,Y|Z) \geq H(X|Z)$

$$H(X,Y|Z) = -\sum_{x\in X}\sum_{y\in Y}\sum_{z\in Z} p(x,y|z)\log p(x,y|z)$$

$$= -\sum_{x\in X}\sum_{Y\in Y}\sum_{z\in Z} p(x,y|z)\log p(x|z) - \sum_{x\in X}\sum_{Y\in Y}\sum_{z\in Z} p(x,y|z)\log p(y|x,z)$$

$$= -\sum_{x\in X}\sum_{y\in Y} p(x|z)\log p(x|z) - \sum_{x\in X}\sum_{y\in Y}\sum_{z\in Z} p(x,y|z)\log p(y|x,z)$$

$$= H(X|Z) + H(Y|X,Z)$$

$$H(X,Y|Z) = H(X|Z) + \underbrace{H(Y|X,Z)}$$

$H(X) \geq 0 \ \forall X \Rightarrow H(Y|X,Y) \geq 0$

## Problem ②

| Y \ X | 0 | 1 | 2 | |
|---|---|---|---|---|
| 0 | 1/8 | 1/8 | 0 | P(Y = 0) = 2/8 |
| 1 | 1/4 | 1/4 | 1/4 | P(Y = 1) = 6/8 |
| | P(X=0) = 3/8 | P(X=1) = 3/8 | P(X=2) = 2/8 | |

$P(X=0|Y=0) = 1/2$ $P(X=0|Y=1) = 1/3$

$P(X=1|Y=0) = 1/2$ $P(X=1|Y=1) = 1/3$

$P(X=2|Y=0) = 0$ $P(X=2|Y=1) = 1/3$

$P(Y=0|X=0) = 1/3$ $P(Y=0|X=1) = 1/3$ $P(Y=0|X=2) = 0$

$P(Y=1|X=0) = 2/3$ $P(Y=1|X=1) = 2/3$ $P(Y=1|X=2) = 1$

- $I(X;Y) = E_{p(x,y)}\left[\log_2 \frac{p(x,y)}{p(x)\cdot p(y)}\right] = \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 \frac{p(x,y)}{p(x)p(y)}$

$\cong 0{,}123$

- **a** $I(X;Y) = H(X) - H(X|Y)$

$= \underbrace{\left(-\sum_{x\in X} p(x)\log_2 p(x)\right)}_{} - \underbrace{\left(-\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x|y)\right)}_{}$

$\cong 0{,}123$

**b** $I(X;Y) = H(Y) - H(Y|X)$

$= \underbrace{\left(-\sum_{y\in Y} p(y)\log_2 p(y)\right)}_{} - \underbrace{\left(-\sum_{y\in Y}\sum_{x\in X} p(y,x)\log_2 p(y|x)\right)}_{}$

$\cong 0{,}123$

## Problem 3

... 3.m file

## Problem 4

**.1** $P_x = \{p, 1-p\}$ et $Q_x = \{q, 1-p\}$ two distributions

$D(P_x \| Q_x) = d(p,q) = p \log \frac{p}{q} + (1-p)\log\frac{1-p}{1-q}$

(i) show that $d(p,q) = 0 \iff p = q$

$d(p,q) = \sum_{x\in[0,1]} P_x \log\frac{P_x}{Q_x} \geq \left(\sum_{x\in[0,1]} P_x\right)\log\left(\sum_{x\in[0,1]} P_x \Big/ \sum_{x\in[0,1]} Q_x\right)$

$= 1 \log \frac{1}{1} = 0$

$\Rightarrow$ Cette égalité est vérifié que si $p = q$

(ii) show that $p = 0 \Rightarrow d(p,q) = -\log(1-q)$

$d(0,q) = \underbrace{0 \log \frac{0}{q}}_{=0} + \underbrace{(1-0)\log\frac{1-0}{1-q}}_{1\ \log\left(\frac{1}{1-q}\right)} = -\log(1-q)$

(iii) Show that $p=1 \Rightarrow d(p,q) = -\log(q)$

$$d(1,q) = \underbrace{1 \log \frac{1}{q}}_{= \log\left(\frac{1}{q}\right)} + \underbrace{(1-1) \log\left(\frac{1-1}{1-q}\right)}_{=0}$$

$$= -\log(q)$$

.2 ...4.m file

.3 ...4.m file

Problem 5

Show that $\min_{q(x)} H(p(x), q(x)) = \min_{q(x)} D_{KL}(p(x) \| q(x))$

$$D_{KL}(p(x) \| q(x)) = E_{p(x)}\left[\log \frac{p(x)}{q(x)}\right] = E_{p(x)}\left[\log p(x) - \log q(x)\right]$$

$$= E_{p(x)}\left[\log p(x)\right] - E_{p(x)}\left[\log q(x)\right]$$

$$= \underbrace{-H(p(x))}_{\text{constante}} + H(p(x), q(x))$$

$$\Rightarrow \min_{q(x)} H(p(x), q(x)) = \min_{q(x)} D_{KL}(p(x) \| q(x))$$