(I)Relevance of Diversification

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Abstract

This paper aims to clarify the role, benefits, and limitations of diversification in modern portfolio construction. While diversification is widely regarded as a fundamental principle of investing, it is often applied superficially especially in retail investors resulting in portfolios that are diversified in appearance but not in substance. By reviewing common misapplications, assessing the reliability of correlation as a diversification tool, and introducing more robust alternatives such as copulas and correlation-sensitive derivatives, the paper provides a practical and theoretical overview of what diversification really means, when it works, and when it fails.

Preface

During my last year on my BSc in Management, I finally started to truly enjoy capital markets. I progressed from thinking "this is quite easy" to now realizing how much I still have to learn- (Dunning–Kruger effect at its finest).

I began investing back in January 2021. Fortunately, I didn't have much money at that time—I was just about to start working as a waiter that summer. I say *"fortunately"* because I never ended up pay the so-called *new joiners, I can beat the market tax*[1]. That was probably because I was cautions, partly due to my parents' constant warnings: *"What if you lose?"* So, I stuck with the "boring" ETFs while dedicating time to study and learning more occasionally.

While I'm grateful for the returns I've achieved without taking on too much risk or experiencing significant drawdowns, while doing nothing besides pressing a button once a while. There's still a bittersweet feeling when I think about the gains I *could* have made, or not.

During the time leading up to the start of my MSc in Finance, I often found myself intrigued by the statement: "Why diversify if you **know** what you're doing?" The point here isn't to dismiss diversification or suggest it's ineffective—in fact, most of my portfolio relies on ETFs and very much likely will remain so. Rather, this is about exploring and continuing *my sailing on the seas of finance.*

Introduction

[1] "tax" were a market´s new joiner thinks he can beat the market but just ends up paying unknowingly big spreads, taking too many risks and blowing up accounts.

Warren Buffet described derivatives as “financial weapons of mass destruction”. While I tend to disagree at some level with his statement. The risk from derivatives comes from not knowing what you´re doing neither recognizing the complexity it involves.
There are other ideas that I agree. For example, he argues that spreading money too thin across mediocre investments just for the sake of diversification is a mistake. Written in this way, I think most of us will, of course, agree!

But **Diversification is essential.** It increases the likelihood that a portfolio will include future outperformers, something that is extremely difficult to predict ahead of time. Efforts to select individual stocks or consistently outperform the market through active management generally fall short over the long term. In fact, most active fund managers underperform their benchmarks.

Let’s not forget that: “*the largest returns come from very few stocks overall — just 86 stocks have accounted for $16 trillion in wealth creation, half of the stock market total, over the past 90 years. All the wealth created can be attributed to the thousand top-performing stocks, while the remaining 96 percent of stocks collectively matched one-month T-bills.*” - *Wealth creation in the U.S. public stock markets 1926 to 2019.*

Although this might seem elucidative, catching these “86” stocks is not easy, in fact is not easy at all! It requires being someone with the right tools so to proper build an industry analysis, valuation techniques, financial modeling and forecast (good luck if it’s a start-up), management quality analysis, macroeconomic context, and the list goes on (*and very on…*).

But let’s imagine you get this all right and you find yourself with a great candidate to invest. Well, I’m sorry but there´s an entire team who did it a lot faster than you and already placed a buy order before you probably even heard from this new company.
Or, we have e.g., noise traders who read on broker headline news the start-up name and invested in it without even doing some research, (brokers who also make money by these types of traders taking unknown risks by giving them recency bias news). So, chances are your, perfect start-up company that you took days for a proper *due diligence*, is already ***priced in***.

Let's just stick with the "main" ETFs to insure more abroad market views.

SP500 seems a fair candidate for the most of us, the five hundred biggest companies by market cap on a country known for being the center of the stock market. And even here from five hundred companies: *"As of March 6, the S&P 500 has gained 7.0% year to date. But 6.5 percentage points of that gain are due to the performance of just five companies."*

24% was what a company[2] alone, accounted for almost a quarter of the S&P 500's gains. This wasn't because so few companies were up—the index was up over 20% for the same year.
The Mag7 helped the SP500 to have one of its greatest bull runs in history, but in 2022:

- S&P: -16.72%
- Apple: -20.81%
- Google: -32.57%
- Amazon: -44.78%
- Block Inc (Square): - 49.19%
- Netflix: -51.1%
- Meta (Facebook): -64.20%
- Tesla: -65%

Diversification

Diversification involves reducing risk by combining various asset classes within a portfolio.

[2] NVIDIA

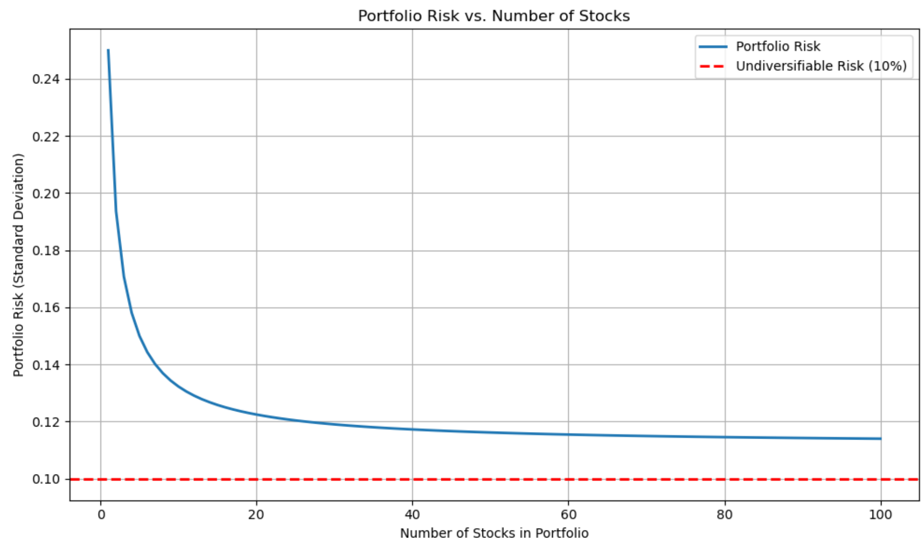

Figure 1 Portfolio Risk vs. Number of Stocks

This image shows in a simple way why adding more stocks decreases the overall risk.

There are two key concepts to understand:

- **Systematic Risk-** This is the portion of total risk that cannot be diversified away[3]. This risk affects all investments to some degree and is represented in the chart by the red dashed line, e.g., sudden nuclear war.
- **Unsystematic Risk-** This is the portion of risk that is specific to individual assets, e.g., poor management decisions. As shown in the chart, unsystematic risk decreases significantly as the number of stocks increases, due to *diversification*.

While diversification is often hailed as the cornerstone of prudent investing, its effectiveness depends entirely on how it is implemented. In practice, what appears to be a well-diversified portfolio may, in fact, offer little protection when it matters most. In the next section, we'll present a example of the most common misapplications of diversification (particularly among retail) starting with the traditional equity/bond split.

Equity/Bonds

[3] While systematic risk cannot be diversified away, it can be hedged using for e.g., derivatives.

It is well known that during strong equity markets, treasury bonds often exhibit the opposite performance, reflecting their role as safe-haven assets.

Using SPY and TLT[4] to show this correlation (2003-2013):

Figure 2 Comparative Returns SPY vs TLT

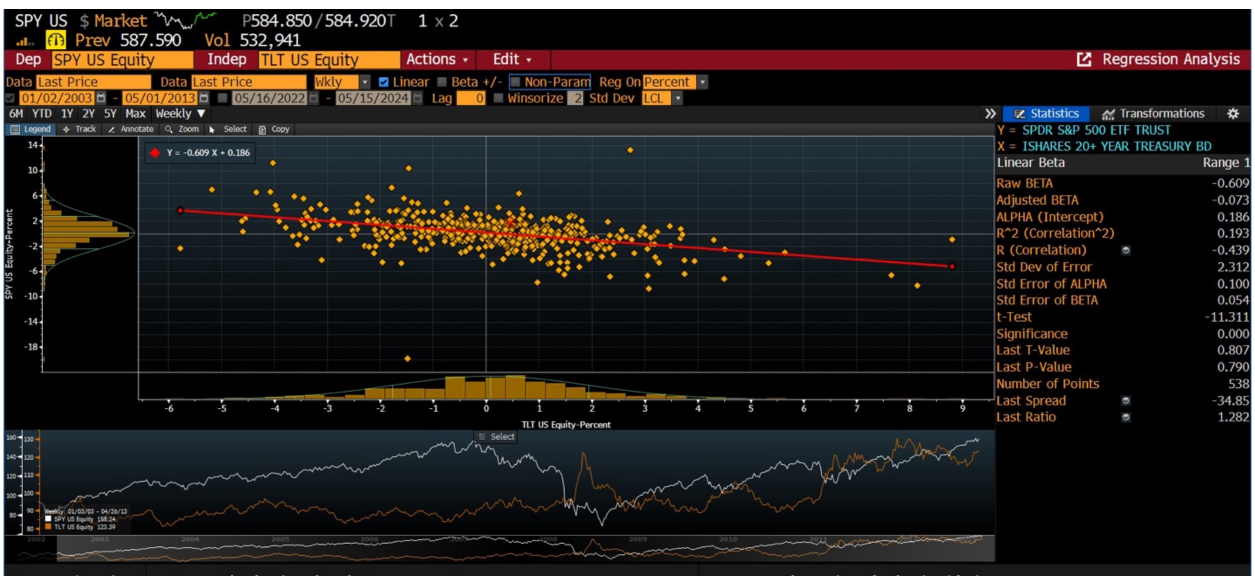

Figure 3 - Spread Analysis SPY vs TLT

The relationship between equities and long-duration government bonds is notably negative over the 2003–2013 period. As shown in Figure 2, SPY and TLT exhibit diverging performance during

[4] ETF that seeks to track the investment results of an index composed of U.S. treasury bonds with remaining maturities greater than twenty years.

periods of market stress. The regression confirms this with a beta of (–0.609) and a correlation of (–0.439), indicating that TLT often moved in the opposite direction of SPY. This negative correlation supports the use of Treasuries as a diversifier in equity-heavy portfolios.

Wrong Diversification

The term *wrong diversification*, isn't widely used per se. But we can find a proper definition to use in this case. So, let's assume wrong diversification as the inclusion of assets that fail to reduce portfolio risk often because they are highly correlated with each other or respond similarly to market shocks.

Take this example (as ETF´s):

- SP500 (SPY) – Representing 500 [5]of the largest publicly traded companies in the U.S.
- MSCI World (IWDA) – Represents large and mid-cap equities across 23 developed markets.

At first glance, combining different ETFs may appear to be a sound diversification strategy. However, if their exposures are highly overlapping, the diversification is only superficial.


SPY US Equity Alert Settings
SPDR S&P 500 ETF Trust Latest Available Portfolio Filing Portfolio Filing Look-Through Creation Unit
Type Fund: ETF Asset Class Equity Cash Pos 491.71M USD Create/Redeem Fee 3000 USD
Historical View Periodicity Quarterly 2024 Q3 - 2025 Q2 Field Position
Group By Country/ Show Asset Type All Currency USD Total Curr Mkt Val 604.9B Num of Holdings 504

Security	Source	Position	Pos Chg	% Out	% Net	Curr MV↓	Rpt MV Fili
	All						All
▸ United States		4,721,056,020	+14.61MLN		97.90	592.24BLN	591.75BLN
▸ Ireland		44,007,826	+100,345		1.26	7.64BLN	7.64BLN
▸ Switzerland		34,089,944	+77,597		.31	1.90BLN	1.90BLN
▸ United Kingdom		5,827,869	+13,301		.25	1.48BLN	1.48BLN
▸ Netherlands		3,077,899	+7,003		.11	650.45MLN	650.45MLN
▸ Bermuda		5,058,447	+11,515		.10	589.77MLN	589.77MLN
▸ Canada		1,356,079	+3,102		.07	428.30MLN	428.30MLN

Figure 4 - SPY Country Exposure

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[5]As illustrated in Figure 1, one might argue that holding a portfolio of 500 companies already achieves substantial diversification. Empirically, most of the unsystematic risk is diversified away after including approximately 20 to 30 assets, beyond which the marginal benefit of adding more securities becomes negligible.

IWDA LN Equity | Alert | Settings

iShares Core MSCI World UCITS ETF | Latest Available | Portfolio Filing | Portfolio Filing Look-Through | Creation Unit

Type Fund: ETF Asset Class Equity Cash Pos 359.39M USD

Historical View Periodicity Quarterly 2024 Q3 - 2025 Q2 Field Position

Group By Country/ Show Asset Type All Currency USD Total Curr Mkt Val 104.9B Num of Holdings 1,374

Security	Source	Position	Pos Chg	% Out	% Net↓	Curr MV	Rpt MV Filing
	All						All
United States		892,027,891	+21.77MLN		69.55	73.76BLN	70.41BLN
Japan		354,484,283	+265,010.01		5.60	5.71BLN	5.70BLN
United Kingdom		374,968,104	+251,165		3.79	3.85BLN	3.85BLN
Canada		58,332,859	+39,835		3.23	3.33BLN	3.29BLN
France		34,533,138	+25,025		2.84	2.89BLN	2.89BLN
Switzerland		41,785,296	+30,725		2.83	2.87BLN	2.88BLN
Germany		34,687,178	+24,980		2.66	2.68BLN	2.71BLN
Australia		110,570,243	+80,230		1.74	1.79BLN	1.77BLN
Netherlands		26,853,931	+19,400		1.27	1.35BLN	1.29BLN
Ireland		12,467,162	+8,995		1.07	1.13BLN	1.09BLN
Sweden		44,808,922	+32,480		.93	965.01MLN	947.55MLN
Spain		65,249,988	+46,655		.78	805.55MLN	793.91MLN
Italy		67,851,044	+49,330		.78	800.62MLN	790.28MLN
Denmark		9,537,642	+6,875		.59	604.43MLN	605.31MLN
Hong Kong		104,336,389	+75,025		.55	572.43MLN	556.97MLN
Singapore		60,597,830	+42,780		.42	439.85MLN	425.57MLN
Finland		21,524,629	+15,585		.26	270.62MLN	267.51MLN
Israel		8,497,034	+1,545		.24	242.54MLN	243.69MLN
Belgium		2,405,882	+1,730		.20	205.60MLN	205.98MLN
Uruguay		66,748	+50		.16	171.14MLN	163.53MLN
Norway		8,349,933	+6,075		.16	158.88MLN	158.56MLN
New Zealand		6,992,687	+3,805		.07	76.66MLN	74.85MLN
Bermuda		586,555	+420		.07	68.81MLN	71.18MLN
Austria		760,425	+545		.05	55.34MLN	54.07MLN
Luxembourg		1,915,193	+1,395		.05	52.94MLN	51.93MLN
Portugal		5,471,609	+3,990		.04	38.33MLN	38.24MLN
Country Not Available		596,005,683	0		.02	0	24.72MLN
Chile		639,740	+465		.01	15.94MLN	14.76MLN
China		3,698,500	+2,500		.01	6.26MLN	6.04MLN
Macau		3,009,070	+2,000		.01	6.18MLN	6.04MLN
Poland		288,629	+215		.01	4.97MLN	5.16MLN

Figure 5 - IWDA Country Exposure

A prime example is the simultaneous allocation to SPY and IWDA. While IWDA is marketed as a global fund, Figures 5 shows that it is heavily weighted toward U.S. equities, with nearly 70% exposure to U.S.-based stocks. As a result, pairing SPY and IWDA leads to significant geographical redundancy since both are essentially "bets" on the same market.

SPY US Equity | Alert | Settings

SPDR S&P 500 ETF Trust | Latest Available | Portfolio Filing | Portfolio Filing Look-Through | Creation Unit

Type Fund: ETF Asset Class Equity Cash Pos 491.71M USD Create/Redeem Fee 3000 USD

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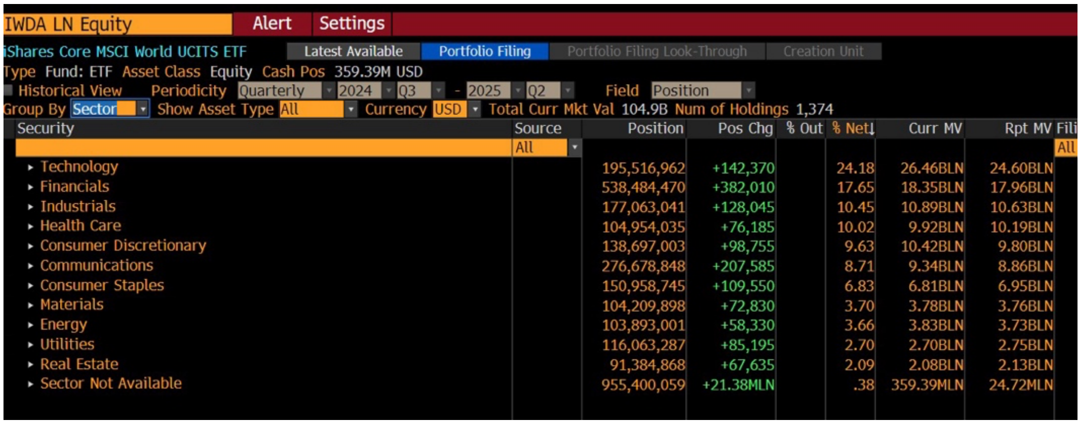
Group By Sector Show Asset Type All Currency USD Total Curr Mkt Val 604.9B Num of Holdings 504

Security	Source	Position	Pos Chg	% Out	% Net	Curr MV↓	Rpt MV Filing
	All						All
Technology		1,055,373,975	+2.41MLN		31.56	190.95BLN	190.95BLN
Financials		575,971,843	+1.31MLN		14.56	88.10BLN	88.10BLN
Communications		470,184,700	+1.07MLN		10.54	63.74BLN	63.74BLN
Consumer Discretionary		400,998,301	+915,043		10.36	62.65BLN	62.65BLN
Health Care		420,326,942	+959,082		9.37	56.68BLN	56.68BLN
Industrials		294,653,288	+672,335		8.19	49.57BLN	49.57BLN
Consumer Staples		364,315,221	+831,242		5.67	34.29BLN	34.29BLN
Energy		243,630,380	+555,822		3.22	19.50BLN	19.50BLN
Utilities		214,411,677	+489,505		2.42	14.67BLN	14.67BLN
Real Estate		145,132,488	+331,444		2.10	12.68BLN	12.68BLN
Materials		137,763,498	+314,242		1.92	11.62BLN	11.62BLN
Sector Not Available		491,711,771	+4.96MLN		.08	491.71MLN	0

Figure 6 - SPY Sector Exposure



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Security	Source	Position	Pos Chg	% Out	% Net	Curr MV	Rpt MV Fili
	All						All
Technology		195,516,962	+142,370		24.18	26.46BLN	24.60BLN
Financials		538,484,470	+382,010		17.65	18.35BLN	17.96BLN
Industrials		177,063,041	+128,045		10.45	10.89BLN	10.63BLN
Health Care		104,954,035	+76,185		10.02	9.92BLN	10.19BLN
Consumer Discretionary		138,697,003	+98,755		9.63	10.42BLN	9.80BLN
Communications		276,678,848	+207,585		8.71	9.34BLN	8.86BLN
Consumer Staples		150,958,745	+109,550		6.83	6.81BLN	6.95BLN
Materials		104,209,898	+72,830		3.70	3.78BLN	3.76BLN
Energy		103,893,001	+58,330		3.66	3.83BLN	3.73BLN
Utilities		116,063,287	+85,195		2.70	2.70BLN	2.75BLN
Real Estate		91,384,868	+67,635		2.09	2.08BLN	2.13BLN
Sector Not Available		955,400,059	+21.38MLN		.38	359.39MLN	24.72MLN

Figure 7 - IWDA Sector Exposure

This overlap becomes even more apparent when examining sector exposures. Figures 6 and 7 reveal that both ETFs are concentrated in technology, financials, and healthcare — further weakening the diversification benefits. When two ETFs are structured around the same dominant companies, any market shock that affects those sectors or geographies will ripple through both holdings simultaneously.

In other words, despite having different tickers and benchmarks, these ETFs are not truly diversified in practice. Without proper adjustment (e.g., weighting by active risk or controlling for beta overlap), investors may unintentionally end up with clustered exposures that behave almost identically under stress.

How much diversification is too much diversification?

Well, it depends.

The entire point of diversification is to reduce portfolio variance. If the assets are highly correlated, then adding more of them does little to reduce overall risk. On the other hand, low or negative correlations (ideally captured via copulas, not just linear correlation).

Diversification is captured by the reduction in portfolio variance:

$$Var\left(R_p\right) = w^{\top}\Sigma\, w \tag{4.1}$$

where R_p is the return of the portfolio, $w \in \mathbb{R}^d$ is the vector of portfolio weights, and $\Sigma \in R^{d \times d}$ is the covariance matrix of asset returns.

So, too much diversification starts becoming "too much" when the marginal asset adds **no** meaningful reduction in portfolio risk:

$$Var\,(w'X) \approx Var(w'X') \tag{4.2}$$

$$for\ portfolios\ X, X'\ with\ and\ without\ the\ new\ asset\ respectively$$

These examples illustrate that diversification is not simply about increasing the number of assets. Beneath the surface of seemingly diversified portfolios lies a dense web of overlapping exposures. But even if asset selection appears independent, a more fundamental issue persists: the reliance on correlation as a measure of risk mitigation. In the following section, we explore why correlation, despite being widely used, is both unstable and misleading, especially during market stress.

Correlation

The book Risk and Asset Allocation from Attilio Meucci is a good reading for this topic.

We assume a properly diversified portfolio if the correlation between those assets is perfectly negative ($\rho = -1$).

The problem here is the *static correlation*. It's a snapshot and it just stays there, fixed!

For instance, Figure 3 shows the historical static inverse correlation between equities and long-term U.S. Treasury bonds, which traditionally supports diversification by reducing portfolio volatility and drawdowns. However, this relationship is not stable all the time. In 2022, the expected inverse correlation broke down: both equities and bonds experienced significant simultaneous drawdowns. This regime shift undermined the classic diversification benefits and exposed the limitations of relying on static correlation metrics.

Figure 8 - Comparative Analysis SPY vs TLT

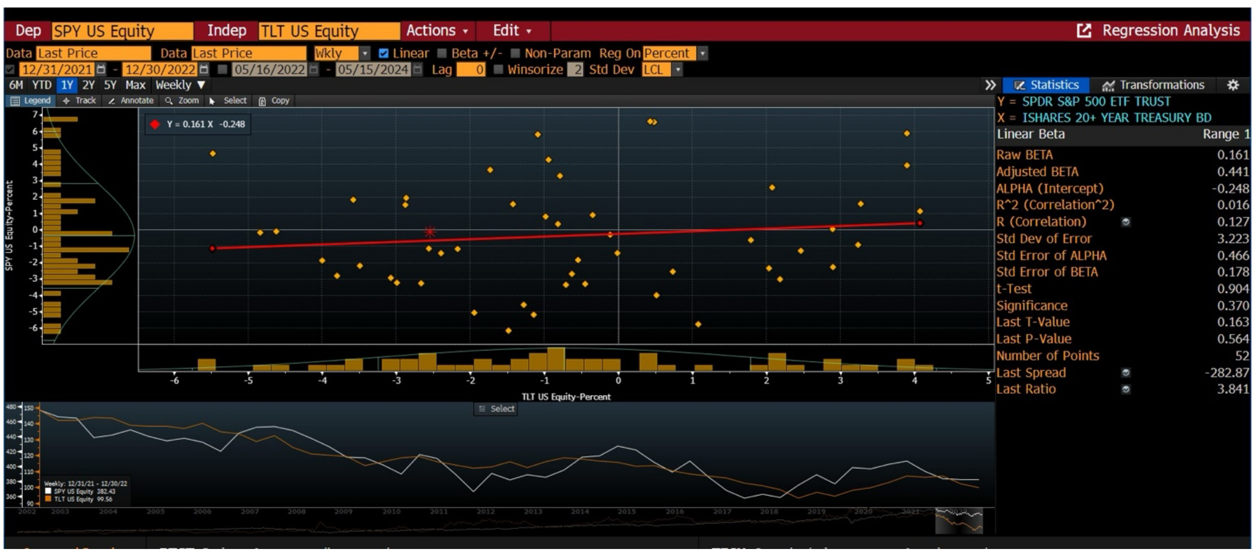

Figure 9 - Regression Analysis SPY vs TLT

Notably, during the period from December 31, 2021, to December 30, 2022, the correlation between SPY and TLT shifted dramatically from –0.439 (as shown in Figure 3) to +0.127. While still weak, this positive correlation indicates a breakdown in the traditionally inverse relationship between equities and long-duration Treasuries, particularly during market stress.

This raises a natural question: *What if we introduced a third asset traditionally known for its ability to preserve capital during turbulent periods?* One such candidate is gold[6] often perceived as a

[6] GLD will be ticker following gold

hedge against inflation, market volatility, and systemic uncertainty. The chart below compares the performance of SPY, TLT, and GLD throughout 2022, highlighting the diversification value (or lack) that each asset brought during a year of a heightened macro stress.

Figure 10 - Comparative Analysis SPY vs TLT vs GLD

Although gold (GLD) emerged as the best performer among the three assets, the overall outcome remained underwhelming. From August to October, all three (SPY, TLT, and GLD) posted simultaneous negative returns, highlighting the fragility of traditional diversification strategies during systemic selloffs.

Correlations in financial time series are notoriously unstable, particularly under stress. Yet, despite this well-known limitation, static correlation continues to be one of the most widely used tools in finance. This is largely due to the absence of a universally accepted alternative, even though many institutional desks have already adopted more sophisticated dependency models.

This leads to a persistent paradox: correlation is both unreliable and indispensable. When markets are calm, it offers intuitive portfolio construction guidance. But during crises precisely when reliable dependence estimates are most needed, static correlations often break down, leading to misleading risk assessments and failed diversification.

This issue becomes especially problematic during periods of market stress when correlations that were previously low or negative suddenly spike. For example, default correlation is a core concept in credit risk modeling, particularly in the pricing of credit default swaps (CDS). However, during the financial crisis, many of these models failed spectacularly. The reason? They relied on fixed assumptions about correlations that did not hold under shifting market regimes. In essence, when correlation estimates are most needed, they tend to disappear or worse, become misleading.

Modeling Conditional Covariances

For a given portfolio, with n assets. The portfolio manager can propose n factors of risk that are the main drivers of risk in the portfolio. In any case, a covariance matrix of dimension n must be estimated. To construct the conditional covariance matrix $\sum_{t+1}^{F}$ directly. The simplest way to model time-varying covariances is to rely on flat moving averages. For the covariance between assets i and j, we can simply estimate:

$$\sigma_{ij,t+1} = \frac{1}{m}\sum_{\tau=1}^{m} R_{i,t+1-\tau}R_{j,t+1-\tau} \tag{5.1}$$

This method is easy to implement but depends heavily on the window size m and does not react quickly to market changes. Alternatively, exponential smoothing can be used:

$$\sigma_{ij,t+1} = (1-\lambda)R_{i,t}R_{j,t} + \lambda\sigma_{ij,t} \tag{5.2}$$

This is more reactive, and commonly $\lambda = 0.94$. However, this method does not include mean reversion, so if today's covariance is high, tomorrow's will also be likely to remain high.

To include mean reversion, a GARCH (1,1) specification is introduced:

$$\sigma_{i,j,t+1} = w_{ij} + \alpha R_{i,t}R_{j,t} + \beta\sigma_{ij,t} \tag{5.3}$$

Here:

- w_{ij} is the long-run average level.
- α captures short-term reaction to returns.

- β controls persistence.
- $\alpha + \beta < 1$[7]

The unconditional (long-run) covariance is:

$$\sigma_{ij} = \frac{w_{ij}}{(1 - \alpha - \beta)} \tag{5.4}$$

To ensure the covariance matrix remains positive semi-definite, it's common to use consistent parameter choices (like the same λ, α, β) across all variances and covariances in models such as EWMA or GARCH (1,1). However, this constraint can be restrictive and may not reflect the heterogeneity of financial assets. This motivates the use of correlation-based models, which avoid the need for such uniform assumptions.

Modeling Conditional Correlations

Instead of modeling covariances directly, we can model correlations, which are easier to interpret and avoid assumptions about long-run persistence. Covariances may vary over time simply because variances change, even if correlations remain constant.

A practical approach is to compute correlation as a residual of the variance-covariance structure:

$$\mathrm{\rho_{ij,t+1}} = \mathrm{\sigma_{i,t+1}}\mathrm{\sigma_{j,t+1}}\sigma_{ij,t+1}$$

$$\Leftrightarrow \sum\nolimits_{t+1} \mathrm{D_{t+1}}\,\Gamma_{\mathrm{t+1}}\mathrm{D_{t+1}} \tag{5.5}$$

The last line in (5.5) corresponds to the vector notation of the earlier expression, where D_{t+1} is the matrix of standard deviations, with $\sigma_{i,t+1}\sigma_{i,t+1}$ on the diagonal and zeros elsewhere and Γ_{t+1} is a correlation matrix with ones on the diagonal and elements $\rho_{ij,t+1}$.

[7] For $\alpha + \beta\ \approx y$; y can help modeling memory.

We now consider models where the dynamics of correlation are modeled directly. Let's assume that volatilities $\sigma_{i,t+1}$ have already been estimated via a GARCH model or any other method.

We can then standardize each return by dividing it by its own conditional standard deviation. This yields standardized returns: $z_{i,t+1} = \frac{R_{i,t+1}}{\sigma_{i,t+1}}$ for every i. These variables $z_{i,t+1}$ will all have conditional variance equal to 1.

The conditional covariance of these standardized returns equals the conditional correlation of the original (raw) returns which shows that modeling the conditional correlation of raw returns is equivalent to modeling the conditional covariance of standardized returns.

Value at Risk

In its most general form, the Value at Risk (VaR) measures the potential loss in value of a risky asset or portfolio over a specified time horizon at a given confidence level. For example, if the VaR on a portfolio is \$25 million over a one-day period at the 99% confidence level, there is only a 1% chance that the value of the portfolio will decline by more than \$25 million on any given day.

Let X denote the portfolio return over a horizon T, and let $\alpha \in (0,1)$ denote the confidence level e.g., 0.95 or 0.99. Then, the Value at Risk at level α is defined as the quantile function of the loss distribution:

$$VaR\ \alpha(X) = \inf\{x \in R: P(-X \leq x) \geq \alpha\} \tag{6.1}$$

Alternatively, if $L = -X$ denotes the loss, then:

$$VaR\ \alpha(L) = inf\{\ell \in R: P(L \leq \ell) \geq \alpha\} \tag{6.2}$$

For parametric (Variance-Covariance) VaR, we assume the return $X \sim N(\mu, \sigma 2)$ is normally distributed. Then the VaR at confidence level α over time horizon T, for an initial portfolio value V, is given by:

$$VaR\ \alpha = V \cdot (-\mu T + z\alpha \cdot \sigma T) \tag{6.3}$$

Where μ is theexpected return per unit time, $z\alpha$ the quantile of the standard normal distribution, i.e., $\Phi - 1(\alpha)$.

For the common assumption of zero mean ($\mu = 0$)[8]:

$$VaR\ \alpha = V \cdot z\alpha \cdot \sigma T \tag{6.4}$$

Einhorm quoted nicely when speaking about VaR's even with 99% c.l.:

"An airbag that works all the time, except when you have a car accident."

Even when performing simulations of a multivariate normal distribution, it won´t produce a forward-looking correlation. The Cholesky decomposition will "just" give you the instant correlation.

The breakdown of traditional correlations during crises reveals a deeper flaw in the risk modeling frameworks that underpin portfolio construction. The limitations of Pearson correlation and its inability to capture non-linear relationships, tail dependence, or dynamic changes demand more robust methods. This is precisely where copulas become relevant. By decoupling marginal behavior from joint dependence, copulas provide a more flexible and accurate framework for understanding how assets co-move, especially under extreme conditions.

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[8] $\mu \neq 0$ typically overstates/understates VaR.

Copulas

Let $X = (X_1, \dots, X_d)$ be a random vector representing *financial* variables (e.g. asset returns, credit spreads). Traditional models capture their dependence via the linear correlation matrix $\rho_{ij} = Corr(Xi, Xj)$, assuming an elliptical distribution (e.g. multivariate normal or t). However, this approach *fails* when:

- Dependence is nonlinear or asymmetric.
- Joint extremes (tail events) behave differently from the center of the distribution.
- Correlation changes over time.

Instead of the Cholesky decomposition we'll use Sklar's Theorem, so for any joint distribution F with continuous marginals $F_1, \dots, F_d$, there exists a unique copula $C: [0,1]^d \rightarrow [0,1]$ such that:

$$F(x_1, \dots, x_d) = C\left(F_1(x_1), \dots, F_d(x_d)\right) \tag{7.1}$$

Conversely, given any copula C and any set of marginal distributions $F_1, \dots, F_d$, we can construct a valid joint distribution F. The copula captures the entire dependence structure, while the F_i capture the marginal behavior.

Uniform margins may be transformed from defining the probability integral transform:

$$U_i \coloneqq F_i(X_i), so\ that\ U_i \sim U(0,1) \tag{7.2}$$

The copula is then the joint distribution function of $U = (U_1, \dots, U_d)$:

$$C(u_1, \dots, u_d) = P(U_1 \leq u_1, \dots, U_d \leq u_d) \tag{7.3}$$

This transformation allows us to separate the modeling of dependence (via C) from the modeling of the marginals (via F_i).

Suppose we observe that dependence between assets changes over time (e.g., increases during crises). We can construct a dynamic copula model where:

$$C_t(u_1, \dots, u_d) = C(u_1, \dots, u_d; \theta_t) \tag{7.4}$$

with θ_t, is a time-varying dependence parameter, potentially driven by latent dynamics or observable covariates (e.g., volatility, liquidity, macro variables). In the case of a Gaussian copula, for example, we let the correlation parameter ρ_t evolve over time:

$$C_\rho^{Ga}(u_1, u_2) = \Phi_\rho\big(\Phi - 1(u_1), \Phi - 1(u_2)\big) \tag{7.5}$$

Where Φ, is the standard univariate normal distribution, and $\Phi_{\rho t}$ is the bivariate normal CDF with correlation ρ_t.

we let the correlation parameter $\rho = \rho_t$ evolve over time, e.g. via ARMA, GARCH or stochastic volatility dynamics. Thus, leading to better forward-looking estimates of the true dependence, especially in nonstationary settings. But we also *need* them for tail events, the lower tail dependence being:

$$\lambda_L = \lim_{u \to 0^+} \mathbb{P}\,(U_2 \le u \mid U_1 \le u) \tag{7.7}$$

While upper being:

$$\lambda_L = \lim_{u \to 1^-} \mathbb{P}\,(U_2 \le u \mid U_1 \le u) \tag{7.8}$$

Different copulas capture this differently: Gaussian copula: $\lambda_L = \lambda_U = 0\ if\ \rho < 1$; t-copula: $\lambda_L, \lambda_U > 0$; Clayton copula: $\lambda_L > 0, \lambda_U = 0$; Gumbel copula: $\lambda_U = 0, \lambda_U > 0$.

Once we fit a copula $C(u; \theta_t)$ and marginals Fi at each time t, we can:

1. Forecast future θ_{t+h} (e.g., using a time series model).
2. Simulate $U^{(m)} \sim C(\cdot; \theta_{t+h})$.
3. Apply quantile transformation: $X_i^{(m)} = F_i^{-1}\left(U_i^{(m)}\right)$.

This generates future joint scenarios with realistic dependence — not limited to linear correlation. We can also extract an implied correlation matrix from the copula (e.g., for Gaussian copula, ρ_t) and compare it with historical sample correlations to see how dependence evolves and how much classical methods under- or overestimate joint risk.

Adapted from Meucci (2005) and standard copula literature

Hedging

PPUT

Is using linear derivatives good? Take an example of e a "low effort" way to tick this box.

"The Cboe S&P 500 5% Put Protection Index (PPUT) tracks the value of a hypothetical portfolio of securities (PPUT portfolio) designed to protect an investor from negative S&P 500 returns. The PPUT portfolio is composed of S&P 500® stocks and of a long position in a one-month 5% out-of-the-money put option on the S&P 500 (SPX Put)."

For a visual representation:

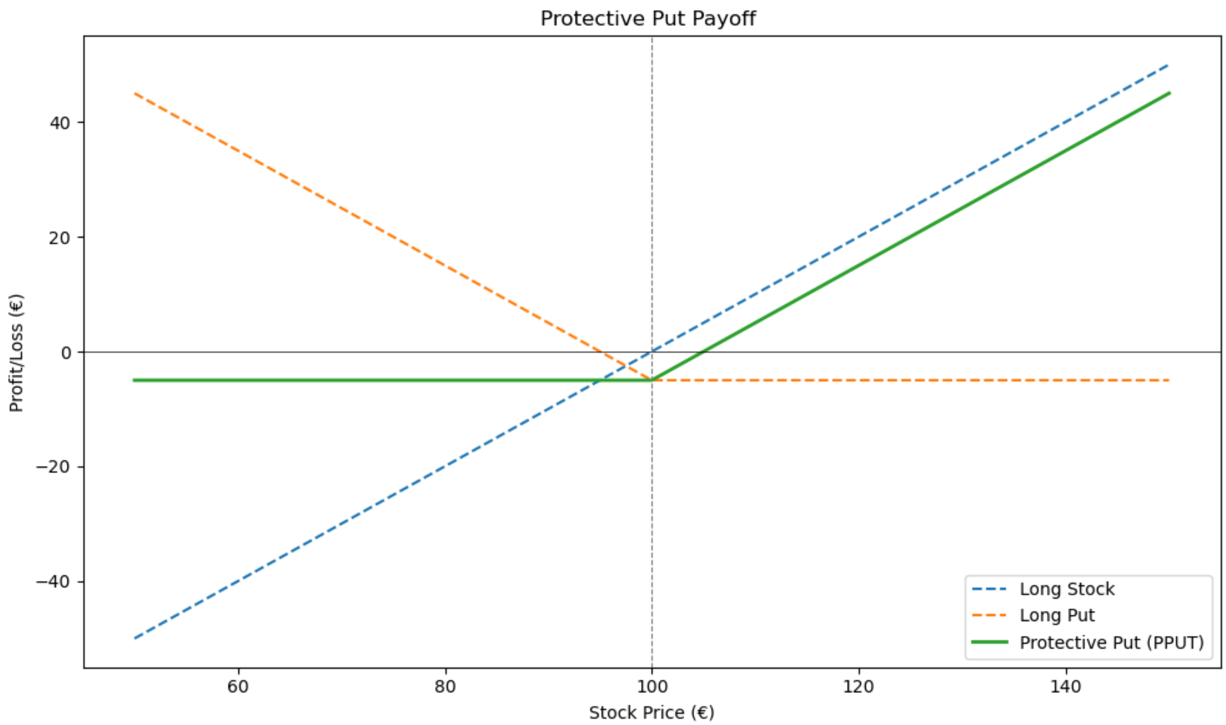


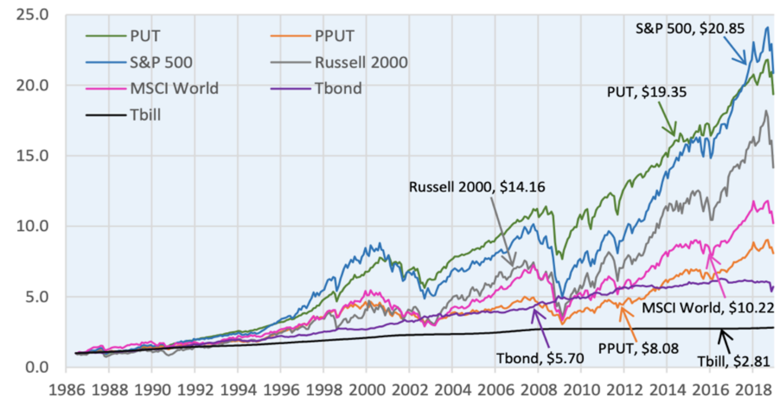
Figure 11 - PPUT Payoff

Combining the payoff by being long stocks with a long put we have a payoff:

$$[-1,1] + [+1,0] = [0,+1]$$

This seems a good way to offer protection in uncertain scenarios, since it targets left-tail risk, but limits gains in bull markets.

EXHIBIT 4 – GROWTH OF BENCHMARK INDICES SINCE JUN 30, 1986



The value of $1 invested in PUT, PPUT, S&P 500, Russell 2000, MSCI World, 30-year Tbond (FTSE), and 30-day Tbill. The period is from Jun 30, 1986 to Dec 31, 2018. Past performance is not predictive of future returns. *Sources: Bloomberg and Cboe Exchange, Inc.*

Figure 12 - Evolution of different assets.

Note: PUT sells puts, while PPUT buys them.

EXHIBIT 5 – MONTHLY STATISTICS (JUN 30, 1986 TO DEC 31, 2018)

	PUT	PPUT	S&P 500	Russell 2000	MSCI World	30-year Tbond	30-day Tbill
Mean Return	0.81%	0.60%	0.88%	0.84%	0.69%	0.59%	0.27%
Compound Return	0.76%	0.54%	0.78%	0.68%	0.60%	0.53%	0.27%
Min Return	-17.65%	-10.60%	-21.54%	-30.63%	-18.96%	-14.61%	0.00%
Standard Deviation	2.87%	3.49%	4.31%	5.54%	4.31%	3.51%	0.21%
Skewness	-2.09	-0.28	-0.81	-0.88	-0.67	0.25	0.24
Kurtosis	12.58	3.52	5.48	6.07	4.79	5.64	1.87
Alpha	0.20%	-0.12%	0.00%	-0.07%	-0.12%	0.38%	0.00%
Beta	0.56	0.74	1.00	1.06	0.89	-0.08	0.00
Sharpe Ratio	0.19	0.10	0.14	0.10	0.10	0.09	
Sortino Ratio	0.25	0.14	0.20	0.14	0.14	0.15	
Stutzer Index	0.18	0.09	0.14	0.10	0.10	0.09	
M-squared	1.08%	0.68%	0.88%	0.71%	0.69%	0.67%	

EXHIBIT 6 – ANNUALIZED STATISTICS (JUN 30, 1986 TO DEC 31, 2018)

	PUT	PPUT	S&P 500	Russell 2000	MSCI World	30-year Tbond	30-day Tbill
Compound Return	9.54%	6.64%	9.80%	8.50%	7.41%	6.60%	3.24%
Standard Deviation	9.95%	12.08%	14.93%	19.18%	14.92%	12.16%	0.74%
Sharpe Ratio	0.65	0.33	0.49	0.36	0.34	0.32	
Sortino Ratio	0.85	0.48	0.70	0.50	0.48	0.51	
Stutzer Index	0.61	0.33	0.48	0.35	0.34	0.33	

Figure 13 - Statistics

Overall, this evidence suggests that PPUT offers a more conservative risk-return profile relative to the S&P 500, making it a potentially attractive choice for investors seeking moderate exposure to equities with reduced volatility. Its performance characteristics indicate suitability as a diversifying component within multi-asset portfolios, particularly for those with a focus on income generation and capital preservation. But targets the price of a single asset and limits gains in stable and bull markets.

Coming back to the risk of correlation itself, can we hedge it?

Correlation Options

The payoff of these multi-asset options depends on the correlation between assets.

Better-of-option (two assets):

$$payoff = \max(S_1, S_2)$$

Worst-of-option (two assets):

$$payoff = \min(S_1, S_2)$$

Exchange option:

$$payoff = \max(0, S_2 - S_1)$$

Spread option (2 assets):

$$payoff = \max[0, (S_2 - S_1) - K]$$

Dual-strike call option:

$$payoff = \max(0, S_1 - K, S_2 - K)$$

Basket option:

$$payoff = \left[\sum_{i=1}^{n} n_i S_i - K, 0\right]$$

The prices are highly to the correlation between S_i, the lower the correlation, the higher the option price[9].

[9] Except for the *worst-of-option* which is the inverse relation and the basket option since it depends on the w_i of the assets.

Correlation Swaps

A correlation swap is a derivative contract that allows one to take a direct position on the realized correlation of a basket of assets.

Let $\rho_{ij}^{realized}$ be the realized pairwise correlation between assets i and j over the life of the swap. For a basket of n assets, the realized average pairwise correlation is:

$$\rho^{realized} = \frac{2}{n(n-1)} \sum_{i<j} \rho_{ij}^{realized} \tag{8.1}$$

Let ρ^{strike} be the fixed (strike) correlation agreed at initiation of the swap. The notional value is N.

Then the payoff to a long correlation swap at maturity is:

$$Payoff = N \cdot (\rho_{realized} - \rho_{strike}) \tag{8.2}$$

While these swaps offer a good direct hedge they come with some limitations. Correlation swaps do not hedge price direction, they rely on co-movement. As such, a portfolio can still experience significant losses if assets fall independently and correlation remains low; in that case, the swap provides no positive payout. Lastly, correlation swaps often have a negative expected carry. This is because the strike level (i.e., the fixed correlation) is typically set above the long-term average realized correlation, making it costly to maintain a long correlation position unless a market dislocation occurs.

Future reading: Faria, G., Kosowski, R., & Wang, T. (2022). *The Correlation Risk Premium: International Evidence.*

Conclusion/Opinion

While diversification remains a foundational principle in finance, its efficacy is often compromised by superficial applications that fail to account for overlapping exposures, dynamic risk structures, and regime-dependent asset behaviors.

Empirical evidence and theoretical insights presented herein highlight the limitations of relying solely on static correlation measures or conventional asset allocations. Particularly during periods of market stress, such approaches tend to underperform, exposing portfolios to unintended systemic risks. The discussion of copula-based models, conditional dependence structures, and derivative-based instruments such as correlation swaps and protective puts, underscores the need for more nuanced and adaptive diversification frameworks.

Ultimately, diversification should not be understood merely as the act of increasing the number of assets in a portfolio, but as a rigorous process of constructing exposures that are genuinely distinct in their risk and return profiles, both in normal and stressed conditions.

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