Data Science

Linear Algebra

Basic and practical reminder

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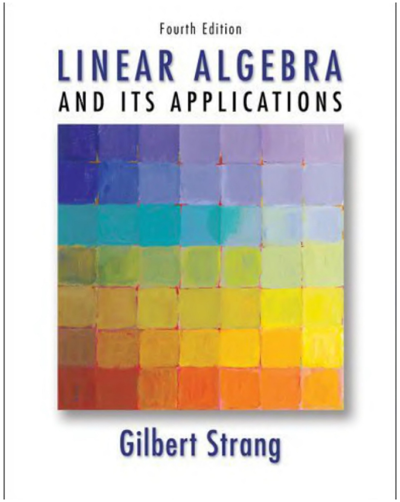
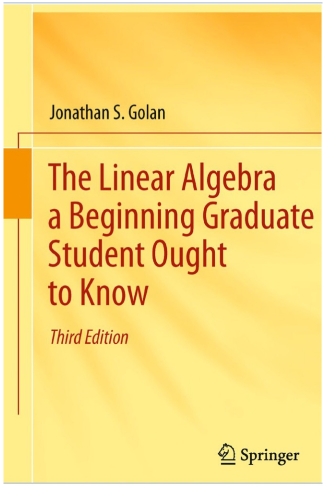
Linear Algebra

- Representation spaces
- Applied Linear Algebra
- Matrix Calculus





Academic Linear Algebra

- A vector space is a space of vectors with linear operations

$u, v \in V \qquad \alpha u + \beta v \in V \quad \alpha, \beta \in K$

Linear mapping $\quad f: V \to V$

$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$

- A vector space can be equipped with a inner product, resulting in a pre-Hilbert space

$u, v \in V \qquad \langle u, v \rangle \in K$

$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$

$\langle u, u \rangle \geq 0 \qquad \langle u, v \rangle = 0 \Rightarrow u \perp v$

Implications

→ Main implication: inner product vs projection

$$\mathrm{Proj}_u(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle} u$$

→ Main implication: coordinates in Shauder basis

$$u \longrightarrow \begin{bmatrix} \langle u, e_1\rangle \\ \langle u, e_2\rangle \\ \vdots \\ \langle u, e_n\rangle \end{bmatrix} \quad \text{in basis } (e_1 \dots e_n) \text{ s.t. } \langle e_i, e_j\rangle = \delta_i^j$$

→ Main implication: representations spaces

- → Vectors → elements
- → Matrices → mapping
- → Matrix calculus

Example

Given a process *f* that transform my input (x,y,z) into (x′,y′,z′) such that:

$$x' = -55x + 11y + 45z$$
$$y' = 130x - 26y - 103z$$
$$z' = -100x + 20y + 81z$$

$v_1 = (x_1, y_1, z_1)$ $v_2 = (x_2, y_2, z_2)$

$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$

Is *f* linear?

What if I apply *f* twice ($f \circ f$) ?

What if I apply *f* 99 times (f^{99})?

What if I apply *f* 100 times (f^{100})?

Apply *f* twice

Search and replace:

$$x'' = -55x' + 11y' + 45z' \qquad = -45x + 9y + 37z$$

$$y'' = 130x' - 26y' - 103z' \qquad = -230x + 46y + 185z$$

$$z'' = -100x' + 20y' + 81z' \qquad = z$$

And continue?

More elegant solution?

f: $R^3 \rightarrow R^3$ represented by matrix

$$A = \begin{bmatrix} -55 & 11 & 45 \\ 130 & -26 & -103 \\ -100 & 20 & 81 \end{bmatrix}$$

And $f \circ f$

$$A^2 = \begin{bmatrix} -45 & 9 & 37 \\ -230 & 46 & 185 \\ 0 & 0 & 1 \end{bmatrix}$$

So let's compute A^{99}

$$A^2 = \begin{bmatrix} -45 & 9 & 37 \\ -230 & 46 & 185 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^3 = \text{😥}$$

Can we rather use Linear Algebra??

Look at A

$$A = \begin{array}{cc} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} & \\ \begin{bmatrix} -55 & 11 & 45 \\ 130 & -26 & -103 \\ -100 & 20 & 81 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$$

What does the facts that

$c1 = -5\ c2$

$c1 + c2 + c3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

tell us? Anything else?

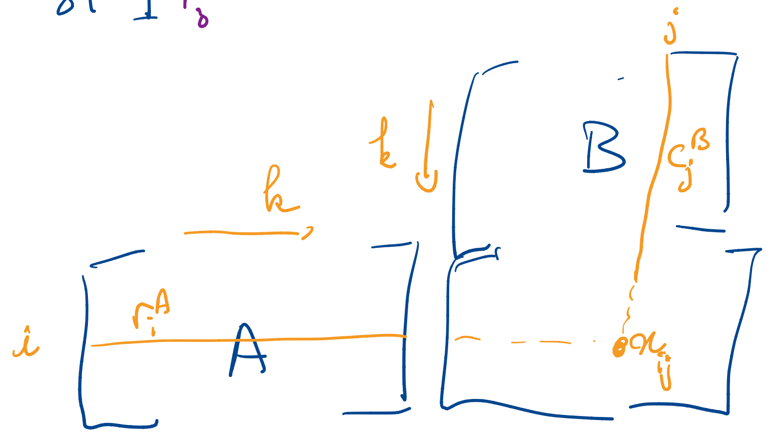
So what is A^{99}? And what is A^{100}?

$$A = \begin{array}{c|c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ & \left[\begin{array}{ccc} -55 & 11 & 45 \\ 130 & -26 & -103 \\ -100 & 20 & 81 \end{array}\right] \end{array} \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array}$$

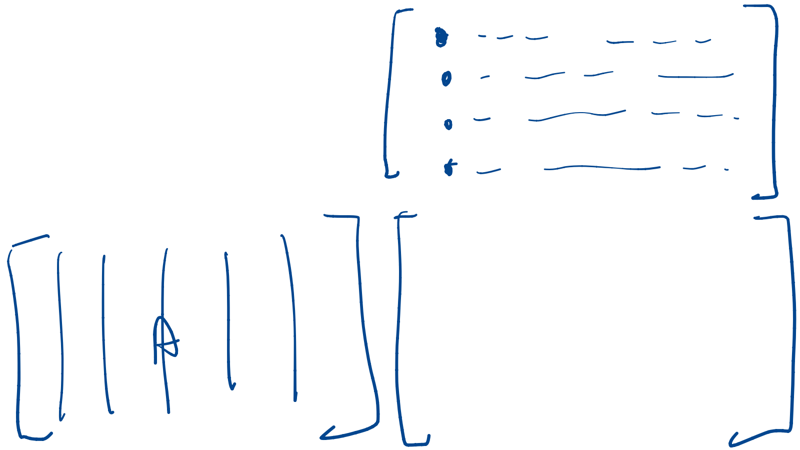
$$X = AB \rightarrow x_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$r_i^A = c_i^{A^T} \qquad c_j^B$$

$$r_i^A = c_i^{A^T}$$



$$Av = \begin{bmatrix} \langle c_1^{A^T}, v \rangle \\ \vdots \\ \langle c_n^{A^T}, v \rangle \end{bmatrix} \Rightarrow A.B = \begin{bmatrix} & & \\ & \langle c_i^{A^T}, c_j^B \rangle & \\ & & \end{bmatrix}$$

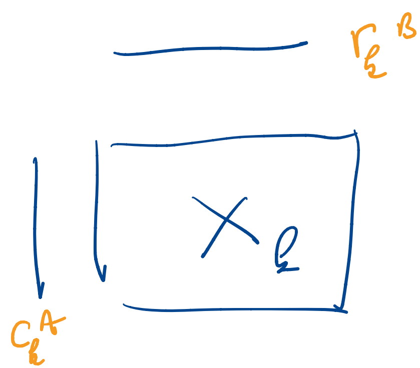


$$a_{ij} = \sum_k a_{ik} b_{kj}$$

$$X = \sum_k X_k$$

$$X_k = a_{\cdot k} \, b_{k \cdot}$$

c_k^A $\quad$ $r_k^B = c_k^{B^T}$



$$A v = \left[\sum_{k=1}^{n} c_k^A \cdot v_k \right] \Rightarrow A \cdot B = \left[\sum_k \overbrace{c_k^A \cdot c_k^{B^T}}^{X_k} \right]$$

"Column-row" view of A (Strang)

$$A = \begin{array}{cc} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} & \\ \begin{bmatrix} -55 & 11 & 45 \\ 130 & -26 & -103 \\ -100 & 20 & 81 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$$

$$\text{rank } A = 2 \quad \left\{ \begin{bmatrix} -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$$

$$A = \begin{bmatrix} 11 & 45 \\ -26 & -103 \\ 20 & 81 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

The Fundamental Theorem of Linear Algebra

Gilbert Strang

$$\dim R(A) = \dim R(A^T) \quad \text{and} \quad \dim R(A) + \dim N(A) = n.$$

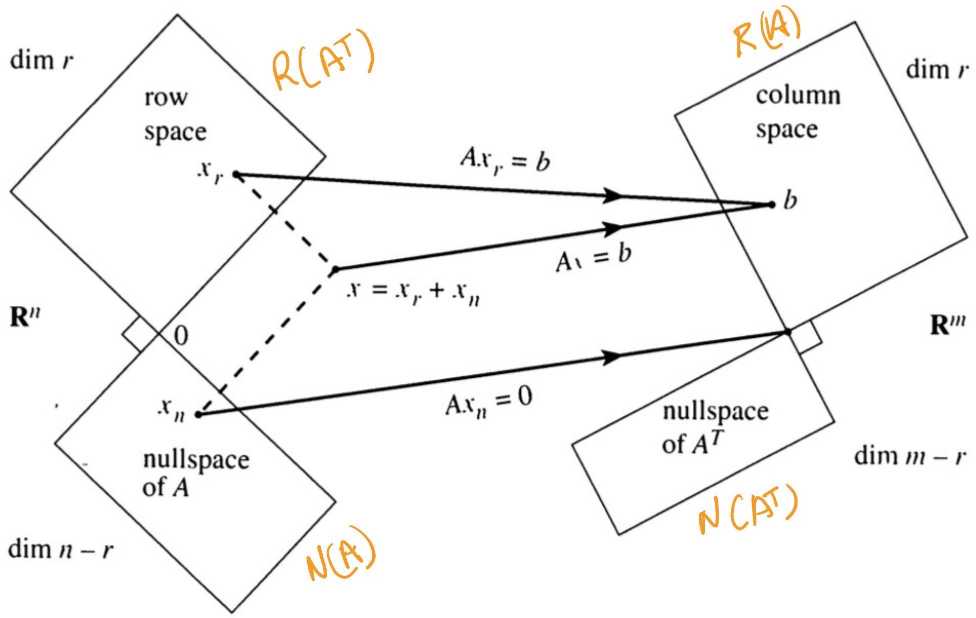


Figure 1. The action of A: Row space to column space, nullspace to zero.

What did we see?

- f can be represented by a matrix A $\rightarrow$ Base change
- A has rank 2 (since $c_2 = -5 c_1$ or $\lambda_2 = 0$)
- U is a orthogonal matrix
- The Trace of A is invariant (so is the Det)
- A has a "eigenvalue" and a "eigenvector"

Ask yourselves:

- Why the above? How can I use that?
- What is the Range of a linear application?
- What is the Kernel of a linear application?
- What are their dimensions?

Any Question? Anything (un) clear?