Analyse et Traitement de l'Information

TP4: Probabilities and communications.

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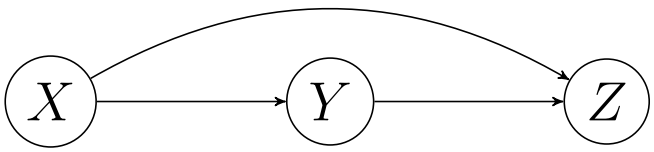
1 Chain rules for probabilities

For each of the 3 following chains of information, give the joint probability using the chain rule:

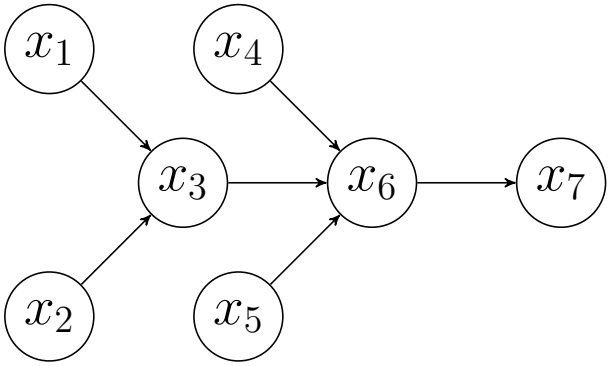
1.

$$p(X, Y, Z) = \ldots?$$
$$p(X, Z) = \ldots?$$
$$p(X, Y) = \ldots?$$
$$p(Y, Z) = \ldots?$$



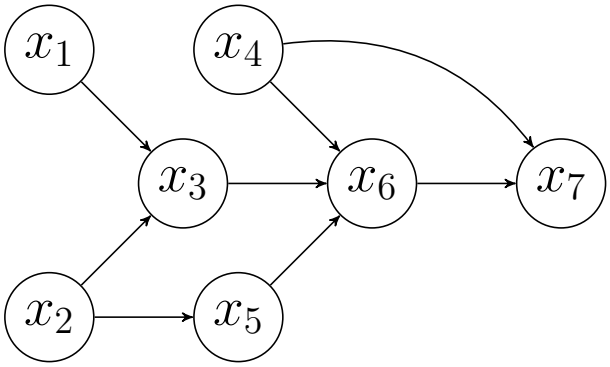
2.

$$p(X_1, X_2, \ldots, X_7) = \ldots?$$
$$p(X_1, X_3, X_5, X_7) = \ldots?$$
$$p(X_2, X_4, X_6, X_7) = \ldots?$$
$$p(X_3, X_6, X_7) = \ldots?$$
$$p(X_1, X_2, X_4, X_5) = \ldots?$$



3.

$$p(X_1, X_2, \ldots, X_7) = \ldots?$$
$$p(X_1, X_3, X_5, X_7) = \ldots?$$
$$p(X_2, X_4, X_6, X_7) = \ldots?$$
$$p(X_3, X_6, X_7) = \ldots?$$
$$p(X_1, X_2, X_4, X_5) = \ldots?$$



2 Communication through noisy channels

In this part we are investigating the problem of data transmission through noisy channels and try to experiment with a very simple channel coding method to combat the errors.

Suppose you have an alphabet of 32 words and you want to communicate a message based on them, i.e. your (random) message consists of the concatenation of these words. Your communication channel is binary, so what you send to the channel as the representative of each word of the alphabet should be a binary sequence as well, i.e. they should consist only of '0's and '1's.

Because you want your communication to be as fast as possible, you want to have your binary sequences for each word as short as possible. As a natural way to choose these binary sequences, assign a 5-bit representative for each of the 32 words in the alphabet and keep them in a database (*in real-world scenarios more advanced coding is used, like Huffman coding*). Therefore, you will have necessarily all possible combinations of '0's and '1' in your database.

As the medium of communication, you have a **Binary Symmetric Channel (BSC)** with probability of bit error equal to 0.1 (BSC, $P_{be} = 0.1$). This means that every bit which is fed to this channel will be flipped in the output to its opposite

due to noise, with probability 0.1. *You can view a simple implementation of BSC here.*

1. Suppose you have a string of 1000 words that you want to send to your destination through this channel (you should use the same string for all experiments). Generate the 5-bit binary representations of these words so that you will have a string of 5000 digits and see the output of the BSC. Convert the output binary string to index numbers. Calculate the fraction of erroneous output words to the total number of communicated words (1000) by comparing the output string to the input string.

2. Redo the above experiment this time by generating binary representations of the words by repeating each bit 3 times, so that you will have 15 bit representations for each word. As an example, suppose the binary representation of the 27th item in the database was '11010' in the previous part. Now by repeating each bit 3 times, this 27th item will be encoded as '111111000111000'. Send a string of 1000 words to the channel (a binary string of length 15000).

   In the output of the channel, divide the received binary sequence to 5000, 3-bit long pieces and for each of these triples perform a majority vote between them. This means that if your received triple is '101', you should decode it as '1', because you have two '1's and one '0'.

   Based on this encoding-decoding scheme, calculate the fraction of erroneous words to the total.

3. Redo the previous part this time by repeating 5, 11 and 23 times and derive the error fraction for each. Your code should have the ability to be easily generalized to different values of repetitions.

4. Calculate the elapsed time for transmission the whole message. *Tip: you could use* `timeit` *magic command or see this example about using* `timeit` *module in python code.*

5. In Information Theory, to quantify how fast a communication is, we define rate of communication as:

$$R = \frac{\log_2 \mathbb{M}}{n}, (0 \leq R \leq 1)$$

   where $\mathbb{M}$ is the size of the alphabet and n is the length of the binary description of the words in the alphabet.

Calculate the rate of communication for each of the experiments above and compare it with the corresponding error fraction. Try to explain what happens when we increase the number of repetitions.

The above scenarios, follow essentially the general idea of **Channel Coding**, a very important branch of Information Theory with lots of achievements.

3 Moments

When X is a continuous real random variable of density f_X , one can define the moment-generating function M_X as:

$$M_X : t \in \mathbb{R} \mapsto \mathbb{E}_{f_X}[e^{Xt}] = \int_{x \in \mathbb{R}} f_X(x) e^{xt} dx$$

It is proven that the characteristic function defines the distribution. Moreover, when the moment-generating function M_X is integrable, we can prove that:

$$f_X : x \in \mathbb{R} \mapsto \frac{1}{2\pi} \mathbb{E}_{\phi_X}[e^{-iXt}] = \frac{1}{2\pi} \int_{t \in \mathbb{R}} M_X(it) e^{-ixt} dt$$

using the relation between moment-generating and characteristic functions: $\varphi_X(t) = M_{iX}(t) = M_X(it)$.

The moment-generating function M_X also allows us to define the moments of X from its successive derivative values when they exist for $t = 0$. For any integer $n \geq 1$, the n-th moment of X is proven to be:

$$\mu_n = \mathbb{E}_{f_X}[X^N] = M_X^{(n)}(0) = \frac{\sum_{k=1}^{N_{\text{samples}}} x_k^n}{N_{\text{samples}}} \tag{1}$$

when N_{samples} of the distribution are given. As a consequence, the moment-generating function M_X can be written as the following Taylor series:

$$M_X(t) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} t^n$$

and can be approximated using the N first moments by

$$M_X(t) = \sum_{n=0}^{N} \frac{\mu_n}{n!} t^n + o(t^N) \tag{2}$$

Questions

With `np.load('jonction.npz')['data']`, load the grayscaled image and make it a flattened vector X. Generate a vector Y with the same dimension as X, which values are randomly sampled from the gaussian distribution of mean 0 and variance 1. For each of these two data, perform the following tasks:

1. Rescale the data to be centered and between -5 and 5:

$$X \longleftarrow 5 \times \frac{\text{float}(X) - \text{mean}(X)}{\max\{|\,\text{float}(X) - \text{mean}(X)|\}} \qquad Y \longleftarrow Y[Y \in [-5; 5]]$$

2. Display and save the **normalized** histogram of the data (to form a probability density).

3. Define and plot the moment-generating functions and approximated by:

$$M_X : t \in [-5; 5] \mapsto \sum_{j=1}^{N_j - 1} f_X(x_j) e^{x_j t}(x_{j+1} - x_j)$$

   where $x_j, f_X(x_j)_{j=1}^{N_j}$ are the values given by the histogram, and:

$$M_Y : t \in [-5; 5] \mapsto \int_{x \in \mathbb{R}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} e^{xt} dx$$

   you can use `scipy.integrate.quad` function to compute the integral.

4. Find the 7 first moments ($n = 0 \dots 6$) of these centered distributions given in Equation 1.

5. For each $N = 0 \dots 6$ define and plot the approximation of the moment-generating function $\phi_X(\cdot)$ as defined in Equation 2. You may need to plot them on a restrained domain such as $[-1; 1]$ because of the errors of approximations far from 0.

6. Legend your figure accordingly.

Please describe your plot and comment your results (*simple pasting a figure is not enough*).

Submission

Please archive your report and codes in "Prénom Nom.zip" (replace "Prénom" and "Nom" with your real name), and upload to "`Upload Tps > Upload TP4`" on `https://moodle.unige.ch` before Monday, November 28 2022, 23:59 PM. Note, the assessment is mainly based on your report, which should include your answers to all questions and the experimental results. *Importance is given on the mathematical explanations of your works and your codes should be commented*

Please use the template from the first TP.