2D Geometric Transformations

Graphics Systems /

Computer Graphics and Interfaces

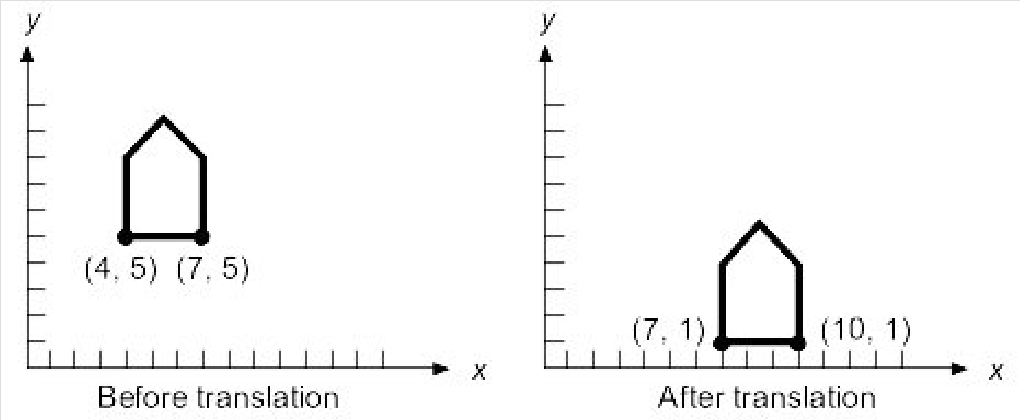
2D Geometric Transformations

The geometric transformations in computer graphics are essential to position, change the orientation, and scale objects in the scene created. The movement is also implemented by the processing parameters vary over time.

Transformations:

- Translation
- Scaling
- Rotation

Translation



$$\begin{cases} x_T = X + T_x \\ y_T = Y + T_y \end{cases}$$

Vertices: (4,5) and (7,5)

$T_x = 3$ $T_y = -4$

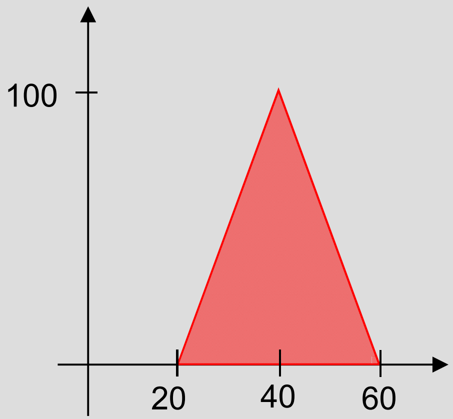
The pair of translation is named "*translation vector*". Each vertex is assigned a displacement **T**:

$$\begin{bmatrix} x_T \\ y_T \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

In the form of matrix product:

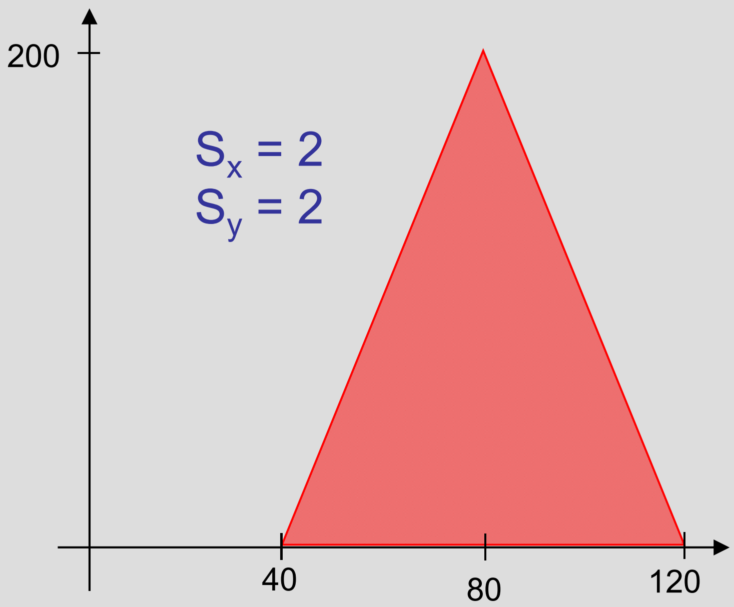
$$\begin{bmatrix} x_T \\ y_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

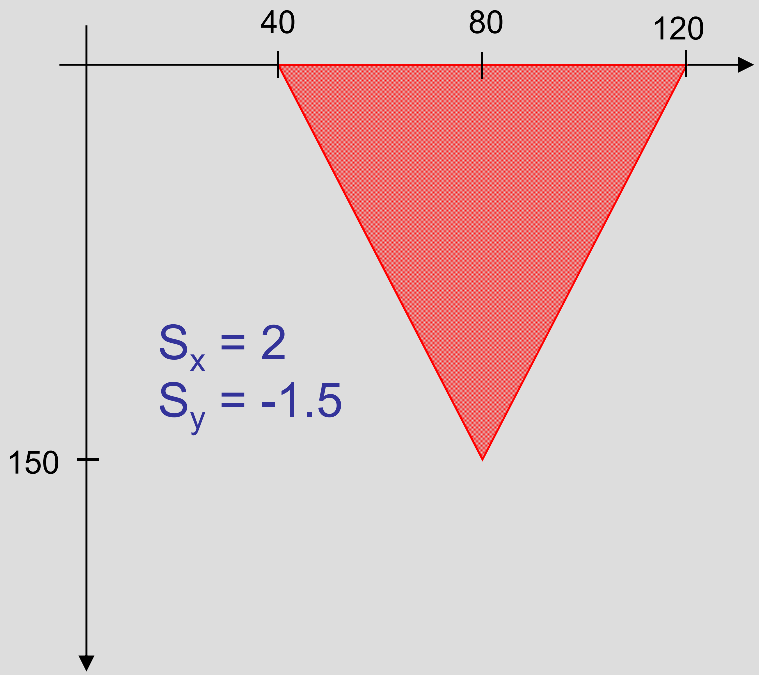
Scaling



$$\begin{cases} x_S = x * S_x \\ y_S = y * S_y \end{cases}$$

Relative to the origin of coordinates.





Scaling

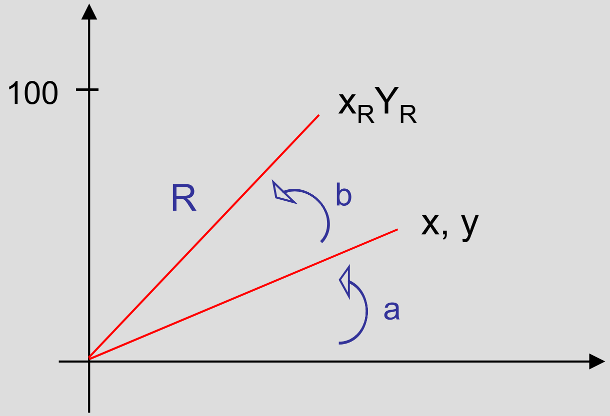
In matrix form:

$$\begin{bmatrix} x_S \\ y_S \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale Factor:

> 1	Increase the object size
<1	Reduces the object size
$S_x = S_y$	Uniform scaling factor → does not distort the object

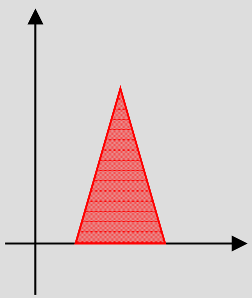
Rotation



$$\begin{cases} x = R.\cos(a) \\ y = R.\sin(a) \end{cases}$$

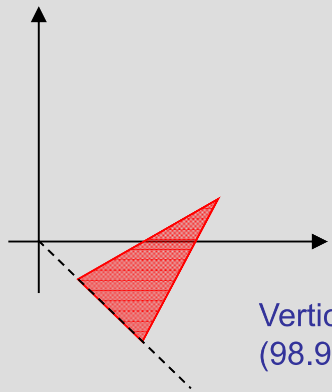
Around the origin.

$$\begin{cases} x_R = R.\cos(a+b) = R.\cos(a).\cos(b) - R.\sin(a).\text{sen}(b) = x.\cos(b) - y.\sin(b) \\ y_R = R.\sin(a+b) = R.\sin(b).\cos(a) + R.\sin(a).\cos(b) = x.\sin(b) + y.\cos(b) \end{cases}$$



Vertices: (20.0) (60.0) (40,100)

Rotation of -45º



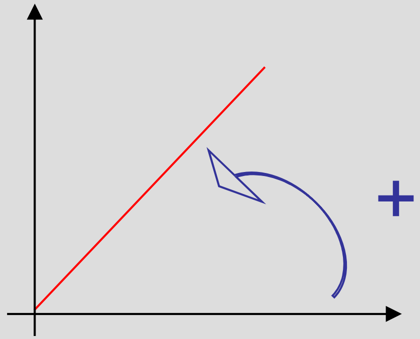
Vertices: (14:14, -14.14), (42.43, -42.43), (98.99, 42.43)

Rotation

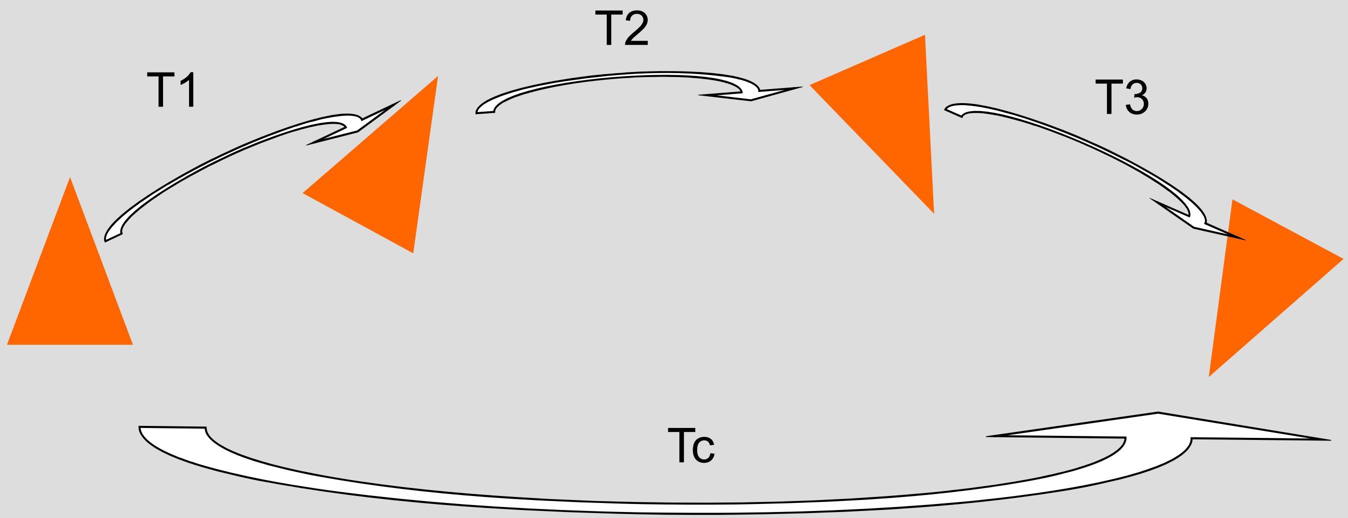
In matrix form:

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: *b* is positive in the counter-clockwise direction.



Composition/Concatenation of Transformations

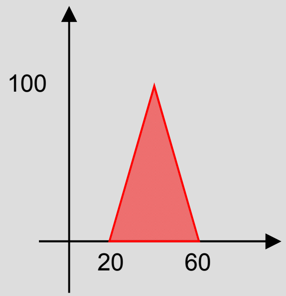


The application of a sequence of operations

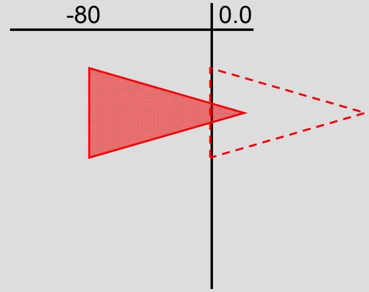
=

1 single transformation: how to calculate?

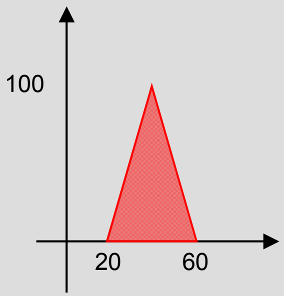
Composition of Transformations



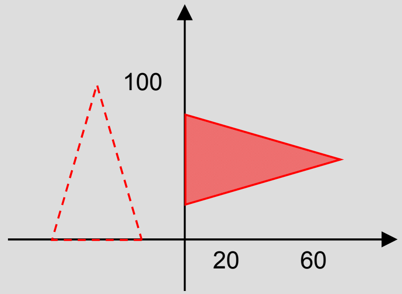
Initial situation	After rotation	After translation
(20.0)	(0, -20)	(-80, -20)
(60.0)	(0, -60)	(-80, -60)
(40,100)	(100, -40)	(20, -40)
	a = - 90 °	Tx = -80, Ty = 0



Swapping the transformations:



Initial situation	After translation	After rotation
(20.0)	(-60.0)	(0.60)
(60.0)	(-20.0)	(0.20)
(40,100)	(-40.100)	(100.40)
	Tx = -80, Ty = 0	a = - 90 °



Conclusion: The application of the transformations is <u>not commutative</u>

Homogeneous coordinates

The previous sequence, rotation followed by translation, then applied to each vertex can be written as:

1 $$\begin{bmatrix} x_{R1} \\ y_{R1} \end{bmatrix} = \begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

2 $$\begin{bmatrix} x_{T2} \\ y_{T2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \end{bmatrix} . \begin{bmatrix} x_{R1} \\ y_{R1} \\ 1 \end{bmatrix}$$

If the matrices representing the transformations were of the same dimensions, they would be able to combinable/multipliable.

However, the above transformations can also be written as (homogeneous coordinates)

$$\begin{bmatrix} x_{R1} \\ y_{R1} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(a) & -\sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = R(a) . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{T2} \\ y_{T2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x_{R1} \\ y_{R1} \\ 1 \end{bmatrix} = T(T_x, T_y) . \begin{bmatrix} x_{R1} \\ y_{R1} \\ 1 \end{bmatrix}$$

We can then write:

$$\begin{bmatrix} x_{T2} \\ y_{T2} \\ 1 \end{bmatrix} = T(T_x, T_y) . R(a) . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The matrix product is:

- **Associative**
- **In general it is Not Commutative**

$$T(T_x, T_y) . R(a) \neq R(a) . T(T_x, T_y)$$

Homogeneous coordinates - summary

Rotation Matrix

$$\begin{bmatrix} \cos(a) & -\sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R(a)$$

Matrix Translation

$$\begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} = T(T_x, T_y)$$

Escalation Matrix

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = S(s_x, s_y)$$

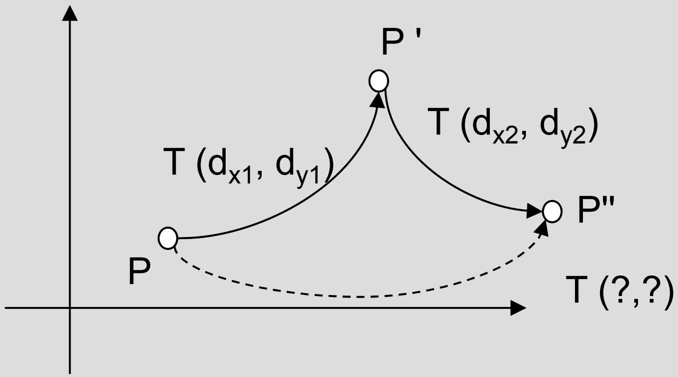
In homogeneous coordinates of an object *n* dimensions is represented in space *n +1* dimensions.

(X, y) → (x.h, y.h, h)

2D 3D

We consider h = 1

Transformations - Examples



$P' = T(d_{x1}, d_{y1}) * P$

$P'' = T(d_{x2}, d_{y2}) * P'$

$P'' = T(d_{x2}, d_{y2}) * T(d_{x1}, d_{y1}) * P$

$$\begin{bmatrix} 1 & 0 & d_{x2} \\ 0 & 1 & d_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & d_{x1} \\ 0 & 1 & d_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_{x2}+d_{x1} \\ 0 & 1 & d_{y2}+d_{y1} \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(d_{x2}, d_{y2}) * T(d_{x1}, d_{y1}) = T(d_{x1}+ d_{x2}, d_{y1}+ d_{y2})$$

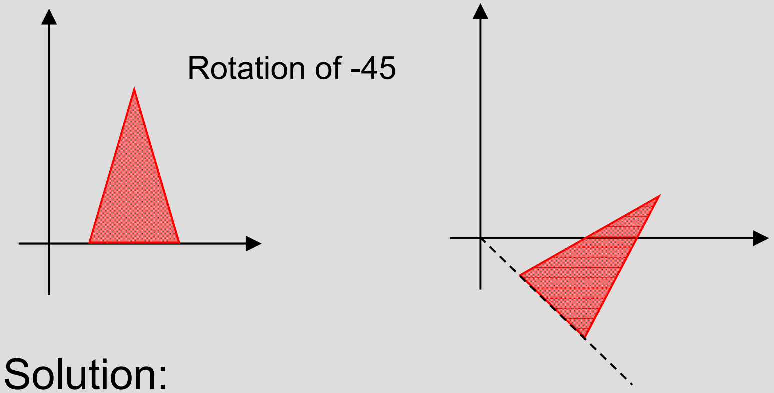
Exercise: verify that

$S(s_{x2},s_{y2})*S(s_{x1},s_{y1})=S(s_{x1}*s_{x2}, s_{y1}*s_{y2})$

$R(a_2)*R(a_1) = R(a_2+a_1)$

Transformations on a arbitrary point (pivot)

Rotation



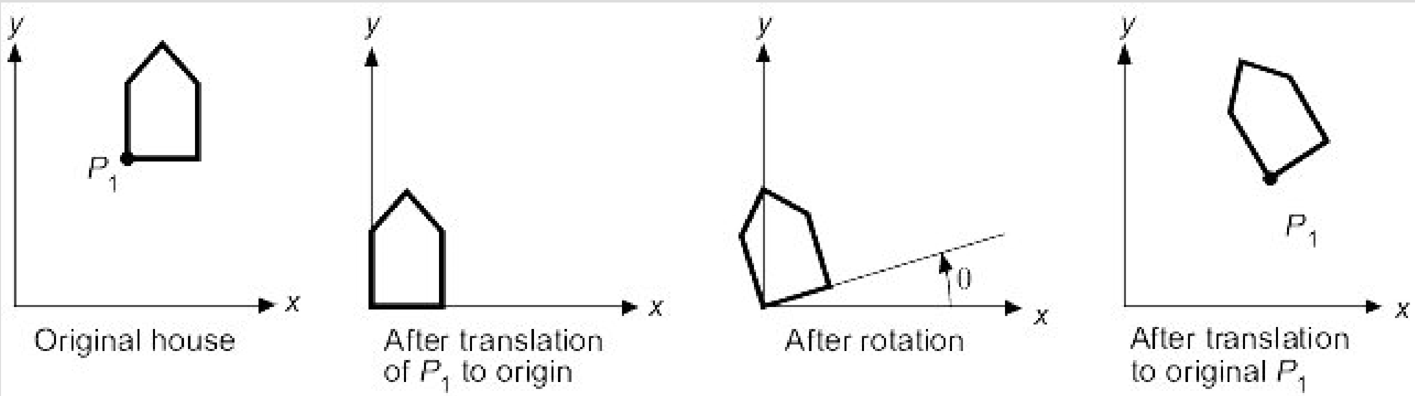
The rotation moves objects around the origin.

Solution:

• Translate the object so that the ***pivot*** point coincides with the origin

• Rotate the object around the origin

• Translate the object so that the **pivot** point returns to the initial position



Transformations on a arbitrary point (pivot)

Transformation matrix

$$\begin{bmatrix}1 & 0 & d_x\\0 & 1 & d_y\\0 & 0 & 1\end{bmatrix}.\begin{bmatrix}\cos(a) & -\sin(a) & 0\\\sin(a) & \cos(a) & 0\\0 & 0 & 1\end{bmatrix}.\begin{bmatrix}1 & 0 & -d_x\\0 & 1 & -d_y\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos(a) & -\sin(a) & d_x(1-\cos(a))+d_y\sin(a))\\\sin(a) & \cos(a) & d_y(1-\cos(a))-d_x\sin(a))\\0 & 0 & 1\end{bmatrix}=$$

$$T(d_x,d_y).R(a).T(-d_x,-d_y)$$

Scaling

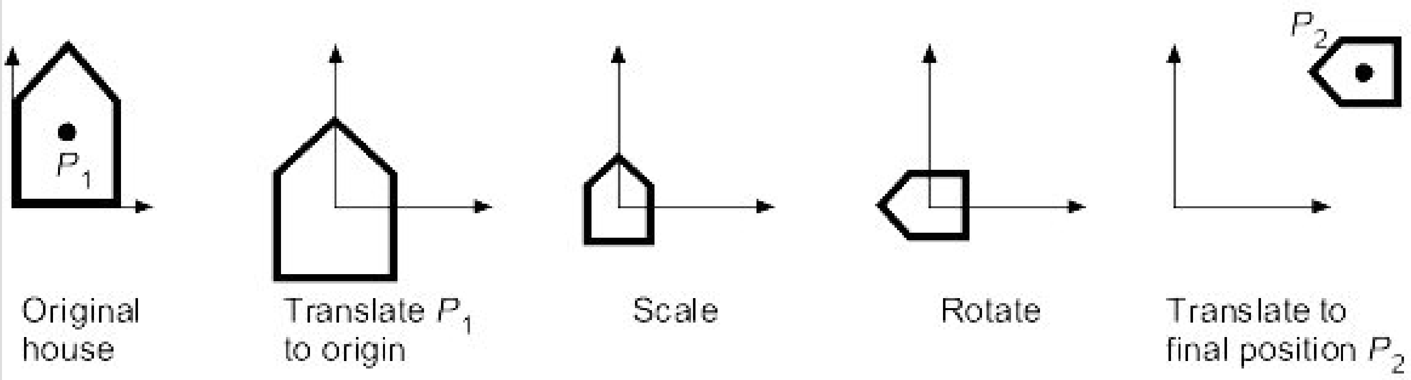
Solution:

- Translate the object so that the ***pivot*** point coincides with the origin
- Scale the object (relative to the origin)
- Translate the object so that the **pivot** point returns to the initial position

$$\begin{bmatrix}1 & 0 & d_x\\0 & 1 & d_y\\0 & 0 & 1\end{bmatrix}.\begin{bmatrix}s_x & 0 & 0\\0 & s_y & 0\\0 & 0 & 1\end{bmatrix}.\begin{bmatrix}1 & 0 & -d_x\\0 & 1 & -d_y\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}s_x & 0 & d_x(1-s_x)\\0 & s_y & d_y(1-s_y)\\0 & 0 & 1\end{bmatrix}=T(d_x,d_y).S(s_x,s_y).T(-d_x,-d_y)$$

Exercise

Determine the transformation matrix for:



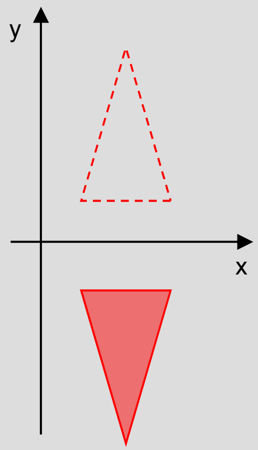
T (-x1,-y1) S (0.5,0.5) R (90) T (x2, y2)

If P_1= (1, 2) and P_2= (3, 3) determine the matrix of the equivalent transformation.

Other transformations

Reflection

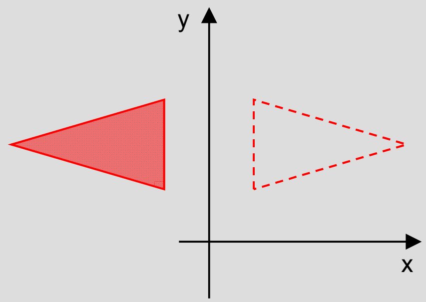
In relation to **axis *x*** corresponds to a "*flip*" around the axis of reflection, which results in a scaling S(1, -1)



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

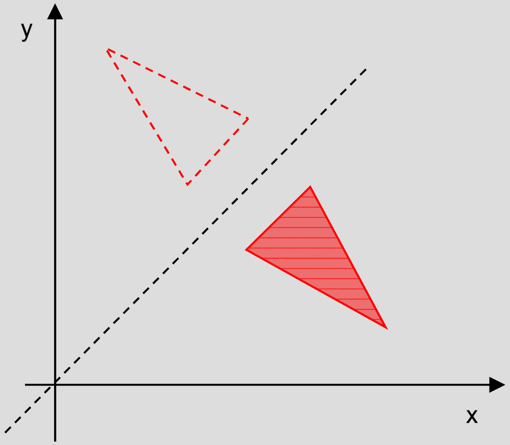
In relation to **axis *y*:**

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Other transformations

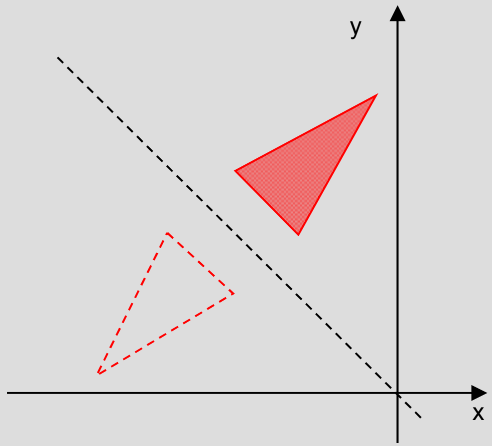
Reflection relative to the line y = x



$$R(45).S(1,-1).R(-45) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection relative to the line y = -x

$$R(45).S(-1,1).R(-45) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Inverse Transformations

If a complex transformation is given by a matrix *M*, then the inverse transformation M^{-1} is the one that puts the object in its initial position.

Once *M* represents one or more transformations, the inverse matrix should be.

$$M.M^{-1} = I$$

For some transformations is easy to find the inverse matrix:

Translation:

$$T^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

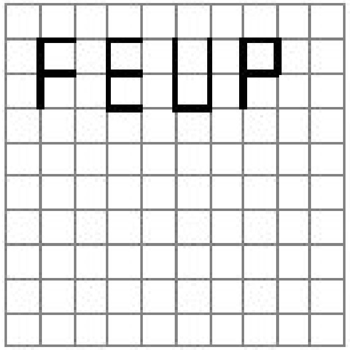
Scaling:

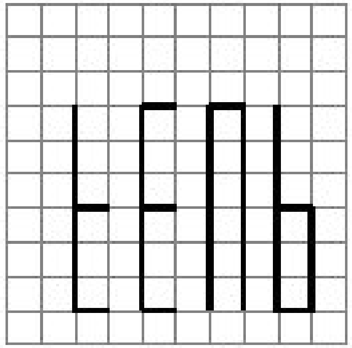
$$S^{-1} = \begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise

2. Da figura seguinte,

a)- Determine a matriz de transformação **2D** necessária para passar a letra *F* da situação da esquerda para a da direita.





b)- Comente a afirmação "A matriz encontrada na alínea anterior é aplicável às restantes três letras".

(Question of the May 23, 2002 test)