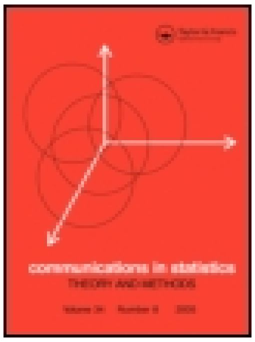

Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: http://www.tandfonline.com/loi/lsta20

Kumaraswamy Marshall-Olkin Exponential Distribution

Roshini George & S. Thobias

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2018, VOL. 0, NO. 0, 1–18
https://doi.org/10.1080/03610926.2018.1440594

Taylor & Francis
Taylor & Francis Group

Kumaraswamy Marshall-Olkin Exponential Distribution

Roshini George and S. Thobias

Department of Statistics, Loyola College (Autonomous), Chennai, Tamil Nadu, India

ABSTRACT
In this paper, a new generalization of the Kumaraswamy distribution namely, the Kumaraswamy Marshall-Olkin Exponential distribution (KwMOE) is introduced and studied. Various properties are explored. The structural analysis includes various aspects such as shape properties, moments, quantiles, mean deviation, Renyi entropy, order statistics and stochastic ordering. Some useful characterizations of the family are also obtained. The method of maximum likelihood is used to estimate the model parameters. Monte Carlo simulation study is being conducted. An application to a real data set is presented for illustrative purposes.

ARTICLE HISTORY
Received 7 June 2017
Accepted 9 February 2018

KEYWORDS
Characterization; Kumaraswamy distribution; Renyi entropy; Simulation; Stochastic ordering.

MATHEMATICS SUBJECT CLASSIFICATION
62E15

1. Introduction

In this paper, we introduce a new family of continuous distributions called Kumaraswamy Marshall-Olkin generalized family of distributions, which we call as Kumaraswamy Marshall-Olkin Exponential distribution (KwMOE). Marshall and Olkin (1997) proposed a flexible semiparametric family of distributions and obtained the survival function $\bar{G}_{MO}(x)$ by introducing an additional parameter $\bar{p}$ such that $p = 1 - \bar{p}$, $\bar{p} > 0$ called the tilt parameter. The probability density function g (x; ξ) and cumulative density function G (x; ξ) of the Marshall-Olkin (MO) family are defined by

$$g_{MO}(x) = \frac{(1-p)\,\mathrm{g}(\mathrm{x};\ \xi)}{\left[1 - p\bar{G}(\mathrm{x}; \xi)\right]^2};\ \ \mathrm{x} \in R \tag{1}$$

and

$$G_{MO}(x) = \frac{\mathrm{G}(\mathrm{x};\ \xi)}{1 - p\bar{G}(\mathrm{x}; \xi)} \tag{2}$$

For p = 0, we have $G_{MO}(x) = \mathrm{G}(\mathrm{x};\ \xi)$.

The Marshall-Olkin extension was applied to several well known distributions such as Pareto (Alice and Jose (2003)), Exponential and Semi-Weibull (Alice and Jose (2004, 2005)), Uniform (Jose and Krishna (2011)), Bivariate Morgenstern Weibull (Jose and Sebastian (2013)), Log-logistic (Gui (2013)), Frechet (Krishna et al. (2013a, b)) etc. Kumaraswamy (1980) proposed and discussed a probability distribution for handling double bounded random processes with varied hydrological applications. The beta and Kumaraswamy distributions share similar properties. The Kumaraswamy's distribution is also referred to

CONTACT Roshini George roshini.thejas@gmail.com Department of Statistics, Loyola College (Autonomous), Chennai, Tamil Nadu, 600 034, India.
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/lsta.

as the minimax distribution and is unimodal or uni-antimodal, increasing, decreasing or constant depending on the values of its parameters. After the paper by Jones (2009) on the tractability properties of the Kumaraswamy's distribution, Cordeiro and de Castro (2011) introduced the Kumaraswamy generated family.

For a baseline random variable having pdf g(x) and cdf G(x) the two-parameter Kw-G distribution has cdf

$$\mathrm{F}\,(\mathrm{x}) = 1 - \left\{1 - G(x)^{a}\right\}^{b} \tag{3}$$

and the corresponding pdf is

$$\mathrm{f}\,(\mathrm{x}) = \mathrm{a\ b\ g}\,(\mathrm{x})\, G(x)^{a-1}\left\{1 - G(x)^{a}\right\}^{b-1} \tag{4}$$

where a > 0 and b > 0 are the two extra shape parameters whose role is to govern skewness and tail weights.

Now we propose a new extension of the Marshall – Olkin family for a given baseline distribution with cdf G(x; ξ) and pdf g(x; ξ) depending on a parameter vector ξ. Using (1) in (3) we have the cdf of the new Kumaraswamy Marshall -Olkin (KwMO) family of distributions due to Alizadeh et al. (2015) is given by

$$\mathrm{F}\,(\mathrm{x}) = 1 - \left\{1 - \left[\frac{G\,(x;\,\xi)}{1 - p\bar{G}\,(\mathrm{x};\,\xi)}\right]^{a}\right\}^{b}, \tag{5}$$

where a > 0, b > 0 and $\bar{p} > 0$ are three shape parameters.

The density function corresponding to (5) is

$$\mathrm{f}\,(\mathrm{x}) = \frac{ab\left(1 - p\right) g\,(x;\,\xi)\, G(x;\,\xi)^{a-1}}{\left[1 - \ p\bar{G}\,(\mathrm{x};\,\xi)\,\right]^{a+1}} \left\{1 - \left[\frac{G\,(x;\,\xi)}{1 - p\bar{G}\,(\mathrm{x};\,\xi)}\right]^{a}\right\}^{b-1} \tag{6}$$

Hereafter a random variable X with density function (6) is denoted by X $\sim$ KwMO-G (a, b, p, ξ).

2. Kumaraswamy Marshall – Olkin – Exponential (KwMO-E) distribution

Let X follows exponential distribution with parameter $\lambda > 0$ having pdf g(x; λ) = $\lambda e^{-\lambda x}$, x > 0 and cdf G(x; λ) = 1 – $e^{-\lambda x}$. Then substituting in (6) we obtain the KwMO-E distribution with density function

$$f_{KwMOE}\,(x) = \frac{ab\lambda\left(1 - p\right) e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{a-1}}{\left(1 - pe^{-\lambda x}\right)^{a+1}} \left\{1 - \left[\frac{1 - \ e^{-\lambda x}}{1 - pe^{-\lambda x}}\right]^{a}\right\}^{b-1} \tag{7}$$

where x > 0; a, b, $\bar{p}$, λ > 0 are shape parameters and ξ = (a, b, p, λ) is the vector of model parameters.

The corresponding survival function is given by

$$\overline{F}\,(\mathrm{x}) = \left\{1 - \left[\frac{1 - e^{-\lambda x}}{1 - pe^{-\lambda x}}\right]^{a}\right\}^{b} \tag{8}$$

For a continuous distribution with pdf and cdf given by (7) and (8), the hazard rate function is

$$\mathrm{h}\,(\mathrm{x}) = \frac{f\,(x)}{\overline{F}\,(\mathrm{x})} = \frac{ab\lambda\left(1 - p\right) e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{a-1}}{\left(1 - pe^{-\lambda x}\right)\left\{\left(1 - pe^{-\lambda x}\right)^{a} - \left(1 - e^{-\lambda x}\right)^{a}\right\}} \tag{9}$$

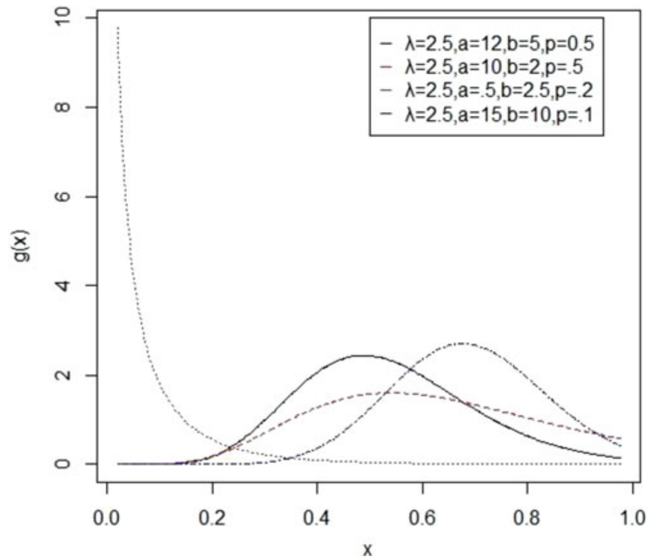

Figure 2.1. The KwMO-E density plots.

2.1. Remarks

1) If p = 0, then (7) reduces to Kumaraswamy Exponential (KwE) distribution.
2) When a = b = 1, then the density yields the Marshall- Olkin Exponential (MOE) distribution.
3) For the case a = b = 1 and p = 0, the pdf reduces to that of exponential distribution.
4) When a = 1 and p = 0, we have an exponential distribution with parameter bλ.
5) When b = 1 and p = 0, we have another exponentiated exponential distribution with parameter a and λ, then the pdf is f(x) = a $\lambda e^{-\lambda x}(1-e^{-\lambda x})^{a-1}$.

Figure 2.1 indicates that the KwMO-E distribution generates distributions with various shapes such as symmetric, left-skewed and reversed-J shaped. Further, Figure 2.2 shows that the KwMO-E family produces flexible hazard rate shapes such as constant, increasing and decreasing. This reveals that the KwMO-E distribution is very useful in fitting wide range of data sets with various shapes.

3. Shape properties

The shapes of the density and hazard rate functions are described analytically. The critical points of the density of KwMO-E model are the roots of the equation $\frac{d\,log[f(x)]}{dx}=0$ which yields,

$$-\lambda+(a-1)\frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x}}-p\,(a+1)\frac{\lambda e^{-\lambda x}}{1-pe^{-\lambda x}}$$
$$=a\,(1-b)\frac{\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{a-1}}{\left[1-pe^{-\lambda x}\right]\left\{\left[1-pe^{-\lambda x}\right]^{a}-\left[1-e^{-\lambda x}\right]^{a}\right\}} \tag{10}$$

$$\text{Let }\lambda\,(x)=\frac{d^2log\left[f\,(x)\right]}{dx^2}=-\,(a-1)\frac{\lambda^2\,e^{-\lambda x}}{\left[1-e^{-\lambda x}\right]^2}+p\,(a+1)\frac{\lambda^2\,e^{-\lambda x}\left[1-2pe^{-\lambda x}\right]}{\left[1-pe^{-\lambda x}\right]^2}$$
$$+a\,(b-1)\frac{\lambda^2 e^{-\lambda x}\left[1-e^{-\lambda x}\right]^{a-1}}{\left[1-pe^{-\lambda x}\right]\left\{\left[1-pe^{-\lambda x}\right]^{a}-\left[1-e^{-\lambda x}\right]^{a}\right\}}$$

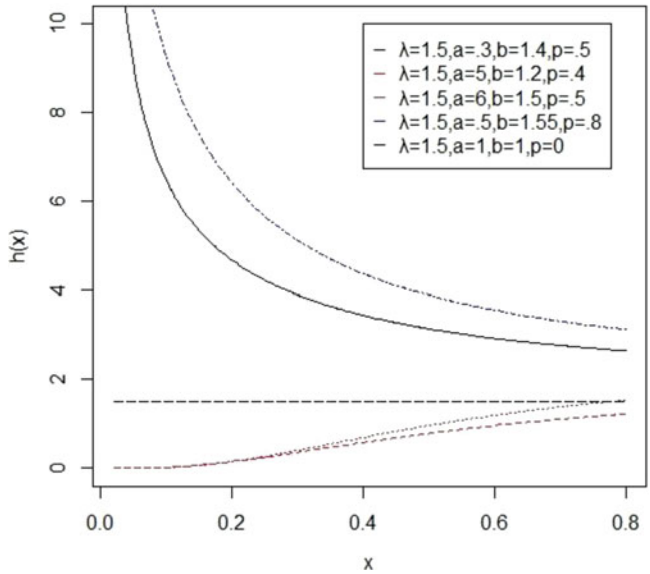

Figure 2.2. The KwMO-E hazard rate plots.

$$-a\,(a-1)\,(b-1)\,\frac{\lambda^2\left[e^{-\lambda x}\right]^2\left[1-e^{-\lambda x}\right]^{a-2}}{\left[1-pe^{-\lambda x}\right]\left\{\left[1-pe^{-\lambda x}\right]^a-\left[1-e^{-\lambda x}\right]^a\right\}}$$
$$+pa\,(b-1)\,\frac{\lambda^2\left[e^{-\lambda x}\right]^2\left[1-e^{-\lambda x}\right]^{a-1}}{\left[1-pe^{-\lambda x}\right]\left\{\left[1-pe^{-\lambda x}\right]^a-\left[1-e^{-\lambda x}\right]^a\right\}}$$
$$-a^2\,(b-1)\,\frac{\lambda e^{-\lambda x}\left[1-e^{-\lambda x}\right]^{a-1}\left\{p\left[1-pe^{-\lambda x}\right]^{a-1}-\left[1-e^{-\lambda x}\right]^{a-1}\right\}}{\left[1-pe^{-\lambda x}\right]\left\{\left[1-pe^{-\lambda x}\right]^a-\left[1-e^{-\lambda x}\right]^a\right\}^2}$$

If $x=x_0$ is the root of (10) then it corresponds to a local maximum if $\lambda(x)>0$ for all $x<x_0$ and $\lambda(x)<0$ for all $x>x_0$. It corresponds to a local minimum if $\lambda(x)<0$ for all $x<x_0$ and $\lambda(x)>0$ for all $x>x_0$. It refers to an inflexion point if either $\lambda(x)>0$ for all $x\neq x_0$ or $\lambda(x)<0$ for all $x\neq x_0$.

The critical point of h(x) is the roots of the equation

$$-\lambda+(a-1)\,\frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x}}-p\frac{\lambda e^{-\lambda x}}{1-pe^{-\lambda x}}=a\lambda e^{-\lambda x}\frac{p\left[1-pe^{-\lambda x}\right]^{a-1}-\left[1-e^{-\lambda x}\right]^{a-1}}{\left[1-pe^{-\lambda x}\right]^a-\left[1-e^{-\lambda x}\right]^a} \tag{11}$$

$$\text{Let}\quad \tau\,(x)=\frac{d^2 log\,[h\,(x)]}{dx^2}=-\,(a-1)\,\frac{\lambda^2\,e^{-\lambda x}}{\left[1-e^{-\lambda x}\right]^2}+p\frac{\lambda^2\,e^{-\lambda x}\left[1-2pe^{-\lambda x}\right]}{\left[1-pe^{-\lambda x}\right]^2}$$
$$+a\lambda^2e^{-\lambda x}\frac{p\left[1-pe^{-\lambda x}\right]^{a-1}-\left[1-e^{-\lambda x}\right]^{a-1}}{\left[1-pe^{-\lambda x}\right]^a-\left[1-e^{-\lambda x}\right]^a}$$
$$-a\,(a-1)\,\lambda^2\left[e^{-\lambda x}\right]^2\frac{p^2\left[1-pe^{-\lambda x}\right]^{a-2}-\left[1-e^{-\lambda x}\right]^{a-2}}{\left[1-pe^{-\lambda x}\right]^a-\left[1-e^{-\lambda x}\right]^a}$$
$$-\left\{a\lambda\,e^{-\lambda x}\,\frac{p\left[1-pe^{-\lambda x}\right]^{a-1}-\left[1-e^{-\lambda x}\right]^{a-1}}{\left[1-pe^{-\lambda x}\right]^a-\left[1-e^{-\lambda x}\right]^a}\right\}^2$$

If $x = x_0$ is the root of (11) then it refers to a local maximum if $\tau(x) > 0$ for all $x < x_0$ and $\tau(x) < 0$ for all $x > x_0$. It corresponds to a local minimum if $\tau(x) < 0$ for all $x < x_0$ and $\tau(x) > 0$ for all $x > x_0$. It refers to an inflexion point if either $\tau(x) > 0$ for all $x \neq x_0$ or $\tau(x) < 0$ for all $x \neq x_0$.

4. Expansion for cumulative and density functions

For any positive real number b and for $|z| < 1$, we have the generalized binomial expansion

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} z^j \tag{12}$$

Applying (12) to (7) gives

$$\begin{aligned} \mathrm{f}(\mathrm{x}; \xi) &= \frac{ab\lambda(1-p)e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{a-1}}{\left(1-pe^{-\lambda x}\right)^{a+1}} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} \left[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}\right]^{aj} = ab\,\lambda(1-p) \\ &\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^{j+l} \frac{\Gamma(b)\Gamma(a(j+1)+k+1)}{j!k!l!\,[a(j+1)+1]\,\Gamma(b-j)\Gamma(a(j+1)-l)} p^k e^{-\lambda x(k+l+1)} \end{aligned} \tag{13}$$

It reveals that the KwMO-E density is a linear combination of exponential density function. Similarly, we can write the cumulative density of KwMO-E as

$$\mathrm{F}(\mathrm{x}; \xi) = 1 - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^{i+k} \binom{b}{i}\binom{ai}{k}\binom{ai+j-1}{j} p^j e^{-\lambda x(j+k)} \tag{14}$$

5. General properties

In this section we discuss various functions associated with the new distribution and their properties.

5.1. Moments and generating function

Moments can be used to study the most important features and characteristics of a distribution.

Let X $\sim$ KwMO-E (a, b, p, λ) for r = 1, 2 … The r^{th} moment is given by

$$\mathrm{E}(X^r) = \int_0^{\infty} x^r f(x)\,dx.$$

From (13), when t = $\lambda(k+l+1)$ and using gamma function we have,

$$\begin{aligned} E(X^r) &= \mathrm{ab}\,\lambda(1-p) \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^{j+l} \\ &\times \frac{\Gamma(b)\Gamma(a(j+1)+k+1)}{j!k!l!\,[a(j+1)+1]\,\Gamma(b-j)\Gamma(a(j+1)-l)} \frac{p^k\, r!}{[\lambda(k+l+1)]^{r+1}} \end{aligned} \tag{15}$$

In particular the mean of KwMO-E distribution is

$$\mu = \mathrm{E}\left(\mathrm{X}\right) = \mathrm{ab}\ \lambda\left(1-p\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l} \times \frac{\Gamma\left(b\right)\Gamma\left(a\left(j+1\right)+k+1\right)}{j!k!l!\left[a\left(j+1\right)+1\right]\Gamma\left(b-j\right)\Gamma\left(a\left(j+1\right)-l\right)}\frac{p^k}{\left[\lambda\left(k+l+1\right)\right]^2} \quad (16)$$

The moment generating function is given by

$$M_X\left(t\right) = E\left(e^{tX}\right) = ab\lambda\left(1-p\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l} \times \frac{\Gamma\left(b\right)\Gamma\left(a\left(j+1\right)+k+1\right)}{j!k!l!\left[a\left(j+1\right)+1\right]\Gamma\left(b-j\right)\Gamma\left(a\left(j+1\right)-l\right)}\frac{p^k}{\left[\lambda\left(k+l+1\right)-\mathrm{t}\right]}$$
$$\mathrm{t} < \lambda\left(k+l+1\right) \quad (17)$$

The characteristic function is

$$\varphi\left(t\right) = \mathrm{ab}\ \lambda\left(1-p\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l} \times \frac{\Gamma\left(b\right)\Gamma\left(a\left(j+1\right)+k+1\right)}{j!k!l!\left[a\left(j+1\right)+1\right]\Gamma\left(b-j\right)\Gamma\left(a\left(j+1\right)-l\right)}\frac{p^k}{\left[\lambda\left(k+l+1\right)-\mathrm{it}\right]} \quad (18)$$

5.2. Quantiles

The q^{th} quantile $x_q = F^{-1}(q) = Q(q)$ of the KwMO-E distribution is given by

$$x_q = \frac{1}{\lambda} log\left\{\frac{1-p\left[1-\left(1-q\right)^{1/b}\right]^{1/a}}{1-\left[1-\left(1-q\right)^{1/b}\right]^{1/a}}\right\}$$

The effects of the shape parameters on the skewness (SK) and kurtosis (KR) can be considered based on quantile measures. The Bowley skewness and Moors kurtosis (based on octiles) can be calculated using the formulae

$$\mathrm{SK} = \frac{Q\left(\frac{3}{4}\right)+Q\left(\frac{1}{4}\right)-2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$$

and

$$\mathrm{KR} = \frac{\left[Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)\right]+\left[Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)\right]}{\left[Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)\right]}$$

5.3. Mean deviations

The amount of speed in a population is measured by the totality of deviations from the mean and median. Let X be a random variable with mean $\mu = E(X)$ and m be the median. The mean deviations about the mean can be obtained as

$$D(\mu) = E|X-\mu| = \int_0^\infty |x-\mu| f(x)\,dx = 2\mu\left\{1-\left[1-\left(\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}\right)^a\right]^b\right\}$$
$$-2\mu + 2\text{ab}\,\lambda\left(1-p\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l}$$
$$\times\frac{\Gamma(b)\,\Gamma\left(a\left(j+1\right)+k+1\right)}{j!k!l!\,\left[a\left(j+1\right)+1\right]\Gamma\left(b-j\right)\,\Gamma\left(a\left(j+1\right)-l\right)}\frac{p^k}{\left[\lambda\left(k+l+1\right)\right]^2}$$

The mean deviation from the median is

$$\mathrm{D}(\mathrm{m}) = E|X-m| = \int_0^\infty |x-m| f(x)\,dx = -\mu + 2\int_m^\infty x f(x)\,dx = -\mu + 2$$

6. Information theory measures

The seminal idea about information theory was pioneered by Hartley (1928) who defined a logarithmic measure of information for communication. Subsequently, Shannon (1948) formalized this idea by defining the entropy and mutual information concepts. The relative entropy notion was proposed by Kullback and Leibler (1951). The Kullback – Leibler's measure can be understood like a comparison criterion between two distributions.

6.1. Renyi entropy

The entropy of a random variable X is a measure of the uncertainty. It is an important concept in many fields of Science, especially communication theory, Statistical Physics and Probability Theory. A popular entropy measure is Renyi entropy. If X has the pdf f(x), then Renyi entropy is defined by

$$I_R(r) = \frac{1}{1-r}log\left\{\int_{-\infty}^{\infty} f^r(x)\,dx\right\}, \quad \text{where } r>0 \text{ and } r\neq 1.$$
$$f^r(x) = \frac{\left[ab\lambda\left(1-p\right)\right]^r \int_0^\infty e^{-\lambda x^r}\left[1-e^{-\lambda x}\right]^{r(a-1)}}{\left[1-pe^{-\lambda x}\right]^{r(a+1)}}\left\{1-\left[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}\right]^a\right\}^{r(b-1)}$$

Therefore,

$$I_R(r) = \frac{r}{1-r}\left[\log a + \log b + \log\lambda + \log\left(1-p\right)\right] + \frac{1}{1-r}\log\sum_j\sum_k\sum_l(-1)^{j+l}$$
$$\times\frac{\Gamma\{\mathrm{r}(\mathrm{b}-1)\}\,\Gamma\left\{\mathrm{r}(\mathrm{a}+1)+\mathrm{aj}+\mathrm{k}\right\}}{j!k!l!\Gamma\left\{r(b-1)+1-j\right\}\Gamma\left\{r(a+1)+aj\right\}}\frac{p^k\Gamma\left\{r(a-1)+aj+1\right\}}{\Gamma\left\{r(a-1)+aj+1-l\right\}}\frac{1}{\lambda(r+k+l)} \tag{19}$$

7. Stochastic ordering

In Probability theory and Statistics, stochastic ordering of positive continuous random variable is an important tool for judging the comparative behaviour. Here are some basic definitions.

A random variable X is said to be less than a random variable Y in the

i) stochastic order (denoted by $X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x.
ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x.
iii) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.
The following implications are well known:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \tag{20}$$

Shaked and Shanthikumar (1994). The KwMO-E distributions are ordered with respect to the strongest likelihood ratio ordering.

Theorem 7.1. *Let $X \sim$ KwMO-E (a, b_1, p, λ) and $Y \sim$ KwMO-E (a, b_2, p, λ).*

If $b_2 < b_1$ then $X \leq_{lr} Y$, $X \leq_{hr} Y$ and $X \leq_{st} Y$.

Proof. The density ratio is given by

$$\mathrm{U}\left(\mathrm{x}\right) = \frac{g_X\left(x\right)}{g_Y\left(x\right)} = \frac{b_1\left\{1-\left[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}\right]^a\right\}^{b_1-1}}{b_2\left\{1-\left[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}\right]^a\right\}^{b_2-1}}$$

Taking the derivative with respect to x,
$U'(x) = \lambda(1-p)a\frac{b_1}{b_2}(b_2-b_1)[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}]^a\{1-[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}]^a\}^{b_1-1}$ is decreasing in x.
Hence $X \leq_{lr} Y$. □

8. Order statistics

Order statistics are the most fundamental tools in non- parametric statistics and inference. Now we derive an explicit expression for the density function of the r^{th} order statistic $X_{r:n}$ in a random sample of size n from KwMO-E distribution. The pdf and cdf of the r^{th} order statistic says $Y = X_{r:n}$ are given by

$$\begin{aligned} f_Y\left(y\right) &= \frac{n!}{(r-1)!\,(n-r)!} F^{r-1}\left(y\right)\left[1-F\left(y\right)\right]^{n-r} f\left(y\right) \\ &= \frac{n!}{(r-1)!\,(n-r)!}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i F^{i+r-1}\left(y\right) f\left(y\right) \end{aligned}$$

and

$$\mathrm{F}\left(\mathrm{y}\right) = \sum_{j=r}^{n}\binom{n}{j}F^j\left(y\right)\left[1-F\left(y\right)\right]^{n-j} = \sum_{j=r}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i F^{j+i}\left(y\right)$$

where f(.) and F(.) are the pdf and cdf of the KwMO-E distribution. Then

$$\begin{aligned} f_Y\left(y\right) = {} & \frac{ab\lambda\left(1-p\right)n!}{(r-1)!\,(n-r)!}\sum_{i=0}^{n-r}\binom{n-r}{i}\sum_j\sum_k\sum_l\sum_m(-1)^{i+j+k+m} \\ & \times\frac{\Gamma\left(i+r\right)\Gamma\left\{b\left(j+1\right)\right\}\Gamma\left\{a\left(k+1\right)+l+1\right\}}{j!k!l!m!\left\{a\left(k+1\right)\right\}\Gamma\left(i+r-j\right)\Gamma\left\{b\left(j+1\right)-k\right\}\Gamma\left\{a\left(k+1\right)-m\right\}}p^l e^{-\lambda y(l+m+1)} \end{aligned}$$

The density function of the order statistics is simply an infinite linear combination of exponential densities. Then

$$F_Y(y) = \sum_{j=r}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left\{1-\left[1-\left(\frac{1-e^{-\lambda y}}{1-pe^{-\lambda y}}\right)^a\right]^b\right\}^{i+j}$$

8.1. Distribution of minimum, maximum and median

Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed random variables from the KwMO-E distribution having first, last and median order probability density function are given by,

$$\begin{aligned} g_{1:n}(x) &= n[1-F(x;\Phi)]^{n-1} f(x;\Phi) \\ &= \frac{nab\lambda\left(1-p\right)e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{a-1}}{\left(1-pe^{-\lambda x}\right)^{a+1}}\left\{1-\left[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}\right]^a\right\}^{bn-1} \end{aligned}$$

$$\begin{aligned} g_{n:n}(x) = n\left[F\left(x_{(n)};\Phi\right)\right]^{n-1} f\left(x_{(n)};\Phi\right) &= \frac{nab\lambda\left(1-p\right)e^{-\lambda x_{(n)}}\left(1-e^{-\lambda x_{(n)}}\right)^{a-1}}{\left(1-pe^{-\lambda x_{(n)}}\right)^{a+1}} \\ &\times\left\{1-\left[\frac{1-e^{-\lambda x_{(n)}}}{1-pe^{-\lambda x_{(n)}}}\right]^a\right\}^{b-1}\left\{1-\left[1-\left[\frac{1-e^{-\lambda x_{(n)}}}{1-pe^{-\lambda x_{(n)}}}\right]^a\right]^b\right\}^{n-1} \end{aligned}$$

$$\begin{aligned} g_{m+1:n}(x) &= \frac{(2m+1)!}{m!m!}F(x)^m(1-F(x))^m f(x) \\ &= \frac{(2m+1)!}{m!m!}\frac{ab\lambda\left(1-p\right)e^{-\lambda x_{(m+1)}}\left(1-e^{-\lambda x_{(m+1)}}\right)^{a-1}}{\left(1-pe^{-\lambda x_{(m+1)}}\right)^{a+1}} \\ &\times\left\{1-\left[\frac{1-e^{-\lambda x_{(m+1)}}}{1-pe^{-\lambda x_{(m+1)}}}\right]^a\right\}^{b(m+1)-1}\left\{1-\left[1-\left[\frac{1-e^{-\lambda x_{(m+1)}}}{1-pe^{-\lambda x_{(m+1)}}}\right]^a\right]^b\right\}^m \end{aligned}$$

8.2. Joint distribution of the i^{th} and j^{th} order statistics

The joint distribution of the i^{th} and j^{th} order statistics from KwMO-E distribution is

$$\begin{aligned} f_{i:j:n}\left(x_i, x_j\right) &= C\,F\left(x_{(i)}\right)^{i-1}\left[F\left(x_{(j)}\right)-F\left(x_{(i)}\right)\right]^{j-i-1}\left[1-F\left(x_{(j)}\right)\right]^{n-j} f\left(x_{(i)}\right) f\left(x_{(j)}\right) \\ &= C\left[1-k_{(i)}^b\right]^{i-1}\left\{\left[1-k_{(j)}^b\right]-\left[1-k_{(i)}^b\right]\right\}^{j-i-1}\left[k_{(j)}^b\right]^{n-j} \\ &\times\left[\frac{ab\lambda\left(1-p\right)e^{-\lambda x_{(i)}}\left(1-e^{-\lambda x_{(i)}}\right)^{a-1}}{\left(1-pe^{-\lambda x_{(i)}}\right)^{a+1}}k_{(i)}^{b-1}\right]\left[\frac{ab\lambda\left(1-p\right)e^{-\lambda x_{(j)}}\left(1-e^{-\lambda x_{(j)}}\right)^{a-1}}{\left(1-pe^{-\lambda x_{(j)}}\right)^{a+1}}k_{(j)}^{b-1}\right] \end{aligned}$$

where $C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$ and $k_{(i)} = 1-\left[\frac{1-e^{-\lambda x_{(i)}}}{1-pe^{-\lambda x_{(i)}}}\right]^a$.

9. Distributions of order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from KwMO-E (a, b, p, λ) distribution. Then the sample minima and maxima are $X_{1:n} = \min\ (X_1, X_2, \ldots, X_n)$ and $X_{n:n} = \max\ (X_1, X_2, \ldots, X_n)$. Here we consider the limiting distribution of sample extremes.

Theorem 9.1. *Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from KwMO-E (a, b, p, λ) distribution. Then*

i) $\lim_{n\to\infty} P(X_{1:n} \le b_n^* t) = 1 - e^{-t^{(a-1)}}$; $t > 0$, *where* $b_n^* = \frac{1}{\lambda a}\log(\frac{1-py}{1-y})$.

ii) $\lim_{n\to\infty} P(X_{n:n} \le b_n t) = e^{-t^{(a-1)}}$; $t > 0$, *where* $b_n = \frac{1}{\lambda a}\log(\frac{1-pu}{1-u})$.

Proof: i) We use the asymptotic result for $X_{1:n}$ and $X_{n:n}$ (Arnold, Balakrishnan and Nagaraja 1992) for the smallest order statistic $X_{1:n}$, we have

$$\lim_{n\to\infty} P(X_{1:n} \le a_n^* + b_n^* t) = 1 - e^{-t^c}, \quad t > 0, c > 0$$

where $a_n^* = F^{-1}(0)$ and $b_n^* = F^{-1}(\frac{1}{n}) - F^{-1}(0)$ if and only if $F^{-1}(0)$ is finite and for all $t > 0$ and $c > 0$

$$\lim_{\epsilon\to 0^+} \frac{F\left(F^{-1}(0) + \epsilon t\right)}{F\left(F^{-1}(0) + \epsilon\right)} = t^c.$$

For a random variable X with the KwMO-E (a,b,p,λ) distribution we have $F^{-1}(0) = 0$ is finite and $\lim_{\epsilon\to 0^+} \frac{F(\epsilon t)}{F(\epsilon)} = t^{a-1}$.

Thus, we have c = a-1, $a_n^* = 0$ and $b_n^* = \frac{1}{\lambda a}\log(\frac{1-py}{1-y})$ where $y = 1 - (1 - \frac{1}{n})^{\frac{1}{b}}$.

ii) For the maximal order statistic $X_{n:n}$ we have $\lim_{n\to\infty} P(X_{n:n} \le a_n + b_n t) = e^{(-t)^{-k}}$, $t > 0, k > 0$

which is Frechet type where $a_n = 0$ and $b_n = F^{-1}(1 - \frac{1}{n})$ if and only if $F^{-1}(1)$ is ∞ and there exists a constant $k > 0$ such that $\lim_{x\to\infty} \frac{1-F(xt)}{1-F(x)} = t^{a-1}$.

Thus, we obtain k = 1-a, $a_n = 0$ and $b_n = \frac{1}{\lambda a}\log(\frac{1-pu}{1-u})$ where $u = 1 - (\frac{1}{n})^{\frac{1}{b}}$.

Remarks 9.2. If p = 0, the results for the Kumaraswamy exponential distribution can be obtained. The norming constants are $b_n^* = \frac{1}{\lambda a}\log(\frac{1}{1-y})$ and $b_n = \frac{1}{\lambda a}\log(\frac{1}{1-u})$.

Let $G^*(t)$ and G(t) denote the limiting distributions of the random variables $(X_{1:n} - a_n^*)/b_n^*$ and $(X_{n:n} - a_n)/b_n$, then for i > 1, the limiting distributions of $(X_{i:n} - a_n^*)/b_n^*$ and $(X_{n-i+1:n} - a_n)/b_n$ are given by

$$\lim_{n\to\infty} P\left(X_{1:n} \le a_n^* + b_n^* t\right) = 1 - \sum_{j=0}^{i-1} (1 - G^*(t)) \frac{\left[-\log\left(1 - G^*(t)\right)\right]^j}{j!}$$

$$\lim_{n\to\infty} P(X_{n-i+1:n} \le a_n + b_n t) = \sum_{j=0}^{i-1} G(t)\,\frac{\left[-\log G(t)\right]^j}{j!}$$

For any infinite i > 1, the limiting distributions of the i^{th} and $(n-i+1)^{th}$ order statistics from the KwMO-E distribution are given by

$$\lim_{n\to\infty} P\left(X_{1:n} \le \frac{1}{\lambda a}\log\left(\frac{1-py}{1-y}\right)\right) = 1 - \sum_{j=0}^{i-1} e^{-t^{(a-1)}} \frac{t^{(a-1)j}}{j!} = 1 - P(Z < i)$$

$$\lim_{n\to\infty} P\left(X_{n-i+1:n} \le 1 - \frac{1}{\lambda a}\log\left(\frac{1-pu}{1-u}\right)\right) = \sum_{j=0}^{i-1} e^{-t^{(a-1)}} \frac{t^{(a-1)j}}{j!} = P(Z < i)$$

where Z follows the Poisson distribution with mean t^{a-1}.

10. Characterization of KwMO-E distribution

Characterization of distributions are important in the applied fields. An investigator will be interested to know whether their model fits the requirements of a particular distribution. For this we will depend on the characterizations of the distribution which provide conditions under which the underlying distribution is indeed that particular distribution. In this section characterizations based on truncated moments and a single function of the random variable are discussed.

10.1. Characterization based on truncated moments

Our characterization results presented here will employ an interesting result due to Glanzel (1987). The advantage of the characterizations given here is that, cdf F need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

Theorem 10.1.1. *Let* (Ω, Σ, P) *be a given probability space and let* $H = [a, b]$ *be an interval for some* $a < b$ a $= -\infty$, $b = \infty$ *might as well be allowed). Let* $X{:}\ \Omega \to$ H *be a continuous random variable with the distribution function F and let* q_1 *and* q_2 *be two real functions defined on H such that* $E[q_1(X)|X \ge x] = E[q_2(X)|X \ge x]\eta(x)$, $x \in H$ *is defined with some real function* η. *Assume that* $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ *and G are twice continuously differentiable and strictly monotone function on the set H. Finally assume that the equation* $q_2\eta = q_1$ *has no real solution in the interior of H. Then G is uniquely determined by the functions* q_1, q_2 *and* η,

$$\mathrm{G}(\mathrm{x}) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,q_2(u) - q_1(u)}\right| \exp(-s(u))\,du$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_2}{\eta q_2 - q_1}$ *and C is a constant, chosen to make* $\int_H dG = 1$.

Theorem 10.1.1 can be stated in terms of two functions q_1 and η by taking $q_2(x) \equiv 1$, which will reduce the condition given in Theorem 10.1.1 to $E[q_1(X)|X \ge x] = \eta(x)$. However adding an extra function will give a lot more flexibility, as far as its application is concerned.

Proposition 10.1.2 ***Let*** $X{:}\ \Omega \to$ R *be a continuous random variable and let* $q_2(x) = \{1 - [\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}]^a\}^{1-b}[1 - e^{-\lambda x}]^{1-a}$ *and* $q_1(x) = q_2(x)[1 - pe^{-\lambda x}]^{-a}$ *for* $x \in R$. *The pdf of X is*

(7) if and only if the function η defined in Theorem10.1.1 has the form

$$\eta(x) = \frac{1}{2}\left\{1 + \left[1 - pe^{-\lambda x}\right]^{-a}\right\}, \quad x \in R.$$

Proof: Let X have density (7), then

$$(1 - F(x))\, E\left[q_2(X)\,|X \geq x\right] = \frac{b(1-p)}{p}\left\{\left[1 - pe^{-\lambda x}\right]^{-a} - 1\right\}, \quad x \in R,$$

and

$$(1 - F(x))\, E\left[q_1(X)\,|X \geq x\right] = \frac{b(1-p)}{2p}\left\{\left[1 - pe^{-\lambda x}\right]^{-2a} - 1\right\}, \quad x \in R$$

and

$$\eta(x)\, q_2(x) - q_1(x) = \frac{1}{2} q_2(x)\left\{1 - \left[1 - pe^{-\lambda x}\right]^{-a}\right\} \neq 0, \quad \text{for } x \in R.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_2(x)}{\eta(x)\, q_2(x) - q_1(x)} = \frac{-ap\lambda e^{-\lambda x}\left[1 - pe^{-\lambda x}\right]^{-(a+1)}}{\left\{1 - \left[1 - pe^{-\lambda x}\right]^{-a}\right\}}, \quad x \in R$$

and hence

$$s(x) = -\ln\left\{1 - \left[1 - pe^{-\lambda x}\right]^{-a}\right\}, \quad x \in R.$$

In view of Theorem 10.1.1, X has density function (7).

Corollary 10.1.3 ***Let*** *X: $\Omega \rightarrow (0, 1)$ be a continuous random variable and let $q_2(x)$ be as in Proposition 10.1.2 the pdf of X is (7) if and only if there exist functions q_1 and η defined in Theorem 10.1.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_2(x)}{\eta(x)\, q_2(x) - q_1(x)} = \frac{-ap\lambda e^{-\lambda x}\left[1 - pe^{-\lambda x}\right]^{-(a+1)}}{\left\{1 - \left[1 - pe^{-\lambda x}\right]^{-a}\right\}}, \quad x \in R.$$

Remarks 10.1.4.

i) The general solution of the differential equation in corollary 10.1.3 is

$$\eta(x) = \left\{1 - \left[1 - pe^{-\lambda x}\right]^{-a}\right\}^{-1}\left\{\int ap\lambda e^{-\lambda x}\left[1 - pe^{-\lambda x}\right]^{-(a+1)}\left[q_2(x)\right]^{-1} q_1(x)\, dx + D\right\}$$

where D is a constant. One set of appropriate functions is given in Proposition10.1.2 with D $= \frac{1}{2}$.

i) Clearly there are other triplets of functions (q_2, q_1, η) satisfying the conditions of Theorem 10.1.1. We presented one such triplet in Proposition 10.1.2.

10.2. *Characterizations based on single function of the random variable*

In this section, we employ a single function ψ of X and state characterization results in terms of $\psi(X)$.

Theorem 10.2.1. *1-$F(x) = [c\psi(x) + d]^e$ if and only if $E[\psi(X)|X \geq x] = \frac{1}{e+1}\left[e\psi(x) - \frac{d}{c}\right]$ where $c \neq 0$, d, $e > 0$ are finite constants.*

Remarks 10.2.2. Taking c = −1, d = 1, e = b, $\psi(x) = \left[\frac{1-e^{-\lambda x}}{1-pe^{-\lambda x}}\right]^a$.

11. Estimation

In this section, we determine the maximum likelihood estimates (MLE's) of the model parameters of the new family from complete samples. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the KwMO-E distribution. Then the loglikelihood function is given by

$$\log l = n\log\left[ab\lambda\left(1-p\right)\right] - \lambda\sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n}\log\left[1-e^{-\lambda x_i}\right] + (b-1)\sum_{i=1}^{n}\log\left\{1-\left[\frac{1-e^{-\lambda x_i}}{1-pe^{-\lambda x_i}}\right]^a\right\} - (a+1)\sum_{i=1}^{n}\log\left[1-pe^{-\lambda x_i}\right] \quad (21)$$

Differentiating the loglikelihood with respect to a, b, p and λ we have

$$\frac{\partial \log l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log\left[\frac{1-e^{-\lambda x_i}}{1-pe^{-\lambda x_i}}\right] + (1-b)\sum_{i=1}^{n}\frac{\left[\frac{1-e^{-\lambda x_i}}{1-pe^{-\lambda x_i}}\right]^a\log\left[\frac{1-e^{-\lambda x_i}}{1-pe^{-\lambda x_i}}\right]}{1-\left[\frac{1-e^{-\lambda x_i}}{1-pe^{-\lambda x_i}}\right]^a}$$

$$\frac{\partial \log l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left\{1-\left[\frac{1-e^{-\lambda x_i}}{1-pe^{-\lambda x_i}}\right]^a\right\}$$

$$\frac{\partial \log l}{\partial p} = \frac{-n}{1-p} + (a+1)\sum_{i=1}^{n}\frac{e^{-\lambda x_i}}{1-pe^{-\lambda x_i}}$$

$$\frac{\partial \log l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}} - p(a+1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1-pe^{-\lambda x_i}} + a(1-b)\left(1-p\right)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{a-1}}{\left[1-pe^{-\lambda x_i}\right]\left\{\left[1-pe^{-\lambda x_i}\right]^a-\left[1-e^{-\lambda x_i}\right]^a\right\}}$$

The solution of these non-linear system of equations gives the estimates of the parameters. To solve these equations, it is usually more convenient to use nonlinear optimization algorithms such as quasi- Newton algorithm to numerically maximize the log likelihood function. In order to compute the standard errors and asymptotic confidence intervals the usual large sample approximation is used, in which the maximum likelihood estimators can be treated as being approximately multivariate normal. Hence as n→ ∞, the asymptotic distribution of the MLE is given by

$$\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{p}\\ \hat{\lambda}\end{pmatrix} = N\left[\begin{pmatrix}a\\ b\\ p\\ \lambda\end{pmatrix}\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34}\\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44}\end{pmatrix}\right]$$

where

$$\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34}\\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44}\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ A_{31} & A_{32} & A_{33} & A_{34}\\ A_{41} & A_{42} & A_{43} & A_{44}\end{pmatrix}^{-1}$$

is the approximate variance – covariance matrix with elements obtained from

$$A_{11} = \frac{-\partial^2 \log L}{\partial a^2}, \quad A_{12} = \frac{-\partial^2 \log L}{\partial a \, \partial b},$$

$$A_{13} = \frac{-\partial^2 \log L}{\partial a \, \partial p}, \quad A_{14} = \frac{-\partial^2 \log L}{\partial a \, \partial \lambda}$$

$$A_{22} = \frac{-\partial^2 \log L}{\partial b^2}, \quad A_{23} = \frac{-\partial^2 \log L}{\partial b \, \partial p}$$

$$A_{24} = \frac{-\partial^2 \log L}{\partial b \, \partial \lambda}, \quad A_{33} = \frac{-\partial^2 \log L}{\partial p^2}$$

$$A_{34} = \frac{-\partial^2 \log L}{\partial p \, \partial \lambda}, \quad A_{44} = \frac{-\partial^2 \log L}{\partial \lambda^2}$$

By solving this inverse dispersion matrix, the solutions will yield asymptotic variance and covariance of these maximum likelihood estimators for $\hat{a}$, $\hat{b}$, $\hat{p}$ and $\hat{\lambda}$. We can obtain approximate 100(1-r) % two-sided confidence intervals for a, b, p and λ which are determined as

$$\hat{a} \pm Z_{r/2}\sqrt{\hat{V}_{11}}, \quad \hat{b} \pm Z_{r/2}\sqrt{\hat{V}_{22}}, \quad \hat{p} \pm Z_{r/2}\sqrt{\hat{V}_{33}}, \quad \hat{\lambda} \pm Z_{r/2}\sqrt{\hat{V}_{44}}$$

where Z_r is the upper r^{th} percentile of the standard normal distribution.

12. Simulation study

In this section, we assess the performance of the KwMOE distribution by conducting various simulations for different sample sizes and parameter values. We generate 5000 samples by using Monte Carlo simulation with sample size n = 25, 50, 75, 100, 150, 200, 300 and the parameter values i) a = 0.5, b = 0.5, p = 0.3, λ = 0.005 ii) a = 1.5, b = 1.5, p = 0.8, λ = 0.05 were considered. The evaluation of the estimates was performed based on the following measures:

1) Average bias of the MLE $\hat{\theta}$ of the parameter θ = (a, b, p, λ):

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta).$$

2) Root Mean Square Error (RMSE):

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta)^2}$$

3) The Average Width (AW) of 95% confidence intervals of the parameter.

The empirical results are given in Table 12.1 and 12.2. We can see that as the sample size increases, the RMSE decreases in all cases. Also, the average confidence width decreases as the sample size increases.

Table 12.1. Average Bias, RMSE and AW of $a = 0.5$, $b = 0.5$, $p = 0.3$ and $\lambda = 0.005$.

Parameter	n	Average Bias	RMSE	AW
a	25	0.10051	0.28195	1.43326
	50	0.11332	0.25939	1.02469
	75	0.09349	0.21916	0.78742
	100	0.07670	0.18486	0.64698
	150	0.05558	0.14394	0.48909
	200	0.04209	0.11713	0.40415
	300	0.03023	0.08771	0.31013
b	25	−0.16800	0.20605	1.48694
	50	−0.16222	0.19413	1.21256
	75	−0.15918	0.18637	1.07944
	100	−0.15831	0.18204	1.00443
	150	−0.15433	0.17441	0.89579
	200	−0.15319	0.17084	0.82497
	300	−0.14771	0.16302	0.73963
p	25	0.25104	0.38507	2.03141
	50	0.25983	0.37130	1.52228
	75	0.24742	0.35212	1.34155
	100	0.23485	0.33390	1.24706
	150	0.22029	0.30749	1.10566
	200	0.20718	0.28982	1.02177
	300	0.19772	0.26198	0.90465
λ	25	0.00235	0.00438	0.04071
	50	0.00227	0.00378	0.03086
	75	0.00223	0.00344	0.02679
	100	0.00225	0.00332	0.02470
	150	0.00218	0.00305	0.02133
	200	0.00217	0.00293	0.01941
	300	0.00208	0.00271	0.01671

Table 12.2. Average Bias, RMSE and AW of $a = 1.5$, $b = 1.5$, $p = 0.8$ and $\lambda = 0.05$.

Parameter	n	Average Bias	RMSE	AW
a	25	0.41499	1.05283	8.26385
	50	0.41868	0.93746	5.59019
	75	0.40311	0.85578	4.39533
	100	0.37295	0.80211	3.73256
	150	0.31386	0.64255	2.66822
	200	0.28299	0.57672	2.19738
	300	0.20363	0.44207	1.56097
b	25	−0.53595	0.72457	7.76685
	50	−0.48711	0.68879	5.57964
	75	−0.47268	0.66608	4.39025
	100	−0.44825	0.65062	3.91930
	150	−0.38062	0.58962	3.30896
	200	−0.35741	0.56797	2.90688
	300	−0.27455	0.50237	2.53802
p	25	−0.12233	0.24979	1.90385
	50	−0.08516	0.20351	1.20974
	75	−0.06238	0.16972	0.89510
	100	−0.05598	0.15947	0.76589
	150	−0.03706	0.12599	0.58955
	200	−0.03128	0.11164	0.50282
	300	−0.02148	0.09059	0.40569
λ	25	0.12831	0.20329	1.79599
	50	0.10641	0.17079	1.11764
	75	0.09303	0.15092	0.80200
	100	0.08521	0.13911	0.68409
	150	0.06469	0.10950	0.47616
	200	0.05855	0.10055	0.40033
	300	0.04109	0.07573	0.28374

13. Application

Here we compare our distribution with Kumaraswamy (Kw), Exponential Kumaraswamy (ExKw) and Marshall-Olkin Kumaraswamy (MOKw) distributions with respect to a real data set. We consider the data from Abouammoh et al. (1994), which represents the lifetime in days of 40 patients suffering from leukaemia from one of the ministry of Health Hospitals in Saudi Arabia.

Data set: 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

We estimate the unknown parameters of each distribution by the method of maximum likelihood estimation. The measures of goodness of fit including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson Darling(A*), Cramer Von Mises(W*) and Kolmogorov-Smirnov (K-S) statistics are computed to compare the fitted models. The statistics A* and W* are described in details in Chen and Balakrishnan (1995). The Cramer- von Mises statistic is

$$W^2 = n\int_{-\infty}^{\infty} \left\{F_n(x) - F\left(x;\hat{\theta}\right)\right\}^2 dF(x;\hat{\theta}) \tag{22}$$

And the Anderson -Darling statistic is

$$A^2 = n\int_{-\infty}^{\infty} \frac{\left\{F_n(x) - F\left(x;\hat{\theta}\right)\right\}^2}{\left\{F\left(x;\hat{\theta}\right)\left(1 - F\left(x;\hat{\theta}\right)\right)\right\}} dF\left(x;\hat{\theta}\right) \tag{23}$$

For a given sample $x_1, x_2, \ldots, x_n$, define $u_i = F(x_i;\hat{\theta}), i = 1, 2, \ldots, n$. Without loss of generality, suppose $x_i{}'s$ and $u_i{}'s$ have been arranged in ascending order. Then

$$W^2 = \sum_{i=1}^{n} [u_i - \{(2i-1)/(2n)\}]^2 + 1/(12n) \tag{24}$$

$$A^2 = -n - n^{-1}\sum_{i=1}^{n} \{(2i-1)\ln(u_i) + (2n+1-2i)\ln(1-u_i)\} \tag{25}$$

To obtain the statistics W* and A*, we proceed as follows:

1) Estimate θ by $\hat{\theta}$ and compute $v_i = F(x_i;\hat{\theta})$ where the $x_i{}'s$ are in ascending order.
2) Compute $y_i = \Phi^{-1}(v_i)$ where Φ is the standard normal cdf and Φ^{-1} its inverse.
3) Compute $u_i = \Phi\{(y_i - \bar{y})/S_y\}$ where $\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$ and $S_y{}^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$
4) Compute W^2 and A^2 according to (24) and (25).
5) Modify W^2 into $W^* = W^2(1 + 0.5/n)$ and A^2 into $A^* = A^2(1 + 0.75/n + 2.25/n^2)$.

In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out using R-language. We compare the fits of the KwMO-E model with the Kw, ExKw and MOKw models. The numerical values of MLE's, AIC, BIC, A^*, W^* and K-S statistics are listed in Table 13.1. It can be seen that Kw distribution gives the worst fit and KwMO-E model provides the best fit among these competing models.

Also, we draw the estimated pdfs, cdfs, Q-Q plots and P-P plots for the fitted models and are presented in Figure 13.1

Table 13.1. MLE's, AIC, BIC, A*, W*, K-S and p-value for the given data.

Distribution	Estimates	AIC	BIC	A*	W*	K-S	p-value
Kw (a, b)	2.564, 196.844	− 124.078	− 120.701	0.979	0.1371	0.12407	0.5693
ExKw (a, b, λ)	2.1419, 122.752, −1.711	− 123.986	− 118.919	0.745	0.0875	0.09621	0.8529
MOKw (λ, a, b)	2.03, 125.59, 3.894	− 125.656	− 120.589	0.6711	0.0762	0.09071	0.8971
KwMOE(a,b,p,λ)	1.299, 5.605, −33.114, 19.143	− 125.518	− 118.762	0.483	0.0674	0.08659	0.9251

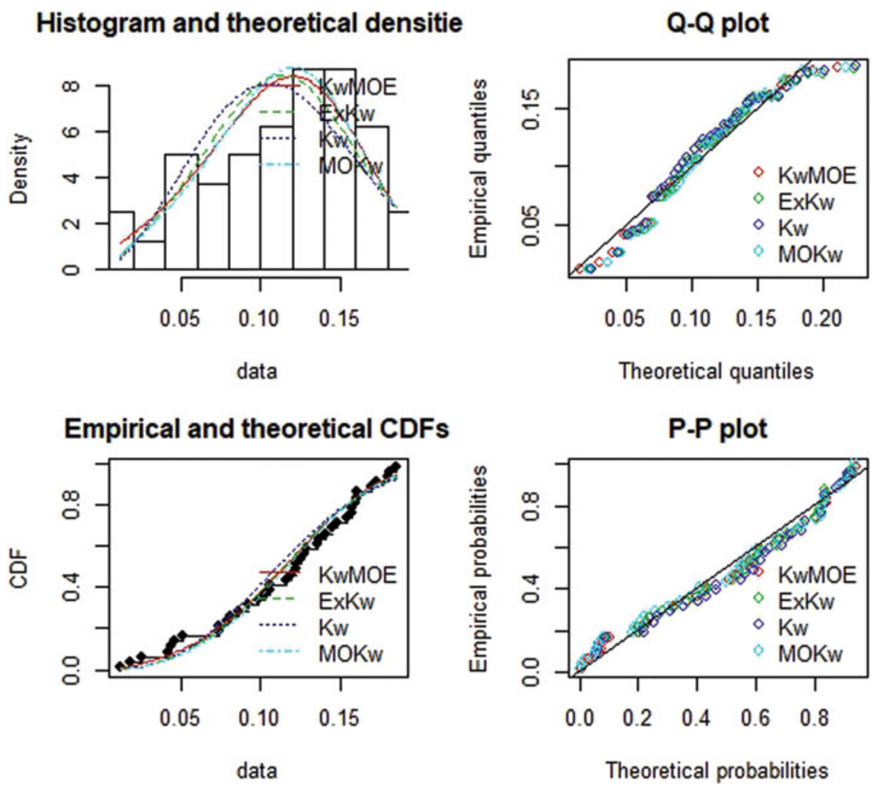

Figure 13.1. Plots of the estimated pdf's, Q-Q plot, estimated cdf's and P-P plots for KwMOE, ExKw, MOKw and Kw models.

14. Conclusion

In this paper, we propose the new Kumaraswamy Marshall-Olkin Exponential family of distributions. We derive various structural properties including expansion for the density function and expression for the moments, mean deviations, quantile function, order statistics and Renyi entropy. The maximum likelihood method is used for estimating the model parameters. We carry out a simulation study which evaluates the precision and accuracy of the maximum likelihood estimates of the KwMO-E distribution. Finally, an empirical application to a real data set is presented and compared with three other competing models. From the study it is very clear that the KwMO-E model provides a better fit than the other competing models. The extended distributions are feasible to develop new models because of the computational and analytical facilities available in programming software's such as R, Matlab and Mathematica.

Acknowledgments

The authors would like to thank the reviewers and the Editor for their valuable suggestions and comments which helped to improve the presentation of the paper significantly.

References

Abouammoh, A. M., S. A. Abdulghani, and I. S. Qamber. 1994. On partial orderings and testing of new better than renewal used classes. *Reliability Engineering & System Safety* 43:37–41. doi:10.1016/0951-8320(94)90094-9.

Alice, T., and K. K. Jose. 2003. Marshall-Olkin Pareto processes. *Far East Journal of Theoretical Statistics* 9 (2):117–32.

Alice, T., and K. K. Jose. 2004. Marshall-Olkin exponential time series modelling. *STARS International Journal* 5 (1):12–22.

Alice, T., and K. K. Jose. 2005. Marshall-Olkin Semi-Weibull minification processes. *Recent Advance Statistical Theory and Applications* 1:6–17.

Alizadeh, M., M. H. Tahir, G. M. Cordeiro, M. Zubair, and G. G. Hamedani. 2015. The Kumaraswamy Marshall-Olkin family of distributions. *Journal of the Egyptian Mathematical Society* (In press). doi:10.1016/j.joems.2014.12.002.

Arnold, B. C., N. Balakrishnan, and H. N. Nagaraja. 1992. *First course in order statistics*. New York: John Wiley.

Chen, G., and N. Balakrishnan. 1995. A general purpose approximate goodness of fit test. *Journal of Quality Technology* 27:154–61.

Cordeiro, G. M., and M. de Castro. 2011. A new family of generalized distributions. *Journal of Statistical Computation and Simulation* 81:883–93. doi:10.1080/00949650903530745.

Glanzel, W. 1987. A characterization theorem based on truncated moments and its application to some distribution families. in: M. L., Puri, P., Revesz, W., Wertz, P., Bauer, F., Konecny (Eds), *Mathematical Statistics and Probability Theory*. B.D. Reidel Publishing Company. 75–84.

Gui, W. 2013. Marshall-Olkin extended log-logistic distribution and its application in minification processes. *Applied Mathematical Sciences* 7 (80):3947–61. doi:10.12988/ams.2013.35268.

Hartley, R. V. L. 1928. Transmission of information. *Bell Systems Technical Journal* 7:535–63. doi:10.1002/j.1538-7305.1928.tb01236.x.

Jones, M. C. 2009. Kumaraswamy's distribution: A beta type distribution with some tractability advantages. *Statistical Methodology* 6:70–81. doi:10.1016/j.stamet.2008.04.001.

Jose, K. K., and E. Krishna. 2011. Marshall-Olkin extended uniform distribution. *Probability and Statistics Forum* 4:78–88.

Jose, K. K., and R. Sebastian. 2013. Marshall-Olkin Morgenstern Weibull distribution: generalisation and applications. *Economic Quality Control* 28 (2):105–16. doi:10.1515/eqc-2013-0018.

Krishna, E., K. K. Jose, T. Alice, and M. M. Ristic. 2013a. The Marshall-Olkin Frechet distribution. *Communication in Statistics- Theory and Methods* 42 (22):4091–107. doi:10.1080/03610926.2011.648785.

Krishna, E., K. K. Jose, and M. M. Ristic. 2013b. Applications of Marshall-Olkin Frechet distribution. *Communication in Statistics-Simulation and Computation* 42 (1):76–89. doi:10.1080/03610918.2011.633196.

Kullback, S., and R. A. Leibler. 1951. On information and sufficiency. *The Annals of Mathematical Statistics* 22 (1):79–86. doi:10.1214/aoms/1177729694.

Kumaraswamy, P. 1980. A generalized probability density function for double- bounded random processes. *Journal of Hydrology* 46:79–88. doi:10.1016/0022-1694(80)90036-0.

Marshall, A. W., and I. Olkin. 1997. A new method for adding a parameter to a family of distribution with application to the exponential and weibull families. *Biometrika* 84 (3):641–52. doi:10.1093/biomet/84.3.641.

Shaked, M., and J. G. Shanthikumar. 1994. *Stochastic orders and their applications*. New York: Academic Press.

Shannon, C. E. 1948. A Mathematical theory of communication. *Bell System Technical Journal* 27:379–432. doi:10.1002/j.1538-7305.1948.tb01338.x.