Prova III

CDI 2001 - B

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① Verifique se o limite $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^3+y^2}$. Em caso afirmativo, calcule o s valor e prove-o.

- $\gamma(t) = (t,t)$, $t \to 0$, $\gamma(t) \to (0,0)$

$$\lim_{t\to 0} \frac{t.t}{t^3+t^2} = \lim_{t\to 0} \frac{t^2}{t^2(t+1)} = \lim_{t\to 0} \frac{1}{t+1} = \boxed{1} \;//$$

- $\mu(t) = (t^2, t)$, $t \to 0$, $\mu(t) \to (0,0)$

$$\lim_{t\to 0} \frac{t^2.t}{t^6+t^2} = \lim_{t\to 0} \frac{t^3}{t^2(t^3+1)} = \lim_{t\to 0} \frac{t}{t^3+1} = \boxed{0} \;//$$

Como $\lim\limits_{t\to 0} f(\gamma(t)) \neq \lim\limits_{t\to 0} f(\mu(t))$,

então $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^3+y^2}$ $\boxed{\text{não existe!}}$

② Encontre os pontos críticos de $f(x,y) = yx^3 + x^2y^2 + 8y$ e classifique-os

$$\frac{df}{dx} = 3x^2y + 2xy^2$$

$$\frac{df}{dy} = x^3 + 2yx^2 + 8$$

$$\frac{df^2}{dx^2} = 6yx + 2y^2$$

$$\frac{df^2}{dy^2} = 2x^2$$

$$3x^2y + 2xy^2 = 0$$

$$y(3x^2 + 2xy) = 0$$

1) $\boxed{y = 0}$

$2xy = -3x^2$

2) $\boxed{y = -\frac{3}{2}x}$

substituir em $x^3 + 2y x^2 + 8 = 0$

$$x^3 + 8 = 0$$

$$x^3 = -8$$

$$x = \sqrt[3]{-8}$$

$$\boxed{x = -2}$$

$$x^3 + 2 \cdot -\frac{3}{2}x \cdot x^2 + 8 = 0$$

$$x^3 - 3x^3 + 8 = 0$$

$$-2x^3 + 8 = 0$$

$$x^3 = 4$$

$$\boxed{x = \sqrt[3]{4}}$$

$\longrightarrow$ como temos $y = -\frac{3}{2}x$, então $\boxed{y = -\frac{3}{2}\sqrt[3]{4}}$

Assim, obtem-se os pontos:

$$(-2, 0) \text{ e } \left(\sqrt[3]{4}, -\frac{3}{2}\sqrt[3]{4}\right)$$

temos: $\boxed{\dfrac{df^2}{dx} = 6xy + 2y^2}$

e

$$H(x,y) = (6xy+2y^2)\,2x^2 - (3x^2+4xy)^2$$

$$\left\{ \frac{df^2}{dxdy} = 3x^2+4xy \right\}$$

$$H(x,y) = 12x^3y + 4x^2y^2 - (9x^4 + 24x^3y + 16x^2y^2)$$

$$\boxed{H(x,y) = -12x^3y - 12x^2y^2 - 9x^4}$$

1) Para o ponto $(-2,0)$

$$H(-2,0) = -9 \cdot 16$$

$$H(-2,0) = -144$$

como $H<0$,

$(-2,0)$ é um ponto de sela!

2) Para o ponto $\left(\sqrt[3]{4}, -\frac{3}{2}\sqrt[3]{4}\right)$

$$H\left(\sqrt[3]{4}, -\tfrac{3}{2}\sqrt[3]{4}\right) = -12 \cdot 4 \cdot -\tfrac{3}{2}\sqrt[3]{4} - 12 \cdot 4^{\frac{2}{3}} \cdot \left(-\tfrac{3}{2}\sqrt[3]{4}\right)^2 - 9 \cdot 4^{4/3}$$

$$H\left(\sqrt[3]{4}, -\tfrac{3}{2}\sqrt[3]{4}\right) = 72 \cdot 4^{\frac{1}{3}} - 9 \cdot 4^{\frac{4}{3}} - 54 \cdot 4^{\frac{4}{3}}$$

$$H\left(\sqrt[3]{4}, -\tfrac{3}{2}\sqrt[3]{4}\right) = -114{,}29$$

como $H<0$,

$\left(\sqrt[3]{4}, -\frac{3}{2}\sqrt[3]{4}\right)$ é um ponto de sela!

3) Utilize a diferencial total para calcular um valor aproximado do módulo do vetor $(6,1 \,;\, 4,8 \,;\, 4 \,;\, 2,1)$

$$f(x,y,z,w) = \sqrt{x^2+y^2+z^2+w^2}$$

$$v = (x_0, y_0, z_0, w_0) = (6,5,4,2)$$

$$\Delta v = (\Delta x, \Delta y, \Delta z, \Delta w) = (0,1 \,;\, -0,2 \,;\, 0 \,;\, 0,1)$$

$$f(v) = \sqrt{6^2+5^2+2^2+4^2} = \sqrt{81} = 9$$

$$f(v+\Delta v) = f(v) + \nabla f(v)\,\Delta v$$

$$\left\{ \nabla f(v) = \left(\frac{df}{dx}(v) \,;\, \frac{df}{dy}(v) \,;\, \frac{df}{dz}(v) \,;\, \frac{df}{dw}(v)\right) \right.$$

$$\frac{df}{dx} = \frac{x}{f(x,y,z,w)} \qquad \frac{df}{dy} = \frac{y}{f(x,y,z,w)}$$

$$\frac{df}{dz} = \frac{z}{f(x,y,z,w)} \qquad \frac{df}{dw} = \frac{w}{f(x,y,z,w)}$$

$$f(v+\Delta v) = 9 + \left(\frac{6}{9}\cdot 0{,}1 + \frac{5}{9}\cdot -0{,}2 + \frac{4}{9}\cdot 0 + \frac{2}{9}\cdot 0{,}1\right) = 9 - \frac{0{,}2}{9} = 8{,}978$$

$$\boxed{\text{Então, } \|(6,1 \,;\, 4,8 \,;\, 4 \,;\, 2,1)\| \approx 8{,}978}$$

④ Sendo: 1) $f(x,y) = x^2y + y^2x$

2) $u(x,y) = x^2 + xy + y^2$

3) $v(x,y) = \sqrt[3]{x^3+y^3}$

4) $\gamma(t) = (e^t \cos t \,;\, e^t \operatorname{sen} t)$

utilizando a regra da cadeia, calcule

a) $\frac{d}{dy} f(u,v) \to$

$$f(u,v) = u^2v + v^2u$$

$$\frac{df}{du}(u,v) = 2uv + v^2 = 2\cdot(x^2+xy+y^2)\sqrt[3]{x^3+y^3} + (x^3+y^3)^{2/3}$$

$$\frac{df}{dv}(u,v) = u^2 + 2vu = (x^2+xy+y^2)^2 + 2\sqrt[3]{x^3+y^3}(x^2+xy+y^2)$$

$$\frac{du}{dy} = x + 2y \qquad \frac{dv}{dy} = \frac{1}{3}\cdot\frac{1}{(x^3+y^3)^{2/3}}\cdot 3y^2 = \frac{y^2}{(x^3+y^3)^{2/3}}$$

$$\frac{d}{dy} f(u,v) = \frac{df}{du}(u,v)\cdot\frac{du}{dy}(u,v) + \frac{df}{dv}(u,v)\cdot\frac{dv}{dy}(u,v)$$

$$\frac{df}{dy}(u,v) = \left(2(x^2+xy+y^2)\sqrt[3]{x^3+y^3} + (x^3+y^3)^{2/3}\right)(x+2y) + \left((x^2+xy+y^2)^2 + 2\sqrt[3]{x^3+y^3}(x^2+xy+y^2)\right)\frac{y^2}{(x^3+y^3)^{2/3}}$$

$$\frac{df}{dy}(u,v) = 2(x^2+xy+y^2)(x^3+y^3)^{1/3}(x+2y) + (x^3+y^3)^{2/3}(x+2y) + \frac{2y^2(x^2+xy+y^2)}{(x^3+y^3)^{1/3}} + \frac{(x^2+xy+y^2)^2y^2}{(x^3+y^3)^{2/3}}$$

b) $\frac{d}{dt} f(\gamma) \to$ $u(t) = e^t \cos t$, $v(t) = e^t \operatorname{sen} t$

$$f(u,v) = u^2v + v^2u$$

$$\frac{df}{du} = 2uv + v^2 = 2e^{2t}\cos t\cdot\operatorname{sen} t + e^{2t}\cos^2 t$$

$$\frac{df}{dv} = 2vu + u^2 = 2e^{2t}\cos t\cdot\operatorname{sen} t + e^{2t}\operatorname{sen}^2 t$$

$$\frac{du}{dt}(t) = e^t\cos t - e^t\operatorname{sen} t$$

$$\frac{dv}{dt}(t) = e^t\operatorname{sen} t + e^t\cos t$$

$$\frac{df}{dt}(\gamma) = \frac{df}{du}(u,v)\cdot\frac{du}{dt}(t) + \frac{df}{dv}(u,v)\cdot\frac{du}{dv}(t)$$

$$\frac{df}{dt}(\gamma) = 2e^{2t}\cos^2 t\operatorname{sen} t + e^{3t}\cos^3 t - 2e^{3t}\operatorname{sen}^2 t\cos t + e^{3t}\operatorname{sen}^3 t + 2e^{3t}\cos^2 t\operatorname{sen} t + e^3\operatorname{sen}^3 t + e^{3t}\cos^3 t + e^{3t}\operatorname{sen}^2 t\cos t$$

$$\frac{df}{dt}(\gamma) = e^t(3\cos^2 t\operatorname{sen} t + \operatorname{sen}^3 t + \cos^3 t + \operatorname{sen}^2 t\cos t)$$

5 e Encontre uma função $f: \mathbb{R}^2 \to \mathbb{R}$, onde $\frac{d^2 f}{dy\,dx} = 6xy^2$ e cuja equação do seu plano tangente no ponto $(1;1;f(1;1))$ é dada por

$$-2x+5y-z-4=0$$

$$\vec{n} = (-2,5,-1) = \left(\frac{dF}{dx}(1,1)\,;\ \frac{dF}{dy}(1,1)\,;\ -1\right)$$

$$\int 6xy^2 = 6x\frac{y^3}{3} = 2xy^3 + ax + c$$

em relação a y

$$2y^3\int x = 2y^3\frac{x^2}{2} = y^3x^2 + ax + by + c$$

em relação a x

$$\boxed{f(x,y) = y^3x^2 + ax + by + c}$$

$$\frac{df}{dx}(1,1) = 2xy^3 + a = 2+a$$

$$\frac{df}{dy}(1,1) = 3y^2x^2 + b = 3+b$$

$$\frac{dF}{dx}(1,1) = 2+a = -2$$

$$\boxed{a=-4}$$

$$\frac{dF}{dy}(1,1) = 3+b = 5$$

$$\boxed{b=2}$$

descobrir o c:

$$f(1,1) = -1 = 1-4+2+c$$

$$\boxed{c=0}$$

assim, a função é:

$$\boxed{f(x,y) = y^3x^2 - 4x + 2y}$$

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