Cálculo de Programas
Algebra of Programming

UNIVERSIDADE DO MINHO
Lic. em Engenharia Informática (3º ano)
Lic. Ciências da Computação (2º ano)

2023/24 - Ficha *(Exercise sheet)* nr. 10

1. A função $concat$, extraída do *Prelude* do Haskell, é o catamorfismo de listas

   The concat function, taken from the Haskell Prelude*, is the list-catamorphism*

$$concat = (\!|\, [\mathsf{nil}\,, \mathsf{conc}] \,|\!) \tag{F1}$$

onde $\mathsf{conc}\ (x, y) = x +\!\!+\ y$ e $\mathsf{nil}\ \_ = [\,]$. Apresente justificações para a prova da propriedade

where $\mathsf{conc}\ (x, y) = x +\!\!+\ y$ *and* $\mathsf{nil}\ \_ = [\,]$. *Provide justifications for proof of property*

$$\mathsf{length} \cdot concat = \mathsf{sum} \cdot \mathsf{map\ length} \tag{F2}$$

que a seguir se apresenta, onde é de esperar que as leis de *fusão*-cata e *absorção*-cata desempenhem um papel importante:

*which is presented below, where the cata-*fusion *and cata-*absorption *laws are expected to play an important role:*

$$
\begin{array}{ll}
& \mathsf{length} \cdot concat = \mathsf{sum} \cdot \mathsf{map\ length} \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \mathsf{length} \cdot (\!|\, [\mathsf{nil}\,, \mathsf{conc}] \,|\!) = (\!|\, [\underline{0}\,, \mathsf{add}] \,|\!) \cdot \mathsf{map\ length} \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \mathsf{length} \cdot (\!|\, [\mathsf{nil}\,, \mathsf{conc}] \,|\!) = (\!|\, [\underline{0}\,, \mathsf{add}] \cdot (id + \mathsf{length} \times id) \,|\!) \\
\Leftarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \mathsf{length} \cdot [\mathsf{nil}\,, \mathsf{conc}] = [\underline{0}\,, \mathsf{add} \cdot (\mathsf{length} \times id)] \cdot (id + id \times \mathsf{length}) \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \left\{ \begin{array}{l} \mathsf{length} \cdot \mathsf{nil} = \underline{0} \\ \mathsf{length} \cdot \mathsf{conc} = \mathsf{add} \cdot (\mathsf{length} \times id) \cdot (id \times \mathsf{length}) \end{array} \right. \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \mathsf{length} \cdot \mathsf{conc} = \mathsf{add} \cdot (\mathsf{length} \times \mathsf{length}) \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& true \\
\Box &
\end{array}
$$

Ver se fiz bem

2. O diagrama genérico de um catamorfismo de gene g sobre o *tipo paramétrico* $\mathsf{T}\ X \cong \mathsf{B}\ (X, \mathsf{T}\ X)$ cuja base é o bifunctor B, bem como a sua propriedade universal, são representados a seguir:

The generic diagram of a catamorphism with gene g over the parametric type $\mathsf{T}\ X \cong \mathsf{B}\ (X, \mathsf{T}\ X)$ *with base* B*, as well as its universal property, are represented below:*

$$\begin{array}{ccc} \mathsf{T}\ X & \xleftarrow{\mathsf{in}} & \mathsf{B}\ (X, \mathsf{T}\ X) \\ {\scriptstyle (\!|\, g \,|\!)}\downarrow & & \downarrow{\scriptstyle \mathsf{B}\ (id, (\!|\, g \,|\!)) = \mathsf{F}\ (\!|\, g \,|\!)} \\ B & \xleftarrow[g]{} & \mathsf{B}\ (X, B) \end{array}$$

$$k = (\!|\, g \,|\!) \quad \equiv \quad k \cdot \mathsf{in} = g \cdot \underbrace{\mathsf{B}\ (id, k)}_{\mathsf{F}\ k}$$

De seguida, apresenta-se uma revisão do inventário de tipos indutivos da questão 6 da ficha anterior, recorrendo agora aos seus functores de base:

Next, a review of the inventory of inductive types of question 6 of the previous exercise sheet is given, now using its base-functors:

(a) Árvores com informação de tipo A nas folhas *(Trees with data in their leaves)* :

$$\mathsf{T} = \mathsf{LTree}\ A \qquad \begin{cases} \mathsf{B}\ (X, Y) = X + Y^2 \\ \mathsf{B}\ (g, f) = g + f^2 \end{cases} \qquad \mathsf{in} = [\mathit{Leaf}\ , \mathit{Fork}]$$

Haskell: $\mathbf{data}\ \mathsf{LTree}\ a = \mathit{Leaf}\ a \mid \mathit{Fork}\ (\mathsf{LTree}\ a, \mathsf{LTree}\ a)$

(b) Árvores com informação de tipo A nos nós *(Trees whose data of type A are stored in their nodes)*:

$$\mathsf{T} = \mathsf{BTree}\ A \qquad \begin{cases} \mathsf{B}\ (X, Y) = 1 + X \times Y^2 \\ \mathsf{B}\ (g, f) = id + g \times f^2 \end{cases} \qquad \mathsf{in} = [\underline{\mathit{Empty}}\ , \mathit{Node}]$$

Haskell: $\mathbf{data}\ \mathsf{BTree}\ a = \mathit{Empty} \mid \mathit{Node}\ (a, (\mathsf{BTree}\ a, \mathsf{BTree}\ a))$

(c) Árvores com informação nos nós e nas folhas *(Full trees — data in both leaves and nodes)*:

$$\mathsf{T} = \mathsf{FTree}\ B\ A \qquad \begin{cases} \mathsf{B}\ (Z, X, Y) = Z + X \times Y^2 \\ \mathsf{B}\ (h, g, f) = h + g \times f^2 \end{cases} \qquad \mathsf{in} = [\mathit{Unit}\ , \mathit{Comp}]$$

Haskell: $\mathbf{data}\ \mathsf{FTree}\ b\ a = \mathit{Unit}\ b \mid \mathit{Comp}\ (a, (\mathsf{FTree}\ b\ a, \mathsf{FTree}\ b\ a))$

(d) Árvores de expressão *(Expression trees)*:

$$\mathsf{T} = \mathsf{Expr}\ V\ O \qquad \begin{cases} \mathsf{B}\ (Z, X, Y) = Z + X \times Y^* \\ \mathsf{B}\ (h, g, f) = h + g \times \mathsf{map}\ f \end{cases} \qquad \mathsf{in} = [\mathit{Var}\ , \mathit{Term}]$$

Haskell: $\mathbf{data}\ \mathsf{Expr}\ v\ o = \mathit{Var}\ v \mid \mathit{Term}\ (o, [\mathsf{Expr}\ v\ o])$

Partindo da definição *genérica* de map associado ao tipo T,

Starting from the generic *definition of* map *associated with the type* T*,*

$$\mathsf{T}\ f = (\!|\, \mathsf{in} \cdot \mathsf{B}\ (f, id) \,|\!)$$

calcule $fmap\ f = \mathsf{T}\ f$ para $\mathsf{T} := \mathsf{BTree}$, entregando o resultado em Haskell sem combinadores *pointfree*. (Repare-se que se tem sempre $\mathsf{F}\ k = \mathsf{B}\ (id, k)$.)

derive $fmap\ f = \mathsf{T}\ f$ for $\mathsf{T} := \mathsf{BTree}$, delivering the result in Haskell without pointfree *combinators. (Note that we always have $\mathsf{F}\ k = \mathsf{B}\ (id, k)$.)*

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3. Recorra à lei da absorção-cata, entre outras, para verificar as seguintes propriedades sobre listas

Use the cata-absorption law, among others, to prove the following properties about lists

$$\mathsf{length} = \mathsf{sum} \cdot (\mathsf{map}\ \underline{1}) \tag{F3}$$

$$\mathsf{length} = \mathsf{length} \cdot (\mathsf{map}\ f) \tag{F4}$$

onde length, sum e map são catamorfismos de listas que conhece. (Recorda-se que o bifunctor de base para listas é $\mathsf{B}\ (f, g) = id + f \times g$, de onde se deriva $\mathsf{F}\ f = \mathsf{B}\ (id, f) = id + id \times f$.)

where length, sum *and* map *they are list-catamorphisms you know. (Remember that the basic bifunctor for lists is* $\mathsf{B}\ (f, g) = id + f \times g$*, yielding* $\mathsf{F}\ f = \mathsf{B}\ (id, f) = id + id \times f$*.)*

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4. Seja dado o catamorfismo

Let catamorphism

$$depth = (\!|\, [\mathsf{one}\ , \mathsf{succ} \cdot umax] \,|\!)$$

que dá a profundidade de árvores do tipo LTree, onde $umax\ (a, b) = max\ a\ b$. Mostre, por absorção-cata, que a profundidade de uma árvore t não é alterada quando aplica uma função f a todas as suas folhas:

be given, which gives the depth of trees of type LTree*, where* $umax\ (a, b) = max\ a\ b$*. Show, by cata-absorption, that the depth of a tree* t *is not changed when you apply a function* f *to all its leaves:*

$$depth \cdot \mathsf{LTree}\ f = depth \tag{F5}$$

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5. Um *anamorfismo* é um *"catamorfismo ao contrário"*, i.é uma função $k : A \to \mathsf{T}$ tal que

An anamorphism *is a* "reverse catamorphism"*, i.e. a function* $k : A \to \mathsf{T}$ *such that*

$$k = \mathsf{in} \cdot \mathsf{F}\ k \cdot g \tag{F6}$$

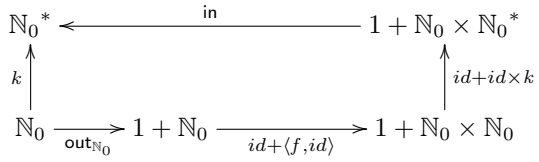
escrevendo-se $k = [\!(g)\!]$. Mostre que o anamorfismo de listas

One writes $k = [\!(g)\!]$*. Show that the list-anamorphism*

$$k = [\!(\, (id + \langle f, id \rangle) \cdot \mathsf{out}_{\mathbb{N}_0} \,)\!] \tag{F7}$$

descrito pelo diagrama

depicted in diagram



é a função

is the function

$$\begin{aligned} k\ 0 &= [\,] \\ k\ (n+1) &= (2\ n+1) : k\ n \end{aligned}$$

para $f\ n = 2\ n + 1$. (Que faz esta função?)

for $f\ n = 2\ n + 1$*. (What does this function do?)*

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6. Mostre que o anamorfismo que calcula os sufixos de uma lista

Show that the anamorphism that computes the suffixes of a list

$$suffixes = [\!(g)\!]\ \mathbf{where}\ g = (id + \langle \mathsf{cons}, \pi_2 \rangle) \cdot \mathsf{out}$$

é a função:

is the function:

$$\begin{aligned} suffixes\ [\,] &= [\,] \\ suffixes\ (h : t) &= (h : t) : suffixes\ t \end{aligned}$$

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7. O formulário desta disciplina apresenta duas definições alternativas para o functor $\mathsf{T}\ f$, uma como *catamorfismo* e outra como *anamorfismo*. Identifique-as e acrescente justificações à seguinte prova de que essas definições são equivalentes:

The reference sheet of this course presents two alternative definitions for functor $\mathsf{T}\ f$, *one as a* catamorphism *and another as an* anamorphism. *Identify them and fill in justifications in the following proof that such definitions are equivalent:*

$$
\begin{array}{ll}
 & \mathsf{T}\, f = \llparenthesis\, \mathsf{in} \cdot \mathsf{B}\ (f, id)\, \rrparenthesis \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \mathsf{T}\, f \cdot \mathsf{in} = \mathsf{in} \cdot \mathsf{B}\ (f, id) \cdot \mathsf{F}\ (\mathsf{T}\, f) \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \mathsf{T}\, f \cdot \mathsf{in} = \mathsf{in} \cdot \mathsf{B}\ (id, \mathsf{T}\, f) \cdot \mathsf{B}\ (f, id) \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \mathsf{out} \cdot \mathsf{T}\, f = \mathsf{F}\ (\mathsf{T}\, f) \cdot \mathsf{B}\ (f, id) \cdot \mathsf{out} \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \mathsf{T}\, f = [\!(\mathsf{B}\ (f, id) \cdot \mathsf{out})\!] \\
\Box &
\end{array}
$$

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8. Mostre que o catamorfismo de listas $\mathsf{length} = \llparenthesis\, [\mathsf{zero}\ , \mathsf{succ} \cdot \pi_2]\, \rrparenthesis$ é a mesma função que o *anamorfismo* de naturais $[\!((id + \pi_2) \cdot \mathsf{out}_{\mathsf{List}})\!]$.

Show that the list catamorphism $\mathsf{length} = \llparenthesis\, [\mathsf{zero}\ , \mathsf{succ} \cdot \pi_2]\, \rrparenthesis$ *is the same function as the* $\mathbb{N}_0$-anamorphism $[\!((id + \pi_2) \cdot \mathsf{out}_{\mathsf{List}})\!]$.

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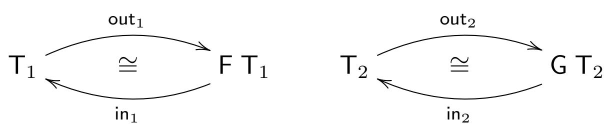
9. O facto de $\mathsf{length} : A^* \rightarrow \mathbb{N}_0$ poder ser definida tanto como *catamorfismo* de listas como *anamorfismo* de naturais (questão 8) pode generalizar-se da forma seguinte: sejam dados dois tipos indutivos

The fact that $\mathsf{length} : A^* \rightarrow \mathbb{N}_0$ *is both a list-*catamorphism *and a* $\mathbb{N}_0$-anamorphism *(question 8) generalizes as follows: let two inductive types be given*



e $\alpha : \mathsf{F}\ X \rightarrow \mathsf{G}\ X$, isto é, α satisfaz a propriedade *grátis*:

and $\alpha : \mathsf{F}\ X \rightarrow \mathsf{G}\ X$, *that is,* α *satisfying the* free *property:*

$$\mathsf{G}\, f \cdot \alpha = \alpha \cdot \mathsf{F}\, f \qquad \text{(F8)}$$

Então $\llparenthesis\, \mathsf{in}_2 \cdot \alpha\, \rrparenthesis = [\!(\alpha \cdot \mathsf{out}_1)\!]$, como se mostra a seguir (complete as justificações):

Then $\llparenthesis\, \mathsf{in}_2 \cdot \alpha\, \rrparenthesis = [\!(\alpha \cdot \mathsf{out}_1)\!]$, *as shown below (complete the justifications):*

$$k = (\!|\, \mathsf{in}_2 \cdot \alpha \,|\!)$$

$\equiv$ $\{$ .... $\}$

$$k \cdot \mathsf{in}_1 = \mathsf{in}_2 \cdot \alpha \cdot \mathsf{F}\ k$$

$\equiv$ $\{$ .... $\}$

$$\mathsf{out}_2 \cdot k = \mathsf{G}\ k \cdot \alpha \cdot \mathsf{out}_1$$

$\equiv$ $\{$ .... $\}$

$$k = [\!(\alpha \cdot \mathsf{out}_1)\!]$$

$\square$

Como aplicação imediata deste resultado, derive a seguinte definição como anamorfismo da função que espelha LTrees:

Use the result above to redefine the function that mirrors LTree*s as the anamorphism:*

$$mirror = [\!((id + \mathsf{swap}) \cdot \mathsf{out})\!] \qquad \text{(F9)}$$