

Introduction to Digital Systems

Part II (4 lectures)

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Combinational Logic Blocks

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Lecture 7 contents

- Block oriented combinational logic design
- Arithmetic Circuits

Addition

- Addition is a very common arithmetic operation in digital systems
- Let's recall some concepts ...

Addition of Binary Numbers

- Addition and subtraction of non-decimal numbers by hand uses the same technique that you know from school for decimal numbers.
- The only catch is that the addition and subtraction tables are different.
- To add two **unsigned binary numbers** X and Y , we add together the least significant bits with an initial carry (c_{in}) of 0, producing carry (c_{out}) and sum (s) bits according to the table. We continue processing bits from right to left, adding the carry out of each column into the next column's sum.

Example:

$$\begin{array}{r}
 1\ 1\ 0\ 0\ 0\ 0\ 1 \\
 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1 \\
 +\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1 \\
 \hline
 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0
 \end{array}$$

c_{in}	x	y	c_{out}	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Subtraction of Binary Numbers

- Binary subtraction is performed similarly, using borrows (b_{in} and b_{out}) instead of carries between steps, and producing a difference bit d .

b_{in}	x	y	b_{out}	d
0	0	0	0	0
0	0	1	1	1
0	1	0	0	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	1	1

Examples:

$$\begin{array}{r}
 0\ 1\ 1\ 1\ 1\ 0\ 0 \\
 1\ 1\ 1\ 0\ 0\ 0\ 1 \\
 -\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1 \\
 \hline
 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0
 \end{array}$$

$$\begin{array}{r}
 1\ 1\ 1 \\
 1\ 0\ 0\ 0 \\
 -\ 0\ 0\ 1\ 1 \\
 \hline
 0\ 1\ 0\ 1
 \end{array}$$



Overflow

- With n bits it is possible to represent **unsigned integer numbers** ranging from 0 to $2^n - 1$.
- If an arithmetic operation produces a result that exceeds the range of the number system, **overflow** is said to occur.
- Overflows can easily be detected by analyzing a **carry or borrow from the most significant bit**.
 - the carry bit c_{out} or the borrow bit b_{out} out of the MSB = 1

Examples:

$n=8$: [0..255]

$$173_{10} + 97_{10} = 270_{10}$$

$$\begin{array}{r} 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1 \\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1 \\ +\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1 \\ \hline 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0 \end{array}$$

overflow

$n=4$: [0..15]

$$4_{10} - 11_{10} = -7_{10}$$

$$\begin{array}{r} 1\ 0\ 1\ 1 \\ 0\ 1\ 0\ 0 \\ -\ 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 1 \end{array}$$

overflow

Representation of Negative Numbers

- There are many ways to represent negative numbers.
- In everyday business we use the **signed-magnitude system** (i.e. reserve a special symbol to indicate whether a number is negative).
- However, most computers use **two's-complement representation**:
 - The **most significant bit (MSB)** of a number in this system serves as the sign bit; a number is negative if and only if its MSB is 1.
 - The weight of the MSB is negative: for an n -bit number the weight is -2^{n-1} .
 - The decimal equivalent for a two's-complement binary number is computed the same way as for an unsigned number, except that the weight of the MSB is negative:
 - $D = d_{n-1}d_{n-2} \dots d_1d_0 = -2^{n-1} + \sum_{i=0}^{n-2} d_i \times 2^i$

Examples:

$$1010_2 = ???_{10}$$

$$1010_2 = -2^3 + 2^1 = -8 + 2 = -6_{10}$$

$$1111_2 = ???_{10}$$

$$1111_2 = -2^3 + 2^2 + 2^1 + 2^0 = -8 + 4 + 2 + 1 = -1_{10}$$

$$0111_2 = ???_{10}$$

$$0111_2 = 2^2 + 2^1 + 2^0 = 4 + 2 + 1 = 7_{10}$$

Two's Complement Representation

- For n bits, the range of representable numbers is $[-2^{n-1}, 2^{n-1}-1]$.
- For $n=4$, the range is $[-8, 7]$:

0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
-8	1	0	0	0
-7	1	0	0	1
-6	1	0	1	0
-5	1	0	1	1
-4	1	1	0	0
-3	1	1	0	1
-2	1	1	1	0
-1	1	1	1	1

Towards implementation

- The immediate approach
 - Digit-wise addition and carry propagation
 - Iterative hardware
 - Building blocks
 - Half-Adder
 - Full-Adder

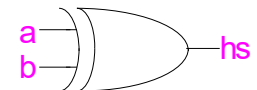
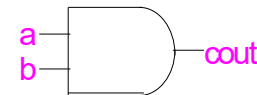
The Half-Adder (HA)

- Inputs: 2 single bit operands (a, b)
- Outputs
 - A 2 bit result:
 - The Half-Sum (hs)
 - The Carry-out (Cout)
 - Note that: $0 \leq (\text{Cout}, \text{hs})_{10} \leq 2$

a	b	c _{out}	hs
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

$$c_{out} = a \cdot b$$

$$hs = a \oplus b$$



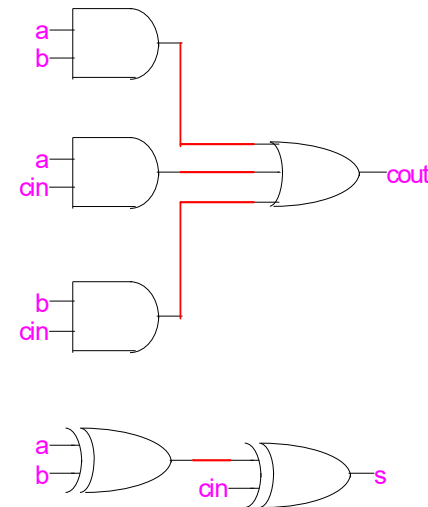
The Full-Adder (FA)

- Inputs: 2 single bit operands (a, b) and a Carry-in bit (C_{in})
- Outputs
 - A 2 bit result:
 - The Sum (S)
 - The Carry-out (C_{out})
 - Note that: $0 \leq (C_{out}, S)_{10} \leq 3$

C_{in}	a	b	C_{out}	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

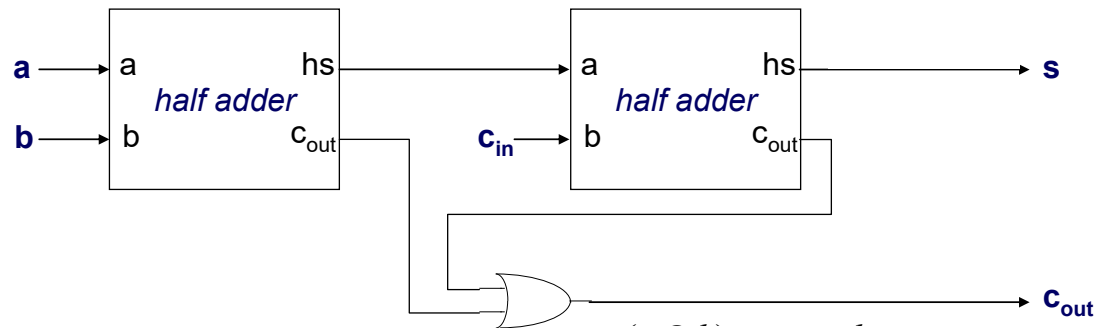
$$c_{out} = a \cdot b + a \cdot c_{in} + b \cdot c_{in}$$

$$s = a \oplus b \oplus c_{in}$$



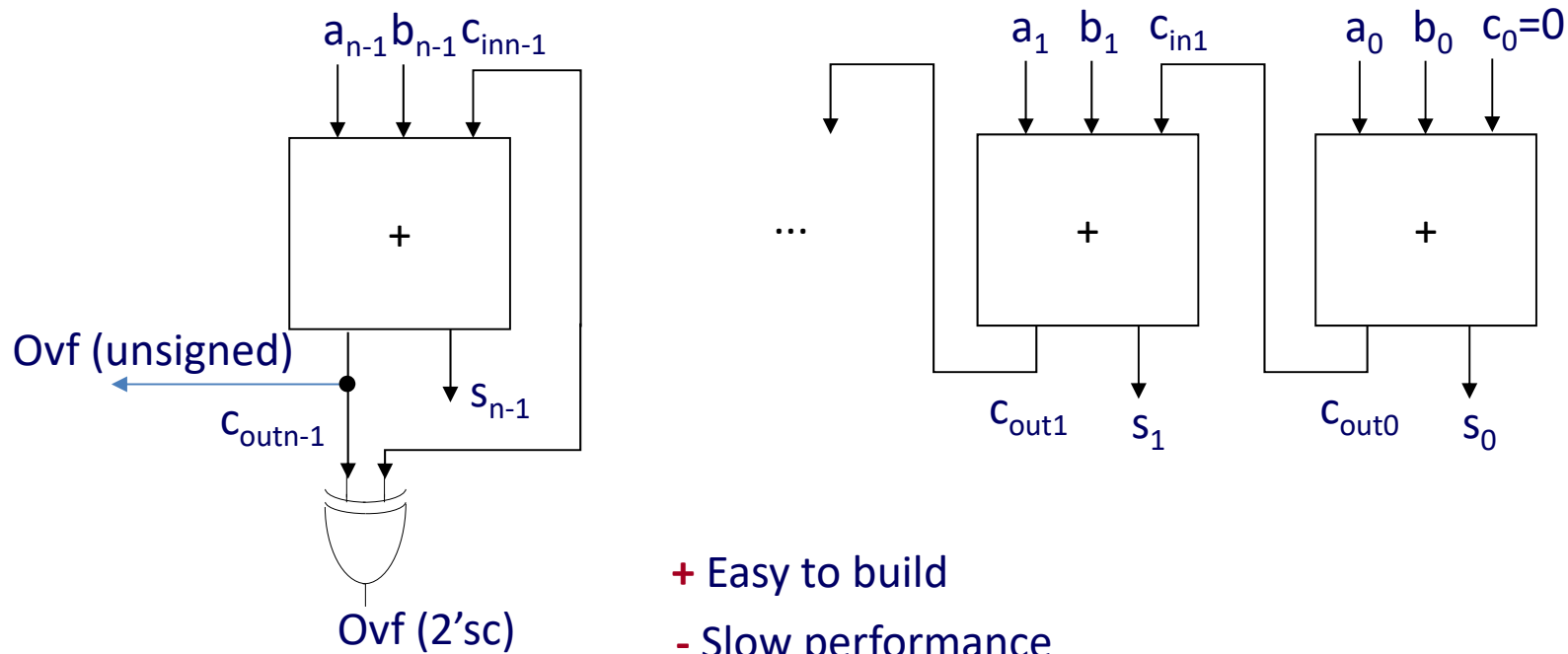
Exercise

- Show that the following circuit is Full-Adder



Ripple Adders

- The bit-wise addition and carry-propagation is implemented by a cascade of Full-Adders
- An iterative circuit paradigm



Full Subtractor

- Inputs: 2 single bit operands (a, b) and a Borrow-in bit (b_{in})

- Outputs

– A 2 bit result:

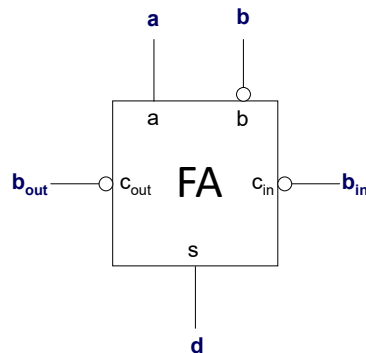
- The difference (d)
- The Borrow-out (b_{out})

b_{in}	a	b	b_{out}	d
0	0	0	0	0
0	0	1	1	1
0	1	0	0	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	1	1

$$b_{out} = \bar{a} \cdot b + \bar{a} \cdot b_{in} + b \cdot b_{in}$$

$$d = a \oplus b \oplus b_{in}$$

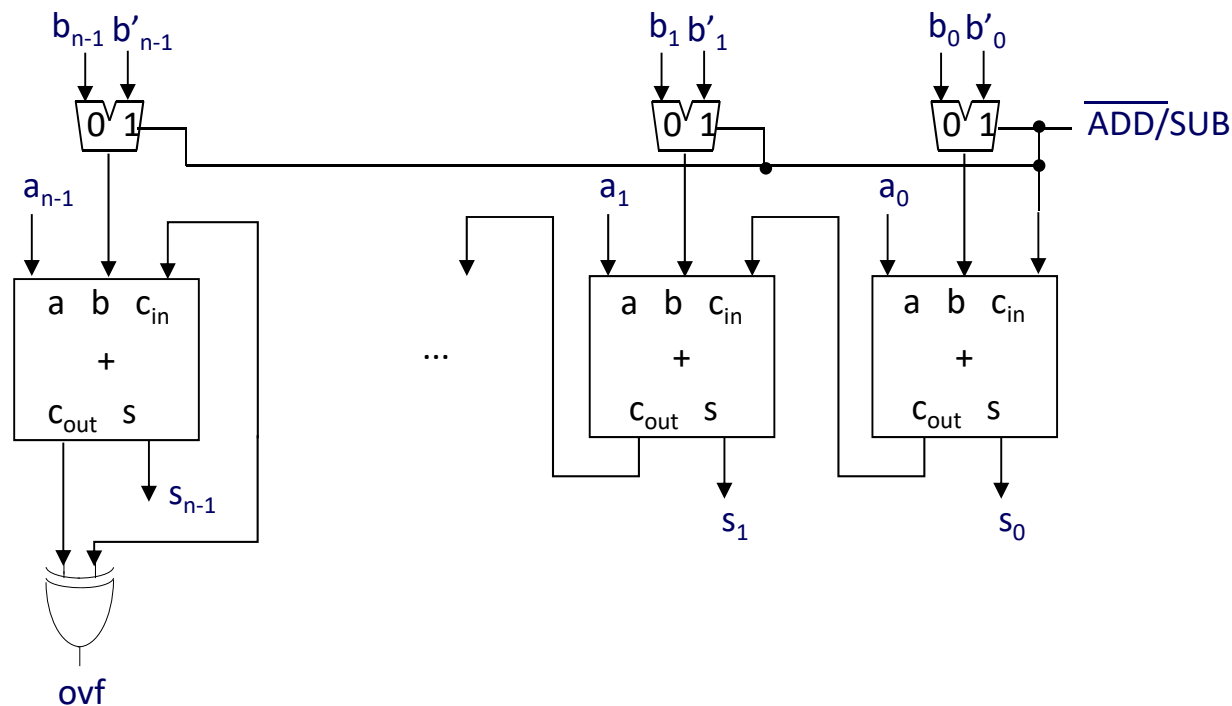
- Exercise



Write the output equations and show that the Full-Adder with modified inputs is a subtractor

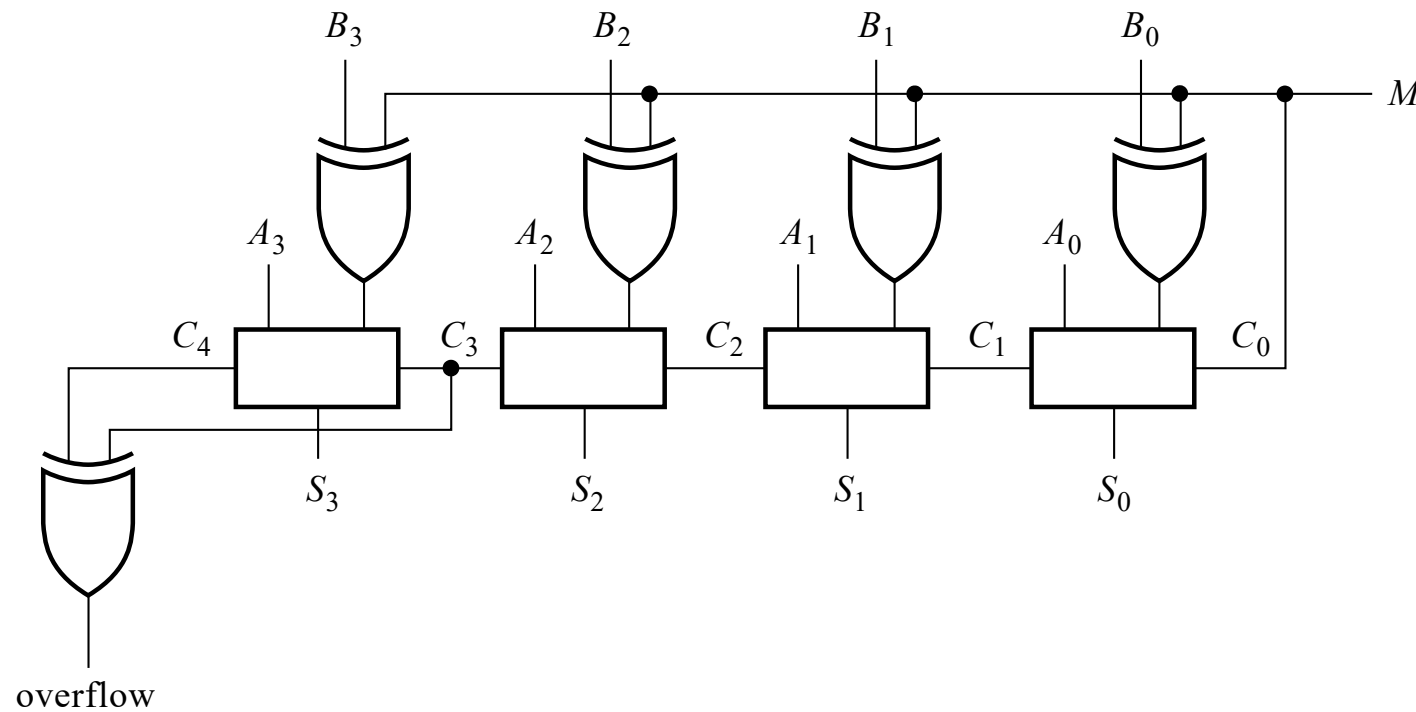
Ripple Adder/Subtractor

- Using 2's complement representation means the same hardware for addition and subtraction
 - “Muxed” b and b' inputs.
 - Initial carry = 1 if subtraction

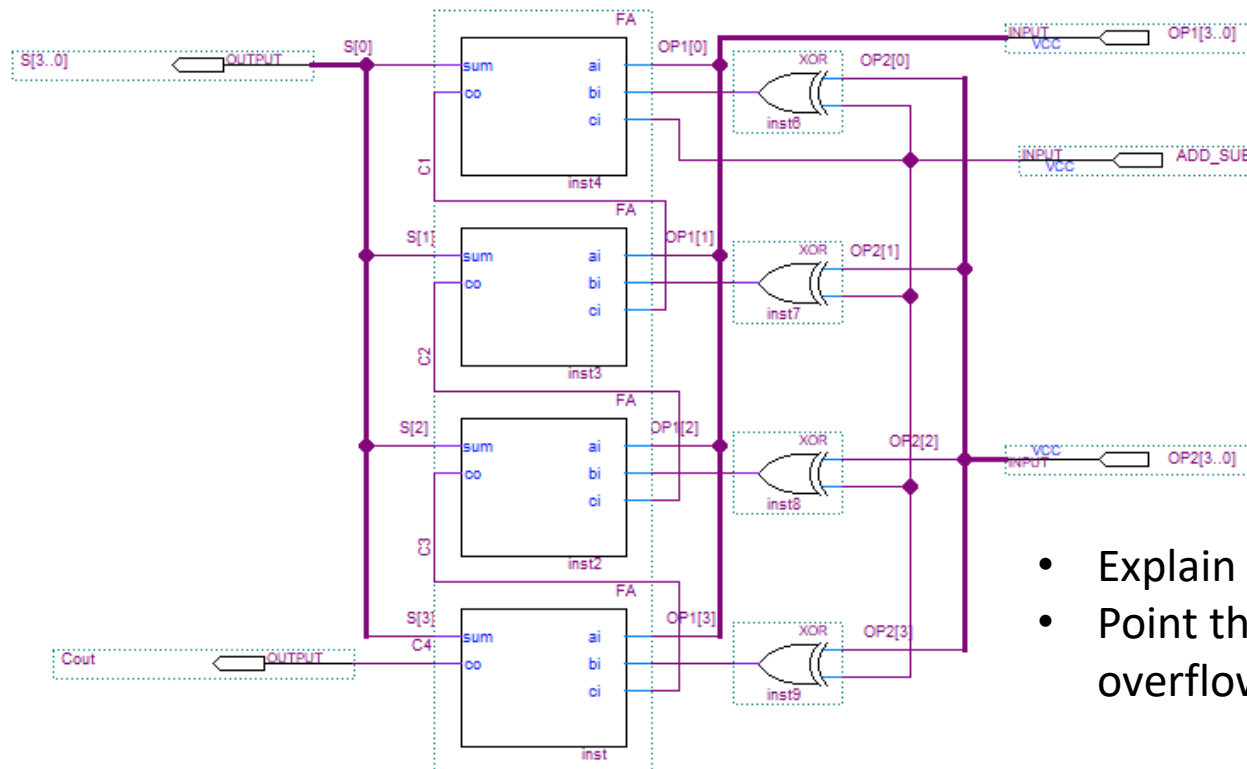


Exercise

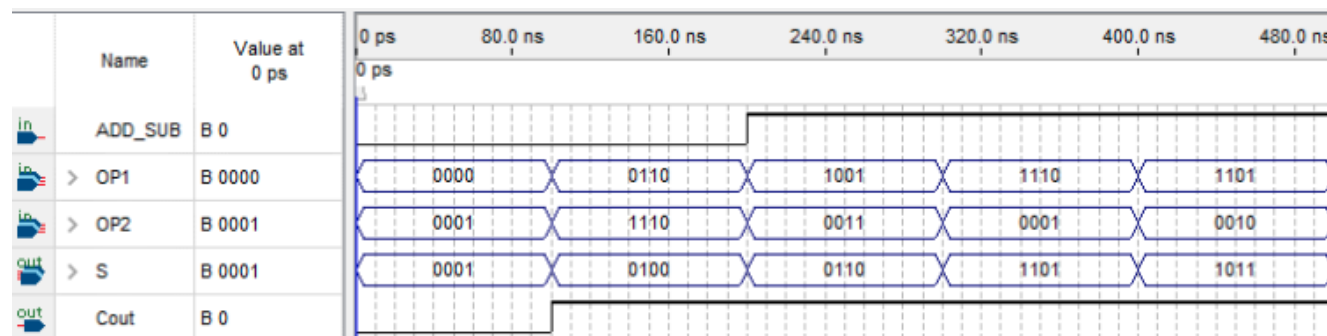
- Verify that the following implementation is equivalent to the adder/subtractor circuit of the previous slide



Exercise



- Explain the timing diagram.
- Point the time intervals where overflow occurs



Carry-Lookahead Adders (CLA)

- The idea: Compute the *ith* carry non-iteratively

- The starting points:

- The usual equations

$$s_i = a_i \oplus b_i \oplus c_i$$

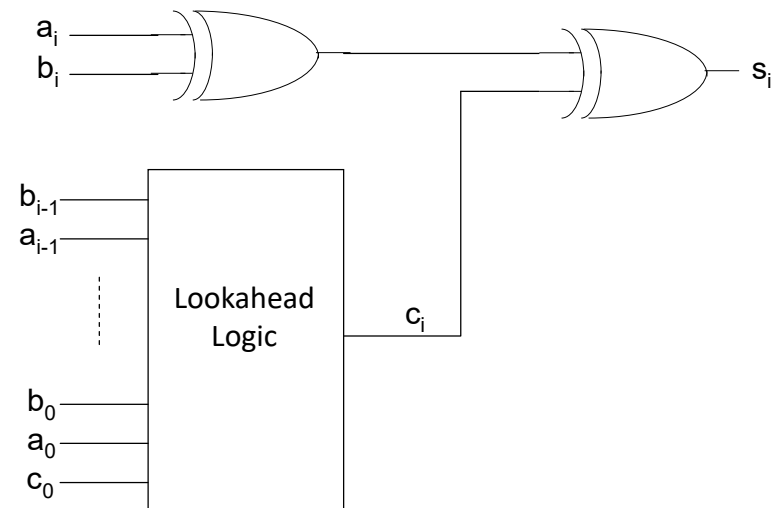
$$c_{i+1} = a_i \cdot b_i + a_i \cdot c_i + b_i \cdot c_i$$

- Intermediate functions

- Carry Generation

- Carry Propagation $g_i = a_i \cdot b_i$

$$p_i = a_i + b_i$$



Remarks about the Carry

- At any stage, we necessarily have a carry generation $C_{i+1} = 1$ whenever $a_i = b_i = 1$.
So $g_i = a_i \cdot b_i$
- If $a_i \neq b_i$ but $a_i + b_i = 1$ the carry propagates since $C_{i+1} = C_i$ so $p_i = a_i + b_i$
- Finally the Carry equation becomes

$$c_{i+1} = g_i + p_i \cdot c_i$$

The 4 bit CLA

- Carry equations

$$c_1 = g_0 + p_0 \cdot c_0$$

$$c_2 = g_1 + p_1 \cdot c_1 = g_1 + p_1 \cdot (g_0 + p_0 \cdot c_0) = g_1 + p_1 \cdot g_0 + p_1 \cdot p_0 \cdot c_0$$

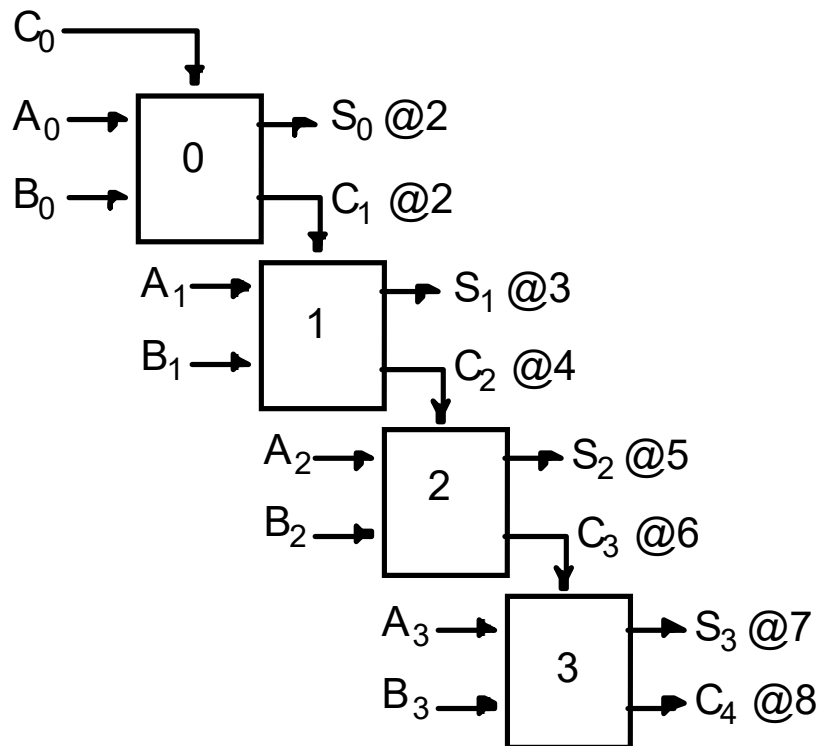
$$\begin{aligned} c_3 &= g_2 + p_2 \cdot c_2 = g_2 + p_2 \cdot (g_1 + p_1 \cdot g_0 + p_1 \cdot p_0 \cdot c_0) = \\ &= g_2 + p_2 \cdot g_1 + p_2 \cdot p_1 \cdot g_0 + p_2 \cdot p_1 \cdot p_0 \cdot c_0 \end{aligned}$$

$$\begin{aligned} c_4 &= g_3 + p_3 \cdot c_3 = g_3 + p_3 \cdot (g_2 + p_2 \cdot g_1 + p_2 \cdot p_1 \cdot g_0 + p_2 \cdot p_1 \cdot p_0 \cdot c_0) = \\ &= g_3 + p_3 \cdot g_2 + p_3 \cdot p_2 \cdot g_1 + p_3 \cdot p_2 \cdot p_1 \cdot g_0 + p_3 \cdot p_2 \cdot p_1 \cdot p_0 \cdot c_0 \end{aligned}$$

- Notice that any Carry is determined after 3 delay levels

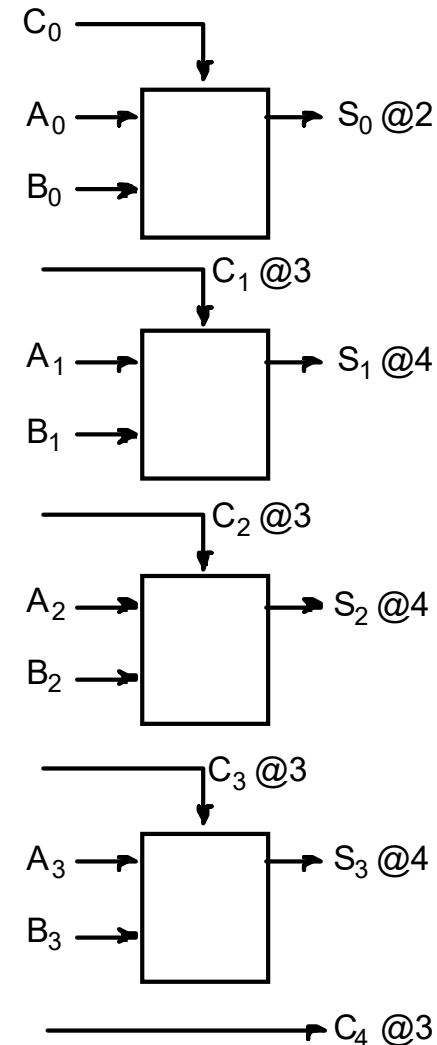
Ripple Adder versus CLA

- 4 bit ripple adder



Final result always after $2 \times 4 = 8$ delays

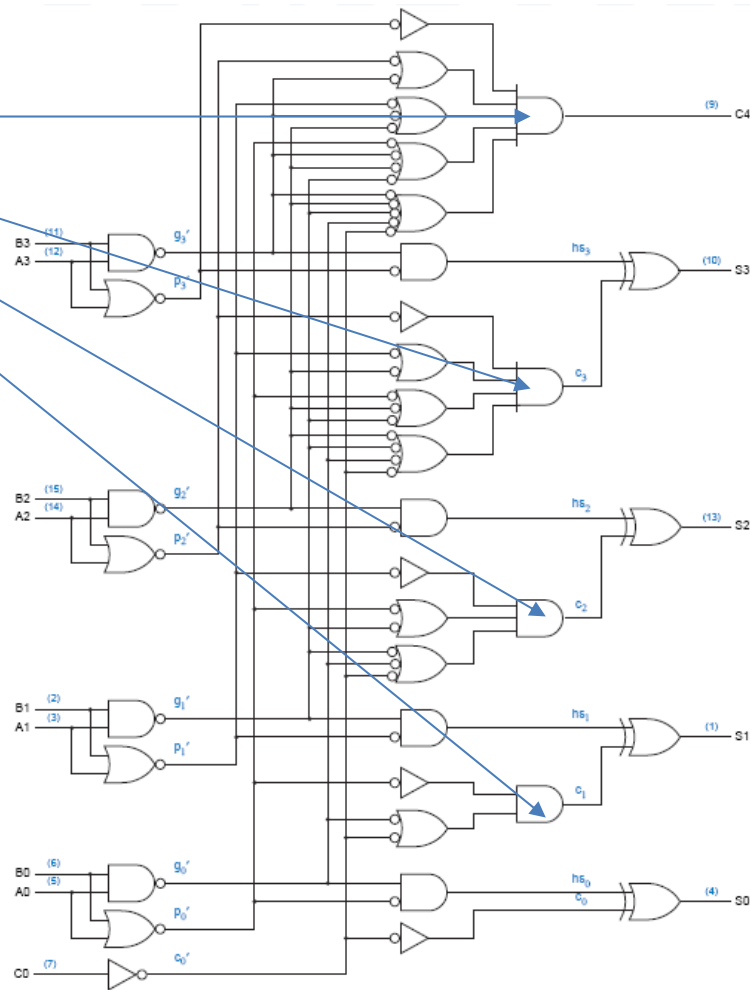
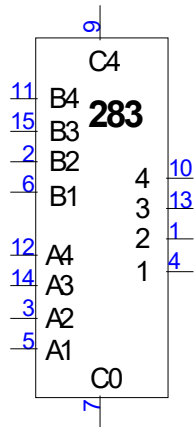
- 4 bit CLA



Final result
always after 4
delays

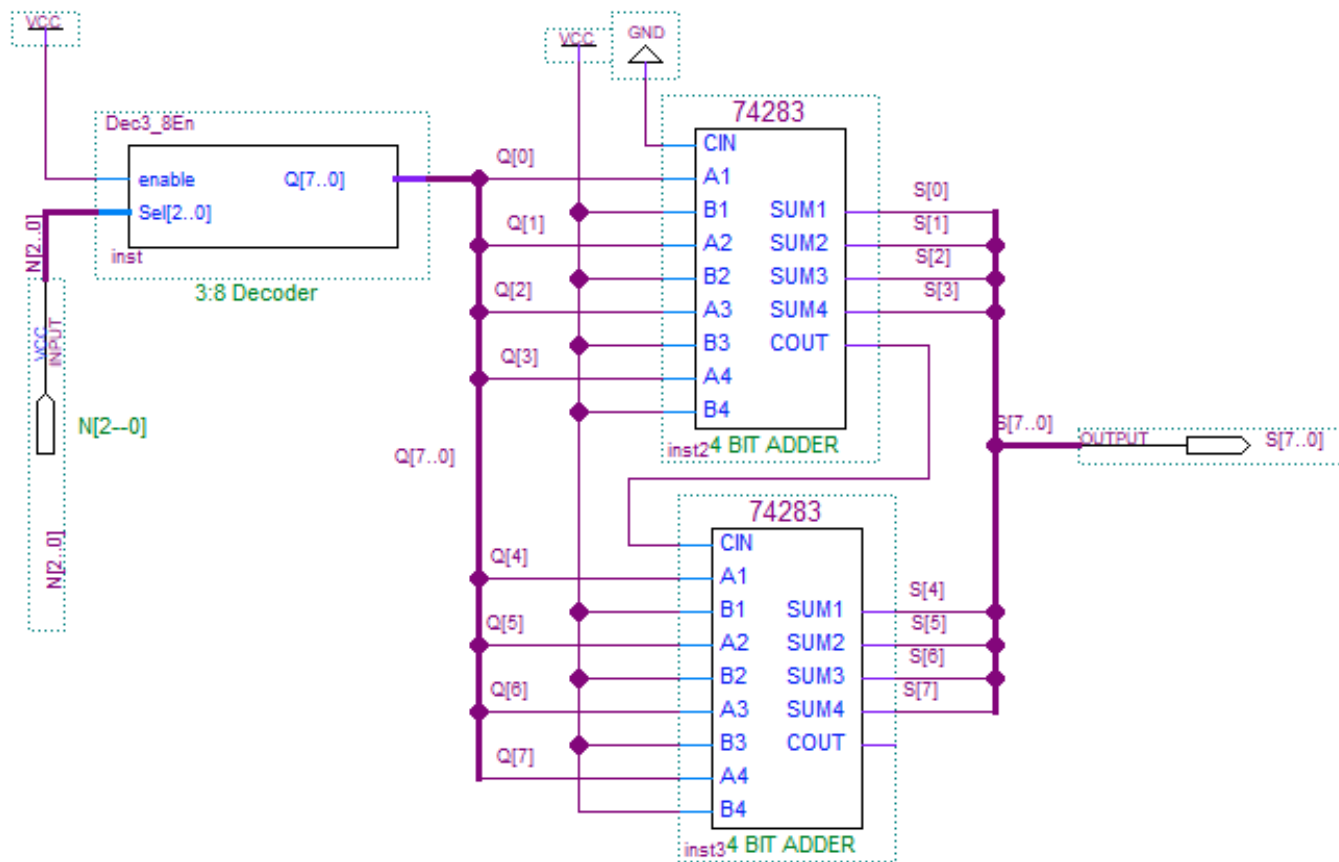
The 74283 model

- CLA logic



Exercise

- In a 2's complement representation, what's the decimal value of the result S with $N = 6_{10}$



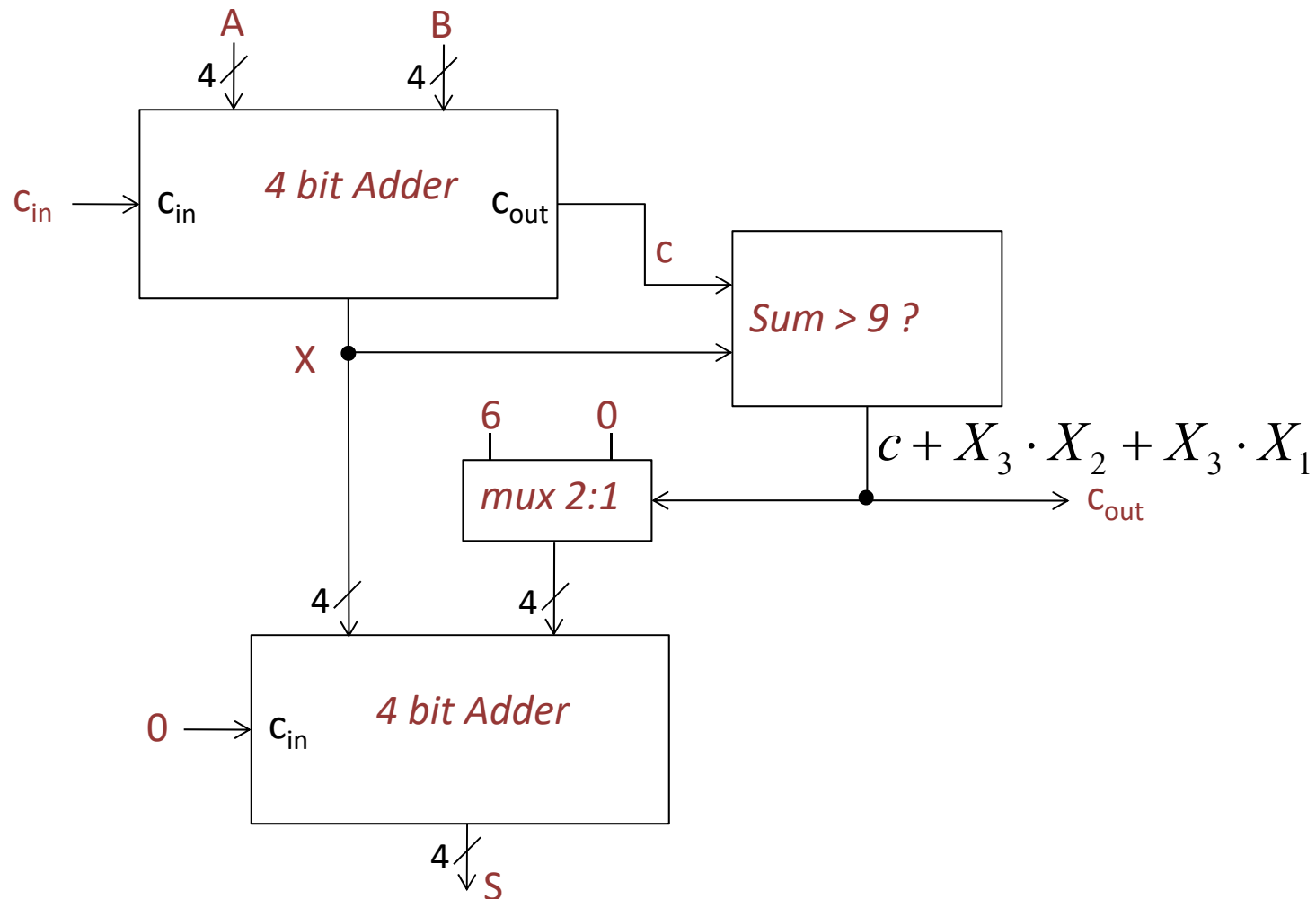
BCD Addition

- Possible results for 2 digit BCD addition with carry

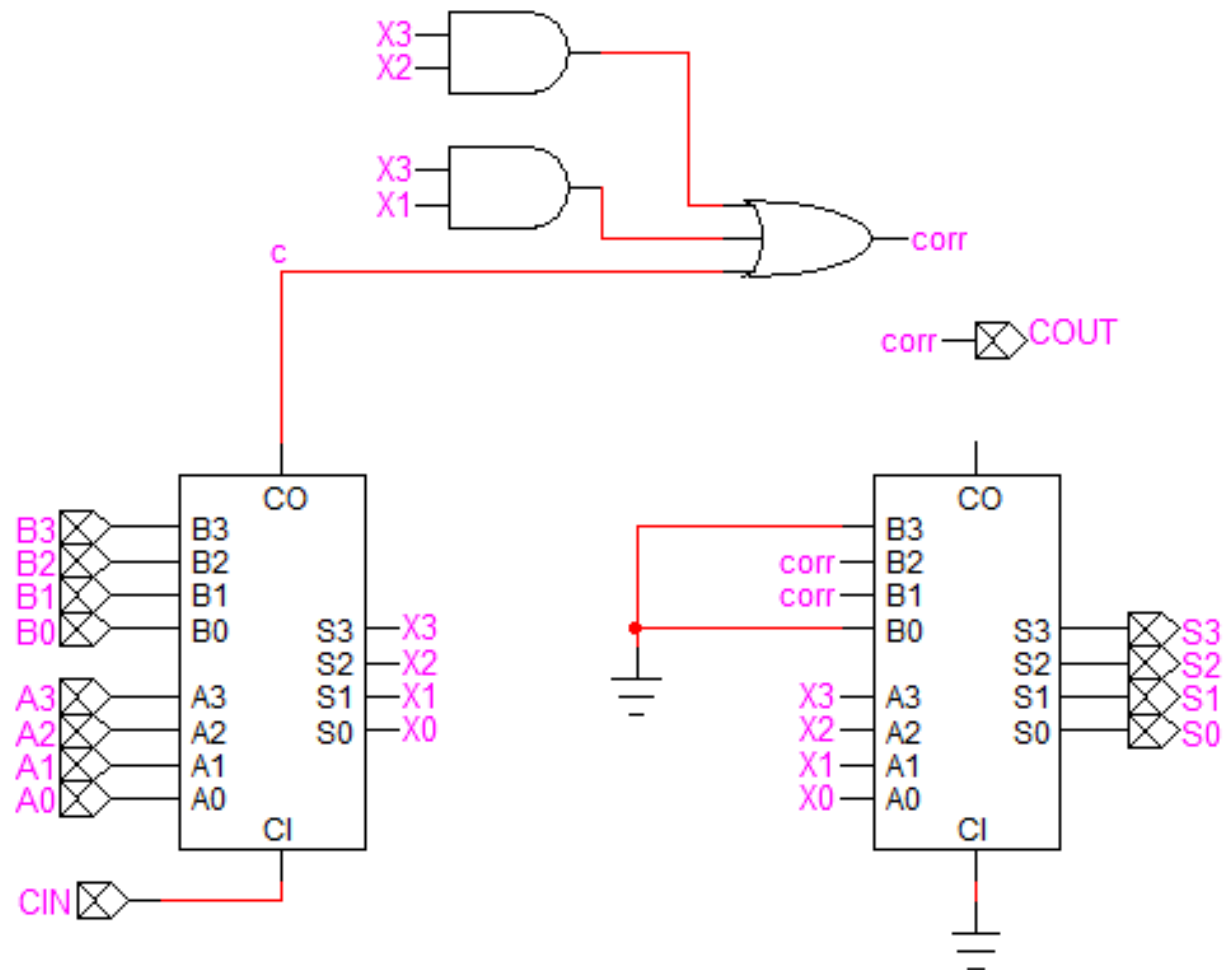
soma	binário		BCD	
	<i>carry out</i>	soma	<i>carry out</i>	soma
0	0	0000	0	0000
1	0	0001	0	0001
2	0	0010	0	0010
3	0	0011	0	0011
4	0	0100	0	0100
5	0	0101	0	0101
6	0	0110	0	0110
7	0	0111	0	0111
8	0	1000	0	1000
9	0	1001	0	1001
10	0	1010	1	0000
11	0	1011	1	0001
12	0	1100	1	0010
13	0	1101	1	0011
14	0	1110	1	0100
15	0	1111	1	0101
16	1	0000	1	0110
17	1	0001	1	0111
18	1	0010	1	1000
19	1	0011	1	1001

Offset correction required:
add 6 to the binary result

BCD addition algorithm



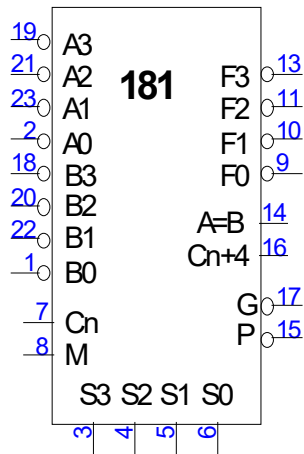
Possible implementation



ALU

- An arithmetic logic unit is a combinational block that executes any logical or arithmetic operation over a pair of b bits operands.
- There is a mode input that chooses between the logical or arithmetic behavior
- There is a op-code set of inputs that choose a particular operation from a limited “operation” set.

The 74181 ALU



S3	S2	S1	S0	M=0 (op. aritm.)	M=1 (op. lógica)
0	0	0	0	$F = A - 1 + \text{CIN}$	$F = \bar{A}$
0	0	0	1	$F = A \text{ AND } B - 1 + \text{CIN}$	$F = \bar{A} \text{ OR } \bar{B}$
0	0	1	0	$F = A \text{ AND } \bar{B} - 1 + \text{CIN}$	$F = \bar{A} \text{ OR } B$
0	0	1	1	$F = 1111 + \text{CIN}$	$F = 1111$
0	1	0	0	$F = A + (A \text{ OR } \bar{B}) + \text{CIN}$	$F = \bar{A} \text{ AND } \bar{B}$
0	1	0	1	$F = A \text{ AND } B + (A \text{ OR } \bar{B}) + \text{CIN}$	$F = \bar{B}$
0	1	1	0	$F = A - B - 1 + \text{CIN}$	$F = A \text{ XOR } \bar{B}$
0	1	1	1	$F = A \text{ OR } \bar{B} + \text{CIN}$	$F = A \text{ OR } \bar{B}$
1	0	0	0	$F = A + (A \text{ OR } B) + \text{CIN}$	$F = \bar{A} \text{ AND } B$
1	0	0	1	$F = A + B + \text{CIN}$	$F = A \text{ XOR } B$
1	0	1	0	$F = A \text{ AND } \bar{B} + (A \text{ OR } B) + \text{CIN}$	$F = B$
1	0	1	1	$F = A \text{ OR } B + \text{CIN}$	$F = A \text{ OR } B$
1	1	0	0	$F = A + A + \text{CIN}$	$F = 0000$
1	1	0	1	$F = A \text{ AND } B + A + \text{CIN}$	$F = A \text{ AND } \bar{B}$
1	1	1	0	$F = A \text{ AND } \bar{B} + A + \text{CIN}$	$F = A \text{ AND } B$
1	1	1	1	$F = A + \text{CIN}$	$F = A$

Unsigned Multiplication

- We follow the same rules of the decimal system

$$12 \times 13 = 156$$

				1	1	0	0	← multiplicand
				×	1	1	0	1 ← multiplier
<hr/>								
					1	1	0	0
					0	0	0	0
			1	1	0	0		
	1	1	0	0				
<hr/>								
1	0	0	1	1	1	0	0	Final product

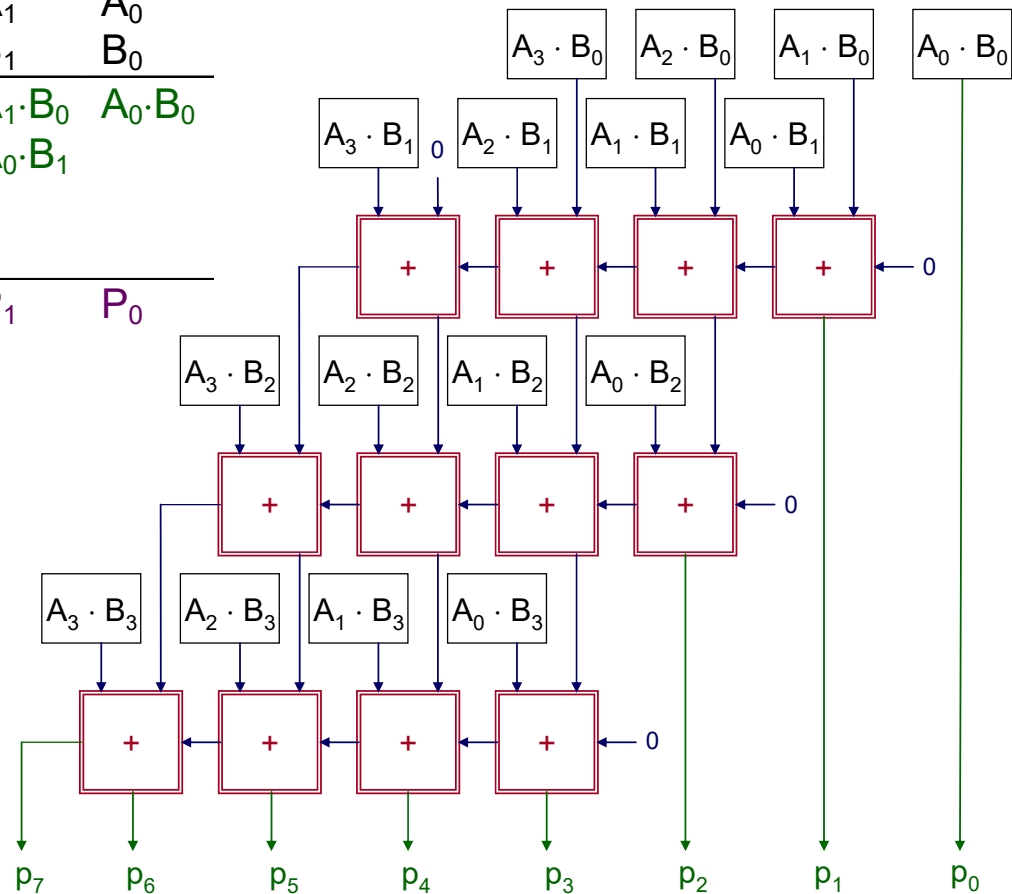
Combinational Multipliers

		A_3	A_2	A_1	A_0
\times		B_3	B_2	B_1	B_0
		$A_3 \cdot B_0$	$A_2 \cdot B_0$	$A_1 \cdot B_0$	$A_0 \cdot B_0$
	$A_3 \cdot B_1$	$A_2 \cdot B_1$	$A_1 \cdot B_1$	$A_0 \cdot B_1$	
	$A_3 \cdot B_2$	$A_2 \cdot B_2$	$A_1 \cdot B_2$	$A_0 \cdot B_2$	
	$A_3 \cdot B_3$	$A_2 \cdot B_3$	$A_1 \cdot B_3$	$A_0 \cdot B_3$	
		P_3	P_2	P_1	P_0

		1	1	0	0
\times		1	1	0	1
		1	1	0	0
	0	0	0	0	0
	1	1	0	0	
	1	1	0	0	
	1	0	0	1	1
	1	0	0	1	1
	1	0	0	1	1
	1	0	0	1	1

16 \times AND-2

12 adders



Final Remarks

- Always recall
 - The block symbol
 - The types of inputs (operands) and outputs
 - Distinguish between iterative and non-iterative solutions
- Design with encapsulated logic requires mastering all the functional details of each block