kml850

Extended Essay in Mathematics

# Finding the surface area of an ellipsoid

RQ: To what degree of accuracy can the surface area of the National Centre for the Performing Arts (Beijing) be calculated?

Word count: 3959

# Content

# 1. Introduction:

I first discovered the National Centre for the Performing Arts on Archdaily, an online architecture magazine, which I read regularly because of my interest in architecture. I found it intriguing because of its unconventional shape in a place of such great historical and symbolical significance, being located in Beijing, 500 metres away from the Forbidden City.

Figure 1 – The National Centre for the Performing Arts. Retrieved from: https://toistudent.timesofindia.indiatimes.com/news/knowledge-bank/national-centre-for-the-performing-arts-china/28501.html (Accessed: March 12, 2023).

Inside its ellipsoidal form, it holds the function of a theatre, opera house, concert hall and exhibition space. This got me interested into how it fit into the architectural debate of whether the form of a building should follow its function or not. Whilst researching about about the relationship between its peculiar form and its function as a musical, theatrical and artistic space, I discovered a peculiarity about its design: the large surface area of its dome meant that if sound attenuation was not carried out the structure-borne sound from the impact of rain would make it so that the whole dome would act like a drum being hit on. This is particularly important for a building like the NCPA in which performances held could be affected by the sound of rain. I found it very exciting to solve a problem for my EE which had such an important real-life application in the design of an arts centre.

The dome of the NCPA has an ellipsoidal shape with a maximum length, width and height of 106.10m, 71.82m and 46.29m, respectively. I am very familiar with ellipses and ellipsoids because of my interest in perspective drawing. When I first thought of solving this problem I already knew, from perspective drawing, that ellipses are symmetrical along two perpendicular axes, their major axis and minor axes which intersect at the centre. Then I looked to develop a more formal understanding of the properties of ellipsoids and learned that ellipsoids are surfaces of revolution of an ellipse or circle along one of its axes meaning that all plane cross sections are ellipses or circles.

Developing a formal understanding of what an ellipsoid actually is led me to approach the problem through double integration as I could approach the problem as finding the surface area of a surface of revolution. It was also what led me to derive my own formula to find an approximation of the surface area by looking at the plane sections of the ellipsoid as ellipses and develop my geometric approach from there. This led me to the following research question:

**To what degree of accuracy can the surface area of the National Centre for the Performing Arts (Beijing) be calculated?**

It is important to bear in mind that the degree of accuracy to which the surface area can be calculated will be impacted by human error in the measurements of the NCPA since this is a real-life applications problem and thus the quoted dimensions given online are approximations. There will also be approximation error impacted by how many units of $\pi$ are used by MATLAB and by rounding, although the MATLAB generated values will be precise.

Since the measurements of the NCPA are given to the nearest 10cm, it is sensible to give the value of the surface area when measured in square meters to 4.d.p.

It is useful to have an estimation of what the maximum and minimum surface area will be. For the minimum and maximum surface area I considered hemi-spheroids of radius 46.29m and 106.1m respectively. This gives the following inequation:

$$13456.6m^2 < Surface\ area\ of\ the\ NCPA < 70695.3m^2$$

This is demonstrated in Figure 2 below:

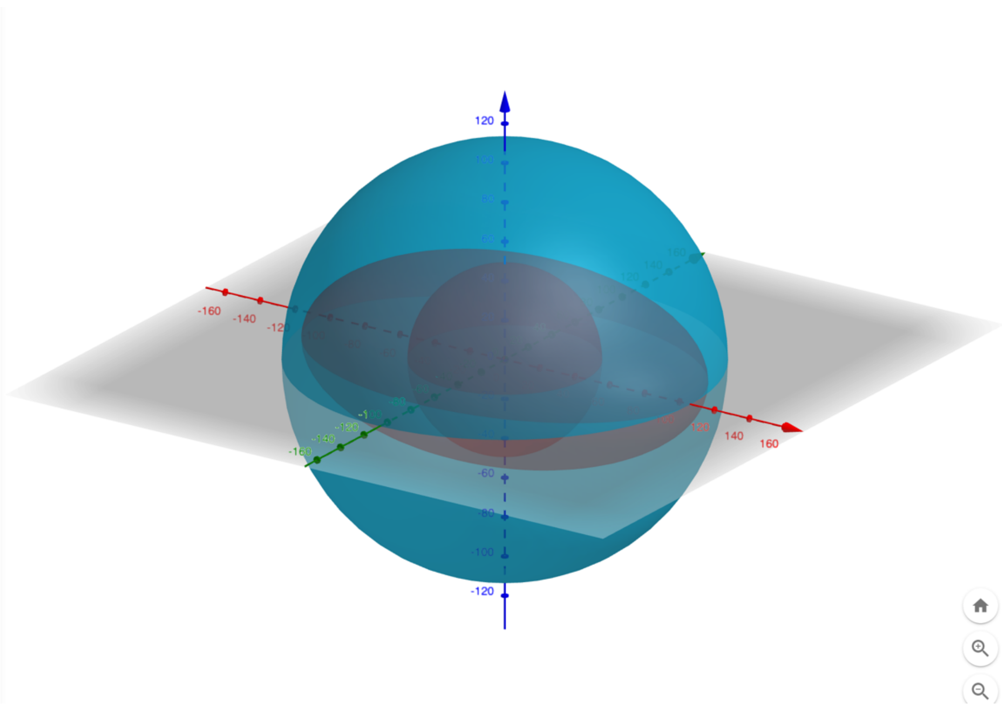


Figure 2 – Candidate's own 3D surface plot of a spheroid of radius 106.10 meters, a spheroid of radius 46.29 metres and an ellipsoid with the dimensions of the NCPA

# 2. Method 1: Double integration approach using polar coordinates

## 2.1. Derivation

To prove the validity of this approach double integration using polar coordinates will be used to calculate the surface area of an ellipsoid whose three axes have the same length, a spheroid.

Let's consider a sphere of radius $a$:

$$x^2 + y^2 + z^2 = a^2$$

$$x^2 + y^2 + z^2 - a^2 = 0$$

$$z^2 = a^2 - x^2 - y^2$$

$$z = \sqrt{a^2 - x^2 - y^2}$$

$F(x, y, z) = x^2 + y^2 + z^2 - a^2$ gives us the $x,\ y\, and\, z$ coordinates of an infinite number of points on the surface of the sphere

We can visualise the hemisphere as being directly above its domain $D$ on the $xy$-plane. The projection of hemisphere, $S$, on the $xy$-plane is a circle, such that:

$$D = \left\{ (x, y) : x^2 + y^2 \leq a^2 \right\}$$

i.e. every single point $(x, y)$ in the circle, which correspond to the image $in\, x\, and\, y$ of every single point $(x, y, z)$ of the hemisphere are contained within or at the radius of the circle, $a$

Figure 3 – Candidate's own drawing for the circular projection of the hemisphere on the $xy$ plane

At the radius $a$, the partial derivative in $z$ is equal to 0 as at the points in which the sphere is on the $xy$- plane the coordinate in $z$ is 0

We can divide $D$ into small rectangles $R_{ij}$ with an area $\Delta A = \Delta x \Delta y$. Let $(x_i, y_i)$ be the corner of $R_{ij}$ closest to the origin and thus $P_{ij}(x_i, y_i, f(x_i, y_i))$ be the point on our hemisphere $S$, directly above $(x_i, y_i)$. Please Figure 4 on page 8. The plane tangent to our hemisphere $S$ at $P_{ij}$ is an approximation of the area of $S$ close to $P_{ij}$. Therefore the area $\Delta T_{ij}$ of this plane tangent to our hemisphere at $P_{ij}$ which is directly above $R_{ij}$ is an approximation of the area $\Delta S_{ij}$ of the part of $S$ which is directly above $R_{ij}$. Therefore, the sum $\sum\sum \Delta T_{ij}$ is an approximation of the total area of $S$. This approximation improves as the number of rectangles increases and becomes exact when this number approaches infinity. There we define the surface area of our hemisphere as:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} \text{ – equation 1}$$

i.e. if we take an infinite number of rectangular planes, of area $\Delta\chi\Delta y$, tangent to our hemisphere $S$ and do the infinite sum of their areas we get the surface area of our hemisphere.

To find a formula to calculate the surface area which is more simple than equation 1 above we let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors starting at $P_{ij}$ such that they are sides of the parallelogram with area $\Delta T_{ij}$ and $\Delta T_{ij} = \left|\boldsymbol{a} \times \mathbf{b}\right|$

$$\boldsymbol{a} = \Delta x\boldsymbol{i} + f_x\left(x_i, y_j\right)\Delta x\boldsymbol{k}$$

$$\boldsymbol{b} = \Delta y\boldsymbol{j} + f_y\left(x_i, y_j\right)\Delta y\boldsymbol{k}$$

$$\boldsymbol{a} \times \mathbf{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \Delta x & 0 & f_x\left(x_i, y_j\right)\Delta x \\ 0 & \Delta y & f_y\left(x_i, y_j\right)\Delta y \end{vmatrix}$$

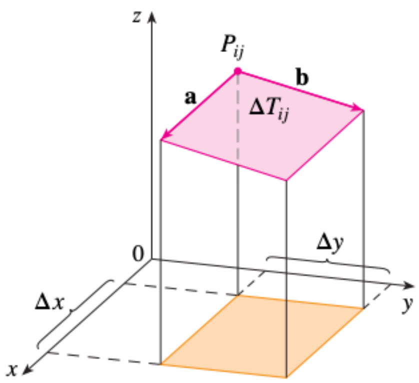



Figure 4 – Area $\Delta T_{ij}$ retrieved from (Stewart et al., 2022)

$$= -f_x\left(x_i, y_j\right)\Delta x\Delta y\boldsymbol{i} - f_y\left(x_i, y_j\right)\Delta x\Delta y\boldsymbol{j} + \Delta x\Delta y\boldsymbol{k}$$

Since $\Delta x \Delta y = \Delta A$,

$$\boldsymbol{a} \times \mathbf{b} = \left[ f_x\left(x_i, y_j\right)\boldsymbol{i} - f_y\left(x_i, y_j\right)\boldsymbol{j} + \boldsymbol{k} \right] \Delta A$$

Therefore, using the formula for the magnitude of a vector,

$$\Delta T_{ij} = \left| \boldsymbol{a} \times \mathbf{b} \right| = \sqrt{\left[ f_x\left(x_i, y_j\right)\right]^2 + \left[ f_y\left(x_i, y_j\right)\right]^2 + 1}\ \Delta A$$

Since $A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} = \iint_D \Delta T_{ij}$

$$A(S) = \iint_D \sqrt{\left[ f_x\left(x_i, y_j\right)\right]^2 + \left[ f_y\left(x_i, y_j\right)\right]^2 + 1}\ \Delta A$$

So now we compute the partial derivatives of the sphere:

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$f_x(x, y) = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}$$

$$f_y(x, y) = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

Since $A(S) = \iint_D \sqrt{\left[ f_x\left(x_i, y_j\right)\right]^2 + \left[ f_y\left(x_i, y_j\right)\right]^2 + 1}\ \Delta A$ only defines the upper hemisphere we double it to get the surface area of a sphere:

$$A(S) = 2\iint_D \sqrt{\left[ f_x\left(x_i, y_j\right)\right]^2 + \left[ f_y\left(x_i, y_j\right)\right]^2 + 1}\ \Delta A$$ – equation 2

$$A(S) = 2\iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2} \ +1}\ \Delta A$$

The region $D$ that we are integrating is the circle, centred at the origin, with radius $a$ such that: $x^2 + y^2 = a^2$. Because of this region, it might ease the integration if we convert to polar coordinates. Substituting in $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$ and bounds $0 \le \theta \le 2\pi$ and $0 \le r \le a$, we have:

$$S = 2\int_0^{2\pi}\int_0^a \sqrt{1+\frac{r^2\cos^2\theta + r^2\sin^2\theta}{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}}\ r\,dr\,d\theta$$

$$S = 2\int_0^{2\pi}\int_0^a \sqrt{1+\frac{r^2(\cos^2\theta + \sin^2\theta)}{a^2 - r^2(\cos^2\theta + \sin^2\theta)}}\ r\,dr\,d\theta$$

$$S = 2\int_0^{2\pi}\int_0^a r\sqrt{1+\frac{r^2}{a^2-r^2}}\ dr\,d\theta$$

$$S = 2\int_0^{2\pi}\int_0^a r\sqrt{\frac{a^2}{a^2-r^2}}\ dr\,d\theta$$

$$S = 2\int_0^{2\pi} a\int_0^a \frac{r}{\sqrt{a^2-r^2}}\ dr\,d\theta$$

We let $u = \ a^2 - r^2$ such that:

$$dr = -\frac{du}{2r}$$

Thus,

$$S = 2\int_0^{2\pi} a\int_0^a \frac{r}{\sqrt{a^2-r^2}}\ dr\,d\theta = 2\int_0^{2\pi} a\int_{a^2}^0 \frac{r}{\sqrt{u}}\ \left(-du/2r\ \right)d\theta$$

$$= 2\int_0^{2\pi}\left(-\frac{a}{2}\right)\int_{a^2}^0 u^{-\frac{1}{2}}\ du\,d\theta$$

$$= 2\int_0^{2\pi} \left(-\frac{a}{2}\right)\left[2u^{\frac{1}{2}}\right]_{a^2}^{0} d\theta$$

$$= 2\int_0^{2\pi} a^2 \, d\theta$$

$$S = 2\int_0^{2\pi} a^2 \, d\theta$$

$S = 4\pi a^2$, where $a = R$, the radius

$$S = 4\pi R^2$$

This result which agrees with the formula for the surface area of a sphere, confirms the validity of the method of using double integration with polar coordinates to calculate surface areas. This result will be used to validate the result of method 2, the approximation, further on in this essay.

The plane projection of the NCPA is an ellipse. The long axis of its plane projection has a length of 212.20 metres. Its short axis in the plane projection has a length of 143.64 metres. The height of the building is 46.285 metres.

The standard equation for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a$ represents the semi-major axis of the ellipse

$b$ represents the semi-minor axis of the ellipse

Proof of equation for an ellipse (Basic, n.d.):

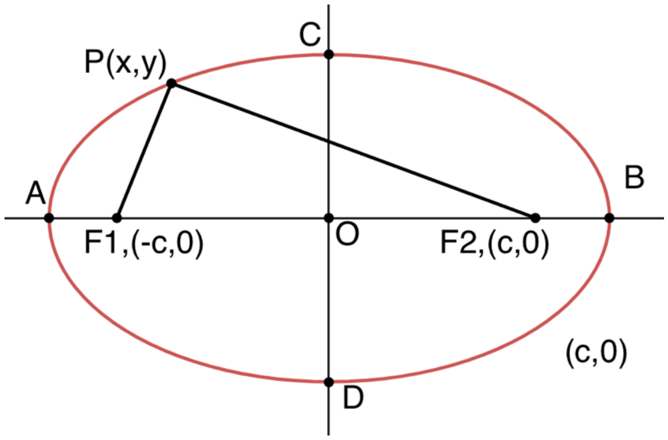



Figure 5 – proof of equation for an ellipse. (Basic, n.d.) Retrieved from: https://www.basic-mathematics.com/derive-the-equation-of-an-ellipse.html

Taking a point on the ellipse such that, $F1P + F2P = 2a$,

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

$$\left[\sqrt{(x+c)^2+y^2}\right] = \left[2a - \sqrt{(x-c)^2+y^2}\right]$$

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

$$x^2+2xc+c^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2-2xc+c^2+y^2$$

$$4xc = 4a^2 - 4a\sqrt{(x-c)^2+y^2}$$

$$xc - a^2 = -a\sqrt{(x-c)^2+y^2}$$

$$x^2c^2 - 2a^2xc + a^4 = a^2\left[(x-c)^2+y^2\right]$$

$$x^2c^2 - 2a^2xc + a^4 = a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2$$

$$x^2c^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$x^2c^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$x^2(-b^2) - a^2y^2 = a^2(-b^2)$$

$$x^2\left(b^2\right) + a^2y^2 = a^2(b^2)$$

$$\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The formula for an ellipsoid is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

We let c = 1, such that:

$$z = \sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}$$

So now, following the method used previously to calculate the surface area of the sphere but this time for the surface area of the ellipsoid, we compute the partial derivatives of the ellipsoid:

$$z = \sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}$$

$$z = \left(1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right)^{\frac{1}{2}}$$

$$f_x(x,y) = \frac{1}{2}\left(1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right)^{-\frac{1}{2}}(-2x)$$

$$f_x(x,y) = \frac{-x}{\sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}}$$

$$f_y(x,y) = \frac{1}{2}\left(1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right)^{-\frac{1}{2}}(-2y)$$

$$f_y(x,y) = \frac{-y}{\sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}}$$

Now on substitution into equation 2 (page 8):

$$A(S) = 2\iint_D \sqrt{\left[\frac{-x}{\sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}}\right]^2 + \left[\frac{-y}{\sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}}\right]^2 + 1}\ \Delta A$$

$$A(S) = 2\iint_D \sqrt{\frac{x^2 + y^2}{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)} + 1}\ \Delta A$$

### 2.2. Calculation

In depth online research will not produce a solution method for this integral. Primary research, via direct e-mail correspondence with a professor at the Faculty of Mathematics at the University of Porto, reveals that in fact today nobody knows how to find the exact surface area of an ellipsoid using method 1. The professor confirms that the double integral can only be solved through elliptical integrals which provide a solution that is only an approximation. Thus, method 1, to find an exact value for the surface area of the ellipsoid using double integration with polar coordinates, does not give a result. The integral is not solvable, even with resort to ICT and the use of elliptical integrals is both beyond the scope the AA HL syllabus and only an approximation of the surface area.

## 3. Method 2: Approximation

### 3.1. Derivation

According to the Professor, the most commonly used approximation for calculating the surface area of ellipsoids is Knud Thomsen's formula:

$$S = 4\pi\left[\frac{\left(a^pb^p + a^pc^p + b^pc^p\right)}{3}\right]^{\frac{1}{p}},\ p = \frac{\ln 2}{\ln\frac{\pi}{2}}$$

The calculator provided by Google to calculate the surface area of ellipsoids uses Thomsen's Formula. Various research papers where it was necessary to calculate the surface area of ellipsoids, such as in Okokon, et al., 2007, and Khairy & Howard, 2011, also use Thomsen's Formula. That said, we can be satisfied that Thomsen's Formula is the best and most accurate way to approximate the surface area of an ellipsoid at this moment in time.

Thomsen's formula relies on elliptical integrals which are way beyond the scope of the HL AA student, yet it is an approximation. It seems possible that another approach could be used to approximate the surface area of an ellipsoid. Thus, it may be possible to create an approximation using calculus concepts with which a HL AA student would be familiar.

An approximation can be created by looking at two different sections through the ellipsoids. These sections will be the $xy$-plane and the $xz$-plane. Any section through an ellipsoid with $a$, $b$ and $c$ values that are not equal will be an ellipse.

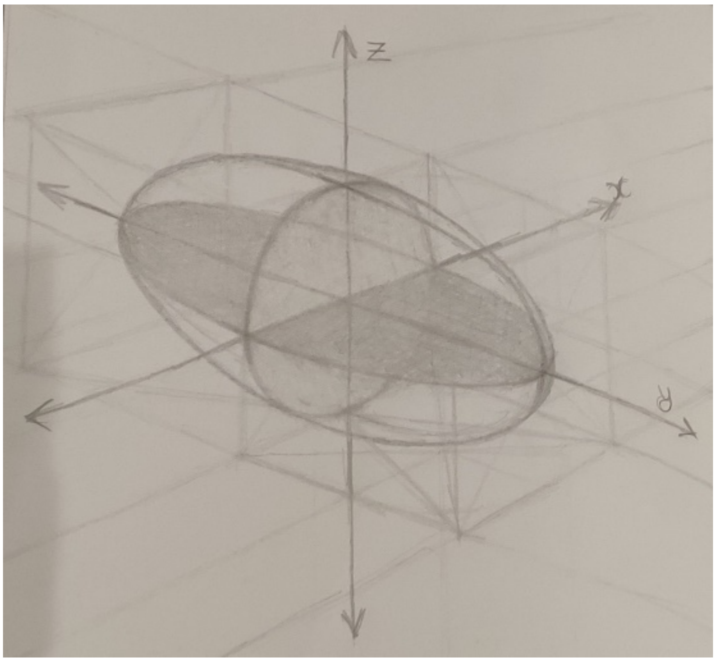



Figure 6 – Candidate's own sketch made to construct reasoning for method 2

For the $xy$-plane elliptical section through the ellipsoid the equation of this ellipse will be:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[b^2\left(1 - \frac{x^2}{a^2}\right)\right]^{-\frac{1}{2}}\left(-\frac{2b^2x}{a^2}\right)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{b^2x^2}{a^4\left(1 - \frac{x^2}{a^2}\right)}$$

Then the calculus concept of using rectangles with an infinitesimally small width $ds$ and height $y$ can be used. A mid-endpoint approximation of the ellipse with rectangles is taken:

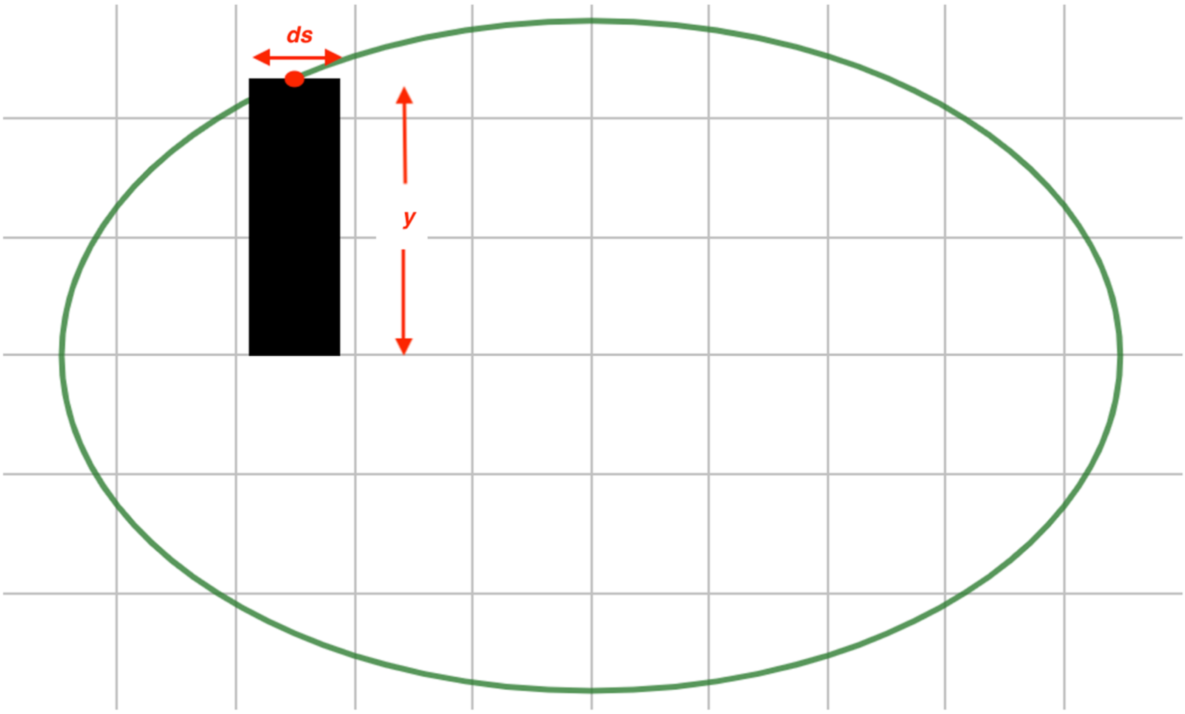



Figure 7 – Candidate's own sketch of midpoint approximation of the ellipse using rectangles of height $y$ and width $ds$

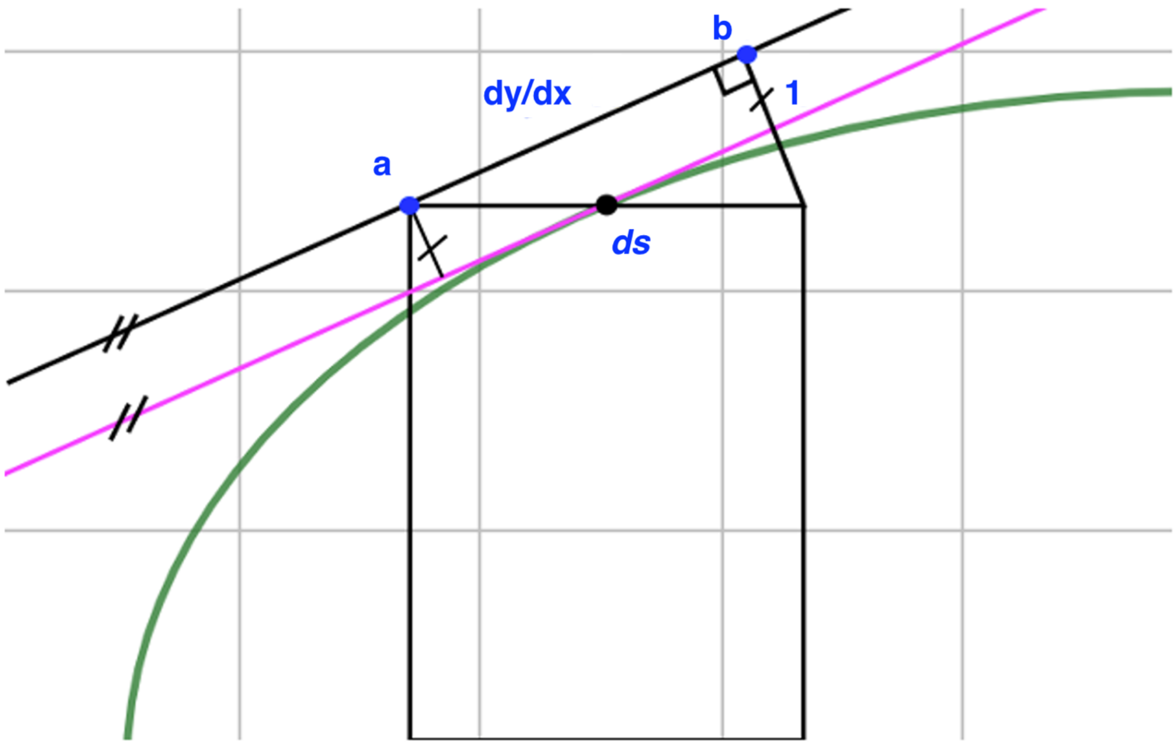



Figure 8 – Candidate's own sketch of Pythagoras' Theorem used to find an expression for $ds$

To find $ds$, the width of the rectangle, Pythagoras' Theorem can be used, as seen above in Figure 8. The length of the hypotenuse is the magnitude of the tangent to the ellipse at the midpoint, restricted between two points $a$ and $b$. For ease of calculation, the shorter other side is assigned a unit value. It is worth noting that obviously the area between the rectangle and the ellipse is not a triangle as an ellipse is curved. However, treating it as a triangle is very useful as it allows us to use Pythagoras' theorem to find the length $ds$ and it does not affect the result because when the number of rectangles approaches infinity, the area of this space approaches 0 and so does the error caused by treating it as a triangle. Thus, for the $xy$-plane elliptical section through the ellipsoid, the length of $ds$ is:

$$ds_{xy} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$ds_{xy} = \sqrt{1 + \frac{b^2x^2}{a^4\left(1 - \frac{x^2}{a^2}\right)}}dx$$

And thus, the area of the rectangle is:

$$Rectangle\,area = ds \times y$$

$$Rectangle\,area = ds \times b\sqrt{1 - \frac{x^2}{a^2}}$$

We can now extend this rectangle so that it wraps all the way around the ellipsoid thus creating a band which rings around the ellipsoid (like an arm band around an arm) and so the area would be found by multiplying the rectangle area by $2\pi$, giving:

$$Band\,area = ds \times 2\pi \times b\sqrt{1 - \frac{x^2}{a^2}},$$

After imagining that these rectangles with an infinitesimally small width are wrapped around the ellipsoid we can take the infinite sum of the areas of the rectangles to find the surface area of the ellipsoid. This method relies on the fact that the rectangles have an infinitesimally small width and so performing the infinite sum of their areas is like taking the infinite sum of lines to form a mesh of lines producing an area:

$$Area\,of\,ellipsoid = \int_{-a}^{a} (band\,area)dx$$

$$Area\,of\,ellipsoid = \int_{-a}^{a} 2\pi\sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}\sqrt{1 + \frac{b^2x^2}{a^4\left(1 - \frac{x^2}{a^2}\right)}}\,dx$$

Now we follow the same process for the $xz$-plane:

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$$

$$z=\sqrt{c^2\left(1-\frac{x^2}{a^2}\right)}$$

$$\frac{dz}{dx}=\frac{1}{2}\left[c^2\left(1-\frac{x^2}{a^2}\right)\right]^{-\frac{1}{2}}\left(-\frac{2c^2x}{a^2}\right)$$

$$\left(\frac{dz}{dx}\right)^2=\frac{c^2x^2}{a^4\left(1-\frac{x^2}{a^2}\right)}$$

$$ds_{xz}=\sqrt{1+\left(\frac{dz}{dx}\right)^2}\,dx$$

$$ds_{xz}=\sqrt{1+\frac{c^2x^2}{a^4\left(1-\frac{x^2}{a^2}\right)}}\,dx$$

$$Band\,area=dA=ds\times 2\pi\times z$$

$$Band\,area=dA=ds\times 2\pi\times c\sqrt{1-\frac{x^2}{a^2}}$$

Then we take the infinite sum of the areas of the bands:

$$Area\,of\,ellipsoid=\int_{-a}^{a}\left(band\,area\right)\,dx$$

$$Area\ of\ ellipsoid = \int_{-a}^{a} 2\pi \sqrt{c^2\left(1-\frac{x^2}{a^2}\right)} \sqrt{1+\frac{c^2x^2}{a^4\left(1-\frac{x^2}{a^2}\right)}}\ dx$$

$$Area\ of\ ellipsoid = \int_{-a}^{a} 2\pi \sqrt{c^2\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{c^2x^2}{a^4\left(1-\frac{x^2}{a^2}\right)}\right)}\ dx$$

Now, the derivation of this approximation relies on a geometric approach. With the two formulae for the area of ellipsoids that are based on the elliptical sections of the $xy$ and $xz$ plane, two intersecting ellipsoids are created, coloured blue and purple in Figure 9 below, one of which represents the revolved elliptical cross-section of the ellipsoid on the $xy$-plane, and the other the revolved elliptical cross-section on the $xz$ -plane. It is helpful to visualize these as two shells. The surface area of the ellipsoid we are trying to find is the midpoint shell of the other two, i.e every point on the surface area of the NCPA is equidistant to the other two shells. Figure 9 enables the principle used in this approach to be more easily seen, if a third shell, the ellipsoid whose surface area we are trying to find, is imagined sitting exactly equidistant between the other two, based on the assumption that two of the three dimensions are equal.

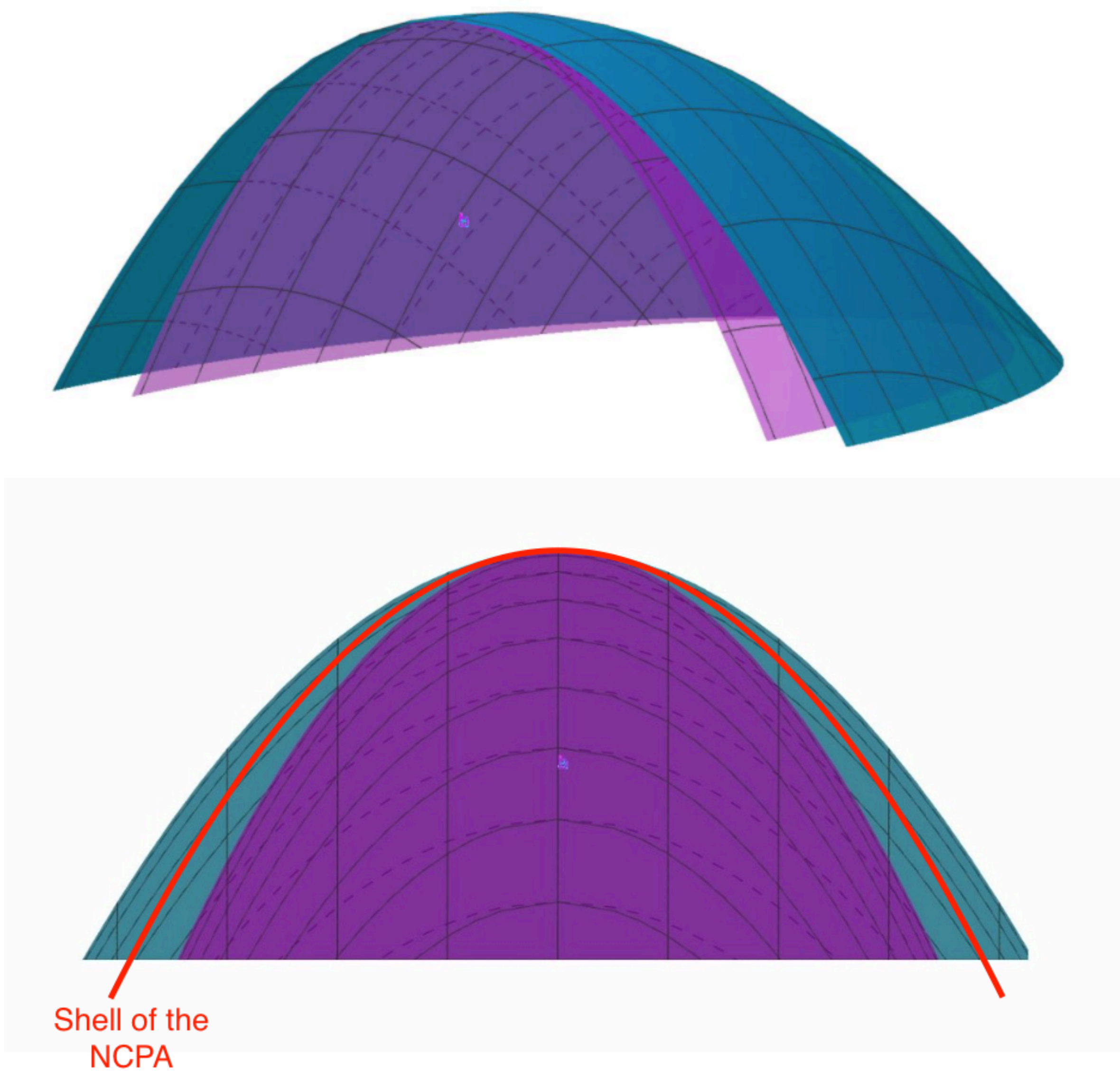



Figure 9 – Candidate's own 3D surface plots created on MATLAB as a visual representation of the method 2

This is the geometric interpretation of this approach.

The algebraic application of this approach is the following:

Let the width $ds$ of the band / rectangle be the average of the widths of the bands around the ellipsoids from the elliptical sections of the $xy$ and $xz$ planes. This will create a third shell that is the midpoint between the other two, it will be equidistant to the other two shells at all points. And so $ds$ will be the width:

$$ds = \frac{1}{2}\left[ds_{xy} + ds_{xz}\right]$$

Now the surface area for third ellipsoid is the infinite sum of the areas of the rectangles with width $ds = \frac{1}{2}\left[ds_{xy} + ds_{xz}\right]$:

$$Surface\ area = \frac{1}{2}\int_{-a}^{a} y \times ds dx$$

$$Surface\ area = \frac{1}{2}\int_{-a}^{a} dA dx$$

$$Surface\ area = \frac{1}{2}\int_{-a}^{a}\left(\sqrt{\left(1 + \frac{b^2x^2}{a^4\left(1 - \frac{x^2}{a^2}\right)}\right)} + \sqrt{\left(1 + \frac{c^2x^2}{a^4\left(1 - \frac{x^2}{a^2}\right)}\right)}\right)\left(b\pi\sqrt{1 - \frac{x^2}{a^2}} + c\pi\sqrt{1 - \frac{x^2}{a^2}}\right)dx$$

$$\boxed{Surface\ area = \frac{1}{2}\int_{-a}^{a}\left(\sqrt{1 + \left(\frac{-bx}{a\sqrt{a^2 - x^2}}\right)^2} + \sqrt{1 + \left(\frac{-cx}{a\sqrt{a^2 - x^2}}\right)^2}\right)\left((b + c)\pi\sqrt{1 - \frac{x^2}{a^2}}\right)dx}$$

Equation 3 - Candidate’s own approximation for the surface area of ellipsoids

## 3.2. Calculation

This complicated integral is more easily solved using a MATLAB program for different values of $a$, $b$ and $c$, which can be seen in Appendix B. Thus, the candidate's own MATLAB code can compute an approximation for the surface area of any ellipsoid using Method 2.

In addition to calculating the surface area for any ellipsoid using Method 2's approximation, the code also calculates the surface area using Thomsen's Formula and uses the input $a$ to calculate the surface area of a sphere with radius $a$. The surface area of a sphere can also be calculated as a reference point, because spheres are a special case of ellipsoids for which the values of $a$, $b$ and $c$ are equal. Thomsen's Formula gives the true surface area for the special case of ellipsoids that are spherical. Below is an example for an ellipsoid (spheroid) whose $a$, $b$ and $c$ values are all 3:

$a$ value= 3

$b$ value= 3

$c$ value= 3

The MATLAB code written by the candidate gives the following output:

Circle Area = 113.0973 $m^2$

Knud Thomsen Formula = 113.0973 $m^2$

Ellipsoid Area = 113.0973 $m^2$

Error = 0 %

Method 2's approximation gives the true surface area for the special case of ellipsoids with equal $a$, $b$ and $c$ values, having a percentage error of 0% relative to Thomsen's formula and 100% degree of accuracy.

Now we run the code using the actual values of $a$, $b\ and\ c$ for the National Centre for the Performing Arts:

$a$ value= 106.10

$b$ value= 71.82

$c$ value= 46.29

Circle Area = 141462.2729 $m^2$

Knud Thomsen Formula = 68388.5848 $m^2$

Ellipsoid Area = 68521.8828 $m^2$

Error = 0.195 %

Since the dome of the National Centre for the Performing Arts is hemi-ellipsoidal, we must divide the value given by the code by 2, giving:

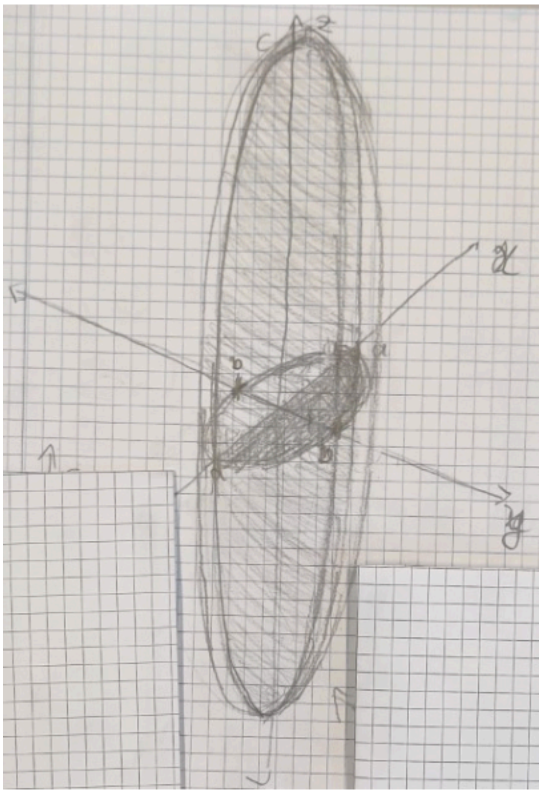
$$\boxed{\begin{array}{c} Surface\ area\ for\ the\ dome\ of\ the\ NCPA\ using\ Knud\ Thomsen's\ Formula\ = 34194.2924m^2 \\ Surface\ area\ for\ the\ dome\ of\ the\ NCPA\ using\ Method\ 2\ approximation\ = 34260.9414m^2 \end{array}}$$

The percentage error for method 2 is calculated relatively to Thomsen's Formula since there is no known formula to find the value for the surface area of ellipsoids to make the percentage error of my approximation relative to. Relative to Thomsen's Formula, method 2 calculates the surface area of the ellipsoid with an error of $0.195\%$. Thomsen's formula yields a relative error of at most ±1.061%, thus the error of the value which method 2 gives for the surface area for the dome of the NCPA is at most ± $1.061 + (0.195\ \times 1.061)\% =\ \pm 1.268\%$

To calculate the surface area of the NCPA, Method 2 is only slightly less accurate but much more accessible to a HL student.

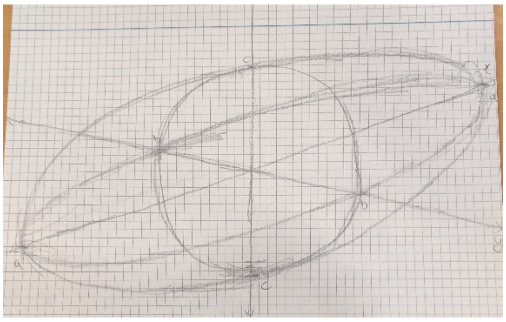
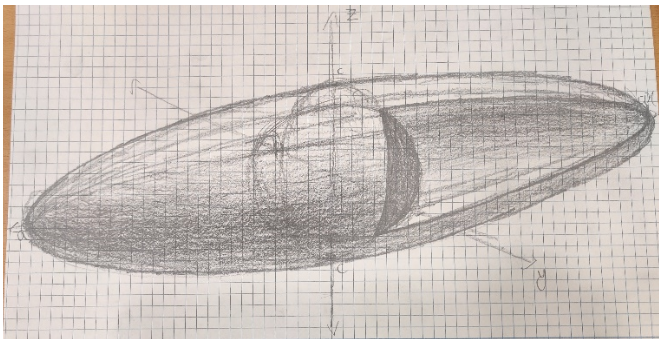
Yet, it is worth noting that given the nature of method, which takes a midpoint shell between two other ellipses as the surface area, its error depends on the extent to which the 3 axes of the ellipsoid whose surface area is being calculated are of equal length. If one or two axes are much bigger than the others then the error of the approximation increases. As the differences in the lengths of the semi-axes of the ellipsoid approach zero, the error of Method 2 approaches zero. It follows from this that when the three axes of the ellipsoid are of equal length, i.e. the ellipsoid is a spheroid, the approximation gives the true surface area. And if the differences in the lengths of the semi-axes is not very large then the error of method 2 will very small relative to Thomsen's Formula.

If one or two axes are much bigger than the others, then the error of the approximation increases, yet the extent to which the error increases is very much dependent on which of the axes is bigger. Since method 2 was designed with an ellipsoid whose semi-axis $a$ was the longest, its error can be very large for ellipsoids whose semi-axis $a$ is not the longest. This becomes very obvious when we view method 2 graphically. For example, when we are using method 2 to calculate the surface area of an ellipsoid whose semi-axis $c$ is much larger than both its semi-axes $a$ and $b$, our approximation gives the following ellipsoid:

| | |
|---|---|
|   | |
| Figure 10 – Candidate's own sketch showing an ellipsoid whose semi-axis $c$'s length is much larger than its semi-axes $a$ and $b$ | Figure 11 – Candidate's own sketch showing the ellipsoid that method 2 uses to approximate the surface area of an ellipsoid whose semi-axis $c$'s length is larger shown in front of said ellipsoid |

The ellipsoid which method 2 creates to approximate the surface area, shown in Figure 11, is obviously not a good fit.

Yet, method 2 is very accurate for ellipsoids whose semi-axes $a$ are larger than the other two semi-axes. This also becomes very evident graphically:

| | |
|---|---|
|  |  |
| Figure 12 – an ellipsoid whose semi-axis $a$'s length is larger than other semi-axes | Figure 13 – half of the ellipsoid that method 2 uses to approximate the surface area of an ellipsoid whose semi-axis $a$'s length is larger shown intersecting said ellipsoid |

The ellipsoid which method 2 creates to approximate the surface area shown in Figure 13 is a much better fit than the one shown in Figure 11.

Using MATLAB, as seen in Appendix C, we can see that using method 2 for ellipsoids whose semi-axis $c$'s length is much larger than their semi-axes $a$ and $b$, can give errors of >1000000%, and it follows that the error approaches infinity as the length of semi-axis $c$ approaches infinity. On the other hand, using method 2 for ellipsoids whose semi-axis $a$'s length is much larger, can be seen to not give errors of over 2.3% relative to Thomsen's Formula, if the other two axes are of equal length.

As the difference in length between the two other semi-axis increases, for the case where the semi-axis $a$'s length is much larger than the other two, then the error of method 2 increases up to 19.7%, relative to Thomsen's Formula. Thus for these cases method 2 gives a the surface area with a degree of accuracy of at least 78.037%.

Nevertheless, this cannot be proven without studying the convergence of the error which is beyond the scope of this essay.

However, it is worth noting that for the case of ellipsoids whose semi-axes $c$ are much larger than the other two semi-axes, for which the error of the formula from Method 2 can be infinite, we can rotate the ellipsoid and treat the semi-axis $c$ as the semi-axis $a$ and thus the error becomes <2.3% or <19.7%.

## 4. Conclusion

For ellipsoids whose three semi-axes are equal in length many formulas can be used to calculate their surface area to a 100% degree of accuracy, including both the formulas created in Methods 1 and 2 and Thomsen's Formula.

There are many ways to calculate the surface area of an ellipsoid, whose three semi-axes are not equal in length to a lesser degree of accuracy, using formulas which give approximations. The NCPA's semi-axes are not of equal length.

To answer my research question, "To what degree of accuracy can the surface area of the National Centre for the Performing Arts (Beijing) be calculated?", at this moment in time the surface area of the dome of the National Centre for the Performing Arts, an ellipsoid whose three semi-axes are not equal in length, cannot be calculated to a 100% degree of accuracy.

The surface area calculated using Thomsen's Formula is $34194.2924m^2$. Thomsen's formula yields a relative error of at most 1.061%, thus it gives the surface area to a degree of accuracy of at least 98.939%.

The surface calculated using the formula created in Method 2 is $34260.9414m^2$. The formula created in Method 2, accessible to a HL Mathematics student, yields an error of <2.3%, relative to Thomsen's Formula, for ellipsoids whose semi-axis $a$ is much larger than their other two semi-axes of equal length, thus for these cases it gives the surface area to a degree of accuracy of at least 95.756%.

We can be satisfied with these answers as they satisfy the inequation in the introduction.

However, it might be that in the future, a mathematician will be able to calculate the surface area of an ellipsoid whose three semi-axes are of different lengths to 100% degree of accuracy. This could be why Thomsen does not disclose any information as to how he derived his approximation since he might be working on this. One can imagine that he is working a formula with 100% accuracy at this very moment in time.

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# 10.Appendix

Appendix A - e-mail from Professor of Mathematics at the University of Porto:

Na página:
https://www.easycalculation.com/formulas/ellisoid-surface-area-formul.html
pode calcular aproximadamente a área dados o a,b e c. Esta fórmula também aparece em

https://en.wikipedia.org/wiki/Ellipsoid#cite_note-3

onde diz approximate formula. O John Cook fornece mais alguma info:

**https://www.johndcook.com/blog/2021/03/24/surface-area-ellipsoid/**
novamente não dão grandes detalhes em como chegam ao S. Os integrais elipticos são dados em:

https://www.johndcook.com/blog/2014/07/06/ellipsoid-surface-area/

Aqui são dados mais detalhes:

https://www.math.auckland.ac.nz/Research/Reports/Series/539.pdf

É realmente um cálculo muito chato mas a aproximação onde se usa o expoente 1.6 parece ser a mais usada e até se consegue controlar o erro.
Espero ter dado uma ajuda.

Appendix B – Candidate's own MATLAB code which takes inputs of the values of the lengths of the semi-axes $a$, $b$ and $c$ and uses them to solve for the surface area of ellipsoids using the integral from Method 2 and Knud Thomsen's formula:

```
 clc;
clear;
a=input('a value= ');
b=input('b value= ');
```

```
c=input('c value= ');

% Ellipsoid Area
   syms x
   g = @(x) (sqrt(1+((-b.*x)./(a.*sqrt(a.^2-x.^2))).^2)+sqrt(1+((-c.*x)./(a.*sqrt(a.^2-
x.^2))).^2)).*(((b+c)*pi.*sqrt(1-(x/a).^2)));

% Integration of g(x)
   I = integral(g,-a,a);
   A=round(0.5*I,4);

% Knud's Ellipsoid Area
   p=log(2)/log(pi/2);
   %p=1.6075
   S=round(4*pi*((a.^p*b.^p+a.^p*c.^p+b.^p*c.^p)/3).^(1/p),4);

% Area of sphere
   W=4*pi.*a^2;

% Error %
   E=round(abs(1-(A/S))*100,4);

%Results
disp(' ')
disp(['        Circle Area = ', num2str(W),' m^2'])
```

```
disp(' ')
disp(['Knud Thomsen Formula = ', num2str(S),' m^2'])
disp(' ')
disp(['      Ellipsoid Area = ', num2str(A),' m^2'])
disp(' ')
disp(['               Error = ',num2str(E),' %'])
```

Appendix C – Candidate's own MATLAB calculations used to find the error of Method 2 relative to Knud Thomsen's Formula:

```
Command Window
a value= 100000
b value= 0.1
c value= 0.1

        Circle Area = 125663706143.5917 m^2

Knud Thomsen Formula = 96491.0629 m^2

     Ellipsoid Area = 98696.044 m^2

              Error = 2.2852 %
>>
```

```
Command Window
a value= 100000000000
b value= 0.001
c value= 0.001

        Circle Area = 1.256637061435917e+23 m^2

Knud Thomsen Formula = 964910628.7818 m^2

     Ellipsoid Area = 986960440.1089 m^2

              Error = 2.2852 %
>>
```

Figure 14 – shows how method 2 gives the surface area of an ellipsoid with a semi-axis $a$ 1 million times larger than its other two semi-axes with an error of only 2.2852% Figure 15 – shows how the error of method 2 does not increase when calculating the surface area of an ellipsoid with a semi-axis $a$ 100 trillion times larger instead of 1 million times larger than its other two semi-axes

```
Command Window
a value= 0.1
b value= 0.1
c value= 100000

        Circle Area = 0.12566 m^2

Knud Thomsen Formula = 96491.0629 m^2

     Ellipsoid Area = 15708010391.9917 m^2

              Error = 16279138.6361 %
>>
```

```
Command Window
a value= 0.001
b value= 0.001
c value= 100000000

        Circle Area = 1.2566e-05 m^2

Knud Thomsen Formula = 964910.6288 m^2

     Ellipsoid Area = 1.570796326842022e+16 m^2

              Error = 1627918980122.578 %
>>
```

Figure 16 – shows how method 2 gives the surface area of an ellipsoid with a semi-axis $c$ 1 million times larger than its other two semi-axes with a much larger