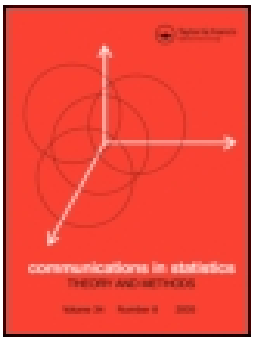

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Distributions and Applications

A Generalization of the Half-Normal Distribution with Applications to Lifetime Data

KAHADAWALA COORAY
AND MALWANE M. A. ANANDA

Department of Mathematical Sciences, University of Nevada,
Las Vegas, Nevada, USA

A two-parameter family of lifetime distribution which is derived from a model for static fatigue is presented. This derivation follows from considerations of the relationship between static fatigue crack extension and the failure time of a certain specimen. The cumulative distribution function (cdf) of this new family is quite similar to the cdf of the half-normal distribution, and therefore this density is referred to as the generalized half-normal distribution (GHN). Furthermore, this GHN family is a special case of the three-parameter generalized gamma distribution. Even though the GHN distribution is a two-parameter distribution, the hazard rate function can form variety of shapes such as monotonically increasing, monotonically decreasing, and bathtub shapes. Some properties of this family are given, and examples are cited to compare with other commonly used failure time distributions such as Weibull, gamma, lognormal, and Birnbaum-Saunders.

Keywords Coverage probabilities; Discriminant analysis; Generalized gamma; Maximum likelihood; Weibull.

Mathematics Subject Classification Primary 62Exx; Secondary 62Fxx.

1. Introduction

From experimental investigations, fatigue crack growth appears as a process with random properties. These random properties seem to vary during crack growth from specimen to specimen. A large number of parametric distribution models which are based on either consideration of the basic characteristics of the fatigue process or data fitting has been proposed to account for the random behavior. However, most of the distributions that were derived under plausible physical consideration are based on fatigue crack growth under variable stress or cyclic load.

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Address correspondence to Malwane M. A. Ananda, Department of Mathematical Sciences, University of Nevada, Las Vegas, NV 89154, USA; E-mail: mananda@ccmail.nevada.edu

This article investigates the requirements of a parametric distribution which are based on static fatigue crack growth under constant applied stress. For this, results from published experimental data are used to confirm the validity of the model.

The derivation of static fatigue model, its properties, functional characteristics, maximum likelihood parameter estimation and coverage probabilities, discriminant analysis, and illustrative examples are discussed, cited, and presented in Secs. 2–7, respectively.

2. A Model for Distribution of Static Fatigue Life

The brittle materials such as glasses, ceramics (Wachtman, 1996), some polymers (Powell, 1983), etc., exhibit delayed fracture when sufficiently stressed in certain environments like corrosive, moist, etc. Under these conditions, time-dependent growth of cracks continue until a critical size is reached and the Griffith criterion for fast fracture becomes operative. Static fatigue is the phenomenon that describes subcritical flaw growth to such critical dimensions.

Consider a set of brittle homogeneous specimens that are subjected to a constant sustained pressure at stress level σ until all had failed. Because of this pressure, the failure times of each specimen may be different from others and is random due to the variation in the material, environmental impact, defective manufacturing, or some other reason. Some of these reasons may accelerate the crack extensions that are initiated due to the constant stress. Here, we assume there are no cracks at the very beginning and cracks will open due to the applied stress.

The strength degradations due to static fatigue can be expressed as follows:

$$\frac{dL}{dt} = AK_I^B, \tag{1}$$

where L is the crack length at time t, K_I is the crack tip stress intensity factor for Mode I (called opening mode, i.e., the failure will take place from the largest flaw normal to the greatest tensile stress), A and B are constants. Equation (1) is called empirical law (power law) of crack velocity.

Let K_0 be the material fracture toughness, and when $K_0 > K_I$ the material is tough enough to withstand the effect of stress concentration about the crack.

The general fracture mechanics relation for the stress intensity factor K_I for a crack of length L under stress σ is

$$K_I = \sigma Y L^{1/2}. \tag{2}$$

The constant Y depends on the crack geometry.

Consider the situation when the stress distribution from the crack remains constant but the crack grows longer.

Assuming for small cracks compared to the specimen dimensions, the change in Y as the crack grows is negligible, i.e., Y is constant.

Replacing K_I by $\sigma Y L^{1/2}$ in the first equation and some simplification gives the following equation for crack length at time t,

$$L = ct^{\alpha} \tag{3}$$

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where $\alpha = 2/(2 - B), B < 2,$ and c depends on the stress level σ and the crack geometry.

Because of the analog relation of "Dominant Crack Length" to "Critical Error", we can assume that fatigue failure is due to the ultimate extension of a dominant crack (L_c) which is a time-dependent continuous random variable with whose distribution may be reasonable to model by the error function of the following form:

$$F_{L_c}(l_c) = 2\Phi(l_c) - 1; \quad l_c \geq 0, \tag{4}$$

where $\Phi(\cdot)$ is the Laplace-Gauss integral.

Specimen failure occurs whenever the crack length is greater than the critical dominant crack, i.e., $L > L_c$

Therefore, we can write

$$\Pr(L > L_c) = F_{L_c}(l) = 2\Phi(l) - 1; \quad l \geq 0. \tag{5}$$

Using the relation in (3) and after some reparameterization, one can formulate the cumulative distribution function (cdf) for the material specimen failure time $t = x$ such that

$$F(x) = 2\Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right] - 1; \quad \alpha, \theta > 0, x \geq 0, \tag{6}$$

where $\Phi(.)$ is the cdf of the standard normal distribution and α, θ are unknown shape and scale parameters of this distribution, respectively. Due to the resemblance of the cdf of the half-normal distribution, the resulting distribution is referred to as the generalized half-normal (GHN) distribution. The probability density function of the GHN distribution is given by:

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\theta}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{7}$$

Notice that

$$F(x) = 2\Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right] - 1 = \Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right] - \Phi\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right] = 1 - 2\Phi\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right]$$

It should be noted that even though this lifetime distribution has been simply derived from a plausible physical consideration of static fatigue, it is a special case of three-parameter generalized gamma distribution (see Eq. (8). So, we analyze and explore the generalized gamma distribution in order to find the inherent properties of this new member.

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3. Importance of GHN Distribution Through the Hazard Function of Generalized Gamma

The generalized gamma distribution introduced by Stacy (1962), and its density function and hazard function can, respectively, be written as

$$f(x; a, b, k) = \frac{b}{\Gamma(k)} \frac{x^{bk-1}}{a^{bk}} \exp\left[-\left(\frac{x}{a}\right)^b\right], \quad a > 0, \ b > 0, \ k > 0; \ x > 0 \tag{8}$$

and

$$h(x; a, b, k) = \frac{b}{\Gamma(k)} \frac{x^{bk-1}}{a^{bk}} \exp\left[-\left(\frac{x}{a}\right)^b\right] \Big/ \left[1 - \Gamma\left(k, \left(\frac{x}{a}\right)^b\right)\right], \quad a > 0, \ b > 0, \ k > 0; \ x > 0, \tag{9}$$

where $\Gamma\left(k, \left(\frac{x}{a}\right)^b\right)$ is the incomplete gamma function defined as $\Gamma(p, q) = \int_0^q t^{p-1} e^{-t} dt$.

From the hazard function of generalized gamma distribution, bathtub-shaped hazard function will arise when $1 < b < 1/k$, $k < 1$, and hump-shaped hazard function will arise when $1/k < b < 1$ and $k > 1$ (see Proof 1 of the Appendix). Furthermore, Lawless (1980) have shown that the lognormal distribution is a limiting case as $k \to \infty$ of the generalized gamma distribution. Hence, the lognormal distribution is a special case of generalized gamma distribution and it possesses flexible and more reliable hump shaped hazard function. For smaller k (<1), so far there is no such a two-parameter simplified distribution, which having a bathtub-shaped hazard function. By considering this region ($k < 1$), the new distribution (when $k = 1/2$) is the only member which has a bathtub-shaped hazard function with rich statistical properties and mathematically tractable two parameter lifetime density in the family of generalized gamma distribution.

3.1. *The Computational Difficulties with Generalized Gamma*

There is difficulty in developing inference procedures with the generalized gamma distribution, especially the maximum likelihood parameter estimation in which the iteration method such as Newton-Raphson did not work. Even with samples of size 200 or 300, for algorithms in determining maximum likelihood estimators fail to converge (Hager and Bain, 1970). Several other authors (Parr and Webster, 1965; Stacy and Mihram, 1965) encountered similar problems with the maximum likelihood estimation. In addition, it was found to be difficult to develop suitable tests or interval estimation procedures. Since the limiting normal distribution approaches very slowly, large sample method based on normal approximation used to estimate the maximum likelihood estimators of a, b, k is unsatisfactory. (Hager and Bain, 1970) found that normal approximations are inefficient even if sample sizes are as large as 400. Moreover, (Klugman et al., 1998) mentioned the problems of finding the parameters of generalized gamma for grouped data. Because of these difficulties practitioners more often use the generalized gamma as a model checking distribution for lifetime data.

Amazingly, the bathtub shape of the hazard function which is one of the prominent properties of the generalized gamma distribution has been inherited only by this GHN distribution which is a family member of this generalized gamma

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distribution. Furthermore, the computational difficulties of generalized gamma will not in anyway affect this GHN distribution.

It is stressed and emphasized that this new density is a derivation of a practical application rather than merely a theoretical derivation. This new promising model derived from the practical or empirical knowledge has surprisingly been a derivation of the theoretical model of the generalized gamma.

4. Some Properties of the GHN Distribution

For the GHN density, one can show the following properties:

- Mode $= \left(\frac{\alpha-1}{\alpha}\right)^{\frac{1}{2\alpha}}\theta$
- Median $= \left\{\Phi^{-1}\left(\frac{3}{4}\right)\right\}^{\frac{1}{\alpha}}\theta$
- Inflection points: $\left\{\frac{4\alpha-3\pm\sqrt{12\alpha^2-12\alpha+1}}{2\alpha}\right\}^{\frac{1}{2\alpha}}\theta$
- kth Moment: $E[X^k] = \sqrt{\frac{2^{k/\alpha}}{\pi}}\Gamma\left(\frac{k+\alpha}{2\alpha}\right)\theta^k$
- Variance $= \frac{2^{1/\alpha}}{\pi}\left\{\sqrt{\pi}\Gamma\left(\frac{2+\alpha}{2\alpha}\right) - \Gamma^2\left(\frac{1+\alpha}{2\alpha}\right)\right\}\theta^2$
- kth limited expected value: $E[(X\wedge x)^k] = \sqrt{\frac{2^{k/\alpha}}{\pi}}\Gamma\left(\frac{k+\alpha}{2\alpha}, \frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}\right)\theta^k + 2\Phi\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right]x^k$

  where $\Gamma\left(\frac{k+\alpha}{2\alpha}, \frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}\right)$ is the incomplete gamma function evaluated at the value of $\frac{k+\alpha}{2\alpha}; \frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}$. Here, the incomplete gamma function is defined as, $\Gamma(x; y) = \frac{1}{\Gamma(x)}\int_0^y e^{-t}t^{x-1}dt$; $x, y > 0$, and $\Gamma(x)$ is the gamma function defined as, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$; $x > 0$. The k^{th} limited expected value is an important measure of interest in the insurance industry and the actuarial sciences (Klugman et al., 1998).

- Inverse of this distribution exists. cdf and probability density function (pdf) of this inverse new model can, respectively, be written as:

$$F(x) = 2\Phi\left[-\left(\frac{\theta_1}{x}\right)^{\alpha_1}\right]; \quad \alpha_1, \theta_1 > 0, \ x \geq 0$$

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(\frac{\alpha_1}{x}\right)\left(\frac{\theta_1}{x}\right)^{\alpha_1} e^{-\frac{1}{2}\left(\frac{\theta_1}{x}\right)^{2\alpha_1}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

  where α_1 and θ_1 are parameters of this inverse model distribution.
  One can write the kth moments of this distribution

$$E[X^k] = \frac{1}{\sqrt{\pi 2^{k/\alpha}}}\Gamma\left(\frac{\alpha-k}{2\alpha}\right)\theta^k, \quad k < \alpha.$$

  Moreover, this inverse distribution may be useful to model lifetime data when $B > 2$ in (1).

- Hazard rate function:

$$h(t) = \frac{\alpha}{\sqrt{2\pi}t}(t/\theta)^{\alpha}\exp\left(-\frac{1}{2}\left(\frac{t}{\theta}\right)^{2\alpha}\right)\Big/\Phi\left[-\left(\frac{t}{\theta}\right)^{\alpha}\right] \tag{10}$$

- Expected Fisher information matrix:

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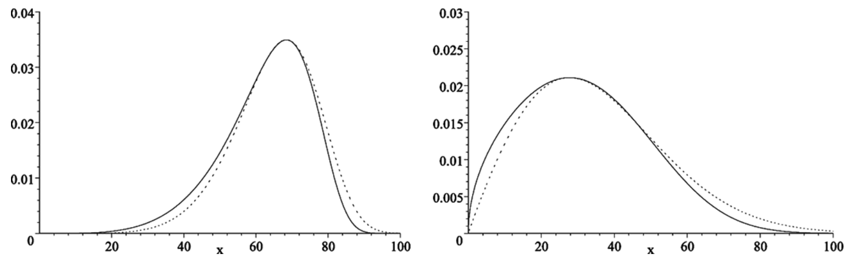

Figure 1. (a) New density ($\alpha = 5, \theta = 70$) and Weibull; (b) New density ($\alpha = 1.5, \theta = 40$) and Weibull.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n, taken from this distribution. Then the Fisher information matrix is

$$I[\theta, \alpha] = \begin{bmatrix} 2n(\frac{\alpha}{\theta})^2 & -\frac{n}{\theta}(2 - \ln 2 - \gamma) \\ -\frac{n}{\theta}(2 - \ln 2 - \gamma) & \frac{n}{2\alpha^2}\left\{\frac{\pi^2}{2} - 2 + (2 - \ln 2 - \gamma)^2\right\} \end{bmatrix}, \quad (11)$$

where γ is Euler's constant ($\gamma = \lim_{m\to\infty}[\sum_{r=1}^{m} \frac{1}{r} - \ln m] = .5772156649$; Abramowitz and Stegun, 1972).

- The GHN density function has a thicker left tail than the Weibull density function (see Appendix), whenever $\alpha > 1$. In order to illustrate this property, one can coincide the two densities at their mode for several values of α and θ. In order to coincide, one needs to pick the parameter τ in the Weibull model in such a way that $\tau > \alpha$. For simplicity here, we considered two different mode values (dotted line indicate Weibull density and solid line indicate new density) Figure 1.

On the other hand, whenever $\tau > \alpha > 1$, $\lim_{x\to 0} \frac{F_n(\alpha,\theta)}{F_w(\tau,\phi)} \to \infty$ will show the GHN density has a thicker left tail than the Weibull. Here, $F_n(\alpha, \theta)$ and $F_w(\tau, \phi)$ are cdf of the proposed new model (6) and Weibull (see Appendix), respectively.

5. Characteristics of the GHN Density and the Hazard Function

We will now examine how the values of the shape parameter α and the scale parameter θ affect the shape of the probability density and the hazard rate function. Even though this is a two-parameter distribution, this density provides negatively and positively skewed densities for different values of shape parameter α. In particular, depending on parameter values, the hazard rate function can form variety of shapes such as monotonically increasing, monotonically decreasing, and bathtub shapes. The following two figures demonstrate the shape of the proposed new density function and its hazard function for different parameter values (Figure 2).

The shape parameter α can be viewed as the slope. This is because the value of α is equal to the slope of the regressed line in a probability plot. As α changes for fixed θ, one can show the following properties by analyzing the density and the hazard function given in (7) and (10), respectively.

Note that some boundary values were calculated numerically.

Range of α	Characteristic of the density and the hazard function for fixed θ
$0 < \alpha < 2.17$	distribution is positively skewed

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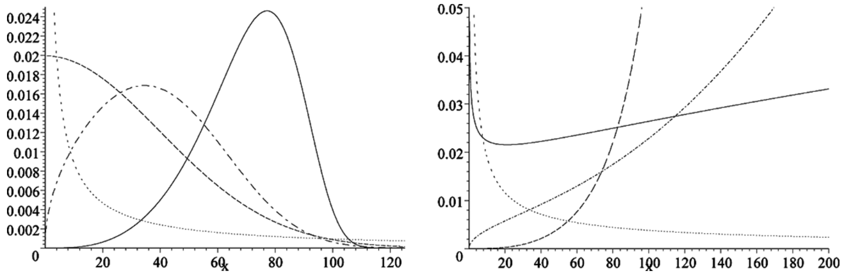

Figure 2. (a) Density curves of proposed new model; (b) Hazard curves of proposed new model.

$\alpha = 2.17$	skewness $= 0$
$2.17 < \alpha$	distribution is negatively skewed
$0 < \alpha < 0.5$	hazard rate decreases monotonically, concave up and approaches zero as $t \to \infty$
$\alpha = 0.5$	hazard rate decreases monotonically, concave up and approaches $1/(2\theta)$ as $t \to \infty$
$0.5 < \alpha < 1.0$	bathtub shape hazard curves (i.e., hazard rate has a local minimum)
$\alpha = 1.0$	hazard rate is nearly close to a straight line
$1.0 < \alpha < 2.0$	hazard rate increases monotonically, concave down and approaches ∞ as $t \to \infty$
$2 \le \alpha$	hazard rate increases monotonically, concave up and approaches ∞ as $t \to \infty$

6. Parameter Estimation under Maximum Likelihood Method

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n, taken from the proposed two-parameter density given in (7).

The loglikelihood function is given by

$$\ln L(x_1, x_2, \ldots, x_n; \theta, \alpha) = \frac{n}{2} \ln\left(\frac{2}{\pi}\right) + n \ln \alpha - n\alpha \ln \theta + (\alpha - 1) \sum_{i=1}^{n} \ln x_i - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{2\alpha}. \tag{12}$$

The maximum likelihood estimate of α, $\hat{\alpha}$ can be calculated by numerically finding the solution for the following equation:

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} \ln x_i - n\left(\sum_{i=1}^{n} x_i^{2\hat{\alpha}} \ln x_i\right)\left(\sum_{i=1}^{n} x_i^{2\hat{\alpha}}\right)^{-1} = 0. \tag{13}$$

The estimate of $\hat{\alpha}$ must be obtained by an iterative method and this is a simple task since the nonlinear equation given in (13) has a unique root.

The maximum likelihood estimate of θ is

$$\hat{\theta} = \left\{\frac{1}{n} \sum_{i=1}^{n} x_i^{2\hat{\alpha}}\right\}^{1/(2\hat{\alpha})}. \tag{14}$$

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Table 1
Actual coverage probabilities based on 100,000 replications for 90% intended

Sample size	$\theta = 1$						$\theta = 2$					
	$\alpha =$	.5	.75	1	2	4	$\alpha =$	.5	.75	1	2	4
10	for α	.8858	.8871	.8869	.8874	.8875	for α	.8864	.8883	.8888	.8873	.8879
	for θ	.8104	.8163	.8231	.8285	.8312	for θ	.8119	.8197	.8212	.8272	.8318
20	for α	.8942	.8945	.8930	.8925	.8922	for α	.8915	.8910	.8931	.8905	.8927
	for θ	.8553	.8589	.8604	.8652	.8664	for θ	.8551	.8596	.8596	.8634	.8680
30	for α	.8949	.8947	.8928	.8962	.8956	for α	.8949	.8947	.8946	.8963	.8949
	for θ	.8682	.8731	.8745	.8773	.8776	for θ	.8677	.8728	.8761	.8761	.8817
50	for α	.8953	.8964	.8965	.8960	.8963	for α	.8965	.8972	.8961	.8973	.8960
	for θ	.8799	.8836	.8833	.8859	.8862	for θ	.8813	.8832	.8853	.8865	.8887

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The confidence intervals for parameters can be obtained via large sample theory by using the expected Fisher information matrix given in (11).

An approximate $100(1-\alpha)\%$ confidence interval for θ and α are, respectively, given by

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\frac{\pi^2/2 - 2 + (2 - \ln 2 - \gamma)^2}{n(\pi^2 - 4)}}\,\frac{\hat{\theta}}{\hat{\alpha}} \tag{15}$$

and

$$\hat{\alpha} \pm z_{\alpha/2}\frac{2\hat{\alpha}}{\sqrt{n(\pi^2 - 4)}}. \tag{16}$$

6.1. *Coverage Probabilities*

The following two tables show the coverage probabilities based on maximum likelihood estimation method. Tables 1 and 2 show the coverages when the intended coverages are at 90 and 95%, respectively. The required confidence intervals are calculated by using (15) and (16).

Clearly, one can see when the sample size increases the actual coverage probabilities for the parameters under the maximum likelihood method gets closer to the intended coverage probabilities. Moreover, these values show that the parameters are not overly estimated under the maximum likelihood estimation method.

7. Discrimination Between the GHN and the Weibull Distribution

The problem of selecting a model from the proposed family and Weibull with unknown parameters are considered. Since the new family and the Weibull distributions can easily be transformed to location-scale parameter distributions, and the distribution of the ratio of maximum likelihoods of this transformed distributions does not depend upon the values of the nuisance location and scale parameters (Dumonceaux et al., 1973). Therefore, this ratio provides a suitable test

Table 2
Actual coverage probabilities based on 100,000 replications for 95% intended

Sample size	$\theta = 1$ $\alpha =$	.5	.75	1	2	4	$\theta = 2$ $\alpha =$	.5	.75	1	2	4
10	for α	.9471	.9475	.9470	.9478	.9475	for α	.9478	.9492	.9491	.9481	.9473
	for θ	.8578	.8704	.8791	.8862	.8893	for θ	.8600	.8727	.8765	.8846	.8896
20	for α	.9485	.9490	.9483	.9476	.9476	for α	.9465	.9475	.9483	.9474	.9473
	for θ	.9030	.9107	.9145	.9194	.9213	for θ	.9027	.9115	.9135	.9195	.9218
30	for α	.9478	.9487	.9476	.9483	.9482	for α	.9487	.9474	.9481	.9492	.9490
	for θ	.9171	.9240	.9262	.9307	.9323	for θ	.9167	.9243	.9280	.9303	.9338
50	for α	.9478	.9489	.9485	.9488	.9477	for α	.9480	.9491	.9482	.9487	.9486
	for θ	.9283	.9343	.9367	.9388	.9393	for θ	.9300	.9337	.9362	.9381	.9407

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statistic for discriminating between two location-scale parameter models when the parameters are unknown.

7.1. *The Likelihood Ratio Test*

Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with densities $f_i(x; a_i, b_i)$, $i = 1, 2$ such that

$$f_i(x; a_i, b_i) = \frac{1}{b_i} g_i\left(\frac{x - a_i}{b_i}\right); \quad -\infty < a_i < \infty, \; b_i > 0, \; i = 1, 2, \tag{17}$$

where a_i, b_i are location and scale parameters of these densities.

Here, our goal is to select f_1 or f_2 as a model for the given observations. We can formulate of this selection as the test of hypothesis such that

$$H_0 : X \sim f_1(x; a_1, b_1) \quad \text{vs.} \quad H_0 : X \sim f_2(x; a_2, b_2).$$

Let the ratio of the maximum likelihood (RML) be defined by

$$\text{RML} = \frac{\text{Max}_{a_1, b_1} \prod_{i=1}^{n} f_1(x_i; a_1, b_1)}{\text{Max}_{a_2, b_2} \prod_{i=1}^{n} f_2(x_i; a_2, b_2)}. \tag{18}$$

If the distribution of RML is independent of the unknown location and scale parameters, a_i and b_i, then it would be a convenient statistic for testing H_0 versus H_1 (Dumonceaux et al., 1973).

7.2. *Critical Values and Power of the RML Test with H_0: Weibull vs H_1: GHN*

Let $f_{GHN}(x; \alpha, \theta)$ be the proposed density given by (7) and let $f_w(x; \tau, \phi)$ be the Weibull density given in the Appendix.

Let $X_1, X_2, \dots, X_n$ be an observed sample to select either $f_{GHN}(x; \alpha, \theta)$ or $f_w(x; \tau, \phi)$ for unknown parameters. One can transform these data by taking

Table 3
Critical values of RML and power of the test for H_0: Weibull and H_1: GHN

	$\alpha = 0.1$		$\alpha = 0.05$		$\alpha = 0.025$	
Sample size (n)	RML_c	Power	RML_c	Power	RML_c	Power
10	1.642	0.175	1.911	0.098	2.188	0.053
20	1.929	0.236	2.399	0.141	2.911	0.082
30	2.138	0.287	2.810	0.177	3.546	0.108
50	2.386	0.373	3.411	0.246	4.620	0.160

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logarithms, say $z_i = \ln x_i$, and choosing between the two location-scale densities like Eq. (10). Hence, one can find RML values by using Eq. (11) (see a similar argument in Dumonceaux and Antle, 1973). We can show that (see Proof 2 in the Appendix) this RML value is same as the RML value in the following equation:

$$\begin{aligned}\mathrm{RML} &= \left(\operatorname*{Max}_{\alpha,\theta}\prod_{i=1}^{n} f_{GHN}(x_i;\alpha_1,\theta_1)\right)\left(\operatorname*{Max}_{\tau,\phi}\prod_{i=1}^{n} f_w(x_i;\tau_2,\phi_2)\right)^{-1}\\ &= \left(\frac{2e}{n\pi}\right)^{n/2}\left(\frac{\hat{\alpha}}{\hat{\tau}}\right)^{n}\left(\frac{\sum_{i=1}^{n} x_i^{\hat{\tau}}}{\sqrt{\sum_{i=1}^{n} x_i^{2\hat{\alpha}}}}\right)\left(\prod_{i=1}^{n} x_i^{\hat{\alpha}-\hat{\tau}}\right), \end{aligned} \tag{19}$$

where $\hat{\alpha}$ and $\hat{\tau}$ are solutions of Eq. (13) and

$$\frac{n}{\hat{\tau}} + \sum_{i=1}^{n}\ln x_i - n\left(\sum_{i=1}^{n} x_i^{\hat{\tau}}\ln x_i\right)\left(\sum_{i=1}^{n} x_i^{\hat{\tau}}\right)^{-1} = 0, \tag{20}$$

respectively.

The distribution of RML as given in Eq. (12), will depend upon n and whether the observations are from $f_{GHN}(x;\alpha,\theta)$ or $f_w(x;\tau,\phi)$, therefore we can use this RML as a test statistic. Table 3 gives critical values for the test statistic, RML, and the power of the test for this problem. Here we reject the Weibull in favor of the new family whenever $\mathrm{RML} \geq \mathrm{RML}_c$.

The values in Table 3 were the result of simulations in which 200,000 samples were used for each sample size.

Here, we select the new family if $\mathrm{RML} \geq \mathrm{RML}_c$ and we select Weibull if $\mathrm{RML} < \mathrm{RML}_c$. This table is also useful to select the size of a sample in order to have satisfactory error probabilities, α (probability of selecting the new family when it is a Weibull) and β (probability of selecting Weibull when it is a new family).

Here we consider Weibull as the null hypothesis, but the experimenter might prefer to use the new family as a null hypothesis. Therefore, we next present a table of critical values with the new family as the null hypothesis.

Table 4
Critical values of RML and power of the test for H_0: GHN and H_1: Weibull

	$\alpha = 0.1$		$\alpha = 0.05$		$\alpha = 0.025$	
Sample size (n)	RML_c	Power	RML_c	Power	RML_c	Power
10	1.441	0.170	1.639	0.092	1.829	0.049
20	1.695	0.233	2.097	0.137	2.525	0.080
30	1.861	0.287	2.432	0.176	3.089	0.106
50	2.052	0.380	2.933	0.254	3.997	0.165

7.3. *Critical Values and Power of the RML Test with H_0: GHN vs H_1: Weibull*

Using the same notation in Sec. 5.2.1, we can reject the new family in favor of Weibull whenever $\text{RML} \geq \text{RML}_c$. The RML value for this test is given by

$$RML = \left(\frac{2e}{n\pi}\right)^{n/2}\left(\frac{\hat{\alpha}}{\hat{\tau}}\right)^{n}\left(\frac{\sum_{i=1}^{n} x_i^{\hat{\tau}}}{\sqrt{\sum_{i=1}^{n} x_i^{2\hat{\alpha}}}}\right)\left(\prod_{i=1}^{n} x_i^{\hat{\alpha}-\hat{\tau}}\right), \tag{21}$$

where $\hat{\alpha}$ and $\hat{\tau}$ are unique solutions of Eqs. (13) and (20), respectively.

The distribution of RML as given in Eq. (14), will depend upon n and whether the observations are from $f_{GHN}(x; \alpha, \theta)$ or $f_w(x; \tau, \phi)$. Therefore, we can use this RML as a test statistic. Table 4 gives critical values for the test statistic, RML, and the power of the test for this problem.

The values in Table 4 were the result of simulations in which 200,000 samples were used for each sample size.

8. Examples

In this section we give several examples using several well-known data sets to demonstrate the flexibility and applicability of the proposed new model over the other leading lifetime parametric models. The two examples are cited to convince the model derived in Sec. 2. In order to compare the models, we used following four criterions:

(a) The value of the log likelihood function at its maximum (LOGLIKE)—large is good
(b) Kolmogorov-Smirnov test statistic (K-S)—small is good
(c) Anderson and Darling test statistic (A-D)—small is good
(d) The p-value from the chi-square goodness-of-fit test—large is good.

Note: the maximum likelihood parameter estimation method was used for all models for the following examples and the density functions of compared distributions are given in the Appendix.

Example 8.1 (Kevlar 49/Epoxy Strands Failure at 90% Stress Level)**.** The following 101 data points represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed,

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Table 5
Estimated values and fitted distributions

Distribution	Parameters	Loglike	K-S	A-D	D.F.	*P*-value
GHN	$\hat{\theta} = 1.223783,\ \hat{\alpha} = 0.710777$	−103.33	0.080	0.835	10	0.58
Gamma	$\hat{\psi} = 1.175293,\ \hat{\gamma} = 0.871996$	−102.82	0.089	1.037	10	0.33
Lognormal	$\hat{\mu} = -0.64866,\ \hat{\sigma} = 1.421309$	−113.30	0.162	3.417	10	0.00
Weibull	$\hat{\phi} = 0.989946,\ \hat{\tau} = 0.925888$	−102.98	0.091	1.122	10	0.27
Birnbaum-Saunders	$\hat{\mu} = 0.336199,\ \hat{\sigma} = 1.069055$	−115.62	0.249	8.191	10	0.00

so that we have complete data with exact times of failure. The failure times in hours are shown below (Andrews and Herzberg, 1985; Barlow et al., 1984):

0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89

Comparison of this proposed new distribution using the criteria explained earlier would be presented in Table 5.

From Table 5, one can clearly see that, even if the values of measures of Kolmogorov-Smirnov statistics, Anderson-Darling statistics, and the *p*-value are better criteria to the proposed new distribution, the loglikelihood value for the Weibull distribution is slightly better than the proposed new distribution. This is because the final data point (7.89) of the given sample may be a possible outlier for this new model. However, the values of measures of Kolmogorov-Smirnov statistics, Anderson-Darling statistics, and especially the *p*-value have convinced and emphasized that the proposed new model as a better fit than the other distributions for the given data in the above example.

The following comparison, Table 6, represents the reanalysis of the above sample without the final data point.

Table 6
Estimated values and fitted distributions

Distribution	Parameters	Loglike	K-S	A-D	D.F.	*P*-value
GHN	$\hat{\theta} = 1.157162,\ \hat{\alpha} = 0.782421$	−93.83	0.070	0.743	10	0.68
Gamma	$\hat{\psi} = 1.034495,\ \hat{\gamma} = 0.924316$	−95.31	0.099	1.247	10	0.26
Lognormal	$\hat{\mu} = -0.67580,\ \hat{\sigma} = 1.402110$	−108.1	0.167	3.681	10	0.00
Weibull	$\hat{\phi} = 0.949168,\ \hat{\tau} = 0.981549$	−95.49	0.093	1.312	10	0.23
Birnbaum-Saunders	$\hat{\mu} = 0.322488,\ \hat{\sigma} = 1.022562$	−109.66	0.257	8.907	10	0.00

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Table 7
Estimated values and fitted distributions

Distribution	Parameters	Loglike	K-S	A-D	D.F.	P-value
GHN	$\hat{\theta} = 10906.98,\ \hat{\alpha} = 1.640670$	−479.56	0.067	0.322	4	0.97
Gamma	$\hat{\psi} = 3161.236,\ \hat{\gamma} = 2.785522$	−483.14	0.112	0.932	4	0.60
Lognormal	$\hat{\mu} = 8.892588,\ \hat{\sigma} = 0.701223$	−487.87	0.142	1.746	4	0.17
Weibull	$\hat{\phi} = 9906.049,\ \hat{\tau} = 2.014980$	−480.85	0.088	0.550	4	0.87
Birnbaum-Saunders	$\hat{\mu} = 6800.546,\ \hat{\sigma} = 62.02113$	−488.43	0.171	2.401	4	0.07

From Table 6, it is clearly evident that the proposed new distribution is a better-fit than the other distributions for the given data. Moreover, the estimated parameter $\hat{\alpha}$ (= 0.782421) of this proposed new model will indicate the bathtub-hazard shape of the given sample.

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Example 8.2 (Kevlar 49/Epoxy Strands Failure at 70% Stress Level). The following 49 data points represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 70% stress level until all had failed, as before we have complete data with exact times of failure. The failure times in hours are shown below (Andrews and Herzberg, 1985; Barlow et al., 1984):

1051, 1337, 1389, 1921, 1942, 2322, 3629, 4006, 4012, 4063, 4921, 5445, 5620, 5817,
5905, 5956, 6068, 6121, 6473, 7501, 7886, 8108, 8546, 8666, 8831, 9106, 9711,
9806, 10205, 10396, 10861, 11026, 11214, 11362, 11604, 11608, 11745, 11762,
11895, 12044, 13520, 13670, 14110, 14496, 15395, 16179, 17092, 17568, 17568

As before, the comparison of this proposed new distribution using the criteria explained earlier would be presented in Table 7.

From Table 7, one can clearly see that, the values of measures of loglikelihood, Kolmogorov-Smirnov statistics, Anderson-Darling statistics, and especially the higher p-value have convinced and emphasized that the proposed new model as a better-fit than the other distributions for the given data in the above example. Moreover, the estimated parameter $\hat{\alpha}$ (= 1.640670) of this proposed new model will indicate that the distribution is positively skewed.

9. Conclusions

As we have seen in previous sections, especially in Secs. 3 and 4, this newly proposed distribution exhibits more exclusive mathematical tractability and statistical attractiveness with a flexible thicker left tail than the other leading lifetime distributions such as Weibull, gamma, lognormal, etc. Positive skewness as well as the negative skewness for different parametric values being a rare property of a distribution has established the better fitness for the lifetime data. On the other hand, the shapes of the hazard functions such as increasing, decreasing, and bathtub shapes have made this new distribution more elegant, applicable, reliable, and a flexible lifetime distribution than the existing parametric distributions.

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Appendix

Probability density functions which are used in the analysis:

Gamma	$f(x;\gamma,\psi)=\frac{1}{x\Gamma(\gamma)}(\frac{x}{\psi})^{\gamma}\exp(-x/\psi),\ \psi>0,\gamma>0,x>0$
Lognormal	$f(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma x}\exp\{-\frac{1}{2}(\frac{\ln x-\mu}{\sigma})^2\},\ -\infty<\mu<\infty,\sigma>0,x>0$
Weibull	$f(x;\tau,\phi)=(\frac{\tau}{x})(\frac{x}{\phi})^{\tau}\exp\{-(x/\phi)^{\tau}\},\ \phi>0,\tau>0,x>0$
Birnbaum-Saunders	$f(x;\mu,\sigma)=(8\pi\sigma^2x^3)^{-1/2}(x+\mu)\exp\{-\frac{1}{2}(\frac{x-\mu}{\sigma\sqrt{x}})^2\},\ \mu>0,\sigma>0,x>0$

Proof 1 (The Hazard Function Behavior of Generalized Gamma). The hazard function can be written as:

$$h(x)=f(x)/(1-F(x)),\quad 0<x<\infty. \tag{22}$$

Suppose local minimum or maximum occurs at the point $x=x_0$, then

$$(1-F(x_0))f'(x_0)+f^2(x_0)=0, \tag{23}$$

where f' denotes the first derivative with respect to x.

Whenever

$$f(x_0)f''(x_0)-(f'(x_0))^2\gtrless 0, \tag{24}$$

the hazard function has a local minimum or local maximum, respectively, where f'' denote the second derivative with respect to x, i.e., the hazard function has a minimum or maximum whenever

$$1-bk+b(1-b)(x_0/a)^b\gtrless 0. \tag{25}$$

Now, if $b>1$, since $x_0>0$, the local minimum occurs whenever $1<b<1/k$, and hence $k<1$, if $b<1$, since $x_0>0$, the local maximum occurs whenever $1/k<b<1$, and hence $k>1$.

Proof 2 (Derivation of (19)). Let's use $Z_i=\ln x_i$ transformations to $f_{GHN}(x;\alpha,\theta)$ and $f_w(x;\tau,\phi)$. Then the GHN density transform into a density function, say f_P,

$$f_P(z_i;a_0,b_0)=\sqrt{\frac{2}{\pi}}\frac{1}{b_0}e^{\left(\frac{z_i-a_0}{b_0}\right)}e^{-\frac{1}{2}e^{2\left(\frac{z_i-a_0}{b_0}\right)}};\quad -\infty<a_0<\infty,\ b_0>0,\ -\infty<x<\infty, \tag{26}$$

and the Weibull density transform into Type I extreme value density (smallest extreme value density, say f_{se}),

$$f_{se}(z_i;a_1,b_1)=\frac{1}{b_1}e^{\left(\frac{z_i-a_1}{b_1}\right)}e^{-e^{\left(\frac{z_i-a_1}{b_1}\right)}};\quad -\infty<a_1<\infty,\ b_1>0,\ -\infty<x<\infty, \tag{27}$$

where

$$a_0=\ln\theta,\ b_0=1/\alpha,\quad \text{and}\quad a_1=\ln\phi,\ b_1=1/\tau. \tag{28}$$

Now (26) and (27) are location-scale family of distributions. Therefore, RML does not depend on parameters (Dumonceaux et al., 1973).

$$\begin{aligned}\mathrm{RML} &= \left(\underset{a_0, b_0}{\mathrm{Max}} \prod_{i=1}^{n} f_P(z_i; a_0, b_0) \right) \left(\underset{a_1, b_1}{\mathrm{Max}} \prod_{i=1}^{n} f_{se}(z_i; a_1, b_1) \right)^{-1} \\ &= \left(\frac{2e}{\pi} \right)^{n/2} \left(\frac{\widehat{b_1}}{\widehat{b_0}} \right)^{n} \exp\left(\sum_{i=1}^{n} \left(\frac{z_i - \widehat{a_0}}{\widehat{b_0}} \right) - \sum_{i=1}^{n} \left(\frac{z_i - \widehat{a_1}}{\widehat{b_1}} \right) \right), \end{aligned} \tag{29}$$

where $\widehat{a_0}$, $\widehat{b_0}$, and $\widehat{a_1}$, $\widehat{b_1}$ are maximum likelihood estimators of P and Type I extreme value distributions, respectively.

Moreover, from (28):

$$\widehat{a_0} = \ln \hat{\theta}, \quad \widehat{b_0} = 1/\hat{\alpha}, \quad \text{and} \quad \widehat{a_1} = \ln \widehat{\phi}, \; \widehat{b_1} = 1/\hat{\tau}.$$

Therefore, from (29) yields (19).

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