Problem ②

upper bound

mq: $\Pr\{Z \geq t\} \leq \frac{1}{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

$$\Pr\{Z \geq t\} = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2}\, dx$$

$$\left.\begin{array}{l} \frac{x}{t} > 1 \\ \forall x \in (t, \infty) \end{array}\right\} \leq \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{x}{t}\, e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{t}\, e^{-t^2/2}$$

lower bound

Soit $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{t^2/2}$, qu'on intègre par partie

$$\frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2}\, dx = \int_t^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx =$$

$$= \int_t^{\infty} \frac{1}{x} \frac{1}{\sqrt{2\pi}}\, x e^{-x^2/2}\, dx =$$

$$= -\frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big|_t^{\infty} - \int_t^{\infty} \left(-\frac{1}{x^2}\right)\left(-\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right) dx =$$

$$= \frac{\phi(t)}{t} - \int_t^{\infty} \frac{\phi(x)}{x^2}\, dx$$

on intègre une deuxième fois:

...

$$\leadsto \quad \frac{\phi(t)}{t} - \left( \frac{\phi(t)}{t^3} - \int_t^{\infty} \frac{\phi(x)}{x^2}\, dx \right) =$$

$$= \frac{\phi(t)}{t} - \frac{\phi(t)}{t^3} + \int_t^{\infty} \frac{3\,\phi(x)}{x^4}\, dx$$

Vu que $\left( \int_t^{\infty} \frac{3\,\phi(x)}{x^4}\, dx \geq 0 \right)$ alors :

$$\frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2}\, dx \geq \frac{\phi(t)}{t} - \frac{\phi(t)}{t^3} = \left( \frac{1}{t} - \frac{1}{t^3} \right) \phi(t)$$

$$= \left( \frac{1}{t} - \frac{1}{t^3} \right) \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}$$

$\square$

Donc

$$P_r\{Z \geq t\} \geq \left( \frac{1}{t} - \frac{1}{t^3} \right) \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}$$

# Problem ①

$p = \frac{1}{2}$, $\leadsto$ $\sigma^2 = N\frac{1}{2}\left(-\frac{1}{2}\right) \wedge \mu_N = \frac{1}{2}N$

L'inéquation de Chebyshev nous donne :

$$Pr\left\{\left|\frac{S_N}{N} - \frac{1}{2}N\right| \geq \frac{3}{4}N\right\} \leq \frac{N\frac{1}{2}\left(-\frac{1}{2}\right)}{N\left(\frac{3}{4}N\right)^2} = \frac{4}{9N^2}$$