# Wilson surfaces for surface knots

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# Introduction

## Knots are interesting in topology...

- knots: embeddings $S^1$ into $S^3$, say.
- $S^1$ lowest dimensional non trivial sphere $\Rightarrow$ 2–knots: embeddings $S^2$ into $S^4$.
- $S^1$ lowest dimensional non trivial closed oriented manifold $\Rightarrow$ surface–knots: embeddings $S_\ell$ into $S^4$, $S_\ell$ genus $\ell$ closed oriented surface.
- Higher knots objects of intense investigation by topologists.

## Knots are interesting in gauge theory...

- In gauge theory, knots $\Rightarrow$ Wilson loops

$$W(\xi) = \mathrm{tr}_R \left[ \mathrm{Pexp} \left( -\int_\xi A \right) \right]. \tag{1.1}$$

- Wilson loops are relevant in:

  confinement in quantum chromodynamics,

  loop formulation of quantum gravity,

  symmetry braking in string theory,

  condensed matter theory,

  knot topology in Chern–Simons theory.

- In higher gauge theory, 2–knots/surface knots $\Rightarrow$ Wilson surfaces

$$W(\Xi) = ? \qquad (1.2)$$

- Wilson surfaces may turn out to be relevant in:

  non perturbative aspects of higher gauge theory,

  brane theory,

  quantum gravity,

  higher knot topology in higher Chern–Simons theory...

- The problem has two parts:

  define surface knot holonomy;

  define higher invariant traces.

- *Warning*: Parallel transport and holonomy related but distinguished.

  holonomy $\Rightarrow$ parallel transport.

- Earlier work on higher parallel transport: Caetano and Picken; Schreiber and Waldorf; Faria Martins and Picken; Chatterjee, Lahiri and Sengupta: Soncini and Z; Abbaspour and Wagemann; Arias Abad and Schaetz; ...

- Earlier work on higher holonomy: Cattaneo and Rossi; Faria Martins and Picken.

**Strategy: describe knots by curves and surface knots by surfaces...**

- Curves $\gamma : p_0 \to p_1$ in a manifold $M$.

- Homotopies $h : \gamma_0 \Rightarrow \gamma_1$ of curves $\Rightarrow$ fundamental 1-groupoid $(M, P^0{}_1M)$.

- Ordinary gauge theory with gauge Lie group $G$ and flat gauge field $A \Rightarrow$ gauge covariant homotopy invariant parallel transport functor

$$F_A : (M, P^0{}_1 M) \to BG, \quad \gamma \to F_A(\gamma).$$

- Knot $\xi$ based at $p$ up to ambient isotopy $\Rightarrow$ curve $\gamma_\xi : p \to p$ up to homotopy $\Rightarrow$ holonomy

$$F_A(\xi) = F_A(\gamma_\xi).$$

- Problems: ensure appropriate base point and isotopy invariance and gauge independence.
- Generalize to higher knots.

  Look for a “gentle” generalization.
- Curves $\gamma : p_0 \to p_1$ and surfaces $\Sigma : \gamma_0 \Rightarrow \gamma_1$ in a manifold $M$.
- Thin homotopies $h : \gamma_0 \Rightarrow \gamma_1$ of curves and homotopies $H : \Sigma_0 \Rrightarrow \Sigma_1$ of surfaces $\Rightarrow$ fundamental 2–groupoid $(M, P_1 M, P^0{}_2 M)$.

- Strict higher gauge theory with gauge Lie crossed module $(G,H)$ and flat higher gauge fields $A,B \Rightarrow$ gauge covariant and (thin) homotopy invariant parallel transport 2–functor

$$F_{A,B} : (M, P_1M, {P^0}_2M) \to B(G,H), \quad \gamma \to F_A(\gamma), \quad \Sigma \to F_{A,B}(\Sigma).$$

- Knot $\xi$ based at $p$ and surface knot $\Xi$ based at a genus dependent fundamental polygon $\tau$ (cutting image of $\Xi$ along standard $a$ and $b$ cycles) up to ambient isotopy $\Rightarrow$ Curve $\gamma_\xi : p \to p$ and surface $\Sigma_\Xi : \iota_p \Rightarrow \tau$ up to (thin) homotopy $\Rightarrow$ holonomy

$$F_A(\xi) = F_A(\gamma_\xi), \quad F_{A,B}(\Xi) = F_{A,B}(\Sigma_\Xi).$$

Problems: higher marking and higher isotopy invariance and gauge independence.

# Curves, surfaces and homotopy

## Closed curves and surfaces describe knots and surface knots...

- *A smooth map $f : S \times \mathbb{R} \to T$ has sitting instants if*

$$f(-,x) = f(-,0) \quad \textit{for } x < \epsilon, \quad f(-,x) = f(-,1) \quad \textit{for } x > 1-\epsilon,$$

  *with* $0 < \epsilon < 1/2$. All maps assumed with sitting instants for each factor $\mathbb{R}$ of their domains.

- For points $p_0, p_1 \in M$, a curve $\gamma : p_0 \to p_1$ in $M$ is a map $\gamma : \mathbb{R} \to M$ s. t.

$$\gamma(0) = p_0, \quad \gamma(1) = p_1.$$

- *For points $p_0, p_1 \in M$ and curves $\gamma_0, \gamma_1 : p_0 \to p_1$ pf $M$, a surface $\Sigma : \gamma_0 \Rightarrow \gamma_1$ of $M$ is a map $\Sigma : \mathbb{R}^2 \to M$ s. t.*

$$\Sigma(0,y) = p_0, \quad \Sigma(1,y) = p_1, \quad \Sigma(x,0) = \gamma_0(x), \quad \Sigma(x,1) = \gamma_1(x)$$

- *Curve operations. For a point* $p$*, the unit curve* $\iota_p : p \to p$

$$\iota_p(x) = p.$$

*For a curve* $\gamma : p_0 \to p_1$*, the inverse curve* $\gamma^{-1\circ} : p_1 \to p_0$

$$\gamma^{-1\circ}(x) = \gamma(1 - x).$$

*For curves* $\gamma_1 : p_0 \to p_1$*,* $\gamma_2 : p_1 \to p_2$*, composed curve* $\gamma_2 \circ \gamma_1 : p_0 \to p_2$

$$\gamma_2 \circ \gamma_1(x) = \gamma_1(2x) \qquad \text{for } x \leq 1/2,$$

$$\gamma_2 \circ \gamma_1(x) = \gamma_2(2x - 1) \qquad \text{for } x \geq 1/2.$$

- *Surface operations. For a curve* $\gamma : p_0 \to p_1$*, the unit surface* $I_\gamma : \gamma \Rightarrow \gamma$

$$I_\gamma(x, y) = \gamma(x).$$

*For a surface* $\Sigma : \gamma_0 \Rightarrow \gamma_1$*, the vertical inverse* $\Sigma^{-1\bullet} : \gamma_1 \Rightarrow \gamma_0$

$$\Sigma^{-1\bullet}(x, y) = \Sigma(x, 1 - y).$$

*For surfaces* $\Sigma_1 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_2 : \gamma_1 \Rightarrow \gamma_2$, *the vertical composition* $\Sigma_2 \bullet \Sigma_1 : \gamma_0 \Rightarrow \gamma_2$

$$\Sigma_2 \bullet \Sigma_1(x,y) = \Sigma_1(x,2y) \qquad \text{for } y \leq 1/2, \tag{2.1}$$

$$\Sigma_2 \bullet \Sigma_1(x,y) = \Sigma_2(x,2y-1) \qquad \text{for } y \geq 1/2. \tag{2.2}$$

*For a surface* $\Sigma : \gamma_0 \Rightarrow \gamma_1$, *the horizontal inverse* $\Sigma^{-1\circ} : {\gamma_0}^{-1\circ} \Rightarrow {\gamma_1}^{-1\circ}$

$$\Sigma^{-1\circ}(x,y) = \Sigma(1-x,y).$$

*For surfaces* $\Sigma_1 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_2 : \gamma_2 \Rightarrow \gamma_3$, *the horizontal composition* $\Sigma_2 \circ \Sigma_1 : \gamma_2 \circ \gamma_0 \Rightarrow \gamma_3 \circ \gamma_1$

$$\Sigma_2 \circ \Sigma_1(x,y) = \Sigma_1(2x,y) \qquad \text{for } x \leq 1/2, \tag{2.3}$$

$$\Sigma_2 \circ \Sigma_1(x,y) = \Sigma_2(2x-1,y) \qquad \text{for } x \geq 1/2. \tag{2.4}$$

Curve and surface operations “nice” only up to homotopy.

- *A homotopy* $h : \gamma_0 \Rightarrow \gamma_1$ *of curves* $\gamma_0, \gamma_1 : p_0 \to p_1$ *of* $M$ *is a map* $h : \mathbb{R}^2 \to M$ *of* $M$ *s. t.*

$$h(0, y) = p_0, \quad h(1, y) = p_1, \quad h(x, 0) = \gamma_0(x), \quad h(x, 1) = \gamma_1(x).$$

*The homotopy is thin if in addition* $\operatorname{rank} dh(x, y) < 2$. *(Thin) homotopy of curves an equivalence relation.*

- *A homotopy* $H : \Sigma_0 \Rrightarrow \Sigma_1$ *of surfaces* $\Sigma_0 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_1 : \gamma_2 \Rightarrow \gamma_3$ *where* $\gamma_0, \gamma_1, \gamma_2, \gamma_3 : p_0 \to p_1$ *is a map* $H : \mathbb{R}^3 \to M$ *s. t.* $\operatorname{rank} dH(x, 0, z)$, $\operatorname{rank} dH(x, 1, z) \leq 1$ *and*

$$H(0, y, z) = p_0, \quad H(1, y, z) = p_1,$$
$$H(x, y, 0) = \Sigma_0(x, y), \quad H(x, y, 1) = \Sigma_1(x, y).$$

*The homotopy is thin if* $\operatorname{rank} dH(x, y, z) \leq 2$. *(Thin) homotopy of surfaces an equivalence relation.*

- $\Pi_1 M$, $\Pi_2 M$ sets of all curves and surfaces of $M$.
- $P_1 M$ and $P^0{}_1 M$ sets of homotopy and thin homotopy classes of curves.
- $(M, P_1 M)$ *and* $(M, P^0{}_1 M)$ *with the operations induced by those of* $\Pi_1 M$ *are groupoids, the path and fundamental groupoids of* $M$.
- $P_2 M$ and $P^0{}_2 M$ sets of homotopy and thin homotopy classes of surfaces.
- $(M, P_1 M, P_2 M)$ *and* $(M, P_1 M, P^0{}_2 M)$ *with the operations induced by those of* $\Pi_1 M$ *and* $\Pi_2 M$ *are 2–groupoids, the path and fundamental 2–groupoids of* $M$.

# Parallel transport

## Holonomy requires parallel transport...

- Gauge theory with gauge Lie group $G$ and Lie algebra $\mathfrak{g}$.
- *$G$–connection on $M$: $\theta \in \Omega^1(M, \mathfrak{g})$. $\theta$ flat if*

$$d\theta + \frac{1}{2}[\theta, \theta] = 0.$$

- *For $\gamma$ a curve of $M$, $F_\theta(\gamma) \in G$ parallel transport along $\gamma$*

$$F_\theta(\gamma) = u(1),$$

*with $u : \mathbb{R} \to G$ the unique solution of*

$$d_x u(x) u(x)^{-1} = -\gamma^* \theta_x(x), \quad u(0) = 1_G.$$

- *Property*: for any point $p$ and curves $\gamma, \gamma_1, \gamma_2$, whenever defined,

$$F_\theta(\iota_p) = 1_G, \quad F_\theta(\gamma^{-1\circ}) = F_\theta(\gamma)^{-1}, \quad F_\theta(\gamma_2 \circ \gamma_1) = F_\theta(\gamma_2) F_\theta(\gamma_1).$$

- *Property*: for $\gamma_0$, $\gamma_1$ thin homotopic curves

$$F_\theta(\gamma_1) = F_\theta(\gamma_0).$$

  Likewise, if $\theta$ is flat and $\gamma_0$, $\gamma_1$ are homotopic.

- *Parallel transport $\Rightarrow$ functor $\bar{F}_\theta : (M, P_1 M) \to BG$ i from the path groupoid $(M, P_1 M)$ of $M$ into $BG$.*
  *For flat $\theta$, parallel transport $\Rightarrow$ functor $\bar{F}^0{}_\theta : (M, P^0{}_1 M) \to BG$ from the fundamental groupoid $(M, P^0{}_1 M)$ of $M$ into $BG$.*

- *$G$–gauge transformation: $g \in \mathrm{Map}(M, G)$.*

- *Gauge transform ${}^g\theta$ of $\theta$:*

$${}^g\theta = \mathrm{Ad}\, g(\theta) - dgg^{-1}.$$

- *If $\theta$ flat, ${}^g\theta$ flat.*

- *Property*: for $\gamma : p_0 \to p_1$ a curve

$$F_{{}^g\theta}(\gamma) = g(p_1) F_\theta(\gamma) g(p_0)^{-1}.$$

- *$g$ encodes a natural transformation $\bar{F}_\theta \Rightarrow \bar{F}_{{}^g\theta}$ of parallel transport functors. For flat $\theta$, $g$ encodes a natural transformation $\bar{F}^0{}_\theta \Rightarrow \bar{F}^0{}_{{}^g\theta}$ of flat parallel transport functors.*
- Generalization to strict higher gauge theory.
- Higher gauge theory with gauge Lie crossed module $H \xrightarrow{t} G \xrightarrow{m} \mathrm{Aut}(H)$ and differential Lie crossed module $\mathfrak{h} \xrightarrow{\dot{t}} \mathfrak{g} \xrightarrow{\widehat{m}} \mathfrak{der}(\mathfrak{h})$.
- *$(G, H)$–2–connection pair on $M$: $(\theta, \Upsilon)$, $\theta \in \Omega^1(M, \mathfrak{g})$, $\Upsilon \in \Omega^2(M, \mathfrak{h})$ s. t.*

$$d\theta + \frac{1}{2}[\theta, \theta] - \dot{t}(\Upsilon) = 0.$$

*(zero fake curvature condition). $(\theta, \Upsilon)$ flat if*

$$d\Upsilon + \widehat{m}(\theta, \Upsilon) = 0.$$

- *For $\gamma$ a curve of $M$, parallel transport $F_\theta(\gamma) \in G$ defined as done earlier for $G$–connection $\theta$.*

*For $\Sigma$ any surface of $M$, $F_{\theta,\Upsilon}(\Sigma) \in H$ parallel transport along $\Sigma$*

$$F_{\theta,\Upsilon}(\Sigma) = E(0,1),$$

*with $E : \mathbb{R}^2 \to H$ the unique solution of the two step differential problem*

$$\partial_x u(x,y)u(x,y)^{-1} = -\Sigma^*\theta_x(x,y), \quad u(1,y) = 1_G,$$

$$\partial_y v(x,y)v(x,y)^{-1} = -\Sigma^*\theta_y(x,y), \quad v(x,0) = 1_G,$$

$$\partial_x(\partial_y E(x,y)E(x,y)^{-1}) = -\dot{m}(v(1,y)^{-1}u(x,y)^{-1})(\Sigma^*\Upsilon_{xy}(x,y))$$

or $$\partial_y(E(x,y)^{-1}\partial_x E(x,y)) = -\dot{m}(u(x,0)^{-1}v(x,y)^{-1})(\Sigma^*\Upsilon_{xy}(x,y))$$

$$E(1,y) = E(x,0) = 1_H$$

*with $u, v : \mathbb{R}^2 \to G$.*

- The two forms of the differential problem for $E$ are equivalent: any solution of one is automatically solution of the other.

- *Property*: for any surface $\Sigma : \gamma_0 \Rightarrow \gamma_1$

$$F_\theta(\gamma_1) = t(F_{\theta,\Upsilon}(\Sigma))F_\theta(\gamma_0).$$

- *Property*: For any point $p$, curves $\gamma, \gamma_1, \gamma_2$ and surfaces $\Sigma, \Sigma_1, \Sigma_2$, if defined,

$$F_\theta(\iota_p) = 1_G, \quad F_\theta(\gamma^{-1\circ}) = F_\theta(\gamma)^{-1}, \quad F_\theta(\gamma_2 \circ \gamma_1) = F_\theta(\gamma_2)F_\theta(\gamma_1);$$

$$F_{\theta,\Upsilon}(I_\gamma) = 1_H, \quad F_{\theta,\Upsilon}(\Sigma^{-1\bullet}) = F_{\theta,\Upsilon}(\Sigma)^{-1},$$

$$F_{\theta,\Upsilon}(\Sigma_2 \bullet \Sigma_1) = F_{\theta,\Upsilon}(\Sigma_2)F_{\theta,\Upsilon}(\Sigma_1),$$

$$F_{\theta,\Upsilon}(\Sigma^{-1\circ}) = m(F_\theta(\gamma_0)^{-1})(F_{\theta,\Upsilon}(\Sigma)^{-1}),$$

$$F_{\theta,\Upsilon}(\Sigma_2 \circ \Sigma_1) = F_{\theta,\Upsilon}(\Sigma_2)m(F_\theta(\gamma_2))(F_{\theta,\Upsilon}(\Sigma_1)),$$

where $\Sigma_1 : \gamma_0 \to \gamma_1$ and $\Sigma_2 : \gamma_2 \to \gamma_3$.

- *Property*: for thin homotopic curves for $\gamma_0$, $\gamma_1$

$$F_\theta(\gamma_1) = F_\theta(\gamma_0).$$

*Property:* for thin homotopic surfaces $\varSigma_0 : \gamma_{00} \Rightarrow \gamma_{01}$, $\varSigma_1 : \gamma_{10} \Rightarrow \gamma_{11}$,

$$F_\theta(\gamma_{10}) = F_\theta(\gamma_{00}), \quad F_\theta(\gamma_{11}) = F_\theta(\gamma_{01}), \quad F_{\theta,\varUpsilon}(\varSigma_1) = F_{\theta,\varUpsilon}(\varSigma_0).$$

Likewise, if $(\theta, \varUpsilon)$ is flat and $\varSigma_0$, $\varSigma_1$ are homotopic.

- *Parallel transport* $\Rightarrow$ *strict* $2$*–functor* $\bar{F}_{\theta,\varUpsilon} : (M, P_1M, P_2M) \rightarrow B(G, H)$ *from the path* $2$*–groupoid* $(M, P_1M, P_2M)$ *of* $M$ *into* $B(G, H)$*.*
  *For flat* $(\theta, \varUpsilon)$*, parallel transport* $\Rightarrow$ *strict* $2$*–functor* $\bar{F}^0{}_{\theta,\varUpsilon} : (M, P_1M, P^0{}_2M)$ $\rightarrow B(G, H)$ *from the fundamental* $2$*–groupoid* $(M, P_1M, P^0{}_2M)$ *of* $M$ *into* $B(G, H)$*.*
- $(G, H)$*–*$1$*–gauge transformation:* $(g, J)$*,* $g \in \mathrm{Map}(M, G)$*,* $J \in \Omega^1(M, \mathfrak{h})$*.*
- *1–gauge transform* $({}^{g,J}\theta, {}^{g,J}\varUpsilon)$ *of* $(\theta, \varUpsilon)$*:*

$${}^{g,J}\theta = \mathrm{Ad}\, g(\theta) - dgg^{-1} - \dot{t}(J),$$

$${}^{g,J}\varUpsilon = \dot{m}(g)(\varUpsilon) - dJ - \frac{1}{2}[J, J] - \widehat{m}(\mathrm{Ad}\, g(\theta) - dgg^{-1} - \dot{t}(J), J).$$

- *If $(\theta,\Upsilon)$ flat, $({}^{g,J}\theta,{}^{g,J}\Upsilon)$ flat.*
- *For $\gamma$ a curve of $M$, $G_{g,J;\theta}(\gamma)\in H$ gauge parallel transport along $\gamma$*

$$G_{g,J;\theta}(\gamma)=\Lambda(0),$$

*with $\Lambda:\mathbb{R}\to H$ the unique solution of the two–step differential problem*

$$d_x u(x)u(x)^{-1}=-\gamma^*\theta_x(x),\quad u(1)=1_G,$$

$$\Lambda(x)^{-1}d_x\Lambda(x)=-\dot m(u(x)^{-1}\gamma^*g(x)^{-1})(\gamma^*J_x(x)),\quad \Lambda(1)=1_H.$$

- *Property:* for any point $p$ and curves $\gamma,\gamma_1,\gamma_2$, whenever defined,

$$G_{g,J;\theta}(\iota_p)=1_H,\quad G_{g,J;\theta}(\gamma^{-1\circ})=m(F_\theta(\gamma)^{-1})(G_{g,J;\theta}(\gamma)^{-1}),$$

$$G_{g,J;\theta}(\gamma_2\circ\gamma_1)=G_{g,J;\theta}(\gamma_2)m(F_\theta(\gamma_2))(G_{g,J;\theta}(\gamma_1)).$$

- *Property*: for thin homotopic curves $\gamma_0$, $\gamma_1$

$$G_{g,J;\theta}(\gamma_1)=G_{g,J;\theta}(\gamma_0).$$

- *Property:* for any curve $\gamma : p_0 \to p_1$,

$$F_{g,J\,\theta}(\gamma) = g(p_1)t(G_{g,J;\theta}(\gamma))F_\theta(\gamma)g(p_0)^{-1}.$$

For any curves $\gamma_0, \gamma_1 : p_0 \to p_1$ and surfaces $\Sigma : \gamma_0 \Rightarrow \gamma_1$,

$$F_{g,J\,\theta,g,J\,\Upsilon}(\Sigma) = m(g(p_1))\big(G_{g,J;\theta}(\gamma_1)F_{\theta,\Upsilon}(\Sigma)G_{g,J;\theta}(\gamma_0)^{-1}\big).$$

- *Gauge parallel transport $\Rightarrow$ pseudonatural transformation $\bar{G}_{g,J;\theta} : \bar{F}_{\theta,\Upsilon} \Rightarrow \bar{F}_{g,J\,\theta,g,J\,\Upsilon}$ of parallel transport $2$–functors.*
*If flat $(\theta,\Upsilon)$, gauge parallel transport $\Rightarrow$ pseudonatural transformation $\bar{G}^0{}_{g,J;\theta} : \bar{F}^0{}_{\theta,\Upsilon} \Rightarrow \bar{F}^0{}_{g,J\,\theta,g,J\,\Upsilon}$ of flat parallel transport $2$–functors.*
- *A $(G,H)$–$2$–gauge transformation is a map $\Omega \in \mathrm{Map}(M,H)$.*
$(G,H)$–$2$–gauge transformations describe gauge transformation of $(G,H)$–$1$–gauge transformations depending on an assigned $(G,H)$–$2$–connection $(\theta,\Upsilon)$, i. e. gauge for gauge symmetry.

They encode modifications $\bar{G}_{g,J;\theta} \Rrightarrow \bar{G}_{\tilde{\Omega}_{g|\theta},\tilde{\Omega}J_{|\theta};\theta}$ of gauge pseudonatural transformations of parallel transport functors.

The apparently have no role in knot holonomy.

# $C-$ and $S-$knots

## Describe 1–, 2–... dimensional knots by curves, surfaces....

- $C$ oriented circle.
- *$C$–marking of $C$: pointing $p_C \in C$ of $C$.*
- *$C$–marking of $M$ (oriented manifold): pointing $p_M \in M$ of $M$.*
- *Marked $C$–knot of $M$: embedding*

$$\xi : C \to M, \quad \xi(p_C) = p_M.$$

- *Ambient isotopic marked $C$–knots $\xi_0, \xi_1$:*

$$F_z \in \mathrm{Diff}_+(M), \quad z \in \mathbb{R}, \quad F_0 = \mathrm{id}_M, \quad \xi_1 = F_1 \circ \xi_0, \quad F_z(p_M) = p_M.$$

- Associate a curve to any marked $C$–knot.

- *Compatible curve in $C$:* $\gamma_C : p_C \to p_C$ *s. t.:*

  $I_C = {\gamma_C}^{-1}(C \setminus p_C)$ open interval in $\mathbb{R}$;

  $\gamma_C|_{I_C} : I_C \to C \setminus p_C$ orientation preserving diffeomorphism.

- *Example*: $C = S^1$, the circle embedded in $\mathbb{R}^2$,

$$s_{S^1}(\vartheta) = (\cos\vartheta, \sin\vartheta),$$

with $\vartheta \in [0, 2\pi)$. $C$–marking $p_{S^1} = (1, 0)$. Compatible curve $\gamma_{S^1} : \mathbb{R} \to S^1$

$$\gamma_{S^1}(x) = s_{S^1}(2\pi\alpha(x)).$$

$\alpha : \mathbb{R} \to [0, 1]$ s. t. $d_x\alpha(x) \geq 0$ and $\alpha(x) = 0$ for $x < \epsilon$ and $\alpha(x) = 1$ for $x > 1 - \epsilon$.

- *Curve of marked $C$–knot $\xi$:* $\gamma_\xi : p_M \to p_M$

$$\gamma_\xi = \xi \circ \gamma_C.$$

- $\gamma_\xi$ *independent of the choice of* $\gamma_C$ *up to thin homotopy.* Note: $p_C$ is fixed.
- *Ambient isotopic marked* $C$*–knots have homotopic curves: if* $\xi_0$, $\xi_1$ *ambient isotopic, then* $\gamma_{\xi_0}$, $\gamma_{\xi_1}$ *homotopic.*
- It should be possible to alter marking data changing $\gamma_\xi$ at most by a (thin) homotopy.
- $\xi_0$, $\xi_1$ *marked* $C$*–knots w. r. t. distinct* $C$*–marking* $p_{M0}$, $p_{M1}$ *of* $M$ *(*$C$*–marking* $p_C$ *of* $C$ *fixed) freely ambient isotopic:*

$$F_z \in \mathrm{Diff}_+(M), \quad z \in \mathbb{R}, \quad F_0 = \mathrm{id}_M, \quad \xi_1 = F_1 \circ \xi_0.$$

- *Freely ambient isotopic marked* $C$*–knots have homotopic curves up to conjugation: if* $\xi_0$, $\xi_1$ *freely ambient isotopic, there is curve* $\gamma_1 : p_{M0} \to p_{M1}$ *s. t.* $\gamma_{\xi_0}$, ${\gamma_1}^{-1\circ} \circ \gamma_{\xi_1} \circ \gamma_1$ *homotopic ("compose rightmost first" convention used here and in the following).*

- *If the embedding $\xi : C \to M$ marked $C-$knot w. r. t. two distinct $C-$markings $p_{C0}, p_{M0}$ and $p_{C1}, p_{M1}$ of $C$ and $M$, there exists curve $\gamma_1 : p_{M0} \to p_{M1}$ in $\xi(C)$ s. t. $\gamma_{\xi|0}, {\gamma_1}^{-1\circ} \circ \gamma_{\xi|1} \circ \gamma_1$ thin homotopic.*
- Find generalization to surface knots.

  Some problems occur for higher genus knots.

  Solution proposed.
- *Spiky $C-$knots:embeddings*

$$\xi : C \to M, \quad \xi(p_C) = p_M.$$

  *smooth on $C \setminus p_C$ with finite derivatives and non zero first derivatives at both ends of $C \setminus p_C$.*
- For spiky $\xi$, $\gamma_\xi$ defined as earlier and smooth anyway.
- $S$ genus $\ell_S$ closed oriented surface.

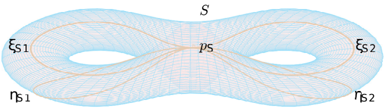
- *$S$–marking of $S$:*

  a $C$–marking $p_S$ of $S$;

  spiky $C$–knots $\zeta_{Si}$ of $S$, $i = 1, \dots, 2\ell_S$, s. t.:

  the $\zeta_{Si}(C)$ intersect only at $p_S$;

  the $\zeta_{Si}$, sorted as $\xi_{Sr}$, $\eta_{Sr}$, represent the standard $a$– and $b$–cycles of $S$.




- *$S$–marking of $M$:*

  a $C$–marking $p_M$ of $M$;

  spiky $C$–knots $\zeta_{Mi}$ of $M$, $i = 1, \dots, 2\ell_S$, s. t.:

  the $\zeta_{Mi}(C)$ intersect only at $p_M$;

  there is an embedding $\Phi : S \to M$ s. t. $\Phi(p_S) = p_M$, $\Phi \circ \zeta_{Si} = \zeta_{Mi}$.

Note: compatible with $S$–marking of $M = S$.

- *Marked $S$–knot of $M$: embedding*

$$\varXi : S \to M, \quad \varXi(p_S) = p_M \ \text{ and } \ \varXi \circ \zeta_{Si} = \zeta_{Mi}.$$

- *Ambient isotopic marked $S$–knots $\varXi_0, \varXi_1$:*

$$F_z \in \mathrm{Diff}_+(M), \quad z \in \mathbb{R}, \quad F_0 = \mathrm{id}_M, \quad \varXi_1 = F_1 \circ \varXi_0,$$
$$F_z(p_M) = p_M, \quad F_z \circ \zeta_{Mi} = \zeta_{Mi}.$$

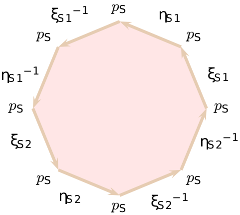
- Associate a surface to any marked $S$–knot.
- View $S$ as a $C$–marked manifold,

$$\gamma_{Si} = \gamma_{\zeta_{Si}} \ \text{ that is } \ \alpha_{Sr} = \gamma_{\xi_{Sr}}, \ \beta_{Sr} = \gamma_{\eta_{Sr}},$$

Fundamental polygon curve:

$$\tau_S = \beta_{S\ell_S}{}^{-1\circ} \circ \alpha_{S\ell_S}{}^{-1\circ} \circ \beta_{S\ell_S} \circ \alpha_{S\ell_S} \cdots \circ \beta_{S1}{}^{-1\circ} \circ \alpha_{S1}{}^{-1\circ} \circ \beta_{S1} \circ \alpha_{S1}$$

If $\ell_S = 0$, $\tau_S = \iota_{p_S}$.




- *Compatible surface in $S$: $\Sigma_S : \iota_{p_S} \to \tau_S$ s. t.:*

  $D_S = {\Sigma_S}^{-1}(S \setminus \cup_i \zeta_{Si}(C))$ open simply connected domain in $\mathbb{R}^2$;

  $\Sigma_S|_{D_S} : D_S \to S \setminus \cup_i \zeta_{Si}(C)$ orientation preserving diffeomorphism.

- *Example*: $S = S^2$, the sphere embedded in $\mathbb{R}^3$,

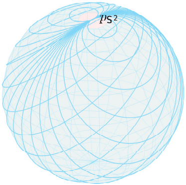
$$S_{S^2}(\vartheta, \varphi) = (\cos\vartheta \sin\vartheta(1 - \cos\varphi), -\sin\vartheta \sin\varphi, 1 - \sin^2\vartheta(1 - \cos\varphi)),$$

$\vartheta \in (0, \pi)$, $\varphi \in [0, 2\pi)$.

$S$–marking $p_{S^2} = (0,0,1)$, Compatible surface $\Sigma_{S^2} : \mathbb{R}^2 \to S^2$

$$\Sigma_{S^2}(x,y) = S_{S^2}(\pi\alpha(y), 2\pi\alpha(x)).$$

$\alpha : \mathbb{R} \to [0,1]$ s. t. $d_x\alpha(x) \geq 0$ and $\alpha(x) = 0$ for $x < \epsilon$ and $\alpha(x) = 1$ for $x > 1-\epsilon$.




- *Example:* $S = T^2$, the torus embedded in $\mathbb{R}^3$

$$S_{T^2}(\vartheta_1, \vartheta_2) = (\cos\vartheta_1(1 + r\cos\vartheta_2), \sin\vartheta_1(1 + r\cos\vartheta_2), r\sin\vartheta_2),$$

$r < 1$ fixed and $\vartheta_1, \vartheta_2 \in [0, 2\pi)$. $S$–marking $p_{T^2} = (1 + r, 0, 0)$ and

$$\xi_{T^2}(\vartheta) = ((1 + r)\cos\vartheta, (1 + r)\sin\vartheta, 0),$$

$$\eta_{T^2}(\vartheta) = (1 + r\cos\vartheta, 0, r\sin\vartheta),$$

$\vartheta \in [0, 2\pi)$. Compatible surface $\Sigma_{T^2} : \mathbb{R}^2 \to T^2$

$$\Sigma_{T^2}(x, y) = S_{T^2}(2\pi c_1(x, y)), 2\pi c_2(x, y)))$$

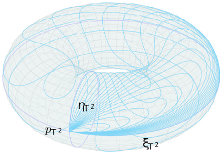
$$c_1(x, y) = \varrho(4\alpha(x), \alpha(y)) - \varrho(4\alpha(x) - 2, \alpha(y)),$$

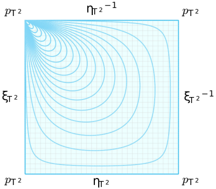
$$c_2(x, y) = \varrho(4\alpha(x) - 1, \alpha(y)) - \varrho(4\alpha(x) - 3, \alpha(y)),$$

with $\alpha : \mathbb{R} \to [0, 1]$ as before and $\varrho : \mathbb{R} \times [0, 1] \to [0, 1]$ defined by

$$\varrho(s, t) = t g_\beta\left(\frac{1 - 2s}{(1 + s - t)(2 - s - t)}\right). \tag{4.1}$$

$g_\beta(w) = 1/(\exp(\beta w) + 1)$, $\beta > 0$, (Fermi–Dirac function).




- View $M$ as a $C$–marked manifold,

$$\gamma_{Mi} = \gamma_{\zeta_{Mi}} \quad \text{that is} \quad \alpha_{Mr} = \gamma_{\xi_{Mr}}, \; \beta_{Mr} = \gamma_{\eta_{Mr}},$$

Fundamental polygon curve:

$$\tau_M = {\beta_{M\ell_S}}^{-1\circ} \circ {\alpha_{M\ell_S}}^{-1\circ} \circ \beta_{M\ell_S} \circ \alpha_{M\ell_S} \cdots \circ {\beta_{M1}}^{-1\circ} \circ {\alpha_{M1}}^{-1\circ} \circ \beta_{M1} \circ \alpha_{M1}$$

If $\ell_S = 0$, $\tau_M = \iota_{p_M}$.

- *Surface of marked $S$–knot $\Xi$:* $\Sigma_\Xi : \iota_{p_M} \Rightarrow \tau_M$,

$$\Sigma_\Xi = \Xi \circ \Sigma_S.$$

Note: $\tau_M = \Xi \circ \tau_S$.

- *Remark:* for a marked $C$–knot $\xi$, $\gamma_\xi : p_M \to p_M$, source, target equal, ok; for a $\ell_S = 0$ marked $S$–knot $\Xi$, $\Sigma_\Xi : \iota_{p_M} \Rightarrow \iota_{p_M}$, source, target equal, ok; for a $\ell_S > 0$ marked $S$–knot $\Xi$, $\Sigma_\Xi : \iota_{p_M} \Rightarrow \tau_M \neq \iota_{p_M}$, source, target different, likely to be a problem in a higher gauge theory of $S$–knots.
- *Proposed solution:* For given $\ell_S$ and $C$–marking of $M$, pick a *reference* marked $S$–knot $\Delta_M$ (e. g. Hosokawa and Kawauchi surface unknot in $S^4$).

- *Normalized surface of marked $S$–knot $\Xi$:* $\Sigma^\sharp{}_\Xi : \iota_{p_M} \Rightarrow \iota_{p_M}$,

$$\Sigma^\sharp{}_\Xi = \Sigma_M{}^{-1\bullet} \bullet \Sigma_\Xi, \qquad \Sigma_M := \Sigma_{\Delta_M}.$$

- $\Sigma^\sharp{}_\Xi$ *independent from the choice of* $\Sigma_S$ *and* $\gamma_C$ *up to thin homotopy.* Note: $(p_S, \zeta_{Si})$ and $p_C$ fixed.

- *Ambient isotopic reference marked $S$–knots yield homotopic normalized marked $S$–knot surfaces: if $\Delta_{M0}$, $\Delta_{M1}$ ambient isotopic, then $\Sigma^\sharp{}_{\Xi|0}$, $\Sigma^\sharp{}_{\Xi|1}$ homotopic for any marked $S$–knot $\Xi$.*

- *Ambient isotopic marked $S$–knots have homotopic normalized surfaces: if $\Xi_0$, $\Xi_1$ ambient isotopic, then $\Sigma^\sharp{}_{\Xi_0}$, $\Sigma^\sharp{}_{\Xi_1}$ homotopic.*

- It should be possible to alter marking changing $\Sigma^\sharp{}_\Xi$ by a (thin) homotopy.

- *$\Xi_0$, $\Xi_1$ marked $S$–knots w. r. t. distinct $S$–marking $(p_{M0}, \zeta_{M0i})$, $(p_{M1}, \zeta_{M1i})$ of $M$ ($S$–marking $(p_S, \zeta_{Si})$ of $S$ fixed) freely ambient isotopic:*

$$F_z \in \mathrm{Diff}_+(M), \quad z \in \mathbb{R}, \quad F_0 = \mathrm{id}_M, \quad \Xi_1 = F_1 \circ \Xi_0.$$

- Two pairs $\Xi_0$, $\Xi_1$ and $\Xi_0{}'$, $\Xi_1{}'$ of freely ambient isotopic marked $S-$knots concordant if ambient isotopies $F_z$ of $\Xi_0$, $\Xi_1$ and $F'{}_z$ of $\Xi_0{}'$, $\Xi_1{}'$ exist s. t. $F_z(p_{M0}) = F'{}_z(p_{M0})$, $F_z \circ \zeta_{M0i} = F'{}_z \circ \zeta_{M0i}$.
- *Freely ambient isotopic marked $S-$knots have homotopic surfaces up to conjugation under concordance with reference knots. Suppose $\Delta_{M0}$, $\Delta_{M1}$ freely ambient isotopic. If $\Xi_0$, $\Xi_1$ freely ambient isotopic concordantly with $\Delta_{M0}$, $\Delta_{M1}$, then there is curve $\gamma_1 : p_{M0} \to p_{M1}$ s. t. $\Sigma^\sharp{}_{\Xi_0|0}$, $I_{\gamma_1}{}^{-1\circ} \circ \Sigma^\sharp{}_{\Xi_1|1} \circ I_{\gamma_1}$ homotopic.*
- *Property :* For two $S-$markings $(p_{S0}, \zeta_{S0i})$, $(p_{S1}, \zeta_{S1i})$ of $S$, there is an orientation preserving ambient isotopy $k_z$ of $S$ s. t. $k_1(p_{S0}) = p_{S1}$, $k_1 \circ \zeta_{S0i} = \zeta_{S1i}$.
- *If the embeddings $\Delta_M, \Xi : S \to M$ simultaneously reference marked $S-$knot and chosen marked $S-$knot w. r. t. two distinct $S-$markings $(p_{S0}, \zeta_{S0i})$, $(p_{M0}, \zeta_{M0i})$ and $(p_{S1}, \zeta_{S1i})$, $(p_{M1}, \zeta_{M1i})$ of $S$ and $M$ and there is an*

*ambient isotopy* $k_z$ *of* $S$ *shifting* $\{p_{S0}, \zeta_{S0i}\}$ *to* $\{p_{S1}, \zeta_{S1i}\}$ *s. t.* $\Xi \circ k_z(p_{S0}) = \Delta_M \circ k_z(p_{S0})$ *and* $\Xi \circ k_z \circ \zeta_{S0i} = \Delta_M \circ k_z \circ \zeta_{S0i}$, *then there is a curve* $\gamma_1 : p_{M0} \to p_{M1}$ *in* $\Xi(S)$ *s. t.* ${\Sigma^\sharp}_{\Xi|0}$, ${I_{\gamma_1}}^{-1\circ} \circ {\Sigma^\sharp}_{\Xi|1} \circ I_{\gamma_1}$ *thin homotopic.*

# Knot holonomy

## Construct holonomy invariants of knots up to conjugation...

- $\theta$ flat $G$–connection on $M$.
- Fix $C$–markings $p_C$ and $p_M$ of $C$ and $M$.
- *Marked $C$–knot holonomy: for $\xi$ marked $C$–knot*

$$F_\theta(\xi) = F_\theta(\gamma_\xi),$$

  *where $\gamma_\xi : p_M \to p_M$ curve of $\xi$, $F_\theta$ parallel transport functor.*
- *$C$–knot holonomy independent of the choice of the compatible curve $\gamma_C$ of $C$.*
- *If $\xi_0$, $\xi_1$ ambient isotopic marked $C$–knots of $M$,*

$$F_\theta(\xi_1) = F_\theta(\xi_0).$$

- Fix $C$–marking $p_C$ of $C$, allow distinct $C$–markings $p_{M0}$, $p_{M1}$ of $M$.
- *If $\xi_0$, $\xi_1$ freely ambient isotopic marked $C$–knots, then there exists $\gamma_1 : p_{M0} \to p_{M1}$ curve of $M$ s. t.*

$$F_\theta(\xi_1) = F_\theta(\gamma_1) F_\theta(\xi_0) F_\theta(\gamma_1)^{-1}.$$

- *If $\xi$ marked $C$–knot w. r. t. two distinct $C$–markings $p_{C0}$, $p_{M0}$ and $p_{C1}$, $p_{M1}$ of $C$ and $M$, then there is $\gamma_1 : p_{M0} \to p_{M1}$ curve in $\xi(C)$ s. t.*

$$F_{\theta|1}(\xi) = F_\theta(\gamma_1) F_{\theta|0}(\xi) F_\theta(\gamma_1)^{-1}.$$

- *Let $\xi$ be a marked $C$–knot of $M$. Then, for $g$ a $G$–gauge transformation,*

$$F_{{}^g\theta}(\xi) = g(p_M) F_\theta(\xi) g(p_M)^{-1}.$$

$C$–knot holonomy $C$–marking and gauge independent and isotopy invariant up to $G$–conjugation.

- Generalize to $S$–knots
- $(\theta,\Upsilon)$ flat $(G,H)$–2–connection pair on $M$.
- Fix $S$-markings $(p_S,\zeta_{Si})$ and $(p_M,\zeta_{SMi})$ of $S$ and $M$.
- Marked $S$–knot holonomy; for a marked $S$–knot $\Xi$,

$$F_{\theta,\Upsilon}(\Xi)=F_{\theta,\Upsilon}({\Sigma^\sharp}_\Xi)=F_{\theta,\Upsilon}(\Sigma_M)^{-1}F_{\theta,\Upsilon}(\Sigma_\Xi),$$

where ${\Sigma^\sharp}_\Xi:\iota_{p_M}\Rightarrow\iota_{p_M}$ normalized surface of $\Xi$, $F_{\theta,\Upsilon}$ parallel transport 2–functor.

- *For a marked $S$–knot $\Xi$, as ${\Sigma^\sharp}_\Xi:\iota_{p_M}\Rightarrow\iota_{p_M}$,*

$$t(F_{\theta,\Upsilon}(\Xi))=1_G,$$

$F_{\theta,\Upsilon}(\Xi)=1_H$ unless $\ker t\neq\{1_H\}$. Further, $F_{\theta,\Upsilon}(\Xi)\in Z_H$.

- *$S$–knot holonomy independent from the choice of compatible surface $\Sigma_S$ of $S$ and curve $\gamma_C$ of $C$.*

- *If the reference marked $S$–knots $\Delta_{M0}$, $\Delta_{M1}$ ambient isotopic, then* for any marked $S$–knot $\Xi$

$$F_{\theta,\Upsilon|0}(\Xi) = F_{\theta,\Upsilon|1}(\Xi).$$

- *If $\Xi_0$, $\Xi_1$ ambient isotopic marked $S$–knots, then*

$$F_{\theta,\Upsilon}(\Xi_1) = F_{\theta,\Upsilon}(\Xi_0).$$

- Fix $S$-markings $(p_S, \zeta_{Si})$ of $S$, allow distinct $S$–marking $(p_{M0}, \zeta_{M0i})$, $(p_{M1}, \zeta_{M1i})$ of $M$.
- *Suppose $\Delta_{M0}$, $\Delta_{M1}$ freely ambient isotopic reference marked $S$–knots. If the marked $S$–knots $\Xi_0$, $\Xi_1$ freely ambient isotopic concordantly with $\Delta_{M0}$, $\Delta_{M1}$, then there is curve $\gamma_1 : p_{M0} \to p_{M1}$ s. t.*

$$F_{\theta,\Upsilon|1}(\Xi_1) = m(F_\theta(\gamma_1))(F_{\theta,\Upsilon|0}(\Xi_0))$$

- If the embeddings $\Delta_M, \Xi : S \to M$ simultaneously reference marked $S$–knots and chosen marked $S$–knots w. r. t. two distinct $S$–markings $(p_{S0}, \zeta_{S0i})$, $(p_{M0}, \zeta_{M0i})$ and $(p_{S1}, \zeta_{S1i})$, $(p_{M1}, \zeta_{M1i})$ of $S$ and $M$ and there is an ambient isotopy $k_z$ of $S$ shifting $\{p_{S0}, \zeta_{S0i}\}$ to $\{p_{S1}, \zeta_{S1i}\}$ s. t. $\Xi \circ k_z(p_{S0}) = \Delta_M \circ k_z(p_{S0})$ and $\Xi \circ k_z \circ \zeta_{S0i} = \Delta_M \circ k_z \circ \zeta_{S0i}$, then there is a curve $\gamma_1 : p_{M0} \to p_{M1}$ in $\Xi(S)$ s. t.

$$F_{\theta,\Upsilon|1}(\Xi) = m(F_\theta(\gamma_1))(F_{\theta,\Upsilon|0}(\Xi))$$

- *If $\xi$ marked $C$–knot, the holonomy*

$$F_\theta(\xi) = F_\theta(\gamma_\xi)$$

  *is defined.*

- *$C$–knot holonomy is still independent from the choice of the compatible curve $\gamma_C$.*

- Since $\theta$ is not flat, unless $\dot{t}(\Upsilon) = 0$, $F_\theta(\xi)$ is not ambient isotopy invariant.
- If $\xi_0$, $\xi_1$ ambient isotopic marked $C$–knots of $M$, then there is a surface $\Sigma : \gamma_{\xi_0} \Rightarrow \gamma_{\xi_1}$ of $M$ s. t.

$$F_\theta(\xi_1) = t(F_{\theta,\Upsilon}(\Sigma))F_\theta(\xi_0).$$

- Fix $C$–marking $p_C$ of $C$, allow distinct $C$–markings $p_{M0}$, $p_{M1}$ of $M$.
- *If $\xi_0$, $\xi_1$ freely ambient isotopic marked $C$–knots, then there exist a curve $\gamma_1 : p_{M0} \to p_{M1}$ and a surface $\Sigma : \gamma_{\xi_0} \Rightarrow {\gamma_1}^{-1\circ} \circ \gamma_{\xi_1} \circ \gamma_1$ of $M$ s. t.*

$$F_\theta(\xi_1) = F_\theta(\gamma_1)t(F_{\theta,\Upsilon}(\Sigma))F_\theta(\xi_0)F_\theta(\gamma_1)^{-1}.$$

- *If $\xi$ marked $C$–knot w. r. t. two distinct $C$–markings $p_{C0}$, $p_{M0}$ and $p_{C1}$, $p_{M1}$ of $C$ and $M$, then there is $\gamma_1 : p_{M0} \to p_{M1}$ curve in $\xi(C)$ s. t.*

$$F_{\theta|1}(\xi) = F_\theta(\gamma_1)F_{\theta|0}(\xi)F_\theta(\gamma_1)^{-1}.$$

- *Let $\Xi$ marked $S$–knot and $\xi$ a marked $C$–knot. For $(g, J)$ $(G, H)$–1–gauge transformation,*

$$F_{g,J\theta,g,J\Upsilon}(\Xi) = m(g(p_M))(F_{\theta,\Upsilon}(\Xi))$$

*and*

$$F_{g,J\theta}(\xi) = g(p_M)t(G_{g,J;\theta}(\gamma_\xi))F_\theta(\gamma)g(p_M)^{-1}.$$

$C$–and $S$–knot holonomy $C$–marking and gauge independent and isotopy invariant up to $(G, H)$–conjugation.

$(G, H)$–conjugation is defined by

$$u' = aua^{-1}t(A), \qquad U' = m(a)(U)$$

with $(u, U), (u', U'), (a, A) \in G \times H$ and is an equivalence relation.

# Invariant traces

## Construct true knot invariants..

- Ordinary gauge theory with gauge group $G$ and flat $G$–connection $\theta$.
- For a $C$–knot $\xi$, the holonomy $F_\theta(\xi)$ $C$–marking and isotopy invariant and gauge independent up to $G$–conjugation

$$F_\theta(\xi) \equiv aF_\theta(\xi)a^{-1}, \quad a \in G.$$

- *A knot invariant is given by the Wilson line*

$$W_{R,\theta}(\xi) = \mathrm{tr}_R(F_\theta(\xi)),$$

*with $R$ a representation of $G$.*

- Strict higher gauge theory with gauge crossed module $(G, H)$ and flat $(G, H)$–2–connection $(\theta, \Upsilon)$.

- For a $C$–knot $\xi$ and an $S$–knot $\Xi$, the holonomy $F_\theta(\xi)$ and $F_{\theta,\Upsilon}(\Xi)$ $C-$ and $S$–marking and isotopy invariant and gauge independent up to $(G,H)$–conjugation

$$F_\theta(\xi) \equiv aF_\theta(\xi)a^{-1}t(A), \quad F_{\theta,\Upsilon}(\Xi) \equiv m(a)(F_{\theta,\Upsilon}(\Xi)) \quad (a,A) \in G \times H.$$

- To obtain knot invariants, one needs traces invariant under $(G,H)$–conjugation.
- Assume $G$, $H$ compact with bi-invariant Haar measures $\mu_G$, $\mu_H$.
- *Pick $R$, $S$ representations of $G$, $H$. Set*

$$\mathrm{tr}_{R,S|b}(u) = \int_H d\mu_H(X)\,\mathrm{tr}_R(ut(X)),$$
$$\mathrm{tr}_{R,S|f}(U) = \int_G d\mu_G(x)\,\mathrm{tr}_S(m(x)(U)),$$

$(u,U) \in G \times H$.

- *Property:* the traces are invariant under $(G, H)$ conjugation,

$$\mathrm{tr}_{R,S|b}(aua^{-1}t(A)) = \mathrm{tr}_{R,S|b}(u),$$
$$\mathrm{tr}_{R,S|f}(m(a)(U)) = \mathrm{tr}_{R,S|f}(U), \quad (a, A) \in G \times H$$

  for $(u, U) \in G \times H$.

- Knot invariants are given by the Wilson line and surfaces

$$W_{R,S,\theta|b}(\xi) = \mathrm{tr}_{R,S|b}(F_\theta(\xi)),$$
$$W_{R,S,\theta,\Upsilon|f}(\varXi) = \mathrm{tr}_{R,S|f}(F_{\theta,\Upsilon}(\varXi)).$$

- *Problem*: the traces may be trivial. F. i. if $t(H) = G$, $\mathrm{tr}_{R,Sb}(u)$ does not depend on $u$ and $\mathrm{tr}_{R,Sf}(U) = \mathrm{tr}_S(U)$ for $U \in \ker t$ (the case of interest for surface knots).

- *Question*: Why does not one use representations of crossed modules on 2–vector spaces to construct invariant traces thereof?

- In ordinary gauge theory with gauge group $G$, a trace $\mathrm{tr} : G \to \mathbb{C}$ must be invariant under

$$a \triangleright u := aua^{-1}, \quad a, u \in G,$$

that is

$$\mathrm{tr}(a \triangleright u) = \mathrm{tr}(u).$$

- What matters is not $G$ itself but its conjugation pointed quandle $G$: a pointed quandle is a set $G$ with an operation $\triangleright : G \times G \to G$ and $1_G \in G$ s. t.

$$a \triangleright a = a, \quad a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c), \quad a, b, c \in G,$$

$a \triangleright \cdot : G \to G$ is invertible for any $a \in G$ and

$$a \triangleright 1_G = 1_G, \quad 1_G \triangleright a = a, \quad a \in G.$$

- If $G$ is compact, $\mathrm{tr}$ reduces to the $\mathrm{tr}_R$ with $R$ irreducible representations of $G$.

- In higher gauge theory with gauge crossed module $(G, H)$, a similar point of view is appropriate. Traces $\mathrm{tr}_b : G \to \mathbb{C}$, $\mathrm{tr}_f : H \to \mathbb{C}$ must be invariant under

$$a \triangleright u := aua^{-1}, \quad A \succ u := ut(A), \quad a \triangleright U := m(a)(U), \tag{6.1}$$
$$a, u \in G, \ A, U \in H,$$

that is

$$\mathrm{tr}_b(a \triangleright u) = \mathrm{tr}_b(u), \quad \mathrm{tr}_b(A \succ u) = \mathrm{tr}_b(u), \quad \mathrm{tr}_f(a \triangleright U) = \mathrm{tr}_f(U).$$

- What matters is not $(G, H)$ itself but its conjugation augmented pointed quandle crossed module $(G, H)$ (Crans and Wagemann, RZ): an augmented pointed quandle crossed module is a pair of sets $G, H$ with operations $\triangleright : G \times G \to G,\ H \times H \to H,\ G \times H \to H$ and $1_G \in G$, $1_H \in H$ s. t.

$$G \text{ a pointed quandle,} \tag{6.2}$$

$$H \text{ a pointed quandle,} \tag{6.3}$$

$$a \triangleright (b \triangleright A) = (a \triangleright b) \triangleright (a \triangleright A), \quad a \triangleright (A \triangleright B) = (a \triangleright A) \triangleright (a \triangleright B), \quad (6.4)$$
$$a, b \in G, \ A, B \in H,$$

for any $a \in G$, $a \triangleright \cdot : H \to H$ invertible,

$$1_G \triangleright A = A, \quad a \triangleright 1_H = 1_H, \quad a \in G, \ A \in H,$$

a quandle morphism $\alpha : H \to G$ (respects $\triangleright$ and $1$) s. t.

$$\alpha(a \triangleright A) = a \triangleright \alpha(A), \quad \alpha(A) \triangleright B = A \triangleright B, \quad a \in G, \ A, B \in H$$

and an augmentation $\succ : H \times G \to G$ s.t.

$$a \triangleright (A \succ b) = (a \triangleright A) \succ (a \triangleright b), \quad a, b \in G, \ A \in H,$$

$A \succ \cdot : G \to G$ is invertible and

$$A \succ 1_G = \alpha(A), \quad 1_H \succ a = a, \quad a \in G, \ A \in H$$

- *Question:* If $G$, $H$ compact, do $\mathrm{tr}_b$, $\mathrm{tr}_f$ reduce to the $\mathrm{tr}_{R,S|b}$, $\mathrm{tr}_{R,S|f}$ with $R$, $S$ irreducible representations of $G$, $H$, respectively?

# Chern–Simons Theory

**To Compute knots invariants in QFT, one needs Chern–Simons theory.**

- This has been known for a long time since Witten's 1988 paper.
- Assume the Lie algebra $\mathfrak{g}$ equipped with a properly normalized invariant non singular bilinear form $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$([z,x],y) + x,[z,y]), \qquad x,y,z \in \mathfrak{g}.$$

- *Chern–Simons action: $M_3$ a 3-dimensional manifold*

$$\mathrm{CS}(\theta) = \frac{k}{4\pi}\int_{M_3}\left(\theta, d\theta + \frac{1}{3}[\theta,\theta]\right)$$

  *with $\theta$ a $G$–connection.*
- Chern–Simons field equations: the flatness condition of $\theta$.

- *Chern–Simons action invariant under gauge transformations* $g$ *mod* $2\pi\mathbb{Z}$

$$\mathrm{CS}({}^g\theta) = \mathrm{CS}(\theta) - 2\pi k \cdot \mathrm{wn}(g)$$

  *with* $\mathrm{wn}(g)$ *winding number of* $g$*.*

- Quantum gauge invariance $\Rightarrow$ level $k$ integer.

- Knot invariants from Chern–Simons Wilson loop correlators $W_{R,\theta}(\xi)$, e. g.

  $G = \mathrm{SU}(2)$, $R = F \Rightarrow$ Jones polynomial;

  $G = \mathrm{SU}(n)$, $R = F \Rightarrow$ HOMFLY polynomial;

  $G = \mathrm{SO}(n)$, $R = F \Rightarrow$ Kauffman polynomial...

- In the Chern–Simons path integral $\theta$ is not flat $\Rightarrow W_{R,\theta}(\xi)$ not ambient isotopy invariant.

  However, the theory somehow localizes on the moduli space of flat connections even though it is not a cohomological topological field theory (proven by Beasley and Witten for $M_3$ Seifert, e. g. $S^1 \times S^2$, $S^3$, ...).

Chern–Simons Wilson loop correlators $W_{R,\theta}(\xi) \Rightarrow$ genuine knot invariants.

**To Compute surface knots invariants in QFT, one needs 2-Chern–Simons theory.**

- Assume the differential Lie crossed module $\mathfrak{h} \xrightarrow{\dot{t}} \mathfrak{g} \xrightarrow{\widehat{m}} \mathfrak{der}(\mathfrak{h})$ equipped with a properly normalized invariant non singular bilinear pairing $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$:

$$(\dot{t}(X), Y) - (\dot{t}(Y), X) = 0,$$

$$([y,x], X) + (x, \widehat{m}(y)(X)) = 0. \qquad x, y \in \mathfrak{g},\ X, Y \in \mathfrak{h}.$$

*Note:* $\dim \mathfrak{g} = \dim \mathfrak{h}$.

- *2-Chern–Simons action:* $M_4$ *4-dimensional manifold*

$$\mathrm{CS}_2(\theta, \Upsilon) = \kappa_2 \int_{M_4} \Big( d\theta + \frac{1}{2}[\theta, \theta] - \frac{1}{2}\dot{t}(\Upsilon), \Upsilon \Big),$$

*with* $(\theta, \Upsilon) \in \Omega^1(M_4, \mathfrak{g}) \times \Omega^2(M_4, \mathfrak{h})$.

*Problem:* $(\theta, \Upsilon)$ not a $(G, H)$–2–connection, as the vanishing fake curvature condition not imposed.

- 2-Chern–Simons field equations:

  the zero fake curvature condition of $(\theta, \Upsilon)$;

  the flatness condition of $(\theta, \Upsilon)$.

- *Problem:* $(\theta, \Upsilon)$ not obeying the zero fake curvature condition in the 2–Chern–Simons path integral

  $\Rightarrow$ definition of Wilson surfaces $W_{R,S,\theta,\Upsilon}(\Xi)$ problematic.

- *2-Chern–Simons action invariant under $(G, H)$–1–gauge transformation $(g, J)$*

$$\mathrm{CS}_2({}^{g,J}\theta, {}^{g,J}\theta\Upsilon) = \mathrm{CS}_2(\theta, \Upsilon).$$

- *Problem:* Apparently all $(G, H)$–1–gauge transformation $(g, J)$ are small $\Rightarrow$ no level quantization.

- Surface knot invariants from Wilson surface 2–Chern–Simons correlator?

## Studying pull–backs of knots may be interesting

- All $f \in \mathrm{Diff}_+(C)$ homotopic to $\mathrm{id}_C$

  $\to$ For a $C$–knot $\xi$ $\gamma_\xi$, $\gamma_{f^*\xi}$ thin homotopic

  $\to$ $\xi$ and $f^*\xi$ have same holonomy.

- There are $f \in \mathrm{Diff}_+(S)$ not homotopic to $\mathrm{id}_S$

  $\to$ For a $S$–knot $\Xi$ ${\Sigma^\sharp}_\Xi$, ${\Sigma^\sharp}_{f^*\Xi}$ not thin homotopic

  $\to$ $\Xi$ and $f^*\Xi$ do not have same holonomy.

- $S$–knot invariants computed using higher gauge theory may have interesting covariance properties under the mapping class group

$$\mathsf{MCG}_+(S) = \mathrm{Diff}_+(S)/\,\mathrm{Diff}_0(S)$$