Loops, Necklaces and Braiding for 3D Topological Phases

Eric Rowell

DMS 1802145 and 1410144.

March 2018

2D Summary

Topological quantum computation (TQC) requires topological phases of matter (TPM)

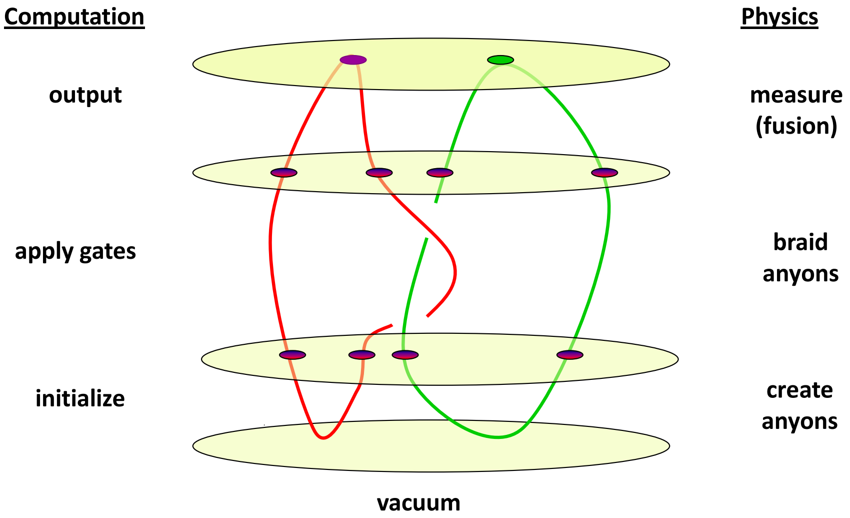
Definition (Nayak, Simon, Stern, Freedman, Das Sarma)

a [2D, bosonic] system is in a topological phase if its low-energy effective field theory is a $[(2+1)D]$ topological quantum field theory (TQFT).

Theorem (Bartlett, Douglas, Schommer-Pries, Vicary and Reshetikhin, Turaev)

$(2+1)$-*TQFTs* $\overset{1-1}{\leftrightarrow}$ *modular categories*.

TQC: Braiding Anyons



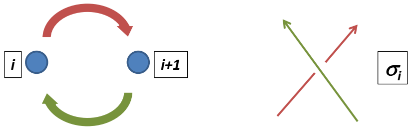
2D Topological Quantum Gates

The braid group $\mathcal{B}_n$ acts on **topological state spaces**:

- Fix anyons x, y
- Braid group representation:

$$\mathcal{B}_n \curvearrowright \mathcal{H}(D^2 \setminus \{z_i\}; y, x, \cdots, x) = \mathrm{Hom}(Y, X^{\otimes n})$$

by particle exchange



We know a lot in 2D:

- Many examples/sources of modular categories: quantum groups $U_q\mathfrak{g}$ at roots of 1, II_1 subfactors, finite groups...
- Rank-finiteness [Bruillard,Ng,R,Wang '16]: finitely many $\mathcal{C}$ with $|Irr(\mathcal{C})| = r$
- Naturally approximate topological invariants: Jones polynomial evaluations etc.

- Theorem (R,Wang '16)

  X is non-abelian if and only if $\dim(X) > 1$.

- Conjecture (R '07)

  $|\rho_X(\mathcal{B}_n)| < \infty$ (X non-universal) if, and only if, $\dim(X)^2 \in \mathbb{Z}$

3D Generalization

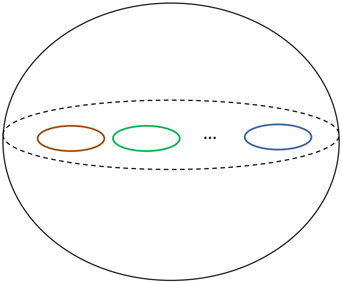
- 3D TPMs?
- Mathematically: a [3D, bosonic] system is in a topological phase if its low-energy effective field theory is a $[(3+1)D]$ topological quantum field theory (TQFT).
- Algebraic characterization? X. Cui: G-crossed braided spherical fusion categories...

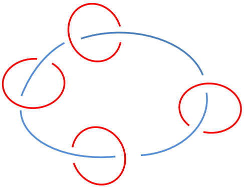
3D Topological Quantum Gates

~~Point-like excitations~~ in $\mathbb{R}^3$ are boring

$MCG(\mathbb{R}^3 - \{p_i\}) = \mathfrak{S}_n$, $\pi_1(\mathbb{R}^3 - \{p_i\}) = 1$

loop-like excitations (linked/knotted vortices)?





Motion Groups! (Damani's talk)

We know precious little in 3D:

- Well-understood examples/sources of (3+1)TQFTs: Twisted Dijkgraaf-Witten and BF (Baez,Crane-Yetter).
- 3D twisted DW-theories: input $(\mathrm{Rep}(G), \omega)$, essentially computes $|\,\mathrm{Hom}(\pi_1(M^3), G)|$
- BF theories: input modular category, essentially computes signature $\sigma(M^4)$
- Major open question: is there a $3+1$TQFT that detects smooth structure?
- Generally: how sensitive are these invariants?

Categorical Approach

- G-crossed braided spherical fusion categories 
- Walker-Wang: unitary pre-modular categories
- expectation: $\mathcal{C}$ unitary pre-modular, and all transparent objects are bosons: Dijkgraaf-Witten+BF.
- Recommended approach: super-modular categories. (Bullivant?)

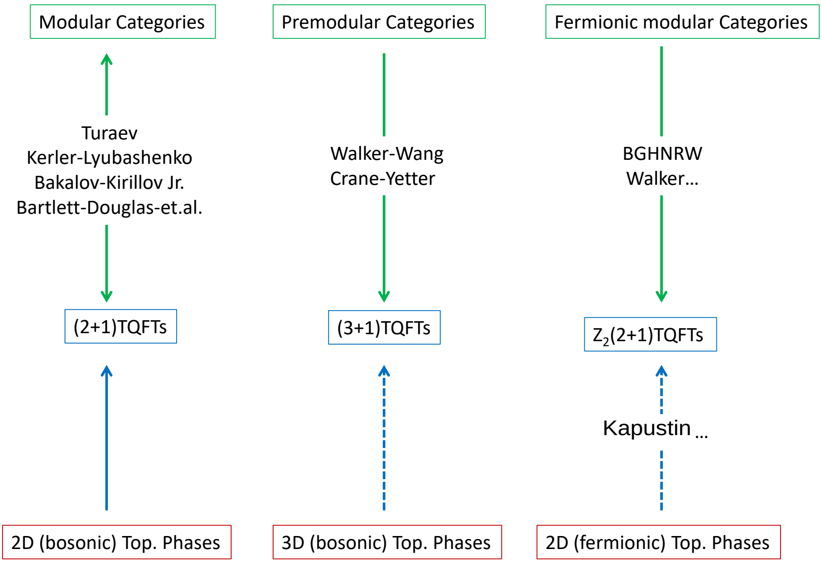
Super-Modular Categories

Definition

A unitary pre-modular category $\mathcal{C}$ is super-modular if has one transparent fermion. I.e. Mueger center $\mathcal{C}' \cong sVec = \mathrm{Rep}(\mathbb{Z}_2, 1)$.

Fact

- *Every unitary pre-modular category is the equivariantization of a modular or super-modular category.*
- *A super-modular category $\rightsquigarrow$ representations of spin mapping class groups.*
- *Condensing the fermion $\rightsquigarrow$ sVec-enriched "fermionic modular category."*


Modular Categories
Premodular Categories
Fermionic modular Categories
Turaev
Kerler-Lyubashenko
Bakalov-Kirillov Jr.
Bartlett-Douglas-et.al.
Walker-Wang
Crane-Yetter
BGHNRW
Walker…
(2+1)TQFTs
(3+1)TQFTs
Z_2(2+1)TQFTs
Kapustin …
2D (bosonic) Top. Phases
3D (bosonic) Top. Phases
2D (fermionic) Top. Phases

Petulant Demands

But I want my 3D topological quantum gates now!

Temporary solution: Lobotomize

$$\text{(3+1)TQFTs} \longmapsto \text{Motion group reps.}$$

$$\begin{tikzcd}
\mathcal{B}_n \arrow[r, hook] & \hat{A}\mathcal{B}_n \arrow[r, hook, "\triangleleft"] \arrow[dr, hook] & C\mathcal{B}_n \arrow[d, two heads] & \\
 & & N\mathcal{B}_n \arrow[d, two heads] & \\
 & & N\mathcal{B}_n/\langle \tau^n \rangle \arrow[r, hook] & L\mathcal{B}_n
\end{tikzcd}$$

Key fact: $\mathcal{B}_n$ survives all maps.

Burning Questions/Problems

- Which $\mathcal{B}_n$ reps lift to $L\mathcal{B}_n$ and/or $N\mathcal{B}_n$?
- Classify lifts.
- Genuinely new ("3D") quantum gates?
- Invariants of surfaces in $\mathbb{R}^4$?
- Classify irreps. (say, of low dimension).
- Local representations/Yang-Baxter extensions?

For $L\mathcal{B}_n$

Theorem (Chang,Plavnik,Hong,Sun,Bruillard,R)

(ρ, V) a rep of $\mathcal{B}_3$, and $\dim(V) \leq 5$*, ρ extends to $L\mathcal{B}_3$ (usually many ways).* $\dim(V) = 6$*, not all extend.*

Definition

If $R \in \mathrm{End}(V \otimes V)$ satisfies the Yang-Baxter equation we say (V, R) is a **Braided Vector Space**.

$\rho^R(\sigma_i) = I^{\otimes i-1} \otimes R \otimes I^{\otimes n-i-1}$ gives $\rho^R : \mathcal{B}_n \to GL(V^{\otimes n})$

Question

For which (V, R) is there an $S \in \mathrm{End}(V \otimes V)$ such that $\rho^R(s_i) = I^{\otimes i-1} \otimes S \otimes I^{\otimes n-i-1}$ is an extension to $\rho^R : \mathcal{LB}_n \to GL(V^{\otimes n})$?

Answer: not always!

Definition

A BVS (V, R) is of (right) **group-type** if there is an ordered basis $X := [x_1, \ldots, x_n]$ of V and $g_i \in GL(V)$ such that $R(z \otimes x_i) = x_i \otimes g_i(z)$ for all i and $z \in V$.

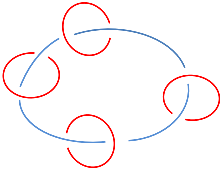
Proposition (Kadar,Martin,R,Wang)

If (V, R) is a BVS of right group-type then $\rho^R(s_i) = I_V^{\otimes i-1} \otimes S \otimes I_V^{\otimes n-i-1}$ with $S(x_i \otimes x_j) := x_j \otimes x_i$ defines an extension of ρ^R to $\mathcal{LB}_n$.

Conjecture (Slingerland, modified)

BVS (V, R) extends only if it is of group-type. "Local representations of $\mathcal{LB}_n$ come from finite groups."

Necklace Braid Group



σ_i: leapfrogging and

τ: cyclic shift. $\sigma_1, \ldots, \sigma_n$, τ satisfying (indices taken modulo n, with $\sigma_{n+1} := \sigma_1$ and $\sigma_0 := \sigma_n$):

(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \neq 1 \pmod{n}$,

(N1) $\tau \sigma_i \tau^{-1} = \sigma_{i+1}$ for $1 \leq i \leq n$

(N2) $\tau^{2n} = 1$

A Smörgåsbord of Results on NB_n Representations

Theorem (Bullivant,Kimball,R,Martin)

- *$\mathcal{B}_n$ completely reducible rep. ρ extends to $N\mathcal{B}_n$, at least in a standard way: $\tau \to D\rho(\sigma_1\sigma_2\cdots\sigma_{n-1})$*
- *Any local representation (R, V) of $\mathcal{B}_n$ extends to $N\mathcal{B}_n$*
- *For $n > 4$, there are no irreducible $N\mathcal{B}_n$ reps V with $1 < \dim(V) < n-2$.*

Quaternionic $N\mathcal{B}_n$ reps

Let $\mathcal{Q}_n$ be the algebra on $t, u_1, \ldots, u_n, v_1, \ldots, v_n$ satisfying:

1. $u_i^2 = v_i^2 = -1$ for all i,
2. $[u_i, v_j] = -1$ if $|i - j| \pmod{n} \in \{0, 1\}$,
3. $[u_i, v_j] = 1$ if $|i - j| \pmod{n} \notin \{0, 1\}$,
4. $[u_i, u_j] = [v_i, v_j] = 1 = t^n$,
5. $tu_it^{-1} = u_{i+1}$, and $tv_it^{-1} = v_{i+1}$ for all i.

Then $\tau \mapsto t$ and $\sigma_i \mapsto (1 + u_i + v_i + u_iv_i)$ is a rep of $N\mathcal{B}_n$.
$\langle u_i, v_i \rangle \cong Q_8$.

Outlook

Question

Are there universal 3D "anyons"? Do (3+1)TQFT give hard invariants?

Thank you!