Transformada Z

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Resumo

Resolução de exercícios do **capítulo 03** do livro "**Introdução ao Processamento Digital de Sinais**"[1] de José Alexandre Nalon.

1 Exercício 01:

A **x[n] =** $4.\sigma[n-1] + 3.\sigma[n] - 2.\sigma[n+1] + 3.\sigma[n+2]$

$\sigma[n] = 1$	
β.**x**[n] = $\beta.x[z]$	(Linearidade)
x[n-k] = $x[z].z^{-k}$	(Deslocamento)

Tabela 1: Transformadas **Z** importantes.

$$\Longrightarrow \mathbf{x[z]} = 4.1.z^{-(-1)} + 3.1 - 2.1.z^{-1} + 3.1.z^{-2} \quad (1a)$$

$$\Longrightarrow \mathbf{x[z]} = 4.z + 3 - 2.z^{-1} + 3.z^{-2} \quad (1b)$$

ROC: $|x[z]| < \infty$, $\forall$ **z** $\neq$ **0** e $\forall$ **z** $\neq$ ∞.

B **x[n]** = $\begin{cases} a^{-n}, & se\ 0 \leq n < N. \\ 0, & fora\ do\ intervalo. \end{cases}$

x[z] = $\sum_{n=-\infty}^{\infty} x[n].z^{-n}$

Tabela 2: Transformadas **Z** importantes.

$$\Longrightarrow \textbf{x[z]} = \sum_{n=0}^{N-1} a^{-n}.z^{-n} \tag{2a}$$

$$\Longrightarrow \textbf{x[z]} = \sum_{n=0}^{N-1} ((a.z)^{-1})^{n} \tag{2b}$$

$$(PG) \Longrightarrow \textbf{x[z]} = \frac{1}{1-(a.z)^{-1}} = \frac{z}{z-a} \tag{2c}$$

ROC: $|x[z]| < \infty$ se, e somente se, $|(a.z)^{-1}| < \mathbf{1} \quad \Longrightarrow \quad |\mathbf{z}| < \frac{\mathbf{1}}{|\mathbf{a}|}$.

C **x[n]** = $\cos(\omega n).u[n]$

$\mathbf{cos}(\theta) = \frac{(e^{j\theta}+e^{-j\theta})}{2}$
x[z] = $\sum_{n=-\infty}^{\infty} x[n].z^{-n}$
u[n] = $\begin{cases} 1, & \forall\ n \geq 0 \\ 0, & caso\ contrario. \end{cases}$

Tabela 3: Propriedades importantes.

$$\Longrightarrow \textbf{x[z]} = \sum_{n=-\infty}^{\infty} \cos(\omega n).u[n].z^{-n} \tag{3a}$$

$$(u[n]) \Longrightarrow \textbf{x[z]} = \sum_{n=0}^{\infty} \cos(\omega n).z^{-n} \tag{3b}$$

$$\Longrightarrow \textbf{x[z]} = \sum_{n=0}^{\infty} \frac{(e^{j\omega n}+e^{-j\omega n})}{2}.z^{-n} \tag{3c}$$

ROC: $|z| > 1$.

D **x[n] =** $\cos(\omega n)\,.\,u[-n-1]$

cos(θ) $= \frac{(e^{j\theta}+e^{-j\theta})}{2}$
x[z] = $\sum_{n=-\infty}^{\infty} \; x[n]\,.\,z^{-n}$
u[n-1] = $\begin{cases} 1, & \forall \; n > 0 \\ 0, & caso \;\; contrario. \end{cases}$
u[-n-1] = $\begin{cases} 1, & \forall \; n < 0 \\ 0, & caso \;\; contrario. \end{cases}$

Tabela 4: Propriedades importantes.

$$\Longrightarrow \; \mathbf{x[z]} = \sum_{n=-\infty}^{\infty} \cos(\omega n)\,.\,u[-n-1]\,.\,z^{-n} \tag{4a}$$

$$(m=-n) \; \Longrightarrow \; \mathbf{x[z]} = \sum_{m=\infty}^{-\infty} \cos(-\omega m)\,.\,u[m-1]\,.\,z^{m} \tag{4b}$$

$$(u[m-1]) \; \Longrightarrow \; \mathbf{x[z]} = \sum_{m=\infty}^{1} \cos(-\omega m)\,.\,z^{m} \tag{4c}$$

$$(m=n) \; \Longrightarrow \; \mathbf{x[z]} = \sum_{n=1}^{\infty} \cos(-\omega n)\,.\,z^{n} \tag{4d}$$

$$\Longrightarrow \; \mathbf{x[z]} = -1 + \sum_{n=0}^{\infty} \cos(-\omega n)\,.\,z^{n} \tag{4e}$$

$$\Longrightarrow \; \mathbf{x[z]} = -1 + \sum_{n=0}^{\infty} \frac{(e^{-j\omega n} + e^{-j(-\omega n)})}{2}\,.\,z^{n} \tag{4f}$$

$$\Longrightarrow \; \mathbf{x[z]} = -1 + \sum_{n=0}^{\infty} \frac{(e^{-j\omega} + e^{j\omega})^{n}}{2}\,.\,z^{n} \tag{4g}$$

$$\Longrightarrow \; \mathbf{x[z]} = -1 + \sum_{n=0}^{\infty} (\cos(\omega)\,.\,z)^{n} \tag{4h}$$

$$(PG) \; \Longrightarrow \; \mathbf{x[z]} = \frac{1}{1-\cos(\omega)\,.\,z} - 1 \tag{4i}$$

ROC: $|x[z]| < \infty$ se, e somente se, $|\cos(\omega)\,.\,\mathbf{z}| < \mathbf{1} \;\; \Longrightarrow \;\; |\mathbf{z}| < \mathbf{1}.$

E **x[n] =** $\frac{1}{2^n} . \cos(\omega n) . u[n]$

$\cos(\theta) = \frac{(e^{j\theta} + e^{-j\theta})}{2}$
$x[z] = \sum_{n=-\infty}^{\infty} x[n] . z^{-n}$
$u[n] = \begin{cases} 1, & \forall\ n \geq 0 \\ 0, & caso\ contrario. \end{cases}$

Tabela 5: Propriedades importantes.

$$\implies \mathbf{x[z]} = \sum_{n=-\infty}^{\infty} \frac{1}{2^n} . \cos(\omega n) . u[n] . z^{-n} \tag{5a}$$

$$(u[n]) \implies \mathbf{x[z]} = \sum_{n=0}^{\infty} \frac{1}{2^n} . \cos(\omega n) . z^{-n} \tag{5b}$$

$$\implies \mathbf{x[z]} = \sum_{n=0}^{\infty} 2^{-n} . \frac{(e^{j\omega n} + e^{-j\omega n})}{2} . z^{-n} \tag{5c}$$

$$\implies \mathbf{x[z]} = \sum_{n=0}^{\infty} (2^{-1} . \frac{(e^{j\omega} + e^{-j\omega})}{2} . z^{-1})^n \tag{5d}$$

$$\implies \mathbf{x[z]} = \sum_{n=0}^{\infty} (\frac{1}{2} . \cos(\omega) . z^{-1})^n \tag{5e}$$

$$(PG) \implies \mathbf{x[z]} = \frac{1}{1 - \frac{\cos(\omega)}{2} . z^{-1}} \tag{5f}$$

ROC: $|\frac{\cos(\omega)}{2} . z^{-1}| < 1 \implies |\mathbf{z}| > \mathbf{\frac{1}{2}}.$

F $\mathbf{x[n]} = a^{|n|}, \ |a| < 1$

$\mathbf{x}[z] = \sum_{n=-\infty}^{\infty} x[n]\,.\,z^{-n}$
$a^{\|n\|} = \begin{cases} a^n, & \forall \ n \geq 0 \\ a^{-n}, & \forall \ n < 0 \end{cases}$

Tabela 6: Propriedades importantes.

$$\Longrightarrow \mathbf{x[z]} = \sum_{n=-\infty}^{\infty} a^{|n|}\,.\,z^{-n} \tag{6a}$$

$$(a^{|n|}) \Longrightarrow \mathbf{x[z]} = \sum_{n=-\infty}^{-1} a^{-n}\,.\,z^{-n} + \sum_{n=0}^{\infty} a^{n}\,.\,z^{-n} \tag{6b}$$

$$(m = -n) \Longrightarrow \mathbf{x[z]} = \sum_{m=\infty}^{1} a^{m}\,.\,z^{m} + \sum_{n=0}^{\infty} a^{n}\,.\,z^{-n} \tag{6c}$$

$$(m = n) \Longrightarrow \mathbf{x[z]} = \sum_{n=1}^{\infty} a^{n}\,.\,z^{n} + \sum_{n=0}^{\infty} a^{n}\,.\,z^{-n} \tag{6d}$$

$$\Longrightarrow \mathbf{x[z]} = -1 + \sum_{n=0}^{\infty} a^{n}\,.\,z^{n} + \sum_{n=0}^{\infty} a^{n}\,.\,z^{-n} \tag{6e}$$

$$\Longrightarrow \mathbf{x[z]} = -1 + \sum_{n=0}^{\infty} (a\,.\,z)^{n} + \sum_{n=0}^{\infty} (a\,.\,z^{-1})^{n} \tag{6f}$$

ROC:

- Convergência 1º somatório: $|a\,.\,z| < 1 \Longrightarrow |\mathbf{z}| < \frac{\mathbf{1}}{|\mathbf{a}|}$.
- Convergência 2º somatório: $|a\,.\,z^{-1}| < 1 \Longrightarrow |\mathbf{z}| > \frac{\mathbf{1}}{|\mathbf{a}|}$.
- **Conclusão:** A função $\mathbf{x[n]} = a^{|n|}$ **não possui** transformada **Z**, pois os somatórios não convergem simultaneamente para os mesmos valores de **z**.

G **x[n]** = $a^{-n}.u[-n]$

x[z] = $\sum_{n=-\infty}^{\infty} x[n].z^{-n}$
u[n] = $\begin{cases} 1, & \forall\ n \geq 0 \\ 0, & caso\ contrrio. \end{cases}$
u[-n] = $\begin{cases} 1, & \forall\ n \leq 0 \\ 0, & caso\ contrrio. \end{cases}$

Tabela 7: Propriedades importantes.

$$\Longrightarrow \mathbf{x[z]} = \sum_{n=-\infty}^{\infty} a^{-n}.u[-n].z^{-n} \tag{7a}$$

$$(m=-n) \Longrightarrow \mathbf{x[z]} = \sum_{m=\infty}^{1} a^{m}.u[m].z^{m} \tag{7b}$$

$$(u[m]) \Longrightarrow \mathbf{x[z]} = \sum_{m=\infty}^{0} a^{m}.z^{m} \tag{7c}$$

$$(m=n) \Longrightarrow \mathbf{x[z]} = \sum_{n=0}^{\infty} a^{n}.z^{n} \tag{7d}$$

$$\Longrightarrow \mathbf{x[z]} = \sum_{n=0}^{\infty} (a.z)^{n} \tag{7e}$$

$$(PG) \Longrightarrow \mathbf{x[z]} = \frac{1}{1-a.z} \tag{7f}$$

ROC: $|a.z| < 1 \quad \Longrightarrow \quad |\mathbf{z}| < \frac{\mathbf{1}}{\mathbf{|a|}}$.

4 Exercício 04:

A **y[n]** = $3.x_1[n] - 2.x_2[n]$

β . **x**[n] = $\beta . x[z]$	(Linearidade)

Tabela 8: Transformadas **Z** importantes.

$$\Longrightarrow \textbf{y[z]} = \; 3.x_1[z] \text{ - } 2.x_2[z] \tag{8a}$$

ROC: $\mathbf{R}_y = R_{x1} \cap R_{x2}$

B **y[n]** = $2^{n-1}.x_2[n-1]$

x[n-k] = $x[z] \, . \, z^{-k}$	(Deslocamento.)
z_0^n . **x**[n] = $x[z/z_0]$	(Mudança de escala.)

Tabela 9: Transformadas **Z** importantes.

$$(m = n-1) \Longrightarrow \textbf{y[z]} = \; 2^m.x_2[m] \tag{9a}$$
$$\Longrightarrow \textbf{y[z]} = \; x_2[z/2] \tag{9b}$$

ROC: $\mathbf{R}_y = 2.R_{x2}$

F **y[n]** = $n.x_1[n]$

n . **x**[n] = $-z . \frac{\partial x[z]}{\partial z}$, **ROC** $= R_x$
(Diferenciação no domínio complexo.)

Tabela 10: Transformadas **Z** importantes.

$$\Longrightarrow \textbf{y[z]} = \; -z . \frac{\partial \; x_1[z]}{\partial z} \tag{10a}$$

ROC: $\mathbf{R}_y = R_{x1}$

5 Exercício 05:

A **x[z] =** $(1+z^{-1}).(1-\frac{1}{2}.z^{-1}).(1+2.z^{-1}),\ R_x : 0 < |z| < \infty$

$$\Longrightarrow \mathbf{x[z]} = 1 + \frac{1}{2}.z^{-1} + \frac{1}{2}.z^{-2} + z^{-3} \tag{11a}$$

$$(Inversa) \Longrightarrow \mathbf{x[n]} = \sigma[n] + \frac{1}{2}.\sigma[n-1] + \frac{1}{2}.\sigma[n-2] + \sigma[n-3] \tag{11b}$$

B **x[z] =** $\frac{1}{1-a.z^{-1}},\ R_x : |z| > |a|$

x[z] = $\frac{1}{1-a.z^{-1}}$, $\mathbf{R}_x : |z| > |a|$	
x[n] = $a^n.u[n]$	(Inversa)

Tabela 11: Inversa das transformadas **Z**.

$$(Inversa) \Longrightarrow \mathbf{x[n]} = a^n.u[n] \tag{12a}$$

C **x[z] =** $\frac{1-b.z^{-1}}{1-a.z^{-1}},\ R_x : |z| < |a|$

x[z] = $\frac{1}{1-a.z^{-1}}$, $\mathbf{R}_x : |z| < |a|$	
x[n] = $-a^n.u[-n-1]$	(Inversa)
x[n-k] = $x[z].z^{-k}$	(Deslocamento)

Tabela 12: Inversa das transformadas **Z**.

$$\Longrightarrow \mathbf{x[z]} = \frac{1}{1-a.z^{-1}} - b.z^{-1}.\frac{1}{1-a.z^{-1}} \tag{13a}$$

$$\Longrightarrow \mathbf{x[z]} = x_1[z] - b.z^{-1}.x_1[z] \tag{13b}$$

$$(Deslocamento) \Longrightarrow \mathbf{x[z]} = x_1[z] - b.x_1[n-1] \tag{13c}$$

$$(Inversa) \Longrightarrow \mathbf{x[z]} = -a^n.u[-n-1] + b.a^{n-1}.u[-n] \tag{13d}$$

Referências

[1] J. A. Nalon, *Introdução ao processamento digital de sinais.* Grupo Gen-LTC, 2000.