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On Extension of Some Exponentiated Distributions with Application

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Abstract

The paper proposes extension of exponentiated weibull, exponentiated exponential, exponentiated lognormal and exponentiated gumble distributions .Two parameter exponentiated weibull is suitable to fit unimodel, monotone and risk functions unlike weibull model. Eexponentiated exponential has shape and scale parameter like a gamma and weibull distribution and can be used as an alternative and has better fit than weibull and gamma in certain cases. Three parameter exponentiated lognormal and two parameter exponentiated gumble are unimodel distributions and can yield better fits. Parameter estimation is done by maximum likelihood estimation. Goodness of fit is carried out for two data sets for illustration.

Keywords: Exponentiated weibull, Exponentiated exponential, Exponentiated lognormal, Exponentiated gumble, Maximum Likelihood Estimators, Goodness of fit

INTRODUCTION

The two parameter gamma and weibull distributions are mostly used in distributions for analyzing life time data. The gamma distribution is less popular than the weibull distribution as in gamma distribution survival function can not be obtained in a closed form unless the shape parameter is an integer. The weibull family of distributions has been widely used in the analysis of survival data emerging from engineering and medical fields. Mudhokar and Srivastava (1993) and Mudhokar et al(1995) have modeled various failure time data sets with the proposed exponentiated weibull distribution The exponentiated weibull (EW) distribution is a special case of general class of exponentiated distribution proposed by Gupta et al (1998).Gupta and Kundu (2001) introduced two parameter exponentiated exponential distribution and analyzed life time data sets and observed that exponentiated exponential (EE) distribution can be used in place of two parameter gamma and weibull distribution. The two parameters of an EE distribution represent the shape and scale parameters like gamma or weibull distribution. Nadarajah (2005) introduced exponential gumble distribution for survival function. He illustrated its use for modeling rainfall data from Orland, Florida .Kakade and Shirke (2006) analyzed real life data sets and observed that exponentiated lognormal distribution fit better than weibull and exponentiated exponential distribution. The exponentiated lognormal(EL) distribution is effective alternative. Kakade and Shirke (2007) studied exponential gumble distribution to analyze positively skewed data The exponentiated gumble(EG) distribution has a better fit and can be used in place of weibull and exponentiated exponential distribution. In this paper we consider all these distributions together and make comparison taking some real life data sets.

1. Exponentiated Weibull Distribution

Probability density function

We define Probability density function(P.d.f) of exponentiated weibull distribution as considered in Mudhokar et al(1995) with parameter $\alpha, \theta, and \sigma$ and life time has a density function given as

$$f(t;\alpha,\theta,\sigma) = \frac{\alpha\theta}{\sigma}\left[1-\exp(-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\exp\left[-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\left(\frac{t}{\sigma}\right)^{\alpha-1}, \nabla t > 0 \ldots\ldots\ldots\ldots(1)$$

Where α >0, θ>0 are shape parameters and σ >0 is a scale parameter.

It is a weibull distribution when θ=1 and the exponential distribution when α=1 and θ=1.

The survival function corresponding to random variable T with exponentiated-weibull density is given as

$$S(t;\alpha,\theta,\sigma) = P(T \geq t) = 1-\left[1-\exp\left(-(\frac{t_i}{\sigma})^{\alpha}\right)\right]^{\theta} \ldots\ldots\ldots\ldots(1.2)$$

The greater flexibility of this model is in fitting survival data.

Maximum Likelihood Estimators

We discuss the maximum likelihood estimators of a three parameter EW distribution as. Let $x_1, x_2, x_3 \ldots\ldots. x_n$ be a random sample from EW the log likelihood can be as

L(α,θ,σ)=n. log (αθ/σ)+

$$(\theta\text{-}1).\sum_{i=1}^{n}\log(g(Ti))-\sum_{i=1}^{n}(Ti/\sigma)^{\alpha}+(\alpha-1).\sum_{i=1}^{n}\log(Ti/\sigma).......(1.3)$$

where

g(Ti)=g(Ti;α,θ)= 1-exp(-T/σ)$^{\alpha}$

We can differentiate(1.1) for with respect to three parameters.

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}+(\theta-1).\sum_{i=1}^{n}g_{\alpha}(Ti)/g(Ti)-\sum_{i=1}^{n}(Ti/\sigma)^{\alpha}.\log(Ti/\sigma)+\sum_{i=1}^{n}\log(T/\sigma)=0...(1.4)$$

$$\frac{\partial L}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log(g(Ti))=0.. (1.5)$$

$$\frac{\partial L}{\partial\sigma}=-\left(\frac{n\alpha}{\sigma}\right)+(\theta-1).\sum_{i=1}^{n}g_{\sigma}(Ti)/g(Ti))+(\alpha/\sigma).\sum_{i=1}^{n}(Ti/\sigma)^{\alpha}........(1.6)$$

where

$$g_{\alpha}(Ti)=\exp(-(Ti/\sigma)^{\alpha}.(Ti/\sigma)^{\alpha}.\log(Ti/\sigma),$$

$$g_{\sigma}(Ti)=-\left[\alpha.\exp(-(Ti/\sigma)^{\alpha}).(Ti/\sigma)^{\alpha}\right]/\sigma$$

From (1.4),(1.5) and(1.6) we obtain the ML Estimates.

2. Exponentiated Exponential

Probability density function

The Probability density function of exponentiated exponential is defined by Gupta and Kunda (2001) with parameters α and λ as

$$f(x,\alpha,\lambda)=\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}.e^{-\lambda x}\ ....(2.1)$$

Where $\alpha,\lambda,x>0$

Here α is the shape parameter and λ is the scale parameter. When α=1 it represent the exponential family.

The survival function corresponding with exponentiated-exponential density is given as

$$S(x,\alpha,\lambda)=1-\left(1-e^{-\lambda x}\right)^{\alpha}\ ....(2.2)$$

The exponentiated exponential represents a parallel system.

Maximum Likelihood Estimators:-

We discuss the maximum likelihood estimators of a two parameter EE distribution as. Let $x_1, x_2, x_3 ....x_n$ be a random sample from EE the log likelihood can be as

L(α, λ)=n In α + n In λ+(α-1) $\sum_{i=1}^{n} In(1-e^{-\lambda xi}) - \lambda \sum_{i=1}^{n} x_i ......(2.3)$

Therefore to obtain the MLE's of α and λ we can directly maximize (2.3) with respect to α and λ or we can solve the non-linear normal equations which are as follows:-

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} In\left(1-e^{-\lambda x}\right) = 0.......\ldots\ldots\ldots\ldots\ldots\ldots..(2.4)$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}} - \sum_{i=1}^{n} x_i = 0.......\ldots\ldots\ldots\ldots(2.5)$$

From (2.4), we obtain the MLE of α as a function of λ, say α $(\lambda^{\wedge})$, as

$$\alpha(\lambda^{\wedge}), = -\frac{n}{\sum_{i=1}^{n} In(1-e^{-\lambda x_i})} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.6)$$

If the scale parameter is known, the MLE of the shape parameter $\alpha^{\wedge}$,can be obtained directly from (2.6).IF both the parameters are unknown, first the estimate of the scale parameter can be obtained by maximizing directly L(α$(\lambda^{\wedge})$, λ) with respect to λ. Once $\lambda^{\wedge}$ is obtained $\alpha^{\wedge}$ can be obtained from (2.6) as α^$(\lambda^{\wedge})$, .

3. Exponentiated Lognormal Distribution

Probability density function

The Probability density function (P.d.f) of exponential Lognormal distribution is defined by with respect to three parameters (α, μ, σ) as

f(x; (α, μ, σ))= $\alpha(\varphi(In(x); \mu, \sigma))^{\alpha-1} . \phi(In(x); \mu, \sigma) . x^{-1}, \ldots\ldots\ldots(3.1)$

x, α >0, $- - \infty < \mu < \infty$

where $\varphi(In(x); \mu, \sigma)$ and $\phi(In(x); \Diamond$ are the c.d.f and p.d.f of the normal distribution with mean and standard deviation as μ and σ.

The survival function corresponding with exponentiated lognormal distribution density is given as

S(x, μ σ. α) =1- ($\varphi(In(x); \mu, \sigma)$)$^{\alpha}$

where x >0

Maximum Likelihood Estimators

Let $x_1, x_2, x_3 ....x_n$ be a random sample from EL distribution the log likelihood function can be as

$$L(\alpha,\mu,\sigma/x) = nIn\alpha - \sum_{i=1}^{n} In(xi) + (\alpha-1)\sum_{i=1}^{n} In\varphi(In(xi);\mu,\sigma) + \sum_{i=1}^{n} In\phi(In(xi);\mu,\sigma) \,. \qquad \ldots\ldots..(3.2)$$

To find the values of the parameters α,μ,σ that maximize L(α,μ,σ /x) we can solve the equations which are as follows:-

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} In\varphi(In(xi);\mu,\sigma) = 0 .... \ldots\ldots\ldots\ldots..(3.3)$$

$$\frac{\partial L}{\partial \mu} = (\alpha-1)\sum_{i=1}^{n} \frac{\varphi_\mu{}^{/}(In(xi);\mu,\sigma)}{\varphi(In(xi);\mu,\sigma)} + \sum_{i=1}^{n} \frac{\varphi_\mu{}^{/}(In(xi);\mu,\sigma)}{\phi(In(xi);\mu,\sigma)} = 0.........(3.4)$$

$$\frac{\partial L}{\partial \sigma} = (\alpha-1)\sum_{i=1}^{n} \frac{\varphi_\sigma{}^{/}(In(xi);\mu,\sigma)}{\varphi(In(xi);\mu,\sigma)} + \sum_{i=1}^{n} \frac{\varphi_\sigma{}^{/}(In(xi);\mu,\sigma)}{\phi(In(xi);\mu,\sigma)} = 0.........(3.5)$$

From (3.3) ,(3.4) and (3.5) MLE of α, µand σ is obtained.

4. Exponential Gumble Distribution

Probability density function

The Probability density function (Pd.f) of exponential gumble distribution was introduced by Nadarajah(2005) with parameters α and σ as

$$f(x;\alpha,\sigma) = \frac{\alpha}{\sigma}\left[\exp\left(-e^{-\frac{x}{\sigma}}\right)\right]^{\alpha} .e^{-\frac{x}{\sigma}}, \ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

Where α and σ>0 and - ∞ < x < ∞

Where α is a shape parameter and σ is scale parameter.

Here when α=1 it reduces to standard gumble distribution. α

The survival function corresponding with exponentiated-gumble density is given as

S(x, α,σ)=1- $\frac{\alpha}{\sigma}\left(\exp(-e^{\frac{x}{\sigma}})\right)^{\alpha}$ ……………………………….(4.2)

The survival function of the exponentiated gumble distribution is the α th power of the survival function of the gumble distribution.

Maximum Likelihood Estimators

Let $x_1, x_2, x_3 ....x_n$ be a random sample from EG distribution the log likelihood function can be as

L (α, σ)=n lnα- nlnσ-$\frac{1}{\sigma}\sum_{i=1}^{n} x_i - \alpha\sum e^{\frac{-xi}{\sigma}}$ .... ……………..(4.3)

Therefore to obtain the MLE's of α and σ we can directly maximize (3.3) with respect to α and σ or we can solve the non-linear normal equations which are as follows:-

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} = 0 \text{.........} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4.4)}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{\alpha}{\sigma^2}\sum_{i=1}^{n} x_i . e^{\frac{-x_i}{\sigma}} = 0 \text{.....} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(45)}$$

From (4.4) and (4.5) MLE of α, and σ is obtained.

Goodness of Fit

Eight models namely gamma, weibull, lognormal, gumble, exponentiated weibull, exponentiated exponential, exponentiated lognormal and exponentiated gumble have been fitted to two real life data sets. The distributions along with Probability density function are given as under:-

<u>Distribution</u>	<u>P.d. f</u>
Gamma	f(x, α, λ) $= \frac{\lambda^{\alpha}}{\Gamma\alpha} x^{\alpha-1} . e^{-\lambda . x}$; α, λ, x >0
Weibull	f(x, α, λ) = αλ $(x\lambda)^{\alpha-1} . e^{(-\lambda x)^{\alpha}}$; α, λ, x >0
Lognormal	f(x, μ, σ) $= \frac{\exp\left(-\frac{1}{2}\left(\frac{In(x)-\mu}{\sigma}\right)^2\right)}{x.\sigma.\sqrt{2\lambda}}$; $-\infty < \mu < \infty, \sigma > 0$
Gumble	f(x, σ) $= \frac{1}{\sigma} \exp^{\frac{-x}{\sigma}} . \exp\left(-e^{\frac{-x}{\sigma}}\right)$; σ >0

Exponentiated weibull

$$f(t;\alpha,\theta,\sigma) = \frac{\alpha\theta}{\sigma}\left[1-\exp(-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\exp\left[-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\left(\frac{t}{\sigma}\right)^{\alpha-1}, \text{t >o}$$

Exponentiated exponential $f(x,\alpha,\lambda) = \alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1} . e^{-\lambda x}, \quad \alpha,\lambda,x > 0$

Exponentiated lognormal f(x, (α,μ,σ))= $\alpha(\varphi(In(x);\mu,\sigma))^{\alpha-1} . \phi(In(x);\mu,\sigma) . x^{-1}$,
x, α >0, $--\infty < \mu < \infty$

Exponentiated gumble $f(x;\alpha,\sigma)=\frac{\alpha}{\sigma}\left[\exp\left(-e^{-\frac{x}{\sigma}}\right)\right]^{\alpha}.e^{-\frac{x}{\sigma}}$, α, σ >0

For the sake of numerical illustrations, we may consider following two data sets.
Data Set1:-The data regarding failure times of the conditioning system of an aero plane:-

23,261,87,7,120,14,62,47,225,71,246,21,42,20,5, 12,120,11,3, 14, 71,11,14,11,16, 90, 1,16,52,95.

Table1:- Showing Distribution along with MLE'S, Log-likelihood and Anderson's value.

Distribution	MLE'S	Log likelihood	Anderson's Value
Gamma	$\hat{\alpha}$=0.8119, $\hat{\lambda}$=0.0136	-152.167	0.078
Weibull	$\hat{\alpha}$=0.8536, $\hat{\lambda}$=0.0183	-151.9.70	0.069
Lognormal	$\hat{\mu}$=3.358, $\hat{\lambda}$=1.3190	-151.621	0.065
Gumble	$\hat{\alpha}$=27.792, $\hat{\lambda}$=55.106	-151.256	0.062
Exponentiated weibull	$\hat{\alpha}$= 3.824, $\hat{\theta}$= 0.1732 $\hat{\sigma}$=82.235	-149.567	0.057
Exponentiated exponential	$\hat{\alpha}$=0.8093, $\hat{\lambda}$=0.0145	-152.206	0.079
Exponentiated lognormal	$\hat{\alpha}$= 0.1543 $\hat{\mu}$= 3.1353 $\hat{\sigma}$=0.3648	-148.659	0.055
Exponentiated gumble	$\hat{\alpha}$=1.9881, $\hat{\lambda}$=49.0638	-148.537	0.054

Data Set 2:-The data regarding runs scored by a cricketer in 27 innings at national level.

28,20,6,4,23,127,25,45,41,67,68,3,17,2, 105,98,55,68,15,3,42,45,7,20,34,9,6,

Table2:- Showing Distribution along with MLE'S, Log-likelihood and Anderson's value.

Distribution	MLE'S	Log likelihood	Anderson's Value
Gamma	$\hat{\alpha}$=0.7235 , $\hat{\lambda}$=0.0127	-125.654	0.956
Weibull	$\hat{\alpha}$=1.040, $\hat{\lambda}$=36.985	-124.021	0.716
Lognormal	$\hat{\mu}$=3.053, $\hat{\lambda}$=1.174	-125.022	0.916
Gumble	$\hat{\alpha}$= 21.432 , $\hat{\lambda}$=25.944	-124.059	0.702
Exponentiated weibull	$\hat{\alpha}$= 3.521, $\hat{\theta}$= 0.1452 $\hat{\sigma}$=67.235	-125.078	0.928
Exponentiated exponential	$\hat{\alpha}$= 0.8126, $\hat{\lambda}$=0.0153	-125.945	0.959
Exponentiated lognormal	$\hat{\alpha}$= 0.578, $\hat{\mu}$=3.836 $\hat{\sigma}$=0.7834	-125.965	0.961
Exponentiated gumble	$\hat{\alpha}$= 1.873 , $\hat{\lambda}$=45.264	-124.843	0.843

Conclusion

This paper discusses the probability density function of exponentiated weibull, exponentiated exponential, exponentiated lognormal and exponentiated gumble distributions and its application for two data sets. Two parameter exponentiated weibull is suitable to fit unimodel, monotone and risk functions unlike weibull model. Exponentiatedexponential has shape and scale parameter like a gamma and weibull distribution and can be used as an alternative and has better fit than weibull and gamma in many cases. Two parameter exponentiated gumble and three parameter exponentiated lognormal are unimodel distributions and can give better fits. In first data set regarding failure times of the conditioning system of an aeroplane. Exponentiated exponential and gamma gives better fit followed by Weibull. Hence gamma weibull and exponentiatedexponential can be replaced as alternative to each other in certain situation. In second data set regarding runs scored by a cricketer exponentiatedlognormal and exponentiatedexponential gives better fit followed by gamma. Hence exponentiatedlognormal, exponentiatedexponential and gamma can be replaced as an alternative with each other. Since the distribution function of exponentiated lognormal distribution can be expressed in terms of normal distribution therefore inference based on the data can be handled easily and can be given preference. However there are other distributions which one could fit and more improvement and flexibility can be achieved regarding their applications.

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