Linear Algebra 101 - Systems of equations

What is it?

A branch of mathematics that:

Provides the tools to represent, analyze, and solve multiple linear equations simultaneously using matrices, and matrix operations.

Solving $Ax = b$ is no more than finding where these objects intersect.

Key concepts (systems of linear equations):

- Matrix [A] - Grid of numbers to represent systems or transformations;
- Linear dependence/Independence - Dependent if one row/column = combo of others;
- Singularity/non-singularity - If the determinant of a matrix is 0, then it is said to be singular;
- Determinant (det) - Scalar that tells if a matrix has a unique solution;
- Rank - Number of independent rows/columns of a matrix. Rank = # rows - Solution space dimension;
- Gaussian Elimination - Step-by-step method to solve equations;
- Row echelon form - Matrix simplified to a staircase pattern;
- Pivots - First nonzero entries in each row.

Matrices as the translation of linear equations

2 possible cases

$$\begin{aligned} a + b &= 0 \\ a + 2b &= 0 \end{aligned} \;\rightarrow\; \begin{array}{cc} a & b \\ \hline 1 & 1 \\ 1 & 2 \end{array}$$

Independent columns ⇒ Non-singular A

Non-singular A ⇒ $\det(A) \neq 0$

$\det(A) \neq 0$ ⇒ **Unique solution** to $Ax = b$ ✅

$$\begin{aligned} a + b &= 0 \\ 2a + 2b &= 0 \end{aligned} \;\rightarrow\; \begin{array}{cc} a & b \\ \hline 1 & 1 \\ 2 & 2 \end{array}$$

Dependent columns ⇒ Singular A

Singular A ⇒ $\det(A) = 0$

$\det(A) = 0$ ⇒ **No or infinite solution** to $Ax = b$

Redundant systems (0x + 0y + 0z = 0) ⇒ infinite solutions. ♾️

Contradictory systems (0x + 0y + 0z != 0) ⇒ no solutions. 🚫

Determinants

$$[M] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \det[M] = \begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 \end{array} \Rightarrow$$

⇒ det[M] = 1x2x2 + 1x1x1 + 1x1x1 - 1x2x1 - 1x1x1 - 2x1x1

main diagonals product (positive) | anti-diagonals product (negative)

det[M] = **1** ⇒ Unique solution

Properties

$$\det(AB) = \det(A)\det(B)$$

$$\det(A^T) = \det(A) \qquad \det(A^{-1}) = \frac{1}{\det(A)}$$

Effect of Row Operations

- Swapping 2 rows: **det()*(−1)**.
- Multiplying a row by a scalar k: **det()*k**.
- Adding a multiple of one row to another: No change.

Gaussian Elimination

Augmented matrix [A | b]

Reduce matrices to row echelon form to solve systems through row operations.

$$\begin{aligned} 2a - b + c &= 1 \\ 2a + 2b + 4c &= -2 \\ 4a + b &= -1 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & -1 \end{array}\right] \rightarrow (\ldots) \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

System of linear equations | Augmented matrix form | Row 1 → 1/2Row 1 | Row 1 → Row 1 - 1/2 Row 2 | Row echelon form/System Solution

Identity Matrix

Solution:

a = 0 b = -1 c = 0