UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES
Département d'informatique

# Université de Genève

Département d'informatique

## Imagerie Numérique

FS and FT

TP Class N° 8

March 10, 2020

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**Notes**: The details of the algebraic calculations must be included in the report. If you do not use an Equation Editor (for Word, Latex or similar) you can easily scan a handwritten page (readable!) and put it in the report.

**Exercise 1. Fourier series expansion** *(2 points)*

We consider the following function (square function):

$$\tilde{f}(t) = \begin{cases} 1 & \text{if } t \in [-1, 1] \\ -1 & \text{if } t \in [1, 3] \end{cases}$$

Repeated periodically on all $\mathbb{R}$.

(a) Compute the periode $T$ and the frequency $w_0$ and show, by computing the integral, that

$$a_k = \begin{cases} (-1)^{(k-1)/2} \cdot \frac{4}{k\pi} & \text{if k is odd} \\ 0 & \text{if k is even} \end{cases}$$

**Hint :** Use the formula on page 42 but choose the interval of integration wisely to cut the integral into two parts. You may also need trigonometric formulas from Theme 7 page 30.

(b) Explain why $b_k = 0$ and compute $F_k$. Write the function $\tilde{f}(t)$ both in sinusoidal form and exponential form.

(c) We write

$$f_N(t) = \frac{1}{2}a_0 + \sum_{k=1}^{N} a_k cos(k\omega_0 t) + b_k sin(k\omega_0 t).$$

Implement the functions $\tilde{f}(t)$ and $f_N(t)$ in Python and plot them on the same graph for $t \in [-5, 5]$. Do several plots for $N = 0, 1, 3, 5, 10, 100$.

(d) Do a small research and explain what is the Gibbs phenomenon.

**Exercise 2. Fourier series and Fourier transform** *(2 points)*

We consider the following "hat" function :

$$h(t) = \begin{cases} 1 - |t| & \text{if } |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute its Fourier transform $\hat{h}(w)$, using integration by part. Why is it a real-valued function ? What is the value $\hat{h}(0)$ ?

**Hint :** Start by decomposing $e^{-itw} = cos(tw) - i \cdot sin(wt)$ in the integral and use symmetry of the functions to show that,

$$\hat{h}(w) = 2 \cdot \int_0^1 (1 - t)\cos(wt)dt$$

(b) Fix a real number $T \geq 2$, and consider $h|_{[-T/2,T/2]}$. Define $h_T$ as the periodic extension of this function on all $\mathbb{R}$.
On separate graphs, visualize $h(t)$, $h_2(t)$, $h_4(t)$ and $h_8(t)$ for $t \in [-10, 10]$.

(c) Using a result seen in the course, compute the Fourier coefficients $F_k$ for $k \in \mathbb{Z}$ from the Fourier transform. Note that your result will depend on the period T.

(d) Plot on the same graph the function $\hat{h}(w)$ and the points $(\frac{2\pi k}{T}, T \cdot F_k)$ for

$$|k| \leq 3T \text{ and } w \in [-20, 20].$$

Do a different graph for each choice of $T = 2, 4, 8, 50, 100$.
Interprete the result based on the theory seen in class.

**Exercise 3. Fourier Transform of Gaussian signal** *(2 points)*
Using complex analysis, one could show that that the Fourier transform of a Gaussian function is again a Gaussian function. Namely :

$$G(t) = e^{-\alpha t^2} \implies \hat{G}(w) = \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{w^2}{4\alpha}}$$

for any real number $\alpha > 0$.

(a) Represent both functions side-by-side on two different plots for $\alpha = 0.1, 1, 2, 10$. Explain the scaling effect that you observe, based on the property 7 of the FT (Theme 7 : p.69).

(b) The energy of a function $f(t)$ is given by

$$E(f) = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Using your knowledge of Gaussian integrals, compute the energy of both $G(t)$ and $\hat{G}(w)$ for a real $\alpha > 0$. Show that $E(G) = \frac{1}{2\pi} E(\hat{G})$.
**Hint :** Starting from $E(G)$ and $E(\hat{G})$, do a change of variable and use the equality :

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

(c) Write a python function *represent_complex_signal*$(x, y)$, where

- x is a real numpy array representing an interval range in $\mathbb{R}$.
- y is a complex numpy array defined by $y = f(x)$.

The function should plot four different representations of the pairs $(x, y)$. Namely :

- Real part : use *plt.plot()* on the pair $(x,$ *np.real*$(y))$
- Imaginary part : use *plt.plot()* on the pair $(x,$ *np.imag*$(y))$
- Magnitude : use *plt.plot()* with $(x,$ *np.abs*$(y))$
- Phase : use *plt.plot()* with $(x,$ *np.angle*$(y))$

(d) For the rest of the exercise, we fix $\alpha = 1$. We define $H(t) = G(t-1)$.

- Compute explicitely $\hat{H}(w)$ using the time shift property of the FT.
- Represent both $\hat{G}(w)$ and $\hat{H}(w)$ using the function you implemented in part (c). Compare similarities and differences and explain them based on the course.

(e) We define $I(t) = H(-t)$.

- Compute explicitely $\hat{I}(w)$ using the correct time-reversal property of the FT.
- Represent both $\hat{I}(w)$ and $\hat{H}(w)$ using part (c). Compare them and explain the differences based on the course.

## Submission

Please archive your report and codes in "Name_Surname.zip" (replace "Name" and "Surname" with your real name), and upload to "`Assignments/TP7 Fourier transform`" on `https://moodle.unige.ch` before Thursday, March 25 2021, 23:59 PM. Note, **the assessment is based not only on your code, but also on your report, which should include your answers to all questions and the experimental results**.