Universidade de São Paulo
Fundação de Amparo à Pesquisa do Estado de São Paulo
Proposta de Projeto de Mestrado A

Assintóticas de Funções Partição em Gases de Coulomb

Proposta vinculada ao Projeto Jovem Pesquisador entitulado

Análise Assintótica de Sistemas de Partículas e Matrizes Aleatórias

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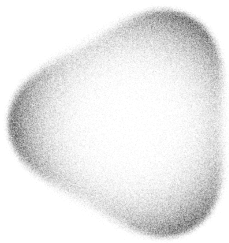
Aluno: João Victor Alcantara Pimenta



Resumo

O estudo de Matrizes Aleatórias demonstra aplicabilidade em uma gama diversa de áreas na matemática e na física, com destaque no estudo de mecânica estatística. No estudo da medida do espectro de autovalores de alguns ensembles de matrizes, analogias físicas à sistemas termodinâmicos se tornam evidentes e algumas motivações físicas podem ser tomadas. Um exemplo disso é a descrição da função partição, que em termodinâmica guarda grande detalhe sobre o sistema e suas propriedades. Desenvolvimentos recentes tem progredido intensamente na expressão para a expansão, no limite termodinâmico, da função partição de alguns sistemas termodinâmicos compatíveis com ensembles de interesse. O objetivo deste trabalho é estudar tais desenvolvimentos e entender suas implicações e importância dentro da teoria de matrizes aleatórias.

University of São Paulo
São Paulo Research Foundation
Proposal of Masters Project A

Asymptotics of Partition Functions on Coulomb Gases

Proposal vinculated to the "Jovem Pesquisador" research proposal entitled
Asymptotic Analysis of Interacting Particle Systems and Random Matrix Theory

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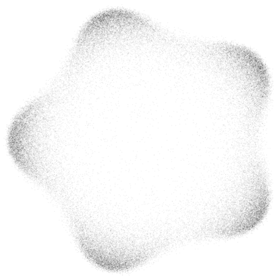

Abstract

The study of Random Matrices demonstrates applicability in a diverse range of areas in mathematics and Physics, emphasizing the study of statistical mechanics. In the study of the spectra of eigenvalue on some random matrices ensembles, an physical analogy regarding thermodynamical systems becomes evident and the use of physical motivations in its study of great importance. An example of such motivation is the study of the partition function, witch holds in thermodynamics a great deal of information on the system as hand and its properties. Recent developments have been intensely changing what is know on the thermodynamical limit asymptotic of these partition functions, especially for systems that regard ensembles of matrices of interest in modern research. The main goal of this work is to understand this developments and its implication and importance in the theory of random matrices.

1 Introduction

A random matrix is a matrix in witch the entries are random variables, not necessarily independent nor equally distributed. The algebraic-geometric properties in a matrix, such as its natural multiplication and spectral decomposition, makes this representation especially useful. It is also important to remember that many complex systems use a matrix representation. Correlation matrices and operators, especially in physics, are big reasons why this representation is important. Studying these matrices we can deduce properties from the system of interest, either that being the eigenvalues and eigenvectors of operators describing an atomic nuclei [Dyson, 1962] or the description of market shares in highly correlated systems [Fabozzi et al., 2010, Chapter 2]. Either way, using an random matrix approach is relevant by the same reason random variables proved themselves crucial: It possibilitates an statistical description of phenomena and systems. That is the purpose of studying Random Matrix Theory (RMT).

The possible random matrices are subdivided by what we call ensembles. Such as in physics, where an ensemble is defined by its possible microstates, in RMT an ensemble is defined by a set of matrices that share some macroscopical properties. We can divide these ensembles in two main characterization, invariance by rotation and independence of entries. We will focus mainly on ensembles we define rationally invariant, that is, for any $\hat{M}$ and $\hat{M}'$ have same probability if they have the same eigenvalues. For each ensemble we can associate a measure to its realizations. For example, the intersection between ensembles invariant by rotation and with independent entries sits uniquely the Gaussian Ensembles. For these ensembles we can write the joint probability density function

$$p(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_{N,\beta}}\, \mathrm{e}^{-\beta_N \mathcal{H}_N(\vec{\lambda})}, \tag{1.1}$$

where $Z_{N,\beta}$ is the canonical partition function, such that p is a measure. The factor $\beta_N = \beta N^2$ is though as the inverse temperature and the Hamiltonian $\mathcal{H}_N$ is expressed

$$\mathcal{H}_N(\vec{\lambda}) = \frac{1}{N}\sum_{i=1}^{N} \frac{\lambda_i^2}{2} + \frac{1}{N^2}\sum_{i<j} \log \frac{1}{|\lambda_i - \lambda_j|}, \quad \lambda_i \mapsto \lambda_i \sqrt{\beta N}.$$

More generally there ia a natural extension of this measure for other invariant ensembles where the Hamiltonian is expressed

$$\mathcal{H}_N(\vec{\lambda}) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N^2}\sum_{i<j}\log\frac{1}{|\lambda_i - \lambda_j|}, \quad V(\lambda_i) \mapsto \beta N V(\lambda_i).$$

Considering such a measure is natural to make an analogy to the well know Coulomb Gas. Under the right conditions, the Coulomb Gas is the Gibbs-Boltzmann probability measuregiven in $(R^d)^N$. This measure p_N models an interacting gas of charged particles, at $x_1, x_2, \ldots, x_N \in \mathbb{S}$ of dimension d in $\mathbb{R}^n$ ambient space, under the influence of an external potential. Its measure its given by

$$p_N(x_1, x_2, \ldots, x_N) = \frac{e^{-\beta N^2 \mathcal{H}_N(x_1, x_2, \ldots, x_N)}}{Z_{N,\beta}}, \tag{1.2}$$

where

$$\mathcal{H}_N(\vec{x}) = \frac{1}{N}\sum_{i=1}^{N} \mathrm{V}(x) + \frac{1}{2N^2}\sum_{i\neq j} \mathrm{g}(x_i - x_j)$$

is the Hamiltonian or energy of the system and $\mathrm{g}(x_i - x_j)$ is the Coulomb Kernel of interaction. This analogy indicates some possible approaches for the study of such systems, for now we can use thermodynamical arguments to describe the configurations of eigenvalues of random matrices. With this in mind we turn to the partition function. In the beginning of its book, Feynmann states [Feynman, 1998] that the key principle of the statistical mechanics is that a system states with energy E, in equilibrium, has probability given by a function $\frac{1}{Z}\,\mathrm{e}^{\frac{-E}{kT}}$ where Z is the partition function. In a more general sense there is a a relation between the partition function and the free energy of the system, that, in itself, relates to the entropy. Knowing well the partition function is a way to describe with great detail the macroproperties of such a system.

There is a great effort in the community of random matrix theory to study expansions of the partition function of systems such of Coulomb Gases. Recently, some advance has been made in the works of Sug-Soo Byun et al. [Byun et al., 2023]. They derived large-N expansions up to the $O(1)$-terms for both Z_N, related to the Coulomb Gas under complex and radially symmetric potential and $\tilde{Z}_N$, for its counterpart in the upper-half plane. They have also notice the expansion dependence on whether the limiting spectrum is an annulus or a disc, witch seems to not have been properly considered in the previous work on the matter. These recent developments are a major step in a continuous effort of more than 20 years in the field and can have many consequences.

As was said, the study of the partition function is a major way to describe thermodynamical systems such that of a Coulomb Gas. With these new developments it is expected that many systems could be better understood and described by the asymptotic given. In that way this poses a great opportunity for development in the field. More than that, by the great connection that the field of random matrices holds with many other fields is possible that by the study of this work some suggestions on the behavior of many other related problems become plausible.

2 Main Goals and Methodology

This MD research proposal has ? main goals.....

1. **Goal 1**;
2. **Goal 2**;
3. **Goal 3**;
4. **Goal 4**;

3 Necessary Background

This project requires some knowledge in a few mathematical areas. Some of them are real and complex analysis, functional analysis, measure theory and correlated courses. All of those are provided at the necessary leal ate the Masters Program in Mathematics at the ICMC - USP. All required background is expected to be gained in the regular courses in the first terms of the program, as usual. Some other disciplines may be coursed during the summer courses at the same institute. Furthermore, te student has had a year of experience in Random Matrix Theory at an undergrad level within a FAPESP fellowship supervised by the same researcher in witch some experience and general knowledge of the field has been gained.

4 Brief Schedule

This M.D. research proposal is schedule to start at August 2024 and run for te usual time for the Masters Program, that is, 24 months. The project will be run with weekly meetings with the supervisor, and sporadic seminar expositions to the other members of the research team associated with the JP Research Grant and correlated groups. We can divide the project executions in the following schedule. The steps are as described in Section 2.

Steps	**Quarters**							
	1	2	3	4	5	6	7	8
1. Master Courses	x	x	x	x	x	x		
2. Concepts			x	x	x	x		
3. Main Article				x	x	x	x	
4. Exploration					x	x	x	
5. Masther Thesis					x	x	x	x

Table 1: Activities planning, te steps are as described in th Section 2. Red color indicates the focus of the quarter that may be divided between activities for some crucial quarters.

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