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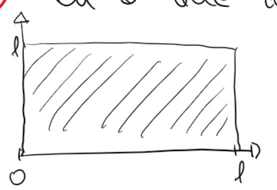
Problem Set 2

problem 1

1.a on a une loi uniforme



Qu'on veut transformer en Normale ou laplace

- Soit $F_x(x)$ ⇒ la CDF de Distribution Normale ou laplace

but : trouver transformation $T$

tq $\underbrace{T(U)}_{\text{uniform}} = X$ → Normale ou laplace

$F_x(x) = P_r(X \leq x) = P_r(T(U) \leq x)$ | $T(U) = X$

$= P_r(T^{-1}(T(U)) \leq T^{-1}(x))$ | on applique inverse

$= Pr(U \leq T^{-1}(x))$ | $T^{-1}(T(U)) = U$

$= T^{-1}(x)$ | du à la nature de la distribution uniforme

⚠ on a prouvé que

$F_x(x) = T^{-1}(x)$

qui peut être réécrit comme

$\boxed{F^{-1}_x(x) = T(x)}$ ⇝ la transformation $T$ = inverse de la CDF

→ Vrai pour $\underbrace{\text{laplace et normale}}_{\text{et toutes les autres}}$ ⚠

## Uniform → Laplace

CDF → $F(x) = \int_{-\infty}^{x} f(u)\,du = \begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right) & , \text{si } x < \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right), & \text{si } x \geq \mu \end{cases}$

$$= \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(x-\mu)\left(1 - \exp\left(-\frac{|x-\mu|}{b}\right)\right)$$

l'inversi de $F(x)$:

$$F^{-1}(p) = \mu - b\,\mathrm{sgn}(p-0{,}5)\ln\left(1 - 2|p-0{,}5|\right)$$
$$= T(U)$$

## Uniform → Gauss

CDF → $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$

l'inverse de $F(x)$:

$$F^{-1}(p) = \mu + \sigma\,\Phi^{-1}(p) = \mu + \sigma\sqrt{2}\,\mathrm{erf}^{-1}(2p-1),$$
$$= T(p) \qquad p \in (0,1) \checkmark$$
$$\sim T(U) \qquad (p \in U)$$

Gauss → laplace

$$\mathcal{L}[f(x)](s) = \int_0^\infty f(x)\, e^{-sx}\, dx$$

$$f(x) = e^{-\pi x^2}$$

$$\mathcal{L}[e^{-\pi x^2}](s) = \int_0^\infty e^{-\pi x^2} e^{-sx}\, dx$$

$$= \int_0^\infty e^{-\pi[(x+s/2)^2 - \frac{s^2}{4}]}\, dx$$

$$= e^{\frac{\pi s^2}{4}} \cdot \int_0^\infty e^{-\pi(x+s/2)^2}\, dx$$

$$= e^{\frac{\pi s^2}{4}} \cdot \int_{s/2}^\infty e^{-\pi u^2}\, du \quad , u = x + s/2$$

$$= e^{\frac{\pi s^2}{4}} \Big[ \underbrace{\int_0^\infty e^{-\pi u^2} du}_{=\frac{1}{2}} - \underbrace{\int_0^{s/2} e^{-\pi u^2} du}_{} \Big]$$

$$\to = \frac{1}{\sqrt{\pi}} \int_0^{\frac{s\sqrt{\pi}}{2}} e^{-t^2}\, dt$$

$$= \frac{1}{2} \operatorname{erf}\left(\frac{s\sqrt{\pi}}{2}\right)$$

et $\operatorname{erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-t^2}\, dt$

$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$

T →

$$\mathcal{L}[e^{-\pi x^2}](s) = e^{\frac{\pi s^2}{4}} \left[ \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{s\sqrt{\pi}}{2}\right) \right]$$

$$= \frac{1}{2} e^{\frac{\pi s^2}{4}} \operatorname{erfc}\left(\frac{s \cdot \sqrt{\pi}}{2}\right)$$

T de gauss → laplace

## Problem (3)

(a) Soit $X$ une V.a. positive et $t>0$

$\leadsto \Pr\{X \geq t\} \leq \frac{E[X]}{t}$

Proof:

$$E[X] = \int_0^\infty x f_X(x)\,dx = \underbrace{\int_0^t x f_X(x)\,dx}_{= \Pr\{0 \leq X \leq t\}} + \underbrace{\int_t^\infty x f_X(x)\,dx}_{= \Pr\{X \geq t\}}$$

$$\geq \int_t^\infty x f_X(x)\,dx \geq t \int_t^\infty f_X(x)\,dx = t \cdot \Pr\{X \geq t\}$$

true since we leave $\int_0^t x f_X(x)dx$ out

$$\boxed{\int_t^\infty f_X(x)\,dx = \Pr\{X \geq t\}}$$

$$E[X] \geq t \cdot \Pr\{X \geq t\} = \frac{E[X]}{t} \geq \Pr\{X \geq t\} \quad \boxtimes$$

## exemple

Soit la V.a. non-neg $X$, avec $\Pr\{X \geq a\} = 1$
$\Pr\{X < a\} = 0$

$$E[X] = a \Pr\{X \geq a\} = a \cdot 1 = a \Rightarrow \Pr\{X \geq a\} = 1 = a/a = \frac{E[X]}{a}$$

(b) Soit $X$ une v.a. à valeurs dans $\mathbb{R}$, $t > 0$

hypothèse $E[X^2] < \infty \rightsquigarrow \forall \varepsilon > 0$, $P(|X - E[X]| > t) \leq \frac{Var(X)}{t^2}$

Soit $E[X] = m$

$$P(|X - m| \geq t) \leq \frac{\sigma^2}{t^2} = \frac{Var(X)}{t^2}$$

<u>Proof</u>: $\Pr\{|X-m| \geq t\} = \Pr\{|X-m|^2 \geq t^2$

$Y = |X - m|^2 \rightsquigarrow Y \geq 0$, $E[Y] = Var(X)$

on utilise Markov:

$$\Pr\{|X-m| \geq t\} = \Pr\{|X-m|^2 \geq t^2\} \leq \frac{Var(X)}{t^2}$$

ⓒ Soit $z_1, \dots, z_n$ $n$ v.a. independentes

mean $= m$
var $= \sigma^2$

$\bar{z}_n = \frac{1}{n}\sum_{i=1}^{n} z_i$

mq: $\Pr\{|\bar{z}_n - m| \geq \varepsilon\} \leq \frac{\sigma^2}{n e^2}$

$\Rightarrow \Pr\{|\bar{z}_n - m| \geq \varepsilon\} \to 0$, si $n \to \infty$

$\bar{z}_n = \frac{z_1 + z_2 + \cdots z_n}{n}$

$m = E[X]$

independentes

$$\Pr\left\{\left(\frac{X_1 + X_2 + \cdots + X_n}{n} - E[X]\right) \geq \varepsilon\right\} \leq \frac{\mathrm{Var}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right)}{\varepsilon^2}$$

$$= \frac{1}{n^2 \varepsilon^2} \mathrm{Var}(X_1 + X_2 + \cdots + X_n)$$

$$= \frac{1}{n^2 \varepsilon^2} \left(\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)\right)$$

$$= \frac{\mathrm{Var}(X)}{n \varepsilon^2} = \frac{\sigma^2}{n \varepsilon^2}$$