The moving average stochastic volatility model: application to inflation forecasting

Chan (2013)

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1 Overview

This project aims to partially replicate the work of Chan (2013) in the application of bayesian methods to inflation forecasting, using time series models with stochastic volatility. The general framework of a q-th order moving average model with stochastic volatility is the following:

$$y_t = \mu_t + \epsilon_t^y, \tag{1}$$

$$\epsilon_t^y = u_t + \psi_1 u_{t-1} + ... + \psi_q u_{t-q}, \qquad u \sim N(0, e^{h_t}), \tag{2}$$

$$h_t = \mu_h + \phi_h h_{t-1} - \mu_h + \epsilon_t^h, \qquad \epsilon_t^h \sim N(0, \sigma_h^2), \tag{3}$$

where $|\psi_h| < 1$ by assumption. The errors u_t and ϵ_t^h are independent of each other for all leads and lags. The author additionally assumes that $u_0 = u_{-1} = ... = u_{-q+1} = 0$. It's important to note that, by choosing a suitable conditional mean process μ_t, the model (1)-(3) includes several nested possible specifications. We follow the author's use of the unobserved components model, which is a popular framework for inflation forecasting. Hence, and because moving average models have a state space representation, the measurement equation is given by

$$\mathbf{y} = \mu + \mathbf{H}_\psi \mathbf{u}, \tag{4}$$

where $\mu = \tau = (\tau_1, ..., \tau_T)'$, $\mathbf{u} = (u_1, ..., u_T)' \sim N(\mathbf{0}, \mathbf{S_y})$, $\mathbf{S_y} = diag(e^{h_1}, ..., e^{h_T})$, and $\mathbf{H}_\psi$ is a $T \times T$ lower triangular matrix with ones on the main diagonal, ψ_1 on the first lower diagonal, ψ_2 on the second lower diagonal, and so forth. The transition equations of the unobserved components model are, in turn, represented by:

$$\tau_t = \tau_{t-1} + \epsilon_t^\tau, \qquad \epsilon_t^\tau \sim N(0, \sigma_\tau^2), \tag{5}$$

$$h_t = \mu_h + \phi_h(h_{t-1} - \mu_h) + \epsilon_t^h, \qquad \epsilon_t^h \sim N(0, \sigma_h^2), \tag{6}$$

with $|\phi_h| < 1$. The transition equation for τ_t represents a Markov chain, which is initialized with the initial condition $\tau_1 \sim N(\tau_0, \sigma_{0\tau}^2)$, where τ_0 and $\sigma_{0\tau}^2$ are some constants. The initial condition for the h_t process is $h_1 \sim N(\mu_h, \sigma_h^2/(1 - \phi_h^2))$. As for the parameters ψ, σ_τ^2, μ_h, ϕ_h and σ_h^2, the author assumed the following independent priors:

$$\begin{gathered} \sigma_\tau^2 \sim IG(v_\tau, S_\tau), \quad \mu_h \sim N(\mu_{h0}, V_{\mu_h}), \\ \phi_h \sim N(\phi_{h0}, V_{\phi_h})\mathbf{1}(|\phi_h| < 1), \quad \sigma_h^2 \sim IG(v_h, S_h), \end{gathered} \tag{7}$$

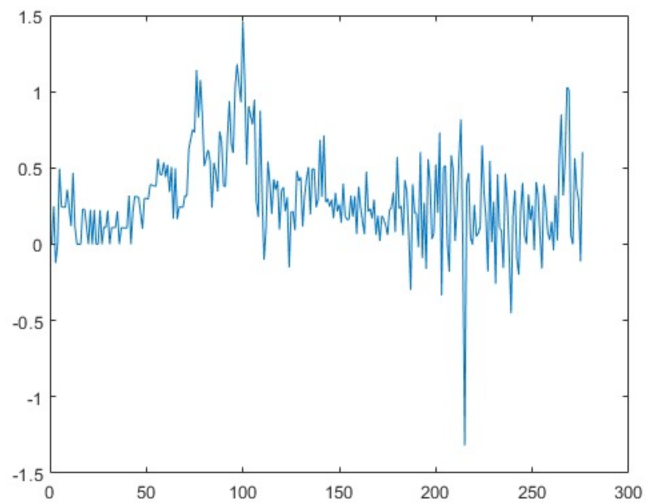
where IG denotes the inverse gamma distribution. The author proceeds to discuss an efficient way to sample from $p(\tau|\mathbf{y}, \mathbf{h}, \psi, \sigma_\tau^{\mathbf{2}})$. Because the model has a state space representation, the likelihood function

$$logp(\tilde{\tau}|\tilde{\mathbf{y}}, \tilde{\mathbf{h}}, \tilde{\psi}, \tilde{\sigma_\tau^{\mathbf{2}}}) \propto -\frac{1}{2}(\tilde{\tau} - \hat{\tau})'\mathbf{D}_{\tilde{\tau}}^{\mathbf{-1}}(\tilde{\tau} - \hat{\tau}) \tag{8}$$

where $\mathbf{D}_{\tilde{\tau}}^{\mathbf{-1}} = (\mathbf{S_y^{-1}} + \mathbf{H}_\psi'\mathbf{\Omega}_\tau^{\mathbf{-1}}\mathbf{H}_\psi)^{\mathbf{-1}}$ could be evaluated using the Kalman filter[1]. This approach is, however, very slow due to matrix multiplications and inversions in, typically, high-dimensional systems. Hence, the author proposes a precision-based method that is substantially more efficient than the Kalman filter, as the same operations take sparse precision matrices as input, rather than dense covariance matrices. While the Kalman filter maintains and updates state estimates and error covariances, the precision-based algorithm focuses on updating precision matrices directly. Moreover, while the former uses the Kalman gain to update state estimates and covariances, the latter updates the precision matrix and mean directly, which simplifies the otherwise required computations.

[1]The author works with $\tilde{\mathbf{y}} = \tilde{\tau} + \mathbf{u}$, where $\tilde{\mathbf{y}} = \mathbf{H}_\psi^{\mathbf{-1}}\mathbf{y}$ and $\tilde{\tau} = \mathbf{H}_\psi^{\mathbf{-1}}\tau$ as it allows a closed form log-likelihood expression, and is therefore more convenient than the matrix representation of (5)-(6).

Figure 1: U.S. quarterly CPI inflation



2 Results

Figure 1 shows our data, which consists of U.S. quarterly CPI inflation from 1947Q1 to 2011Q3. In order to model inflation, I will focus on two specifications presented by Chan (2013), which are the unobserved components model represented in (5)-(6), (i) with stochastic volatility in the measurement equation, and (ii) with stochastic volatility in both the measurement and transition equations. In the latter variation, the variance of ϵ_t^τ is allowed to be time-varying: $\epsilon_t^\tau \sim N(0, \sigma^{g_t})$. The log-volatility g_t, in turn, evolves as a stationary AR(1) process:

$$g_t = \mu_g + \phi_g(g_{t-1} - \mu_g) + \epsilon_t^g, \qquad \epsilon_t^g \sim N(0, \sigma_g^2), \tag{9}$$

where $|\phi_g| < 1$. In each of the two models, we follow Chan (2013) by including the MA-SV variants as specified in (1)-(3). We refer to these models as **UC-MA** and **UCSV-MA**, respectively. Taking into consideration the setup in the previous section, there are three elements that we understand as necessary to assess each of the two models' performances. First, we want to look at the estimated posterior means of the underlying inflation τ_t for both models. We report these estimates in figures 2 and 3. The first noticeable difference

Figure 2: Posterior mean of underlying inflation using UC-MA

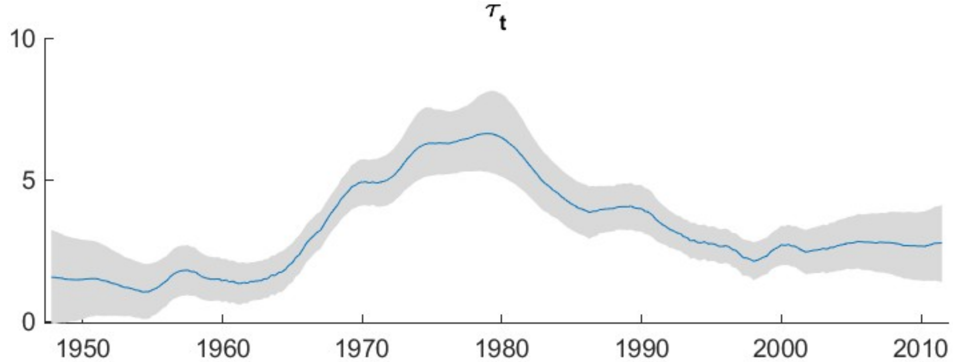


Figure 3: Posterior mean of underlying inflation using UCSV-MA

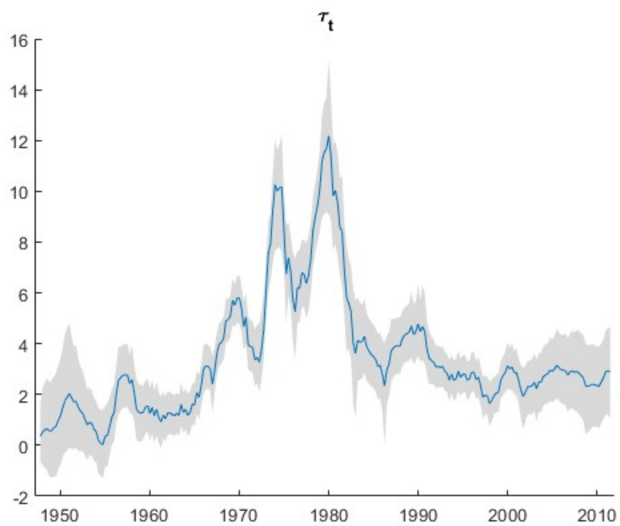
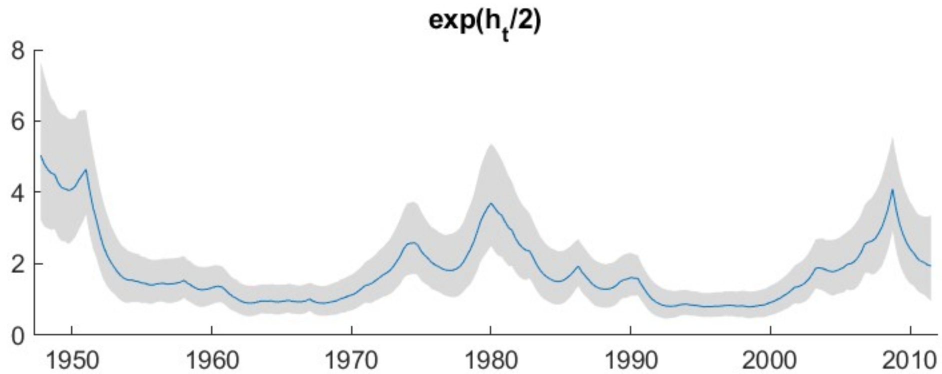


Figure 4: Posterior estimates of inflation volatility using UC-MA



between the two plots is how the posterior obtained through UC-MA is relatively smoother over time than the UCSV-MA estimates. Hence, by allowing time-varying volatility in the transition equation, the posterior means of underlying inflation become much more volatile than with the UC-MA. This means that the moving average errors in the measurement equation (5) play less of a role in the volatility estimates than the process (9) for g_t. Chan (2013) suggests a preference for the specification that yields the smoother inflation estimates.

Second, we want to look at how the inflation volatility estimates differ between the two specifications. To that end, we report the estimates for $exp(h_t/2)$ for both models, and $exp(g_t/2)$ for the UCSV-MA in figures 4 and 5. The posterior estimates differ significantly between the two models, and we can see that, in contrast to the posterior mean estimates, the output is smoother for the UCSV-MA than for the UC-MA. This suggests that the UCSV-MA framework might have higher flexibility in capturing time-varying uncertainty. Moreover, the different output shows how posterior inflation volatility estimates are sensitive to modeling assumptions.

Finally, we're interested in the MA coefficient ψ_1 for both models. This is an important step to provide validity to both frameworks, as a posterior density for ψ_1 concentrated around zero would indicate that the MA component is not necessary. As it stands, and as we report in figures 6 and 7, the posterior density for ψ_1 is concentrated to the right of zero in both models.

Figure 5: Posterior estimates of inflation volatility using UCSV-MA

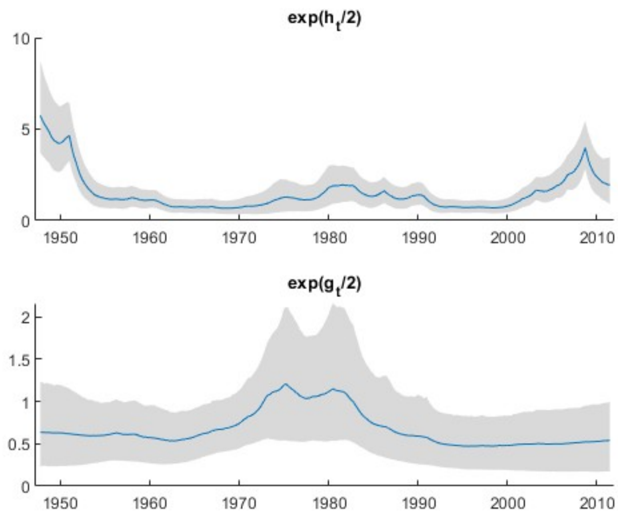


Figure 6: Posterior distribution of MA parameter in UC-MA

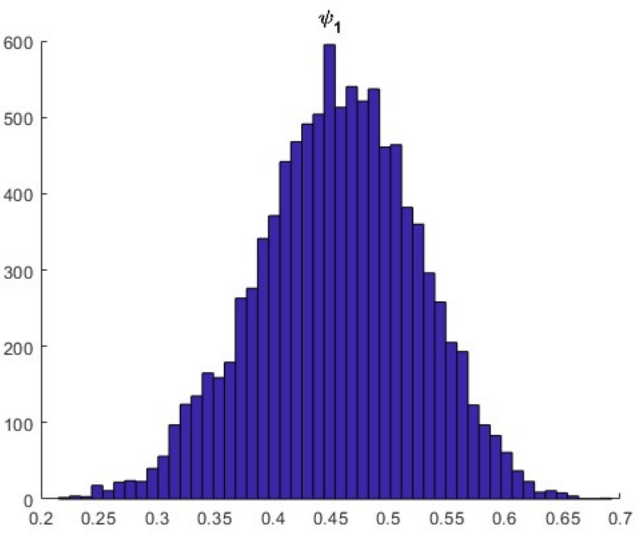
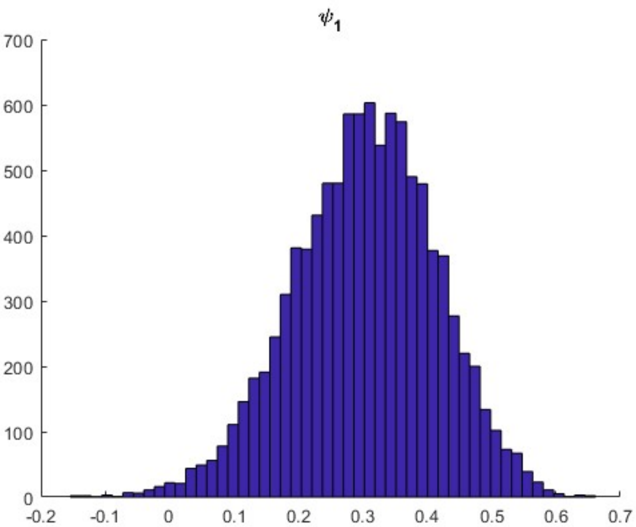


Figure 7: Posterior distribution of MA parameter in UCSV-MA



References

[1] Chan, Joshua, (2013). Moving average stochastic volatility models with application to inflation forecast, *Journal of Econometrics, 176, issue 2*, p. 162-172.