Brazilian Journal of Probability and Statistics
2025, Vol. 39, No. 2, 272–286
https://doi.org/10.1214/25-BJPS635
© Brazilian Statistical Association, 2025

Simulating simple random walks with a deck of cards

Raphael Alves[1,a], Sabrina Estácio[2,b], Susana Frómeta[3,c], Milton Jara[4,d], Rodrigo Marinho[5,e,f], Luiz F. S. Marques[6,g] and João V. A. Pimenta[7,h]

[1]*University of São Paulo, Ribeirão Preto, Brazil,* [a]*raphael.alves.duarte@usp.br*
[2]*Federal University of Minas Gerais, Belo Horizonte, Brazil,* [b]*estacio@ufmg.br*
[3]*Federal University of Rio Grande do Sul, Porto Alegre, Brazil,* [c]*susana.frometa@ufrgs.br*
[4]*Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil,* [d]*mjara@impa.br*
[5]*Federal University of Santa Maria, Santa Maria, Brazil,* [e]*rodrigo.marinho@ufsm.br,*
[f]*https://marinhor.weebly.com/*
[6]*Federal University of Paraíba, João Pessoa, Brazil,* [g]*lfsmarquess@gmail.com*
[7]*University of São Paulo, São Carlos, Brazil,* [h]*joaovictorpimenta@usp.br*

Abstract. When we want to simulate the realization of a symmetric simple random walk on $\mathbb{Z}^d$, we use $(2d)$-side fair dice to decide to which neighbor it jumps at each step if $d \geq 2$ or we simply use a fair coin when $d = 1$. Assume that instead of using a dice or a coin we want to do a simulation using a well shuffled deck with K cards of each of the $2d$ suits. In the first step the probability of jumping to each neighbor is $(2d)^{-1}$, but from the second step it becomes biased. Of course if we continue performing this simulation, the total variation distance between its law and the law of the random walk will increase until all cards are used. In this paper we investigate the minimum number of cards $N = 2dK$ that a deck must contain so that the total variation distance between the law of a n-step simulation and the law of a n-step realization of the random walk is smaller than a chosen threshold $\varepsilon \in (0, 1)$. More generally, we prove that when $N = cn$ this distance converges, as $n \to \infty$, to a Gaussian profile which depends on $c \geq 2d$. Furthermore, our analysis shows that this Gaussian profile vanishes as $c \to \infty$, proving the convergence of a multivariate hypergeometric distribution to a multinomial distribution in total variation.

1 Introduction

The discrete time symmetric simple random walk (SSRW) on $\mathbb{Z}^d$ is one of the simplest examples of discrete time Markov chains: when at a vertex $x \in \mathbb{Z}^d$ it jumps to one of its $(2d)$-equally-likely neighbors. Despite its simplicity, there are several studies involving random walks in biology, financial modeling, statistical physics and other sciences.

It is very common to use the SSRW on $\mathbb{Z}$, starting at the origin, to illustrate the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT). The idea is that the position of the SSRW after n steps can be seen as the sum of n i.i.d. random variables taking values in $\{-1, 1\}$, and therefore, the average of this random variables converges almost surely to zero and the distribution of the aforementioned sum converges to a Gaussian distribution. Since we can associate i.i.d. random variables taking values in $\{-1, 1\}$ with independent fair coin tosses, this example can be easily explained to college students through a practical experiment, as well as Buffon's Needle Experiment, see Ramaley (1969). For the SSRW on $\mathbb{Z}^d$ one can adapt this experiment using d fair coins at each step, which would not be very practical if d is too large, or a $(2d)$-sided dice.

In this work, we are interested in a practical experiment to simulate the SSWR on $\mathbb{Z}^d$, but using a well shuffled deck of cards instead of coins or dices. If $d = 1$, we can color half of the

---

Key words and phrases. Cards, mixing times, simple random walks, total variation limit profile.
Received November 2024; accepted August 2025.

cards in black and the other half in red and, after mixing the deck, we remove the cards—one by one—and update the process according to the color of the card. More precisely, if the card is black then the process jumps to the right, otherwise it jumps to the left. When the first card is removed, the probability of jumping to each direction is 1/2, but from the second step it becomes biased. Thus, it seems that if we continue performing this simulation, its law and the law of the SSRW will become more distant. The same happens for $d \geq 2$, dividing the cards into $2d$ suits of the same size. Our goal is to determine the minimum number of cards N that a deck, divided into $2d$ suits, must contain so that the total variation distance between the law of a n-step simulation and the law of a n-step realization of the SSRW is smaller than a chosen threshold $\varepsilon \in (0, 1)$. More generally, we prove that when $N = cn$ this distance converges, as $n \to \infty$, to a Gaussian profile which depends on $c \geq 2d$, see Theorem 2.1.

The Gaussian shape of the profile derived in this paper is actually expected. Indeed, the analogy with coins made for the SSWR on $\mathbb{Z}$ allows us to associate the position of the process after n steps with a Binomial random variable. Similarly, we can associate the position of the d-dimensional process after n steps with a Multinomial random variable. Therefore, the CLT assures that, independently of the dimension, the SSRW approaches a Gaussian distribution. Furthermore, since there are N cards in the deck and exactly $N(2d)^{-1}$ of the cards correspond to each suit, the position of the simulation of interest depends only on the numbers of cards of each suit among the n removed cards. Therefore, we can associate it with a Multivariate Hypergeometric random variable, which by the CLT also approaches a Gaussian distribution. Given this explanation, the scaling $N = cn$ is the one for which we can see the convergence of the discrete random variables to the Gaussian ones. Moreover, as can be seen in Theorem 2.1, the Gaussian profile converges to zero as $c \to \infty$, showing the convergence, in total variation, of the Multivariate Hypergeometric distribution to the Multinomial distribution. Although well known, the proof of this classical limit could not be found in the literature for $d \geq 2$, not even for weaker types of convergence.

The paper is organized as follows: in Section 2 we rigorously define the SSRW on $\mathbb{Z}^d$ and its simulation. We also introduce some notation and we enunciate our main result, Theorem 2.1. In Section 3, we discuss the simulation when using only the colors of the cards ($d = 1$) and when using only four suits ($d = 2$), explaining, algorithmically, how to proceed in higher dimensions. In each particular case we explain how far one can go in a simulation with a fifty-two-card deck. Last, but not least, we prove Theorem 2.1 in Section 4.

2 Definitions and main result

Let $(e_i)_{i=1}^d$ denote the canonical basis of $\mathbb{R}^d$. Let $(\xi_j)_{j\in\mathbb{N}}$ be an independent and identically distributed sequence of random variables with $\mathbb{P}(\xi_j = e_i) = \mathbb{P}(\xi_j = -e_i) = \frac{1}{2d}$ for all $i \in \{1, \dots, d\}$. A symmetric simple random walk (SSRW) on $\mathbb{Z}^d$ can be defined as a sequence $(X_k)_{k\geq 0}$ where X_0 is the origin and $X_k = \sum_{j=1}^k \xi_j$.

Denote the euclidean norm of a vector $v = (v_1, \dots, v_m) \in \mathbb{R}^m$ by

$$\|v\| = \left(\sum_{i=1}^m |v_i|^2\right)^{1/2} \tag{1}$$

for any $m \in \mathbb{N}$. Notice that, for a fixed $n \in \mathbb{N}$, $(X_k)_{0\leq k\leq n}$ takes values in the set Γ_n equal to

$$\left\{x = (x_0, \dots, x_n) \in \mathbb{Z}^{d(n+1)} : x_0 = (0, \dots, 0) \text{ and } \|x_j - x_{j-1}\| = 1 \text{ for all } j = 1, \dots, n\right\}.$$

By the independence of the random variables $(\xi_j)_{1\leq j\leq n}$, we have

$$\mathbb{P}\big((X_k)_{0\leq k\leq n} = x\big) = \frac{1}{|\Gamma_n|} = \frac{1}{(2d)^n}$$

for any $x \in \Gamma_n$. That is, $(X_k)_{0 \le k \le n}$ is uniformly distributed in Γ_n. Also notice that there exists a bijection between Γ_n and

$$\Omega_n := \{0, 1, \ldots, 2d-1\}^n.$$

Let us consider, for $N \in \mathbb{N}$, a deck of N cards defined by the set $\Lambda_N = \{1, \ldots, N\}$ where each $j \in \Lambda_N$ represents a card. We will assume that $N = 2dK$ for $K \in \mathbb{N}$. Therefore we can suppose that each card is labeled not only with a number but also with one of $2d$ suits, and that there are exactly K cards of each suit. We denote by S_N the set of permutations of Λ_N, that is

$$S_N = \big\{\big(\sigma(1), \ldots, \sigma(N)\big) : \sigma : \Lambda_N \to \Lambda_N \text{ is a bijection}\big\}.$$

Let us consider the suit function $\varphi : \Lambda_N \to \mathbb{Z}/(2d\mathbb{Z})$ defined by

$$\varphi(j) = r \quad \text{if } j \equiv r \ (\text{mod } 2d).$$

For $n < N$, the function $X : S_N \to \Omega_n$, defined by

$$X(\sigma) = \big(\varphi(\sigma_1), \ldots, \varphi(\sigma_n)\big),$$

represents a simulation of the random walk with size n by using a deck of N cards.

Let us consider $\mathbb{Q}_N$ and $\mathbb{P}_n$ uniform distributions in $(S_N, \mathcal{P}(S_N))$ and $(\Omega_n, \mathcal{P}(\Omega_n))$, respectively. The function $X : S_N \to \Omega_n$ induces a probability measure μ_X in $(\Omega_n, \mathcal{P}(\Omega_n))$ given by

$$\mu_X(\omega) = \mathbb{Q}_N(X = \omega), \quad \omega \in \Omega_n.$$

We are interested in quantifying how close the simulation defined by X is to the SSRW on $\mathbb{Z}^d$. For that, we use the total variation distance. That is, we would like to estimate the total variation distance between the probabilities μ_X and $\mathbb{P}_n$, which is given by

$$d_n(N) = \sum_{\omega \in \Omega_n} \big[\mu_X(\omega) - \mathbb{P}_n(\omega)\big]^+ = \sum_{\omega \in \Omega_n} \big[\mu_X(\omega) - (2d)^{-n}\big]^+. \tag{2}$$

Above we use the notation $[f(\omega)]^+ = f(\omega) \cdot \mathbf{1}_{\{f(\omega)>0\}}$.

In order to obtain an expression for $\mu_X(\omega)$ and, consequently, for $d_n(N)$ we need to recall the definition of multinomial coefficients: for each $\ell, m \in \mathbb{N}$ define

$$\Pi_\ell^m := \left\{\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{R}^\ell; \lambda_i \in \mathbb{N} \cup \{0\} \text{ for all } i \in \{1, 2, \ldots, \ell\} \text{ and } \sum_{i=1}^{\ell} \lambda_i = m\right\}.$$

The multinomial coefficient is defined as

$$\binom{m}{\lambda_1, \ldots, \lambda_\ell} = \frac{m!}{\prod_{i=1}^{\ell} \lambda_i!}$$

for each $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \Pi_\ell^m$.

For each $i \in \mathbb{Z}/(2d\mathbb{Z})$, let λ_ω^i be cardinality of $\{j \in \{1, \ldots, n\} : \omega_j = i\}$. We can compute

$$\mu_X(\omega) = \frac{\binom{N-n}{K-\lambda_\omega^1, \ldots, K-\lambda_\omega^{2d}}}{\binom{N}{K, \ldots, K}},$$

since the denominator counts the equiprobable elements of Ω_N with exactly K terms equal to each element of $\mathbb{Z}/(2d\mathbb{Z})$, and the numerator counts all the sequences whose first n terms coincides with those of ω.

For each $\omega \in \Omega_n$ define $\lambda_\omega = (\lambda_\omega^0, \ldots, \lambda_\omega^{2d-1})$. Notice that $\mu_X(\omega) = \mu_X(\omega')$ whenever $\lambda_\omega = \lambda_{\omega'}$. Then we can rewrite equation (2) as

$$d_n(N)$$

$$= \sum_{\lambda_1=0}^{n} \sum_{\lambda_2=0}^{n-\lambda_1} \cdots \sum_{\lambda_{2d}=0}^{n-\sum_{i=1}^{2d-1}\lambda_i} \left[\frac{\binom{N-n}{K-\lambda_1,\ldots,K-\lambda_{2d}}\binom{n}{\lambda_1,\ldots,\lambda_{2d}}}{\binom{N}{K,\ldots,K}} - \frac{1}{(2d)^n}\binom{n}{\lambda_1,\ldots,\lambda_{2d}} \right]^+ . \quad (3)$$

The above expression shows that $d_n(N)$ actually is the the total variation distance between a multivariate hypergeometric $H_n \sim H(N, K_1 = K, \ldots, K_{2d} = K, n)$ and a multinomial distribution $B_n \sim B(n, p_1 = 1/2, \ldots, p_{2d} = 1/2)$. It is well known that for n fixed, $d_n(N) \to 0$ as $N \to \infty$ (see the one-dimensional case in Murteira (1979)). However, we are interested in a finer estimate that provides also information about the behavior of $d_n(N)$ for n and N large.

Notice that $d_n(N) \in [0, 1]$. We are interested in finding the smallest number of cards N that a deck must contain so that a simulation with n iterations presents $d_n(N)$ smaller than a chosen threshold $\varepsilon \in (0, 1)$. This problem, which has similarities to the problem of determining mixing times of Markov chains (see Wilmer, Levin and Peres (2009) for an introduction to the subject), is equivalent to determining the largest number n of iterations that one can perform on a simulation with a N-card deck so that $d_n(N)$ is smaller than ε.

From now on we use the notation $a_n = \mathcal{O}(b_n)$ if there exists a positive constant C such that $|a_n| \le C|b_n|$ for sufficiently large n. In this case, we say that a_n is of order b_n. Our main result states that, for $N = cn$, the total variation distance is of the order of a constant that depends on c, which is explicitly given, plus an error term of order $n^{-1/2}$.

Theorem 2.1. *Let $c \ge 2d$. Let us define $r = r(c) = \sqrt{(4d^2 - 2d)(c-1)\log(\frac{c}{c-1})}$. Let $B_r^m = \{X \in \mathbb{R}^m; \|X\| \le r\}$ be the m-ball that is centered at the origin and whose radius is equal to r. Let $\varPi_m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m; \sum_{i=1}^m x_i = 0\}$ be the hyperplane that is orthogonal to the vector $(1, \ldots, 1)$ and contains the origin. Let $\mathcal{G} : B_r^{2d} \cap \varPi_{2d} \to [0, 1]$ be the Gaussian profile defined as*

$$\mathcal{G}(X) = (\pi^{1/2-d} 2^{1-2d} d^{1/2-d}) \left[\left(\frac{c}{c-1} \right)^{d-1/2} \exp\left(-\frac{c\|X\|^2}{4d(c-1)} \right) - \exp\left(-\frac{\|X\|^2}{4d} \right) \right]$$

for every $X \in B_r^{2d} \cap \varPi_{2d}$. Then

$$d_n(cn) = \int_{B_r^{2d} \cap \varPi_{2d}} \mathcal{G}(X)\, dX + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right).$$

3 Discussion of the main result

In this section, we present some elementary computations that lead us to a simpler formulation of Theorem 2.1 for the particular, and most natural, cases $d = 1$ and $d = 2$. Before we do so, let us define

$$S_r^m = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^m; \left(\sum_{i=1}^m x_i \right)^2 + \sum_{i=1}^m x_i^2 \le r^2 \right\}. \quad (4)$$

Observe that every vector in $B_r^{2d} \cap \varPi_{2d} \subset \mathbb{R}^{2d}$ presents the form

$$(x_1, \ldots, x_{2d-1}, x_{2d}) = \left(x_1, \ldots, x_{2d-1}, -\sum_{i=1}^{2d-1} x_i \right).$$

Thus, the application $\varphi : S_r^{2d} \subset \mathbb{R}^{2d-1} \to B_r^{2d} \cap \Pi_{2d}$ defined by

$$\varphi(x_1,\dots,x_{2d-1}) = \left(x_1,\dots,x_{2d-1}, -\sum_{i=1}^{2d-1} x_i\right)$$

is a parametrization of $B_r^{2d} \cap \Pi_{2d}$. Therefore,

$$\int_{B_r^{2d}\cap\Pi_{2d}} \mathcal{G}(X)\,dX = \int_{S_r^{2d-1}} \mathcal{G}\big(\varphi(x_1,\dots,x_{2d-1})\big)\sqrt{\det(J_\varphi^{\mathrm{T}} J_\varphi)}\,dx_1\cdots dx_{2d-1},$$

where J_φ is the Jacobian matrix of φ and J_φ^{T} is its transpose. From the definition of φ and since $\det(J_\varphi^{\mathrm{T}} J_\varphi) = 2d$, we conclude that

$$\int_{B_r^{2d}\cap\Pi_{2d}} \mathcal{G}(X)\,dX = \sqrt{2d}\int_{S_r^{2d-1}} \mathcal{G}\left(x_1,\dots,x_{2d-1}, -\sum_{i=1}^{2d-1} x_i\right) dx_1\cdots dx_{2d-1} \qquad (5)$$

We will use the above identity to analyze the cases $d = 1$ and $d = 2$.

3.1 Using only the colors of cards

When $d = 1$, the SSRW on $\mathbb{Z}$ jumps to the left or to the right according to the color of the card which was removed from the deck. Observe that S_r^1 is the interval $[-r/\sqrt{2}, r/\sqrt{2}]$. Therefore, by (5), $\int_{B_r^2\cap\Pi_{2d}} \mathcal{G}(X)\,dX$ is equal to

$$\begin{aligned}
&\sqrt{2}\int_{-r/\sqrt{2}}^{r/\sqrt{2}} \frac{1}{2\sqrt{\pi}}\left[\sqrt{\frac{c}{c-1}}\exp\left(-\frac{c\|(x,-x)\|^2}{4(c-1)}\right) - \exp\left(-\frac{\|(x,-x)\|^2}{4}\right)\right]dx\\
&= \int_{-r/\sqrt{2}}^{r/\sqrt{2}} \frac{1}{\sqrt{2\pi}}\left[\sqrt{\frac{c}{c-1}}\exp\left(-\frac{cx^2}{2(c-1)}\right) - \exp\left(-\frac{x^2}{2}\right)\right]dx.
\end{aligned}$$

Taking $\xi = x\sqrt{c/(2(c-1))}$ and $\eta = x/\sqrt{2}$, we obtain

$$\int_{B_r^2\cap\Pi_{2d}} \mathcal{G}(X)\,dX = \frac{1}{\sqrt{\pi}}\int_{-r\sqrt{c/(4(c-1))}}^{r\sqrt{c/(4(c-1))}} e^{-\xi^2}\,d\xi - \frac{1}{\sqrt{\pi}}\int_{-r/2}^{r/2} e^{-\eta^2}\,d\eta.$$

Since in this case $r = \sqrt{2(c-1)\log(\frac{c}{c-1})}$, when $d = 1$ Theorem 2.1 can be enunciated as follows:

Theorem 3.1. *If* $d = 1$ *then for all* $c \geq 2$,

$$d_n(cn) = \operatorname{erf}\left(\sqrt{\frac{c\log(c/c-1)}{2}}\right) - \operatorname{erf}\left(\sqrt{\frac{(c-1)\log(c/c-1)}{2}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

where $\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\,dt$ *is the Gauss error function.*

Figure 1 shows the graph of the function given in Theorem 3.1 which is the asymptotic profile of $d_n(cn)$ when $d = 1$.

Table 1 informs us that with a 52-card deck one can simulate 26-steps of the SSRW on $\mathbb{Z}$ and $d_{26}(52)$ will be close to 0.160. Similarly, $d_{17}(52)$ and $d_9(52)$ (when using 17 and 9 cards, respectively) are close to 0.100 and 0.050, respectively.

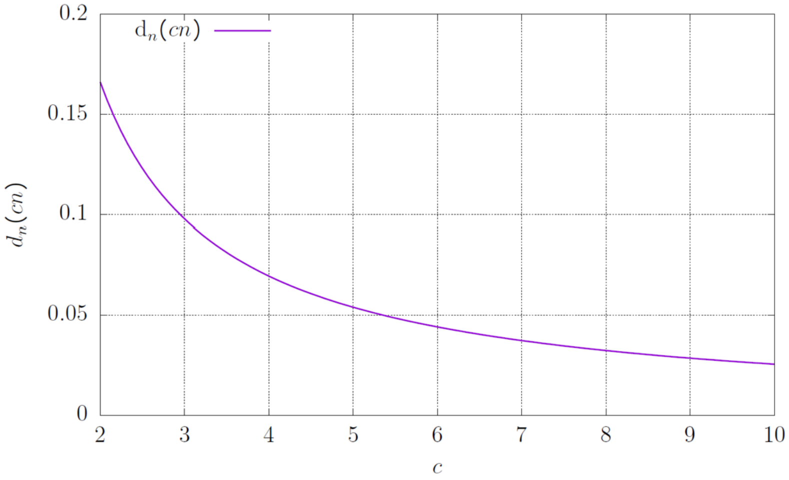

Figure 1 *Graph of the asymptotic profile of $d_n(cn)$ when $d = 1$.*

Table 1 *Values of $d_n(cn)$ for given values of c. The value $c = 2$ is the smallest possible choice; the other values were chosen so that $d_n(cn)$ assumed values which are often chosen*

c	$d_n(cn)$
2.00	0.160
2.94	0.100
5.35	0.050
24.70	0.010
48.89	0.005
242.47	0.001

3.2 Using a deck with four suits

When $d = 2$, the SSRW on $\mathbb{Z}^2$ jumps to one of the 4 directions according to the suit of the card which was removed from the deck. For this case, observe that

$$S_r^3 = \{x^2 + y^2 + z^2 + xy + xz + yz \leq r^2/2\}.$$

Therefore, by (5), $\int_{B_r^4 \cap \Pi_4} \mathcal{G}(X)\, dX$ is equal to

$$\begin{aligned}
&\sqrt{4} \int\int\int_{S_r^3} \frac{1}{\sqrt{512\pi^3}} \left[\left(\frac{c}{c-1}\right)^{3/2} \exp\left(-\frac{c\|(x, y, z, -x-y-z)\|^2}{8(c-1)}\right) \right. \\
&\quad \left. - \exp\left(-\frac{\|(x, , y, z, -x-y-z)\|^2}{8}\right)\right] dx\, dy\, dz \\
&= \int\int\int_{S_r^3} \frac{1}{\sqrt{128\pi^3}} \left[\left(\frac{c}{c-1}\right)^{3/2} \exp\left(-\frac{c(x^2 + y^2 + z^2 + xy + xz + yz)}{4(c-1)}\right) \right. \\
&\quad \left. - \exp\left(-\frac{(x^2 + y^2 + z^2 + xy + xz + yz)}{4}\right)\right] dx\, dy\, dz. \qquad (6)
\end{aligned}$$

We can find a change of variables $\tilde{\varphi}(u, v, w) = (x, y, z)$ that maps S_r^3 onto the unit ball B_1^3 so that we can apply spherical coordinates in the above integral. Indeed, we follow the algorithm explained in the proof of Sylvester's theorem in Bueno (2006). Let $q : \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form defined by

$$q(x, y, z) = x^2 + y^2 + z^2 + xy + xz + yz$$

for any $x \in \mathbb{R}^3$. Isolating the variable x, we obtain

$$q(x, y, z) = x^2 + (y + z)x + y^2 + z^2 + yz = \left(x + \frac{y + z}{2}\right)^2 + \frac{3}{4}y^2 + \frac{3}{4}z^2 + \frac{1}{2}yz.$$

Taking $s = x + \frac{y+z}{2}$ and isolating the variable y, we obtain

$$q(x, y, z) = s^2 + \frac{3}{4}\left(y + \frac{z}{3}\right)^2 + \frac{2}{3}z^2.$$

Taking $t = y + \frac{z}{3}$ we obtain

$$q(x, y, z) = s^2 + \frac{3}{4}t^2 + \frac{2}{3}z^2.$$

Therefore, the change of variables

$$\begin{cases} u = \dfrac{\sqrt{2}}{r}s = \dfrac{\sqrt{2}}{r}x + \dfrac{\sqrt{2}}{2r}y + \dfrac{\sqrt{2}}{2r}z, \\ v = \dfrac{\sqrt{3}}{r\sqrt{2}}t = \dfrac{\sqrt{3}}{r\sqrt{2}}y + \dfrac{\sqrt{3}}{3r\sqrt{2}}z, \\ w = \dfrac{2}{r\sqrt{3}}z \end{cases} \tag{7}$$

implies that

$$q(x, y, z) = \frac{r^2}{2}(u^2 + v^2 + w^2)$$

for any $(x, y, z) \in \mathbb{R}^3$ and maps the spheroid S_r^3 onto B_1^3. Solving the linear system (7), we obtain that

$$\begin{cases} x = \dfrac{\sqrt{2}}{2}ru - \dfrac{\sqrt{6}}{6}rv - \dfrac{\sqrt{3}}{6}rw, \\ y = \dfrac{\sqrt{6}}{3}rv - \dfrac{\sqrt{3}}{6}rw, \\ z = \dfrac{\sqrt{3}}{2}rw. \end{cases}$$

Therefore, we have

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \dfrac{\sqrt{2}}{2}r & -\dfrac{\sqrt{6}}{6}r & -\dfrac{\sqrt{3}}{6}r \\ 0 & \dfrac{\sqrt{6}}{3}r & -\dfrac{\sqrt{3}}{6}r \\ 0 & 0 & \dfrac{\sqrt{3}}{2}r \end{vmatrix} = \frac{r^3}{2}.$$

Hence, we can rewrite identity (6) as

$$\int_{B_r^4 \cap \Pi_4} \mathcal{G}(X)\,dX = \int\int\int_{B_1^3} \frac{r^3}{16\sqrt{2\pi^3}}\left[\left(\frac{c}{c-1}\right)^{3/2} \exp\left(-\frac{cr^2(u^2 + v^2 + w^2)}{8(c-1)}\right) - \exp\left(-\frac{r^2(u^2 + v^2 + w^2)}{8}\right)\right] du\,dv\,dw.$$

Now, changing the variables u, v, w in the above integral to the spherical coordinates

$$\begin{cases} u = \rho \sin\phi \cos\theta, \\ v = \rho \sin\phi \sin\theta, \\ w = \rho \cos\phi \end{cases}$$

with $\rho \in [0, 1]$, $\phi \in [0, \pi]$ and $\theta \in 2\pi$, we obtain

$$\int_{B_r^4 \cap \Pi_4} \mathcal{G}(X)\, dX = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \frac{r^3 \rho^2 \sin\phi}{16\sqrt{2\pi^3}} \Big[\Big(\frac{c}{c-1} \Big)^{3/2} \exp\Big(-\frac{cr^2\rho^2}{8(c-1)} \Big) - \exp\Big(-\frac{r^2\rho^2}{8} \Big) \Big] d\rho\, d\phi\, d\theta.$$

By Fubini's theorem, $\int_{B_r^4 \cap \Pi_4} \mathcal{G}(X)\, dX$ equals

$$\begin{aligned} &\int_0^{\pi} \int_0^1 \frac{r^3 \rho^2 \sin\phi}{8\sqrt{2\pi}} \Big[\Big(\frac{c}{c-1} \Big)^{3/2} \exp\Big(-\frac{cr^2\rho^2}{8(c-1)} \Big) - \exp\Big(-\frac{r^2\rho^2}{8} \Big) \Big] d\rho\, d\phi \\ &= \frac{r^3}{8\sqrt{2\pi}} \Big(\int_0^{\pi} \sin\phi d\phi \Big) \int_0^1 \rho^2 \Big[\Big(\frac{c}{c-1} \Big)^{3/2} \exp\Big(-\frac{cr^2\rho^2}{8(c-1)} \Big) - \exp\Big(-\frac{r^2\rho^2}{8} \Big) \Big] d\rho \\ &= \frac{r^3}{4\sqrt{2\pi}} \int_0^1 \rho^2 \Big[\Big(\frac{c}{c-1} \Big)^{3/2} \exp\Big(-\frac{cr^2\rho^2}{8(c-1)} \Big) - \exp\Big(-\frac{r^2\rho^2}{8} \Big) \Big] d\rho. \end{aligned}$$

Finally, taking $\xi = r\rho\sqrt{c/(8(c-1))}$ and $\eta = r\rho/\sqrt{8}$ and using the fact that the function e^{-x^2} is even in variable $x \in \mathbb{R}$, we conclude that

$$\begin{aligned} \int_{B_r^4 \cap \Pi_4} \mathcal{G}(X)\, dX &= \frac{cr^2}{2(c-1)\sqrt{\pi}} \int_0^{r\sqrt{c/(8(c-1))}} e^{-\xi^2}\, d\xi - \frac{r^2}{2\sqrt{\pi}} \int_0^{r/\sqrt{8}} e^{-\eta^2}\, d\eta \\ &= \frac{cr^2}{4(c-1)\sqrt{\pi}} \int_{-r\sqrt{c/(8(c-1))}}^{r\sqrt{c/(8(c-1))}} e^{-\xi^2}\, d\xi - \frac{r^2}{4\sqrt{\pi}} \int_{-r/\sqrt{8}}^{r/\sqrt{8}} e^{-\eta^2}\, d\eta \\ &= \frac{cr^2}{4(c-1)} \operatorname{erf}\Big(\sqrt{\frac{cr^2}{8(c-1)}} \Big) - \frac{r^2}{4} \operatorname{erf}\Big(\sqrt{\frac{r^2}{8}} \Big). \end{aligned}$$

Since in this case $r = \sqrt{12(c-1)\log(\frac{c}{c-1})}$, when $d = 2$ Theorem 2.1 can be enunciated as the following theorem.

Theorem 3.2. *If $d = 2$, then for all $c \geq 4$,*

$$\begin{aligned} d_n(cn) = 3c \log\Big(\frac{c}{c-1} \Big) \operatorname{erf}\Big(\sqrt{\frac{3c\log(c/(c-1))}{2}} \Big) \\ - 3(c-1) \log\Big(\frac{c}{c-1} \Big) \operatorname{erf}\Big(\sqrt{\frac{3(c-1)\log(\frac{c}{c-1})}{2}} \Big) + \mathcal{O}\Big(\frac{1}{\sqrt{n}} \Big). \end{aligned}$$

Figure 2 shows the graph of the function given in Theorem 3.2 which is the asymptotic profile of $d_n(cn)$ when $d = 2$. We can conclude that using a typical 52-card deck is not very good to simulate the SSRW on $\mathbb{Z}^2$ as it is to simulate the SSRW on $\mathbb{Z}$. In that case, one must use a deck with more cards.

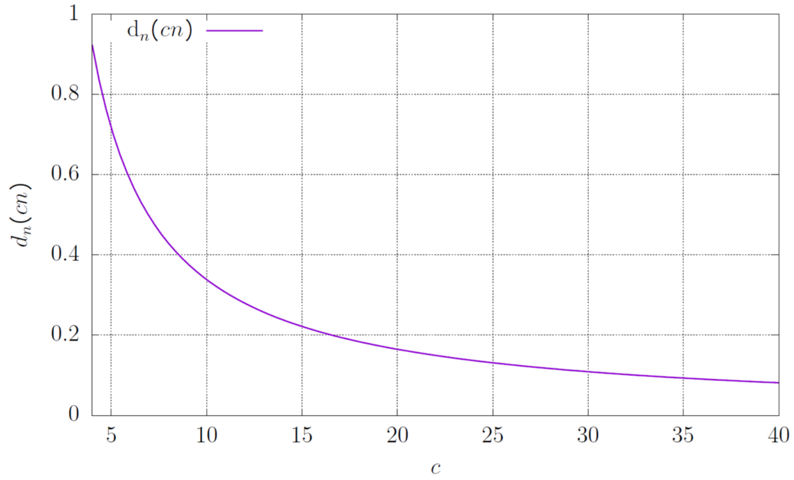

Figure 2 *Graph of the asymptotic profile of* $d_n(cn)$ *when* $d = 2$.

Remark (Using a deck with more suits). There are many different ways to calculate $\int_{B_r^{2d} \cap \Pi_{2d}} \mathcal{G}(X)\, dX$. Although the procedure used to obtain the change of variables (7) is very exhausting, it can be used for any $d \in \mathbb{N}$ and so can the spherical coordinates (see Bartle (1964)). This explains why we decided to analyze the bi-dimensional problem in this way.

4 Proof of the main result

For $(\lambda_1, \dots, \lambda_{2d}) \in \Pi_{2d}^n$, and recalling that $K = N/2d$, we introduce the notation

$$f(\lambda_1, \lambda_2, \dots, \lambda_{2d}, N, n) := (2d)^n \frac{\binom{N-n}{K-\lambda_1, \dots, K-\lambda_{2d}}}{\binom{N}{K, \dots, K}}. \tag{8}$$

Notice that this function represents the Radon–Nikodym derivative of H_n with respect to B_n. We can now rewrite equation (3), for $N = cn$, as

$$d_n(cn) = \sum_{\lambda_1=0}^{n} \sum_{\lambda_2=0}^{n-\lambda_1} \cdots \sum_{\lambda_{2d}=0}^{n-\sum_{i=1}^{2d-1} \lambda_i} \frac{1}{(2d)^n} \binom{n}{\lambda_1, \dots, \lambda_{2d}} \left[f(\lambda_1, \lambda_2, \dots, \lambda_{2d}, cn, n) - 1 \right]^+. \tag{9}$$

The positivity that appears in the terms of (9) is inconvenient for our task of estimating $d_n(cn)$. The first step of this proof is to get rid of it. This is done by localizing the vectors in

$$\Delta_n = \left\{ \lambda = (\lambda_1, \dots, \lambda_{2d}) \in \Pi_{2d}^n ;\ f(\lambda_1, \lambda_2, \dots, \lambda_{2d}, cn, n) \geq 1 \right\}. \tag{10}$$

The next lemma, which is more of an observation, states a monotonicity property satisfied by the function f.

Lemma 4.1. *Let* $\lambda = (\lambda_1, \dots, \lambda_{2d}) \in \Pi_{2d}^n$. *For* $\lambda_i < \lambda_j$ *let us define* $\lambda^{i,j} = (\lambda_1^{i,j}, \dots, \lambda_{2d}^{i,j})$ *as*

$$\lambda_\ell^{i,j} = \begin{cases} \lambda_i + 1, & \text{if } \ell = i, \\ \lambda_j - 1, & \text{if } \ell = j, \\ \lambda_\ell, & \text{if } \ell \notin \{i, j\}. \end{cases} \tag{11}$$

Then

$$f(\lambda, N, n) \leq f\left(\lambda^{i,j}, N, n\right).$$

Proof. We only need to prove that

$$\binom{N-n}{K-\lambda_1,\dots,K-\lambda_{2d}} \le \binom{N-n}{K-\lambda_1^{i,j},\dots,K-\lambda_{2d}^{i,j}},$$

which is equivalent to

$$(K-\lambda_i^{i,j})!(K-\lambda_j^{i,j})! \le (K-\lambda_i)!(K-\lambda_j)!.$$

This holds if and only if $\lambda_i + 1 \le \lambda_j$, and this completes the proof. □

We will need some technical lemmas in order to find the location of the points in Δ_n. We start by the classical Stirling's approximation (see, for instance, Franco (2020)):

$$m! = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m (1+\mathcal{O}(1/m)). \tag{12}$$

As a direct consequence of the Stirling's approximation, we also have

$$\binom{m}{k,\dots,k} = \frac{\sqrt{2}d^d(2d)^m}{\pi^{d-1/2}m^{d-1/2}}(1+\mathcal{O}(1/m)) \tag{13}$$

for $k = m/2d$.

Recall the definition of the euclidean norm given in (1). Denote the ℓ_∞-norm of a vector $v = (v_1,\dots,v_m) \in \mathbb{R}^m$ by $\|v\|_\infty = \max_{1\le i\le m}|v_i|$. The next lemma gives us an asymptotic expression for $\binom{m}{k_1,\dots,k_{2d}}$ when the ℓ_∞-norm of $(k_1,\dots,k_{2d}) \in \Pi_{2d}^m$ is not too far from $(\frac{m}{2d},\dots,\frac{m}{2d})$. The result is standard but we will present the proof for the seek of completeness.

Lemma 4.2. *Let $L_m = (\ell_m^1,\dots,\ell_m^{2d}) \in \mathbb{R}^{2d}$ define a sequence satisfying $\sum_{i=1}^{2d}\ell_m^i = 0$ and $\lim_{m\to\infty}\|L_m\|_\infty/m^{2/3} = 0$. If $k_i = (m+\ell_m^i)/(2d)$ for all $i \in \{1,\dots,2d\}$ then*

$$\binom{m}{k_1,\dots,k_{2d}} = \frac{\sqrt{2}d^d(2d)^m}{\pi^{d-1/2}m^{d-1/2}}\exp\left\{-\frac{\|L_m\|^2}{4dm}\right\}\left(1+\mathcal{O}\left(\frac{\|L_m\|_\infty^3}{m^2}\right)\right).$$

Proof. For each $i \in \{1,\dots,2d\}$ define

$$\varepsilon_i = \frac{\ell_m^i}{m},$$

and notice that, since $\ell_m^i/m^{2/3} \to 0$ as $m \to \infty$, we have, in particular, that $\varepsilon_i \to 0$ as $m \to \infty$. Therefore, by Stirling's formula, we have

$$\begin{aligned} k_i! &= (2\pi)^{1/2}\left(\frac{m}{2d}\right)^{1/2}\left(\frac{m}{2de}\right)^{k_i}(1+\varepsilon_i)^{k_i}\left(1+\frac{\ell_m^i}{m}\right)^{1/2}\left(1+\mathcal{O}\left(\frac{1}{m}\right)\right)\\ &= (2\pi)^{1/2}\left(\frac{m}{2d}\right)^{1/2}\left(\frac{m}{2de}\right)^{k_i}(1+\varepsilon_i)^{k_i}\left(1+\mathcal{O}\left(\frac{\|L_m\|_\infty}{m}\right)\right). \end{aligned}$$

Since $\sum_{i=1}^{2d}k_i = m$, we have

$$\prod_{i=1}^{2d}k_i! = (2\pi)^d\left(\frac{m}{2d}\right)^d\left(\frac{m}{2de}\right)^m\left(\prod_{i=1}^{2d}(1+\varepsilon_i)^{k_i}\right)\left(1+\mathcal{O}\left(\frac{\|L_m\|_\infty}{m}\right)\right).$$

Hence, by (12) we have

$$\begin{aligned} \binom{m}{k_i,\dots,k_{2d}} &= \frac{m!}{\prod_{i=1}^{2d}k_i}\\ &= \frac{\sqrt{2}d^d(2d)^m}{\pi^{d-1/2}m^{d-1/2}}\left(\prod_{i=1}^{2d}(1+\varepsilon_i)^{k_i}\right)^{-1}\left(1+\mathcal{O}\left(\frac{\|L_m\|_\infty}{m}\right)\right). \end{aligned} \tag{14}$$

Now observe that

$$\prod_{i=1}^{2d}(1+\varepsilon_i)^{k_i} = \exp\left\{\sum_{i=1}^{2d} k_i \log(1+\varepsilon_i)\right\}$$
$$= \exp\left\{\frac{m}{2d}\sum_{i=1}^{2d}(1+\varepsilon_i)\log(1+\varepsilon_i)\right\}.$$

Since $(1+\varepsilon)\log(1+\varepsilon) = \varepsilon + \frac{\varepsilon^2}{2} + \mathcal{O}(|\varepsilon|^3)$ when $\varepsilon \to 0$ and $\sum_{i=1}^{2d} \varepsilon_i = 0$, we have

$$\prod_{i=1}^{2d}(1+\varepsilon_i)^{k_i} = \exp\left\{\left(\frac{m}{2d}\sum_{i=1}^{2d}\varepsilon_i\right) + \left(\frac{m}{4d}\sum_{i=1}^{2d}|\varepsilon_i|^2\right) + \mathcal{O}\left(\frac{\|L_m\|_\infty^3}{m^2}\right)\right\}$$
$$= \exp\left\{\left(\frac{m}{4d}\sum_{i=1}^{2d}\left|\frac{\ell_m^i}{m}\right|^2\right) + \mathcal{O}\left(\frac{\|L_m\|_\infty^3}{m^2}\right)\right\}$$
$$= \exp\left\{\left(\frac{1}{4dm}\sum_{i=1}^{2d}|\ell_m^i|^2\right) + \mathcal{O}\left(\frac{\|L_m\|_\infty^3}{m^2}\right)\right\}$$
$$= \exp\left\{\left(\frac{1}{4dm}\sum_{i=1}^{2d}|\ell_m^i|^2\right)\right\}\left(1 + \mathcal{O}\left(\frac{\|L_m\|_\infty^3}{m^2}\right)\right).$$

The proof is completed putting the above estimate in (14) and noticing that we can replace $(1 + \mathcal{O}(\|L_m\|_\infty^3/m^2))^{-1}$ and $(1 + \mathcal{O}(\|L_m\|_\infty/m))$ by $(1 + \mathcal{O}(\|L_m\|_\infty^3/m^2))$ since $\|L_m\|_\infty/m^{2/3} \to 0$. □

Notice that the obtained asymptotic expression suggests that the parametrization given by $L_m = \sqrt{m}A_m$, $A_m = (a_m^1, \ldots, a_m^{2d}) \in \mathbb{R}^{2d}$, could be useful. That way we get

$$\binom{m}{\frac{m+a_m^1\sqrt{m}}{2d}, \ldots, \frac{m+a_m^{2d}\sqrt{m}}{2d}}$$
$$= \frac{\sqrt{2}d^d(2d)^m}{\pi^{d-1/2}m^{d-1/2}}\exp\left\{-\frac{\|A_m\|^2}{4d}\right\}\left(1 + \mathcal{O}\left(\frac{\|A_m\|_\infty^3}{\sqrt{m}}\right)\right), \tag{15}$$

when $\|A_m\|_\infty/m^{1/6} \to 0$.

We are now in condition to return to the task of characterizing the vectors in Δ_n.

Lemma 4.3. *If* $A_n = (a_n^1, \ldots, a_n^{2d}) \in \mathbb{R}^{2d}$ *defines a sequence satisfying* $\sum_{i=1}^{2d} a_n^i = 0$ *and* $\lim_{n\to\infty}\|A_n\|_\infty/n^{1/6} = 0$, *then*

$$f\left(\frac{n+a_n^1\sqrt{n}}{2d}, \ldots, \frac{n+a_n^{2d}\sqrt{n}}{2d}, cn, n\right)$$
$$= \left(\frac{c}{c-1}\right)^{d-\frac{1}{2}}\exp\left\{-\frac{\|A_n\|^2}{4d(c-1)}\right\}\left(1 + \mathcal{O}\left(\frac{\|A_n\|_\infty^3}{\sqrt{n}}\right)\right).$$

Proof. Taking $N = cn$ and $\lambda_i = \frac{n+a_n^i\sqrt{n}}{2d}$ for every $i \in \{1, \ldots, 2d\}$, we have

$$\binom{N-n}{K-\lambda_1, \ldots, K-\lambda_{2d}}$$

$$= \binom{(c-1)n}{\frac{(c-1)n - (a_n^1/\sqrt{c-1})\sqrt{(c-1)n}}{2d}, \dots, \frac{(c-1)n - (a_n^{2d}/\sqrt{c-1})\sqrt{(c-1)n}}{2d}} \tag{16}$$

and

$$\binom{N}{K, \dots, K} = \binom{cn}{cn/(2d), \dots, cn/(2d)}. \tag{17}$$

By identity (16), and using (15) for $m = N - n = (c-1)n$ we obtain

$$\binom{N-n}{K-\lambda_1, \dots, K-\lambda_{2d}}$$
$$= \frac{\sqrt{2}d^d (2d)^{(c-1)n}}{\pi^{d-1/2}[(c-1)n]^{d-1/2}} \exp\left\{-\frac{\|A_n\|^2}{4d(c-1)}\right\}\left(1 + \mathcal{O}\left(\frac{\|A_n\|_\infty^3}{\sqrt{n}}\right)\right).$$

By identities (13) and (17), we have

$$\binom{N}{K, \dots, K} = \frac{\sqrt{2}d^d (2d)^{cn}}{\pi^{d-1/2}(cn)^{d-1/2}}(1 + \mathcal{O}(1/n)).$$

The result follows from the definition of f given in (8). □

Recall the definition of Δ_n given in (10). Let us define

$$\Phi_n = \left\{A_n = (a_n^1, \dots, a_n^{2d}) \in \mathbb{R}^{2d}; \left(\frac{n + a_n^1\sqrt{n}}{2d}, \dots, \frac{n + a_n^{2d}\sqrt{n}}{2d}\right) \in \Delta_n\right\}.$$

Notice that

$$\sum_{i=1}^{2d} a_n^i = 0 \quad \text{for every } A_n = (a_n^1, \dots, a_n^{2d}) \in \Phi_n$$

Observe that defining $\psi : \Delta_n \to \Phi_n$ by

$$\psi(\lambda_1, \dots, \lambda_{2d}) = \left(\frac{2d\lambda_1}{\sqrt{n}} - \sqrt{n}, \dots, \frac{2d\lambda_{2d}}{\sqrt{n}} - \sqrt{n}\right), \tag{18}$$

we obtain a bijection between Δ_n and Φ_n. Therefore, characterizing Δ_n and Φ_n are equivalent tasks. In order to accomplish these tasks, we prove the following lemma.

Lemma 4.4. *The set $\bigcup_{n=1}^{\infty} \Phi_n$ is bounded.*

Proof. Notice that, for each n, the set Φ_n is finite. It will be enough to prove that any sequence defined by $A_n = (a_n^1, \dots, a_n^{2d})$, such that $A_n \in \Phi_n$ for all n, is bounded. If $(A_n)_n$ were unbounded, then there would be a subsequence $(A_{n_k})_k$ such that $\|A_{n_k}\|_\infty \to \infty$ as $k \to \infty$.

We will split our analysis in two cases: $\|A_{n_k}\|_\infty / n_k^{1/6} \to 0$ or $\|A_{n_k}\|_\infty / n_k^{1/6} \nrightarrow 0$ as $k \to \infty$.

In the first case, we use Lemma 4.3 to conclude that

$$\lim_{k\to\infty} f\left(\frac{n_k + a_{n_k}^1\sqrt{n_k}}{2d}, \dots, \frac{n_k + a_{n_k}^{2d}\sqrt{n_k}}{2d}, cn_k, n_k\right) = 0,$$

which contradicts the fact that $A_{n_k} \in \Phi_{n_k}$ for all k.

In the second case, there would exist $\varepsilon > 0$ and a subsequence of $(A_{n_k})_k$, which, for simplicity, we will still call $(A_{n_k})_k$, such that $\|A_{n_k}\|_\infty > \varepsilon n_k^{1/6}$ for all k.

Now we will modify the vector $\lambda_{n_k} = (\lambda^1_{n_k}, \dots, \lambda^{2d}_{n_k}) \in \Pi^{n_k}_{2d}$, where $\lambda^\ell_{n_k} = \frac{n_k + a^\ell_{n_k}\sqrt{n_k}}{2d}$ in the following way: consider $\lambda^i_{n_k}$ and $\lambda^j_{n_k}$ two coordinates that assume, respectively, the smallest and the biggest value among all coordinates $\lambda^\ell_{n_k}$, then we replace the vector λ_{n_k} by the vector $\lambda^{i,j}_{n_k}$ defined in (11). By Lemma 4.1 we will have

$$f(\lambda_{n_k}, cn_k, n_k) \le f\big(\lambda^{i,j}_{n_k}, cn_k, n_k\big).$$

We repeatedly execute the same procedure described above, always considering at each step the smallest and the biggest of all current coordinates, until we reach a vector $\tilde{\lambda}_{n_k} = (\tilde{\lambda}^1_{n_k} \dots, \tilde{\lambda}^{2d}_{n_k})$ that is close enough to the vector $(\frac{n_k}{2d}, \dots, \frac{n_k}{2d})$ in the ℓ_∞-norm, but still with the difference in the ℓ_∞-norm being bounded away from zero. More specifically, for a fixed $0 < \delta < 1/6$, we will ask that $n_k^{1/6-\delta} \le \|\tilde{A}_{n_k}\|_\infty \le 2n_k^{1/6-\delta}$, where $\tilde{A}_{n_k} = (\tilde{a}^1_{n_k}, \dots, \tilde{a}^{2d}_{n_k})$ for $\tilde{a}^\ell_{n_k}$ defined by $\tilde{\lambda}^\ell_{n_k} = \frac{n_k + \tilde{a}^\ell_{n_k}\sqrt{n_k}}{2d}$. Therefore, again by Lemma 4.1,

$$f(\lambda_{n_k}, cn_k, n_k) \le f(\tilde{\lambda}_{n_k}, cn_k, n_k). \tag{19}$$

By Lemma 4.3, we have

$$\begin{aligned} f(\tilde{\lambda}_{n_k}, cn_k, n_k) &= \left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{\|\tilde{A}_{n_k}\|^2}{4d(c-1)}\right)\left(1 + \mathcal{O}\left(\frac{\|\tilde{A}_{n_k}\|^3_\infty}{\sqrt{n}}\right)\right) \\ &\le \left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{n_k^{1/3-2\delta}}{4d(c-1)}\right)\left(1 + \mathcal{O}\left(\frac{1}{n_k^{3\delta}}\right)\right). \end{aligned}$$

The last line above goes to zero as $k \to \infty$. By (19) we have that $f(\lambda_{n_k}, cn_k, n_k)$ will go to zero as well, and this again contradicts the fact that of $A_{n_k} \in \Phi_{n_k}$ for all k. □

Let $B^m_r = \{X \in \mathbb{R}^m; \|X\| \le r\}$ be the m-ball that is centered at the origin and whose radius is equal to r. Let $\Pi_m = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m; \sum_{i=1}^m x_i = 0\}$ be the hyperplane that is orthogonal to the vector $(1, \dots, 1)$ and contains the origin.

Notice that the points in Φ_n form a discrete hyperplane (lattice) of dimension $2d - 1$ in $\mathbb{R}^{2d}$, in which the distance between two neighbouring points is always equal to $2\sqrt{2}d/\sqrt{n}$ and the volume of its fundamental parallelotopes is equal to $\sqrt{2d}(2d/\sqrt{n})^{2d-1}$. In the next proposition, we provide a precise characterization of the set Φ_n.

Proposition 4.5. *Let* $r = \sqrt{(4d^2 - 2d)(c-1)\log(\frac{c}{c-1})}$. *The set* Φ_n *induces a partition of the compact set* $B^{2d}_r \cap \Pi_{2d}$ *for which the volume of each partition parallelotope is equal to* $\sqrt{2d}(2d/\sqrt{n})^{2d-1}$.

Proof. Let $A_n = (a^1_n, \dots, a^{2d}_n)$ be a sequence in Φ_n. From the definition of Φ_n and by Lemma 4.3, we have

$$\begin{aligned} 1 &\le f\left(\frac{n + a^1_n\sqrt{n}}{2d}, \dots, \frac{n + a^{2d}_n\sqrt{n}}{2d}, cn, n\right) \\ &= \left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{\|A_n\|^2}{4d(c-1)}\right)(1 + \mathcal{O}(\|A_n\|^3_\infty/\sqrt{n})). \end{aligned}$$

In Lemma 4.4, we proved that the set $\bigcup_{n=1}^\infty \Phi_n$ is bounded, then the asymptotic bound in the right-hand side above does not depend on the sequence A_n. That is,

$$1 \le \left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{\|A_n\|^2}{4d(c-1)}\right)(1 + \mathcal{O}(1/\sqrt{n})).$$

Isolating the term $\|A_n\|$ above, we see that $\|A_n\| \le r + \mathcal{O}(1/\sqrt{n})$. □

Proof of Theorem 2.1. By (9), and since $f(\lambda_n^1, \dots, \lambda_n^{2d}, cn, n) - 1$ is nonnegative when $(\lambda_n^1, \dots, \lambda_n^{2d}) \in \Delta_n$, we can rewrite the total variation distance (9) as

$$d_n(cn) = \sum_{(\lambda_n^1,\dots,\lambda_n^{2d})\in\Delta_n} \frac{1}{(2d)^n} \binom{n}{\lambda_n^1, \dots, \lambda_n^{2d}} (f(\lambda_n^1, \lambda_n^2, \dots, \lambda_n^{2d}, cn, n) - 1).$$

Recall the definition of $\psi : \Delta_n \to \Phi_n$ given in (18). Thus, we can rewrite the right-hand side above as

$$\sum_{A_n\in\Phi_n} \frac{1}{(2d)^n} \binom{n}{\frac{n+a_n^1\sqrt{n}}{2d}, \dots, \frac{n+a_n^{2d}\sqrt{n}}{2d}} \left(f\left(\frac{n+a_n^1\sqrt{n}}{2d}, \dots, \frac{n+a_n^{2d}\sqrt{n}}{2d}, cn, n\right) - 1\right).$$

By Lemma 4.3, we have

$$\begin{aligned} d_n(cn) &= \sum_{A_n=(a_n^1,\dots,a_n^{2d})\in\Phi_n} \frac{1}{(2d)^n} \binom{n}{\frac{n+a_n^1\sqrt{n}}{2d}, \dots, \frac{n+a_n^{2d}\sqrt{n}}{2d}} \\ &\quad\times \left[\left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{\|A_n\|^2}{4d(c-1)}\right) - 1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right] \\ &= \sum_{A_n=(a_n^1,\dots,a_n^{2d})\in\Phi_n} \frac{1}{(2d)^n} \binom{n}{\frac{n+a_n^1\sqrt{n}}{2d}, \dots, \frac{n+a_n^{2d}\sqrt{n}}{2d}} \\ &\quad\times \left[\left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{\|A_n\|^2}{4d(c-1)}\right) - 1\right] + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

where in the last line above it was used that

$$\sum_{A_n\in\Phi_n} \frac{1}{(2d)^n} \binom{n}{\frac{n+a_n^1\sqrt{n}}{2d}, \dots, \frac{n+a_n^{2d}\sqrt{n}}{2d}} = \sum_{(\lambda_n^1,\dots,\lambda_n^{2d})\in\Delta_n} \frac{1}{(2d)^n} \binom{n}{\lambda_n^1, \dots, \lambda_n^{2d}} \leq 1.$$

Also, by (15), for $A_n = (a_n^1, \dots, a_n^{2d}) \in \Phi_n$ we have

$$\binom{n}{\frac{n+a_n^1\sqrt{n}}{2d}, \dots, \frac{n+a_n^{2d}\sqrt{n}}{2d}} = \frac{\sqrt{2}d^d(2d)^n}{\pi^{d-1/2}n^{d-1/2}} \exp\left\{-\frac{\|A_n\|^2}{4d}\right\}(1 + \mathcal{O}(1/\sqrt{n})).$$

Then,

$$\begin{aligned} d_n(cn) &= \sum_{A_n\in\Phi_n} (\pi^{1/2-d}2^{1-2d}d^{1/2-d})\sqrt{2d}\left(\frac{2d}{\sqrt{n}}\right)^{2d-1} \\ &\quad\times \left[\left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{c\|A_n\|^2}{4d(c-1)}\right) - \exp\left(-\frac{\|A_n\|^2}{4d}\right)\right] + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

By Proposition 4.5, the right-hand side above represents a Riemann sum of the Gaussian profile $\mathcal{G} : B_r^{2d} \cap \Pi_{2d} \to [0, 1]$ defined as

$$\mathcal{G}(X) = (\pi^{1/2-d}2^{1-2d}d^{1/2-d})\left[\left(\frac{c}{c-1}\right)^{d-1/2} \exp\left(-\frac{c\|(X)\|^2}{4d(c-1)}\right) - \exp\left(-\frac{\|X\|^2}{4d}\right)\right].$$

The order of the error in the transition from the Riemann sum to the integral is equal to the volume of each of the partition parallelotopes. Hence, we have

$$d_n(cn) = \int_{B_r^{2d}\cap\Pi_{2d}} \mathcal{G}(X)\, dX + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

□

Acknowledgments

Part of this work was carried out during the undergraduate research program "Jornadas de Pesquisa em Matemática do ICMC 2023" held at the Instituto de Ciências Matemáticas e de Computação (ICMC)—Universidade de São Paulo (USP), and which was partially supported by the Centro de Ciências Matemáticas Aplicadas à Indústria (CeMEAI—CEPID) under FAPESP Grant # 2013/07375-0, and by the FAPESP Grant # 2019/16061-2. We thank the hospitality of ICMC-USP during the program.

References

Bartle, R. G. (1964). *The Elements of Real Analysis, Vol. 2*. New York: Wiley. MR0393369

Bueno, H. P. (2006). *Álgebra Linear—Um Segundo Curso*. *Textos Universitários.* **06**.

Franco, T. (2020). *Princípios de Combinatória e Probabilidade*. Coleção Matemática Universitária.

Murteira, B. J. F. (1979). *Probabilidades e Estatística, Volume I*. Lisboa: McGraw-Hill de Portugal, Lda.

Ramaley, J. F. (1969). Buffon's noodle problem. *The American Mathematical Monthly* **76**, 916–918. MR0254893 https://doi.org/10.2307/2317945

Wilmer, E. L., Levin, D. A. and Peres, Y. (2009). *Markov Chains and Mixing Times*. Providence: American Mathematical Society. MR2466937 https://doi.org/10.1090/mbk/058