Trabalho CDI2 Séries

1) a) $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$

$= \frac{\ln(1)}{1} + \frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln(4)}{4} + \dots$

$= 0 + \frac{1}{2}\ln(2) + \frac{1}{3}\ln(3) + \frac{1}{4}\ln(4) + \dots$

$f(x) = \frac{\ln(x)}{x}$, decrescente para $x \geq 1$

$\int_1^{\infty} \frac{\ln(x)}{x} dx$

$\lim_{n \to \infty} \int_1^n \frac{\ln(x)}{x} dx$

$u = \ln(x) \quad du = \frac{1}{x}$

$\lim_{n \to \infty} \int_1^n u \, du$

$\lim_{n \to \infty} \left(\frac{u^2}{2}\right)\Big|_1^n$

$\lim_{n \to \infty} \frac{\ln(x)^2}{2}\Big|_1^n$

$\lim_{n \to \infty} \frac{\ln(n)^2}{2} - \cancel{\frac{\ln(1)^2}{2}}$

$\lim_{n \to \infty} \frac{\ln(n)^2}{2} = \infty \to$ Diverge

Se a integral diverge então a série também//

b) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

$= \frac{1}{1} + \frac{2}{4} + \frac{6}{9} + \frac{24}{256} + \dots$

Pelo teste da razão:

$p = \lim_{k \to} \left|\frac{a_{k+1}}{a_k}\right|$

$p < 1 \to$ converge, $p > 1 \to$ diverge, $p = 1 \to$ inconclusivo

$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}}\right| = \left|\frac{\cancel{(k+1)} \cdot \cancel{k!}}{\cancel{(k+1)} \cdot (k+1)^k} \cdot \frac{k^k}{\cancel{k!}}\right| = \left|\frac{k^k}{(k+1)^k}\right|$

$p = \lim_{k \to \infty} \left|\frac{k^k}{(k+1)^k}\right| = \frac{\infty}{\infty} \to$ transformar

$= \lim_{k \to \infty} \left|\left(\frac{k}{(k+1)}\right)^k\right| \quad t = k+1$

$= \lim_{t \to \infty} \left|\left(\frac{t-1}{t}\right)^{t-1}\right|$

$= \lim_{t \to \infty} \left|\left(\frac{t}{t} - \frac{1}{t}\right)^{t-1}\right|$

$= \lim_{t \to \infty} \left|\left(1 - \frac{1}{t}\right)^{t \cdot t-1/t}\right|$

$= \lim_{t \to \infty} \left|\left(\left(1 - \frac{1}{t}\right)^t\right)^{\frac{t-1}{t}}\right|$

$= \lim_{t \to \infty} \left|\left(\left(1 - \frac{1}{t}\right)^t\right)^{\frac{t}{t} - \frac{1}{t}}\right|$

$= \lim_{t \to} \left|\left(\left(1 - \frac{1}{t}\right)^t\right)^{1 - \frac{1}{t}}\right| \qquad \lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = e^{-1}$

$= \left|\left(e^{-1}\right)^1\right| = \frac{1}{e}$, $p < 1$ logo a expressão converge

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2) a) $\sum_{k=0}^{\infty} 2\cdot\frac{3^k}{5^k} + 7\cdot\frac{4^k}{10^k}$

Pelo teste da razão $\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$

$\rho < 1 \rightarrow$ converge ; $\rho > 1 \rightarrow$ diverge

$$\rho = \lim_{k\to\infty} \left|\frac{2\cdot\frac{3^{k+1}}{5^{k+1}} + 7\cdot\frac{4^{k+1}}{10^{k+1}}}{2\cdot\frac{3^k}{5^k} + 7\cdot\frac{4^{k+1}}{10^{k+1}}}\right|$$

como $k \geq 0$, podemos tirar o absoluto

$$\left(\frac{2\cdot\frac{3^{k+1}}{5^{k+1}} + 7\cdot\frac{4^{k+1}}{10^{k+1}}}{2\cdot\frac{3^k}{5^k} + 7\cdot\frac{4^k}{10^k}}\right)$$

$$\left(\frac{2\cdot\frac{3^{k+1}}{5^{k+1}} + 7\cdot\frac{\cancel{4}^{k+1}}{2^{k+1}\cdot 5^{k+1}}}{2\cdot\frac{3^k}{5^k} + 7\cdot\frac{\cancel{4}^k}{2^k\cdot 5^k}}\right)$$

$$\left(\frac{\frac{2\cdot 3^{k+1}}{5^{k+1}} + \frac{7\cdot 2^{k+1}}{5^{k+1}}}{\frac{2\cdot 3^k}{5^k} + \frac{7\cdot 2^k}{5^k}}\right)$$

$$\left(\frac{2\cdot 3^{k+1} + 7\cdot 2^{k+1}}{5^{k+1}} \cdot \frac{5^k}{2\cdot 3^k + 7\cdot 2^k}\right)$$

$$\left(\frac{6\cdot 3^k + 14\cdot 2^k}{5\cdot \cancel{5^k}} \cdot \frac{\cancel{5^k}}{2\cdot 3^k + 7\cdot 2^k}\right)$$

$$\left(\frac{6\cdot 3^k + 14\cdot 2^k}{10\cdot 3^k + 35\cdot 2^k}\right)$$

$$\left(\frac{3^k\cdot\frac{6\cdot\cancel{3^k}}{\cancel{3^k}} + 3^k\cdot\frac{14\cdot 2^k}{3^k}}{3^k\cdot\frac{10\cdot\cancel{3^k}}{\cancel{3^k}} + 3^k\cdot\frac{35\cdot 2^k}{3^k}}\right)$$

$$\left(\frac{\cancel{3^k}\cdot\left(6 + 14\cdot\left(\frac{2}{3}\right)^k\right)}{\cancel{3^k}\cdot\left(10 + 35\cdot\left(\frac{2}{3}\right)^k\right)}\right)$$

$$\rho = \lim_{k\to\infty}\left(\frac{6 + 14\cdot\left(\frac{2}{3}\right)^k}{10 + 35\cdot\left(\frac{2}{3}\right)^k}\right) = \frac{6 + 14\cdot 0}{10 + 35\cdot 0} = \frac{6}{10} < 1$$

Como $\rho < 1$, a série converge //

$$\sum_{k=0}^{\infty} 2\cdot\frac{3^k}{5^k} + 7\cdot\frac{4^k}{10^k}$$

$$= \sum_{k=0}^{\infty} 2\cdot\left(\frac{3}{5}\right)^k + 7\cdot\left(\frac{4}{10}\right)^k$$

$$= \sum_{k=0}^{\infty} 2\cdot\left(\frac{3}{5}\right)^k + \sum_{k=0}^{\infty} 7\cdot\left(\frac{4}{10}\right)^k$$

$$= 2\cdot\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k + 7\cdot\sum_{k=0}^{\infty}\left(\frac{4}{10}\right)^k$$

$\frac{3}{5} < 1$ e $\frac{4}{10} < 1$

$$= 2\cdot\left(\frac{1}{1-(3/5)}\right) + 7\cdot\left(\frac{1}{1-(4/10)}\right)$$

$$= 2\cdot\frac{5}{2} + 7\cdot\frac{10}{6} = 5 + \frac{70}{6} = 5 + \frac{35}{3} = \frac{50}{3}$$

$\lim_{k \to \infty} \left| -\frac{5}{4k^2+6k+2} \right|$

2) b) $\sum_{k=0}^{\infty} \frac{(-5)^k}{(2k)!}$

Teste da razão

$$p = \lim_{k \to \infty} \left| \frac{\frac{(-5)^{k+1}}{(2\cdot(k+1))!}}{\frac{(-5)^k}{(2k)!}} \right|$$

$$\left(\frac{(-5)^{k+1}}{(2\cdot(k+1))!} \cdot \frac{(2k)!}{(-5)^k} \right)$$

$$\left(\frac{\cancel{(-5)^k} \cdot (-5)}{(2k+2)!} \cdot \frac{(2k)!}{\cancel{(-5)^k}} \right)$$

$$\left(\frac{-5 \cdot \cancel{(2k)!}}{(2k+2)\cdot(2k+1)\cdot\cancel{(2k)!}} \right)$$

$$\left(-\frac{5}{4k^2+6k+2} \right)$$

$p = \lim_{k \to \infty} \left| -\frac{5}{4k^2+6k+2} \right| = 0 < 1$, logo converge //

3) $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k^2 5^k}$

Teste da razão

$p = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$, $p < 1 \to$ converge

$p = \lim_{k\to\infty} \left|\frac{\frac{(x-3)^{k+1}}{(k+1)^2 5^{k+1}}}{\frac{(x-3)^k}{k^2 5^k}}\right|$

$\left|\frac{(x-3)^{k+1}}{(k+1)^2 5^{k+1}} \cdot \frac{k^2 \cdot 5^k}{(x-3)^k}\right|$

$\left|\frac{(x-3)\cdot \cancel{(x-3)^k}}{(k+1)^2 \cdot 5 \cdot \cancel{5^k}} \cdot \frac{k^2 \cdot \cancel{5^k}}{\cancel{(x-3)^k}}\right|$

$\left|\frac{(x-3)\cdot k^2}{(k+1)^2 \cdot 5}\right| = \left|\frac{(x-3)}{5} \cdot \left(\frac{k}{(k+1)}\right)^2\right|$

$= \lim_{k\to\infty} \left|\frac{(x-3)}{5}\right| \cdot \left|\left(\frac{k}{k+1}\right)^2\right|$

$= \left|\frac{(x-3)}{5}\right| \cdot \lim_{k\to\infty} \left|\left(\frac{k}{k\cdot(1+\frac{1}{k})}\right)^2\right|$

$= \left|\frac{(x-3)}{5}\right| \cdot \lim_{k\to\infty} \left|\left(\frac{1}{1+\frac{1}{k}}\right)^2\right|$

$p = \left|\frac{(x-3)}{5}\right| \cdot |1^2|$, para ser convergente $p < 1$, logo

$\left|\frac{(x-3)}{5}\right| < 1 \to \frac{|x-3|}{|5|} < 1 \to \frac{|x-3|}{5} < 1 \to |x-3| < 5$ //

Sendo assim o raio de convergencia é 5 e o intervalo é]-2, 8[

4) a) $f(x) = \frac{x}{4+x^2}$

$f(0) = \frac{0}{4+0} = 0$

$f'(x) = \frac{4-x^2}{(4+x^2)^2}$

$f'(0) = \frac{4-0}{(4+0)^2} = \frac{4}{16} = \frac{1}{4}$

$f''(x) = x \cdot \left(\frac{8x^2}{(4+x^2)^3} - \frac{2}{(4+x^2)^2} \right) - \frac{4x}{(4+x^2)^2}$

$f''(0) = 0 - \frac{4 \cdot 0}{(4+0)^2} = 0$

$f'''(x) = 3 \cdot \left(\frac{8x^2}{(4+x^2)^3} - \frac{2}{(4+x^2)^2} \right) + x \cdot \left(\frac{24x}{(4+x^2)^3} - \frac{48x^3}{(4+x^2)^4} \right)$

$f'''(0) = 3 \cdot \left(0 - \frac{2}{4^2} \right) + 0 = -\frac{3}{8}$

$\sum_{n=0}^{\infty} \frac{f(0)}{n!} x^n = \frac{0}{0!} \cdot x_0 + \frac{\frac{1}{4}}{1!} \cdot x_1 + \frac{0}{2!} \cdot x_2 + \frac{-\frac{3}{8}}{3!} \cdot x_3 + \dots$

$= 0 + \frac{1}{4} \cdot x^1 + 0 + \left(-\frac{1}{16}\right) x^3 + \dots$

$= \frac{x}{4} - \frac{x^3}{16} + \dots$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n-1}}{4^n}, \quad |x| < 2$

4b) $\int \frac{\cos(x) - 1}{x} \, dx$

$\int \frac{1}{x} \cdot \underbrace{\cos(x) - \cancel{1}}_{\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} \, dx$

$\int \frac{1}{x} \cdot \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right)$

$\int -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots \, dx$

$\int -\frac{x}{2!} + \int \frac{x^3}{4!} - \int \frac{x^5}{6!} + \cdots$

$-\frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \frac{x^6}{6 \cdot 6!} + \cdots$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n) \cdot (2n)!}$