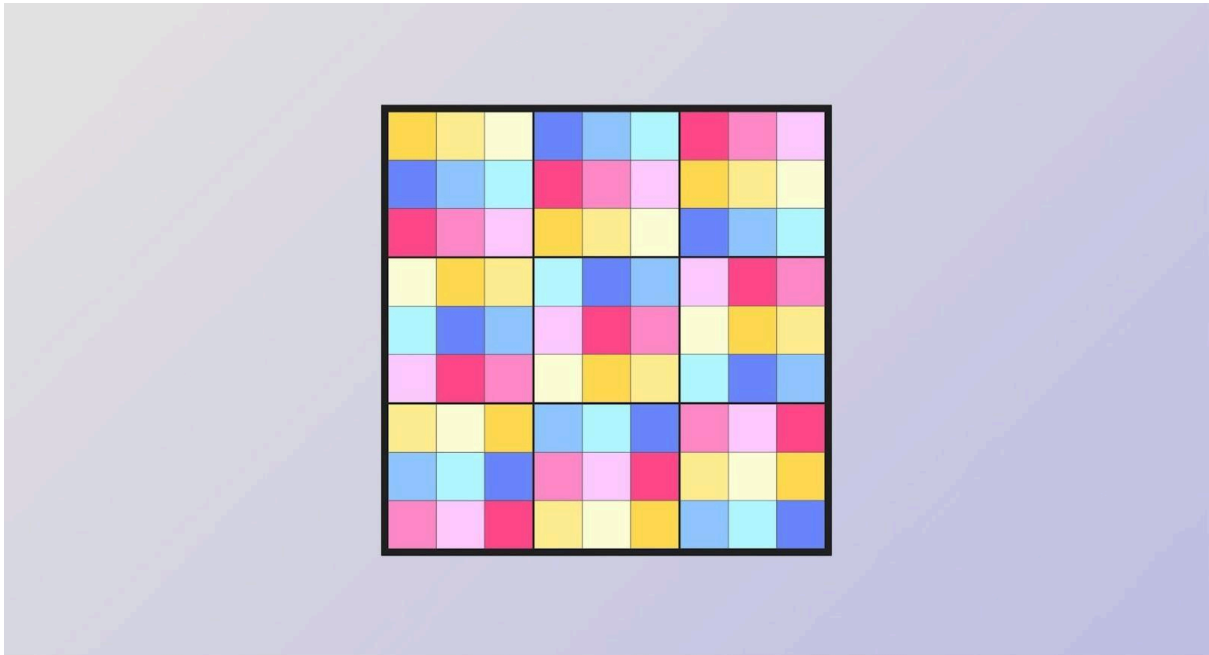


Classification of Sudoku Patterns and Transformations



TERMINOLOGY

Sudoku configuration: disposition of the digits in the Sudoku board.

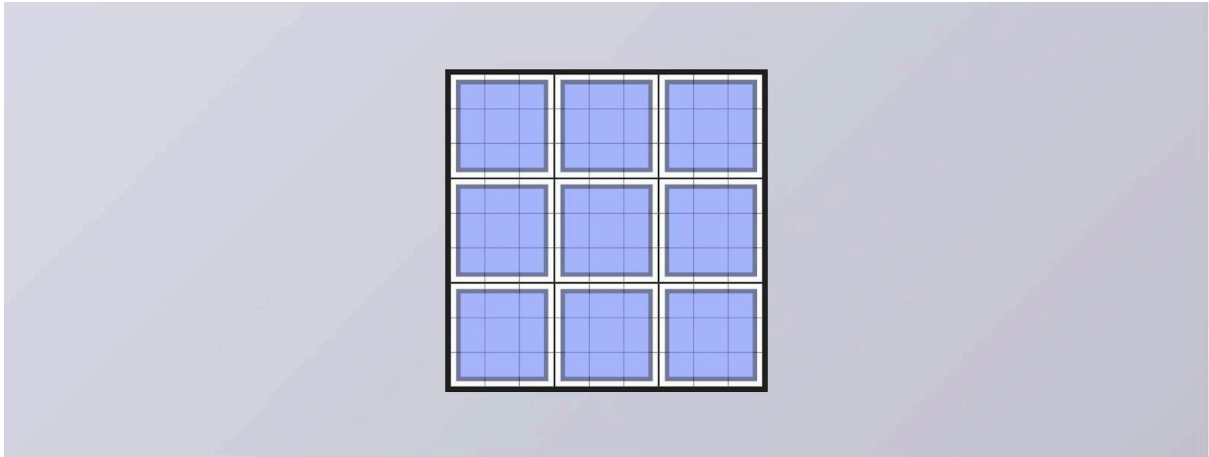
Sudoku constraints: rules of common Sudoku puzzles that state that there can't be repeated digits in any column, row or 3x3 box.

Valid configuration: Sudoku configuration that respects Sudoku constraints.

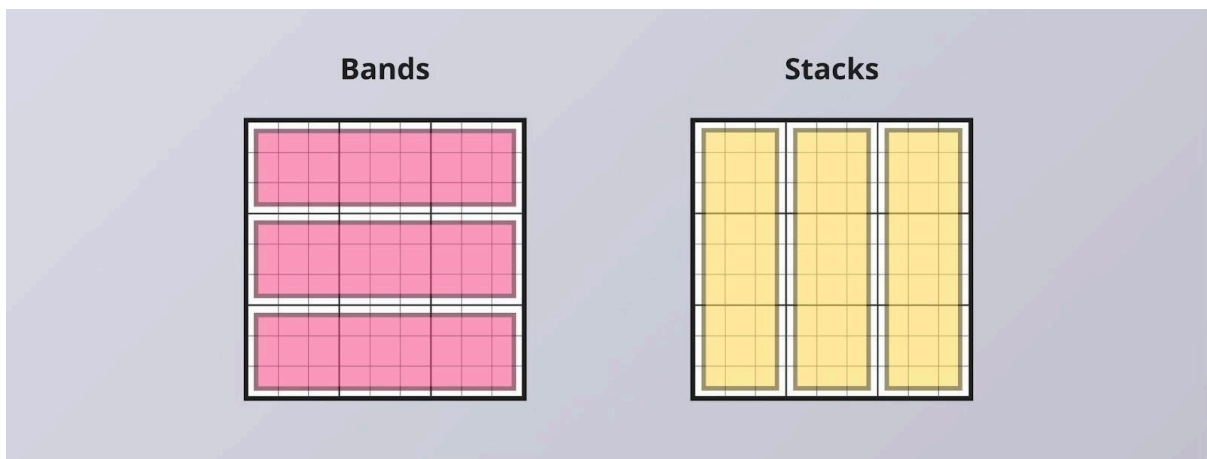
Complete Sudoku: Sudoku board filled with digits, respecting Sudoku constraints.

Transformation: Specific modifications in the disposition of the digits in a complete Sudoku that result in a new valid configuration.

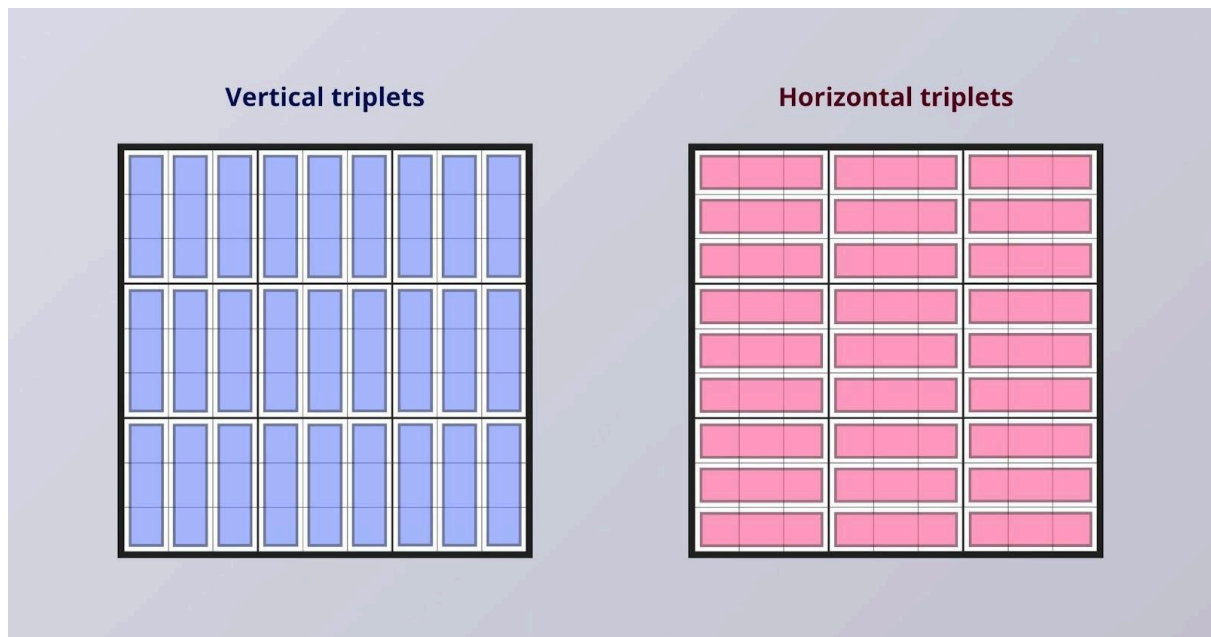
Box: Block of 3x3 squares. The Sudoku board has 9 boxes: top-left, top-middle, top-right, center-left, center-middle, center-right, bottom-left, bottom-middle, and bottom-right.



Bands and stacks: Bands are a set of 3 boxes horizontally aligned. Stacks are a set of 3 boxes vertically aligned. A Sudoku board has 3 bands and 3 stacks.

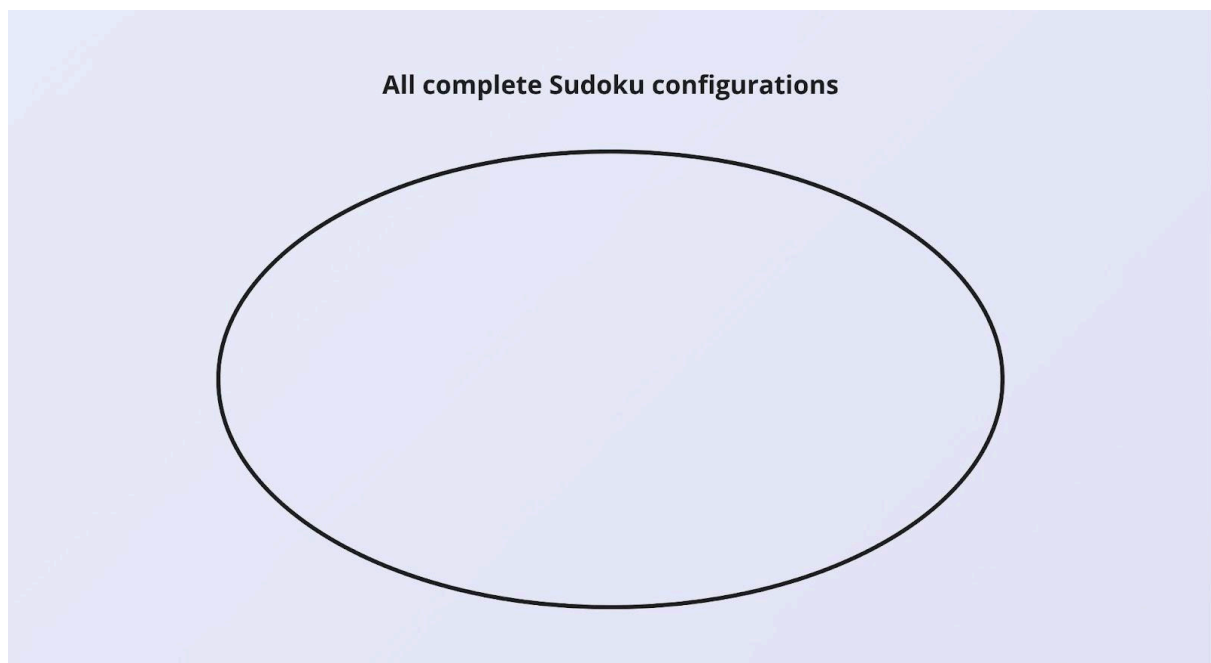


Triplet: Each box has 3 columns and 3 rows with a height / length of 3 squares. Each one of these columns is a vertical triplet, and each row is a horizontal triplet. Each box has 6 triplets.



CLASSIFICATION OF PATTERNS

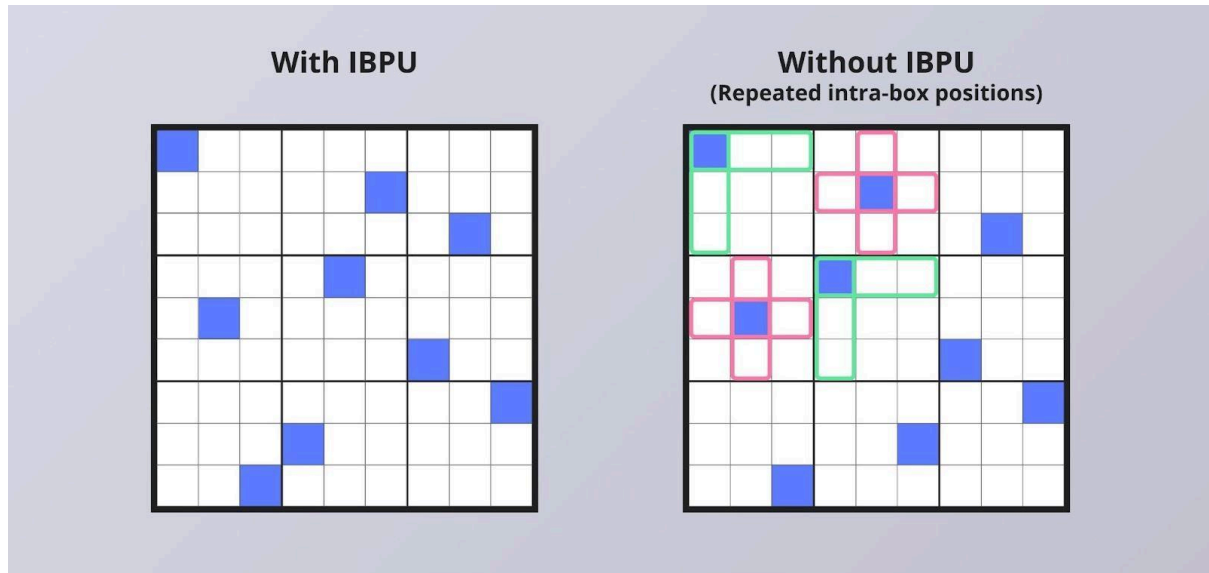
First, we have the set of all complete Sudoku configurations. The configurations in this set don't follow any particular pattern, other than respecting the Sudoku constraints.



Inside this set, there are sub-sets of configurations that follow specific patterns.

Intra-Box Positional Uniqueness (IBPU)

The squares inside each 3x3 box have one of 9 positions: top-left, top-middle, top-right, center-left, center-middle, center-right, bottom-left, bottom-middle, and bottom-right. The Intra-Box Positional Uniqueness pattern (IBPU) is present in a configuration when there are no digits that occupy the same intra-box position more than once.



For example, if in a box there is a “7” at top-left, then there can’t be another “7” at top-left in any other box. The same applies to all the digits, in all the positions.

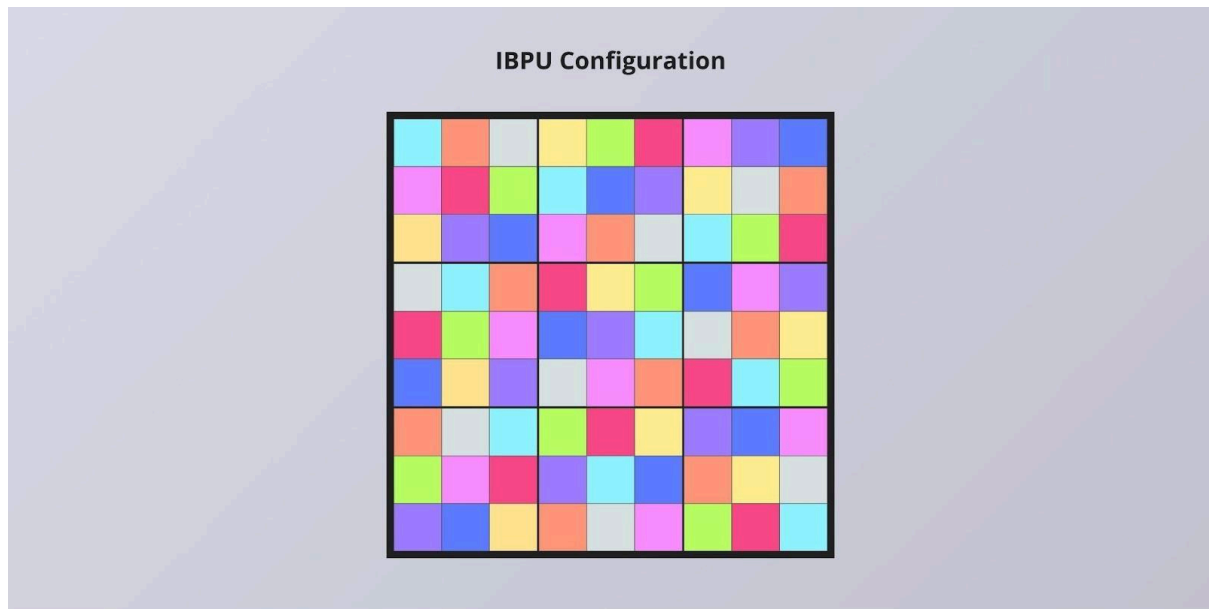
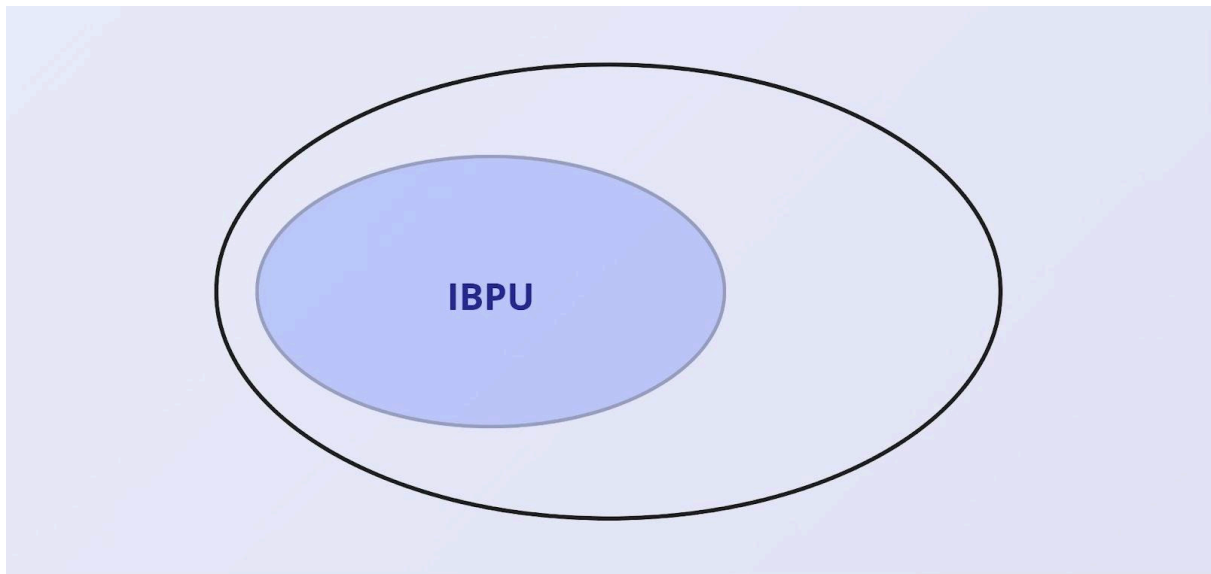


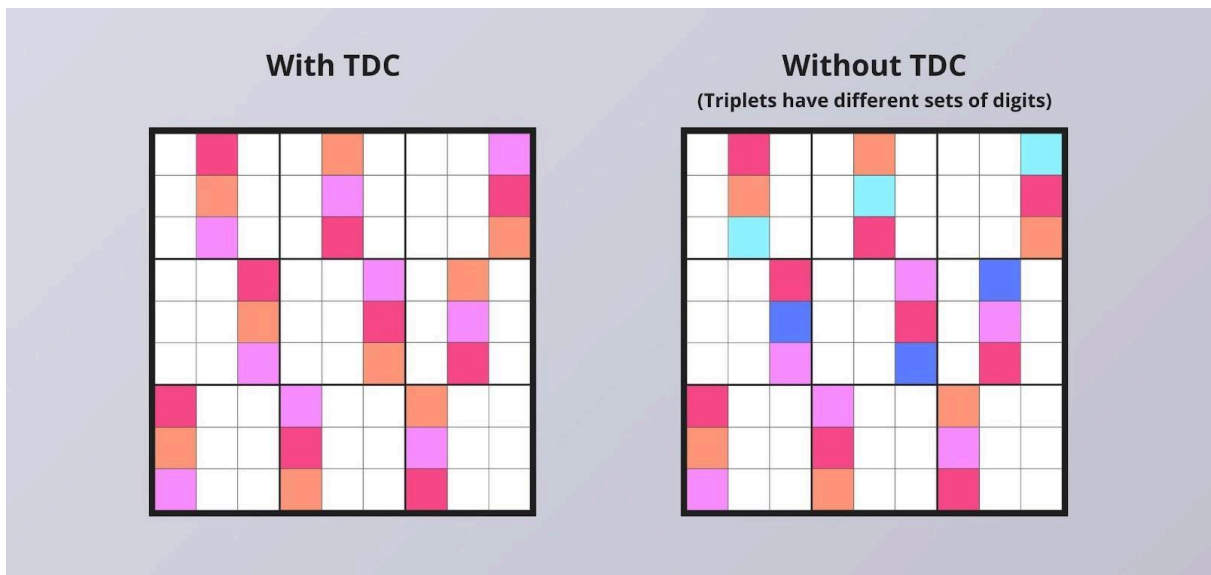
Diagram with the set of all IBPU configurations:



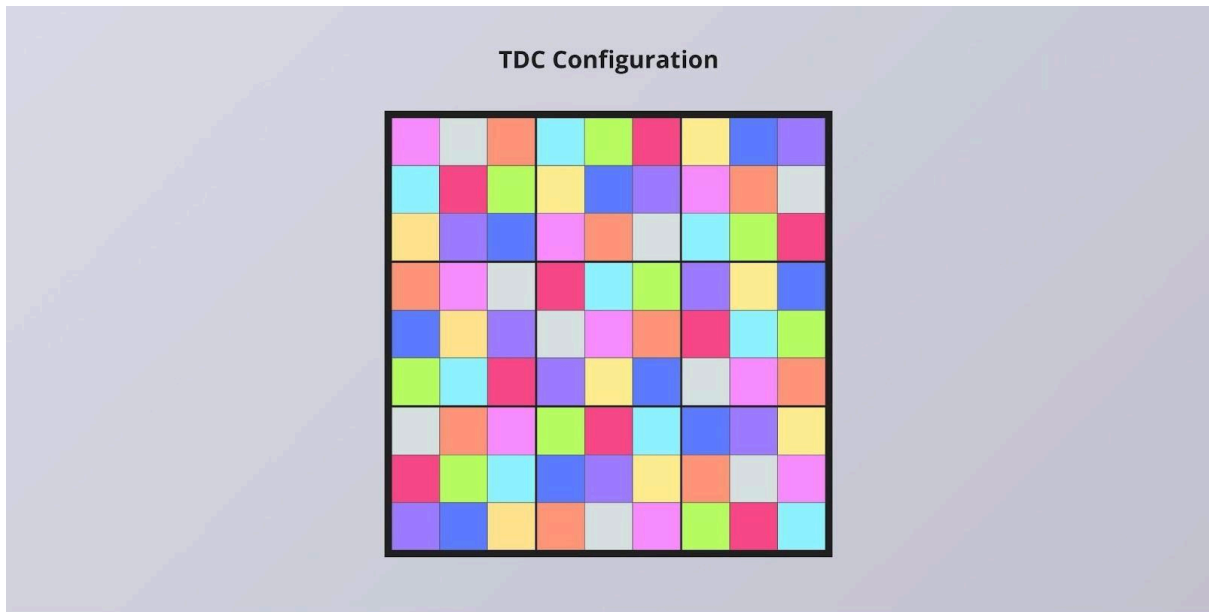
Triplet Digit Consistency (TDC)

In a valid configuration, each one of the 6 triplets inside a box always has a different set of 3 digits.

The Triplet Digit Consistency pattern (TDC) is present when the set of digits of the 6 triplets inside a box coincides with the set of digits of the 6 triplets inside all the other boxes.



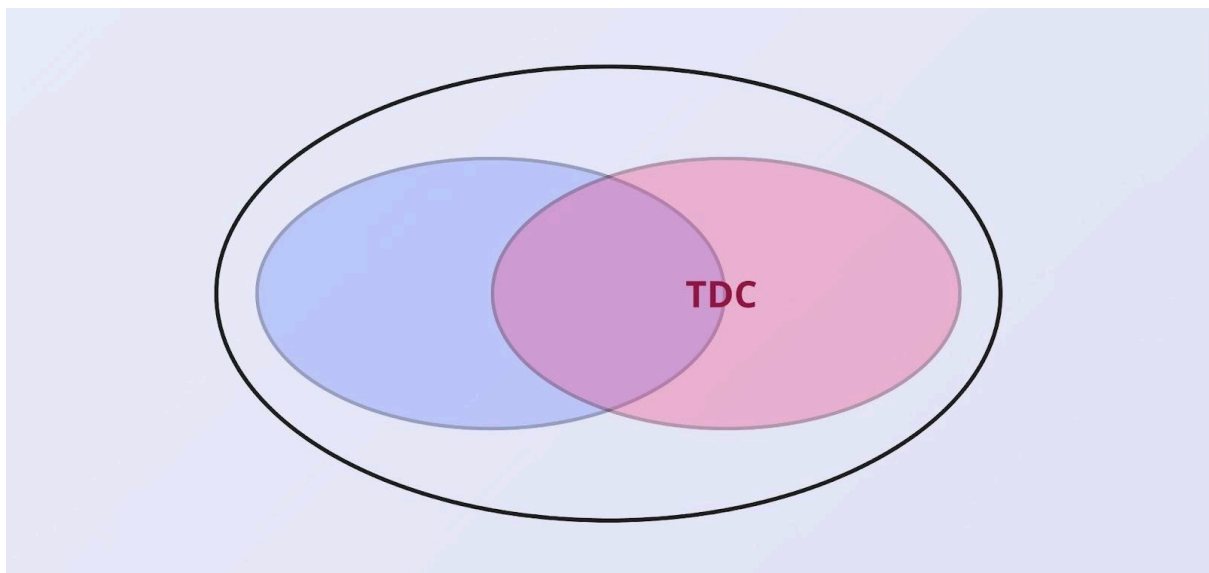
For example, if in box 1 there is a triplet with the set of digits 3, 5 and 8, then all the other boxes must have a triplet with the digits 3, 5 and 8. This applies to the 6 triplets of every box.



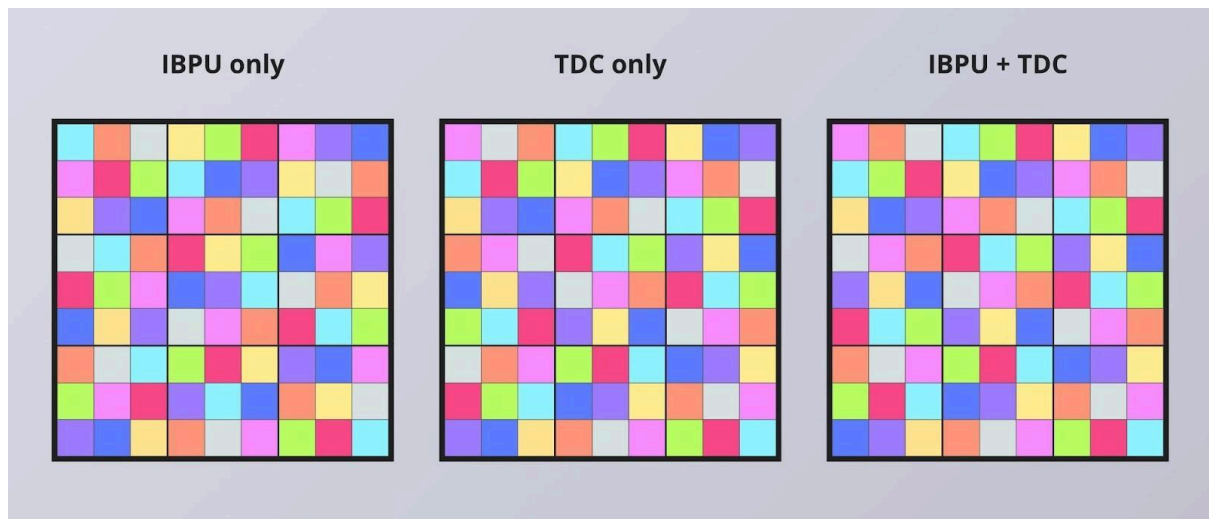
The order / position of the digits doesn't have to be the same in every triplet, only the set of digits must be the same.

All the triplets that share a set of digits must have the same orientation (horizontal or vertical). If not, it means that Sudoku constraints were violated.

Diagram with the set of all TDC configurations:

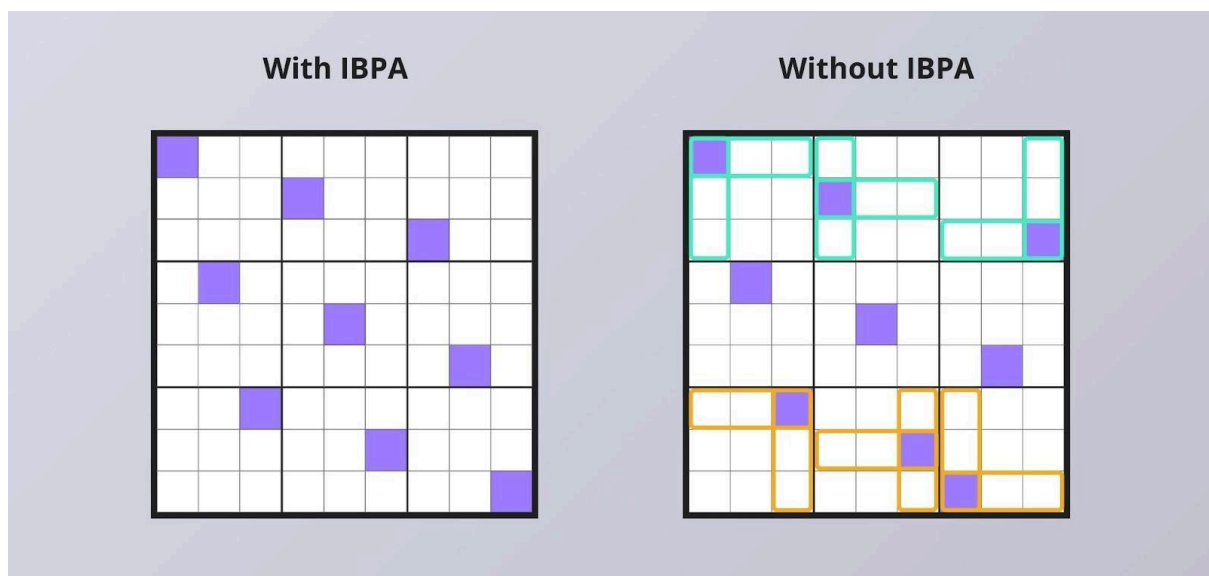


Some configurations follow only one of the patterns, while others follow both.

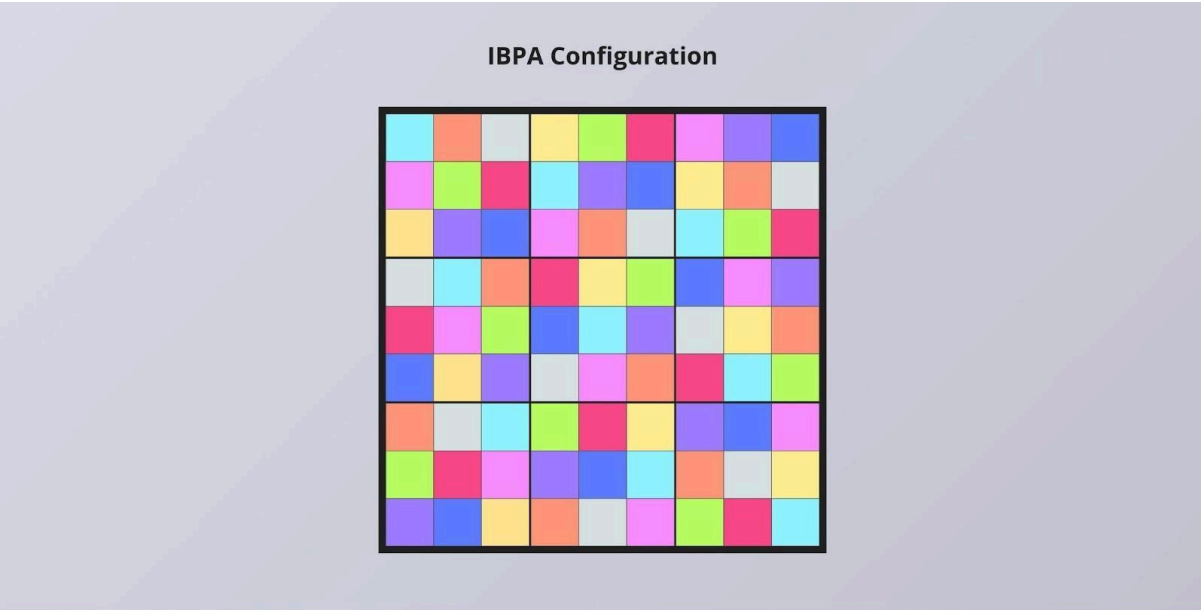


Intra-Box Positional Alignment (IBPA)

The Intra-Box Positional Alignment pattern (IBPA) is present when the digits along a band occupy the same horizontal intra-box position, and the digits along a stack occupy the same vertical intra-box position.

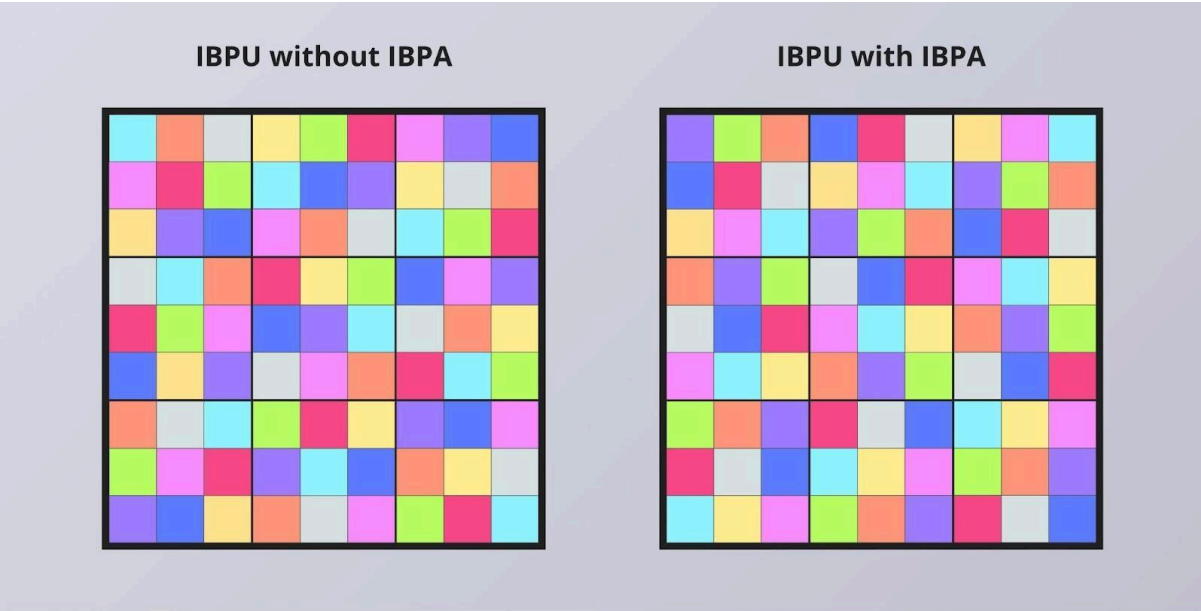


For example, if I have a band with box 1, box 2 and box 3, and in box 1 there is a “9” at the bottom-left, then the digits “9” of box 2 and box 3 also have to be positioned in the left side of the boxes (the left triplet). If I have a stack with box 2, box 5 and box 8, and in box 2 there is a “7” at the center-right, then the digits “7” of box 5 and box 8 also have to be positioned in the center row of each box (the center triplet). This applies for all the digits and all the boxes in every band / stack.

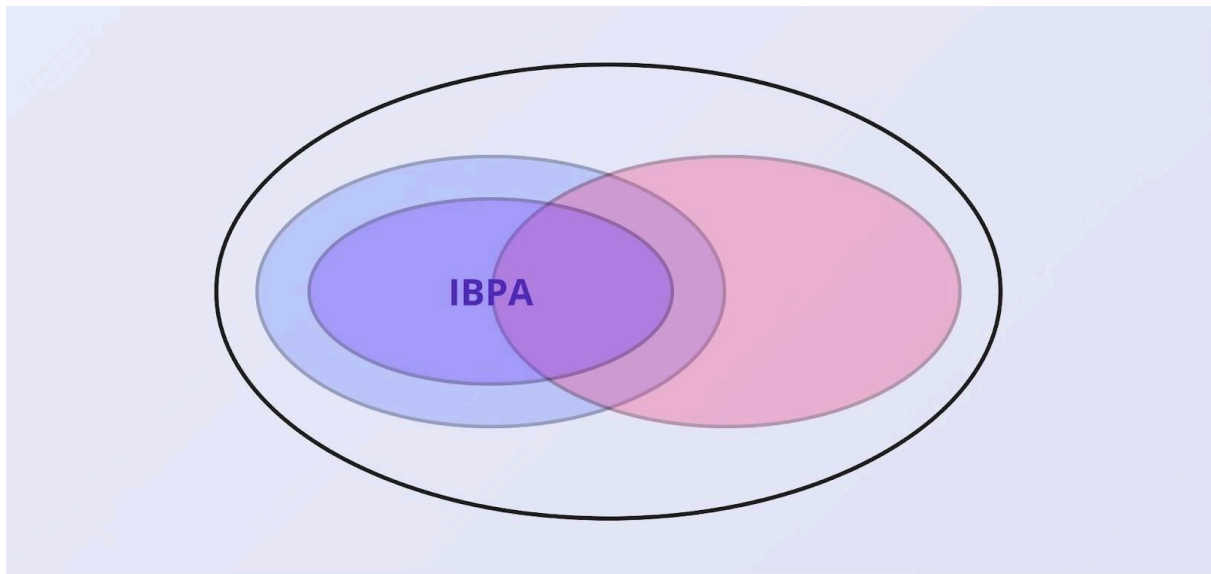


While in a band the horizontal intra-box position of the digits is the same, the vertical intra-box position must be different to comply with Sudoku constraints. The same applies to stacks: while the vertical intra-box position of the digits is the same, the horizontal intra-box position must be different.

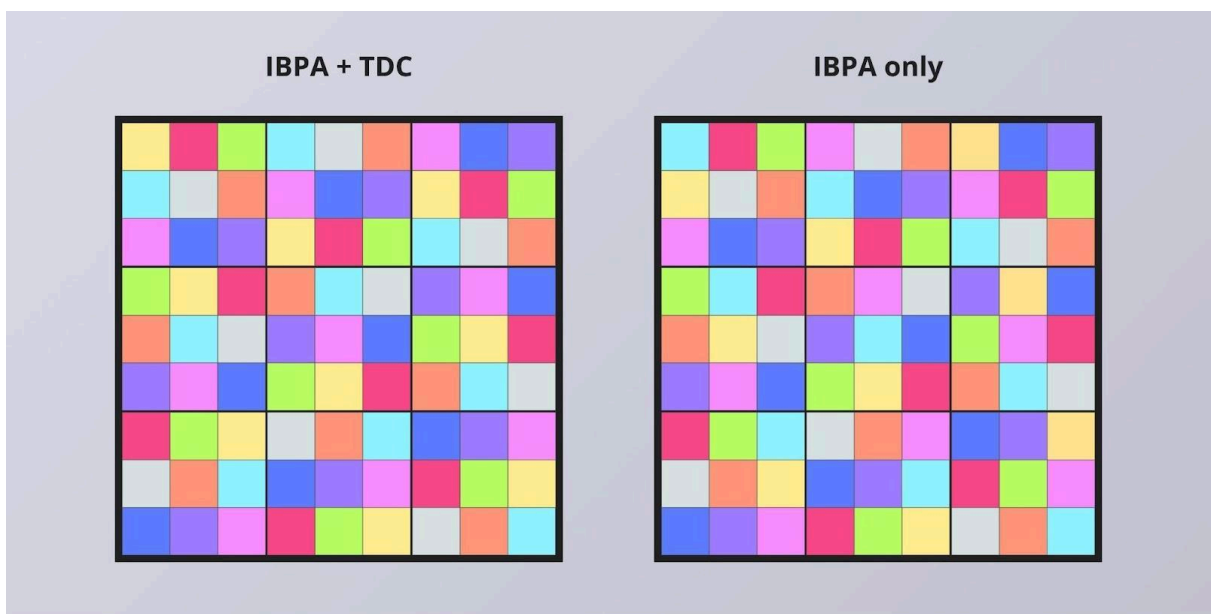
This means that there can never be a digit in the same intra-box position more than once, and that is the definition of the IBPU pattern. So, in consequence, every configuration that follows the IBPA pattern also follows the IBPU pattern. However, not every configuration that follows the IBPU pattern also follows the IBPA pattern.



In other words, the IBPA configurations form a sub-set of the set of IBPU configurations.



There are IBPA configurations that follow the TDC pattern, and other IBPA configurations that don't.

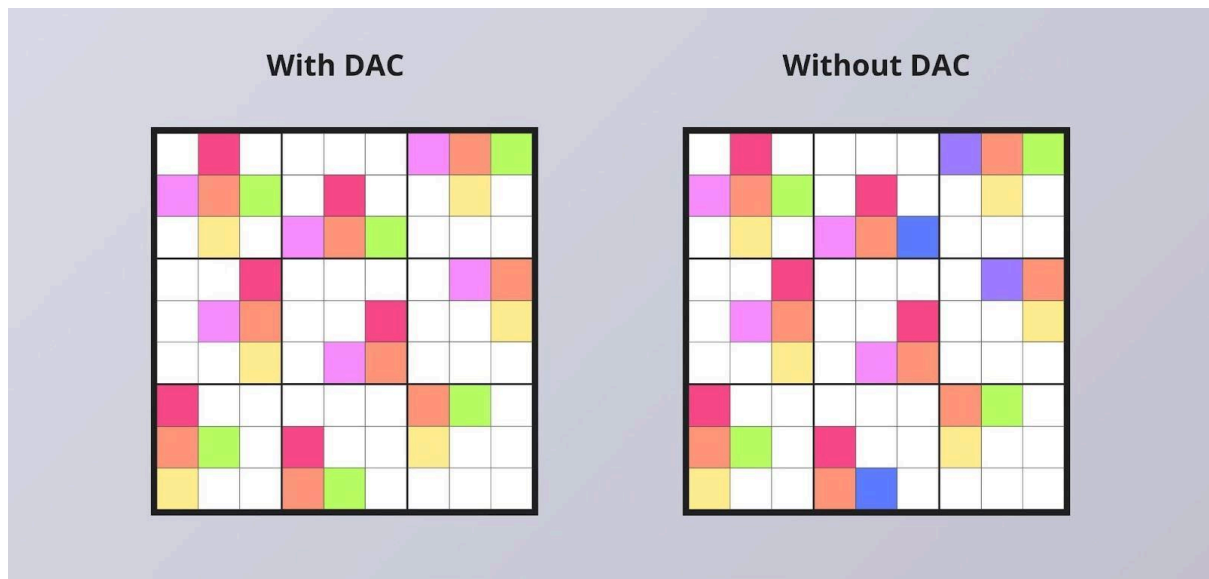


Digit Adjacency Consistency (DAC)

The Digit Adjacency Consistency pattern (DAC) is present if inside each box, the digits adjacent to any other digit are always the same.

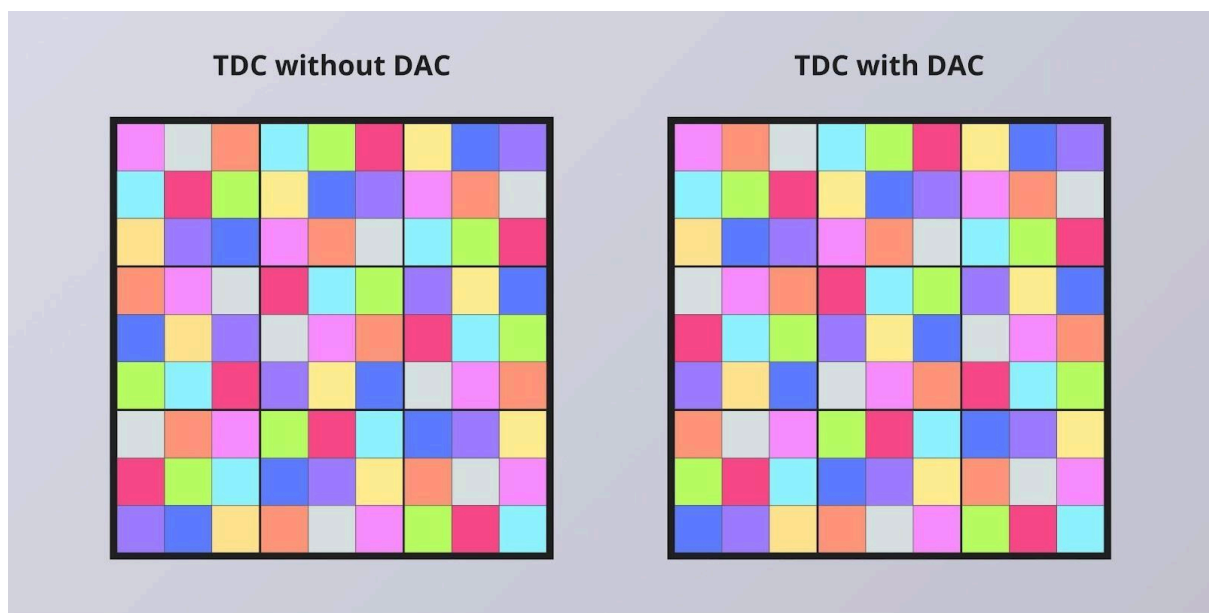
For example, If in one box the “8” is in the center-middle, and it has a “1” in the square above, a “4” in the square to the right, a “2” in the square below, and a “6” in the square to the left, then in all other boxes the “8”s must have a “1” in the square

above, a “4” in the square to the right, a “2” in the square below, and a “6” in the square to the left.

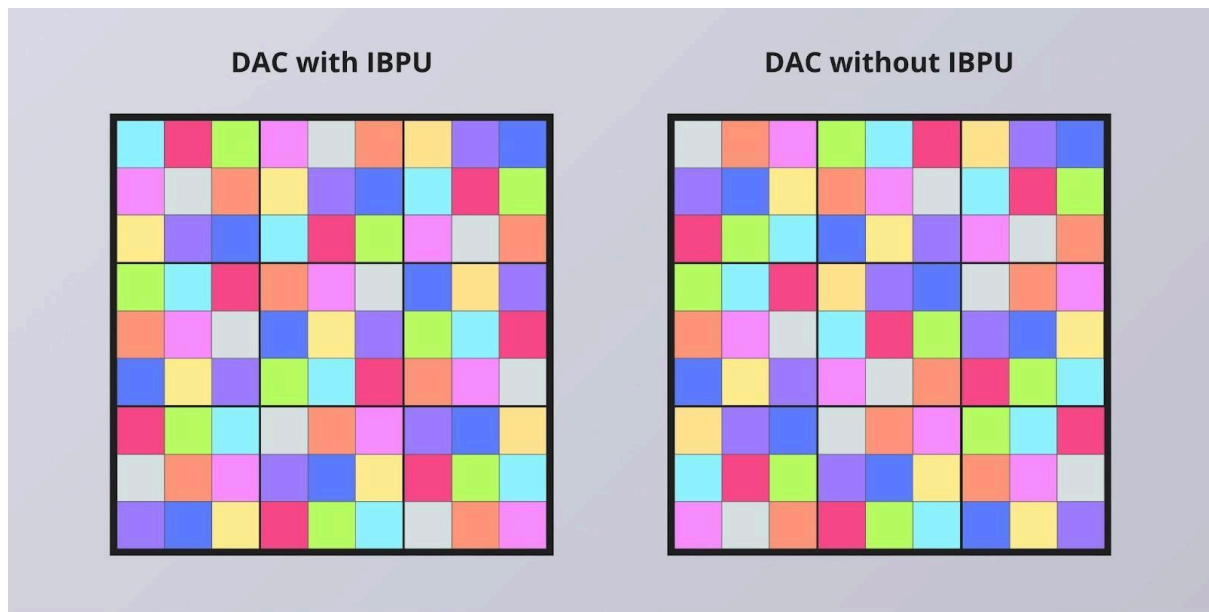


It is important to specify that this only counts inside each 3x3 box, so if there is a box with an “8” in the bottom-center, the square below it doesn’t have to be a “2”, because that square is not from the same box.

This pattern implies that every digit always has the same position relative to the other digits inside its box. So, every vertical and horizontal triplet must always have the same set of digits, which is the TDC pattern. Therefore, every configuration that satisfies the DAC pattern also satisfies the TDC pattern. However, not every TDC configuration satisfies the DAC pattern.



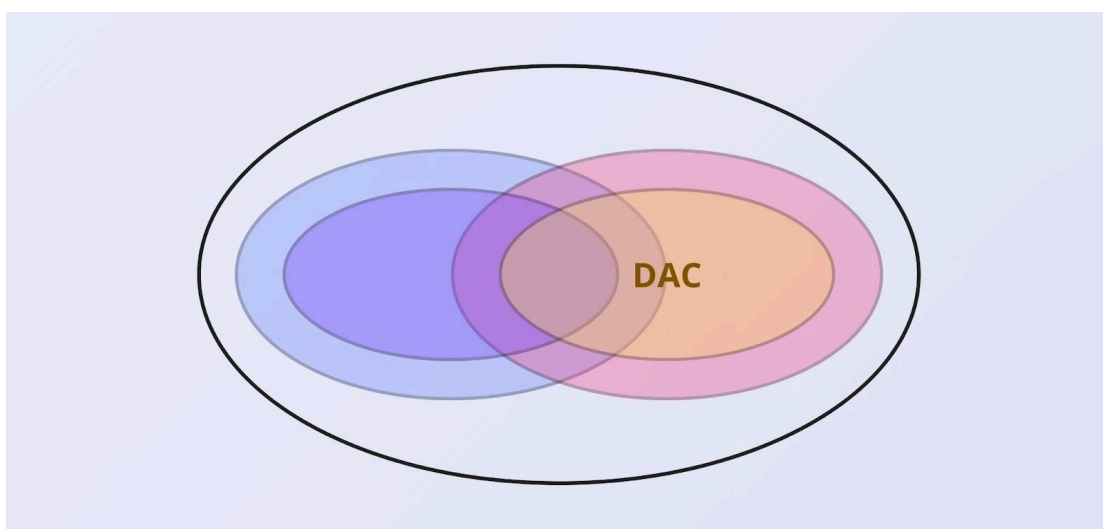
Also, there are some DAC configurations that follow the IBPU pattern, and some that don't. The ones that follow IBPU also follow IBPA.



So far, I haven't found any DAC configurations that satisfy IBPU without also satisfying IBPA. I suspect DAC + IBPU (without IBPA) configurations don't exist, but haven't found proof of it yet.

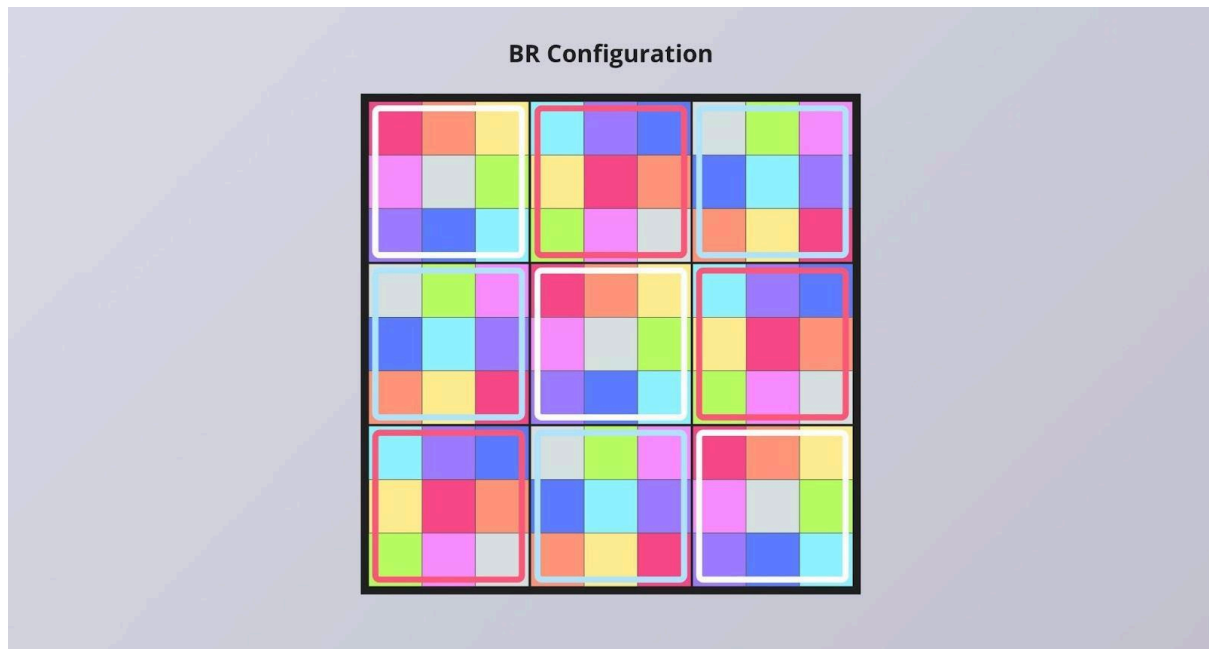
I also believe that all TDC + IBPA configurations are DAC configurations, but haven't found a way to prove it.

In conclusion, the DAC configurations form a sub-set of the set of TDC configurations. Some DAC configurations satisfy IBPU and some don't.

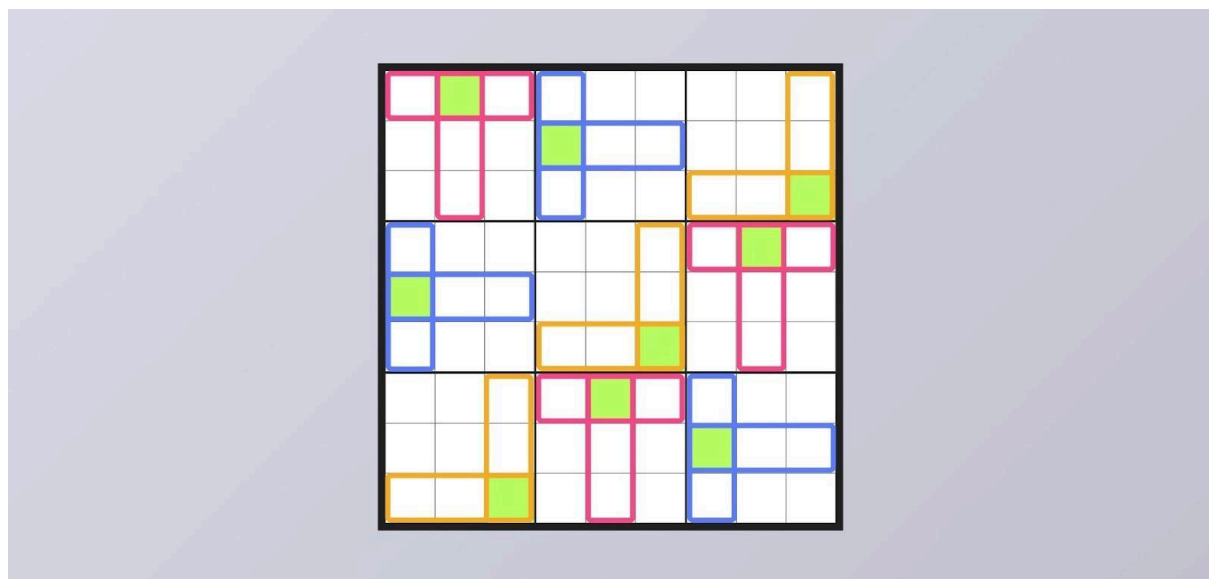


Box Repetition (BR)

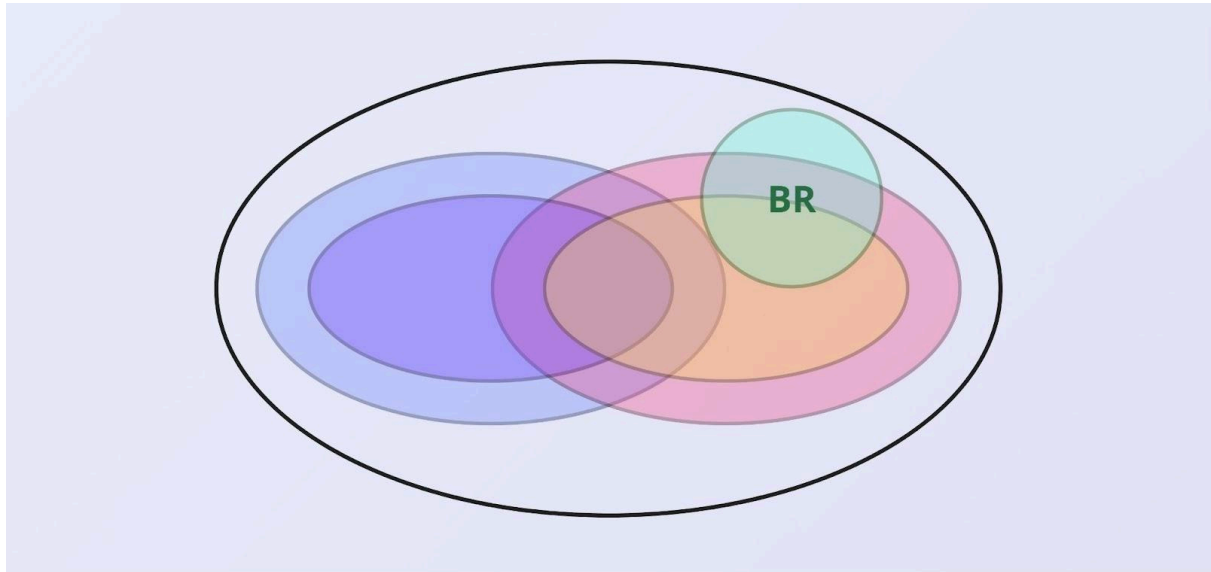
Every band and stack has copies of the same 3 boxes, with their digits in the exact same intra-box positions. So, there are only 3 unique 3x3 boxes in the whole Sudoku board, repeated in each band and stack.



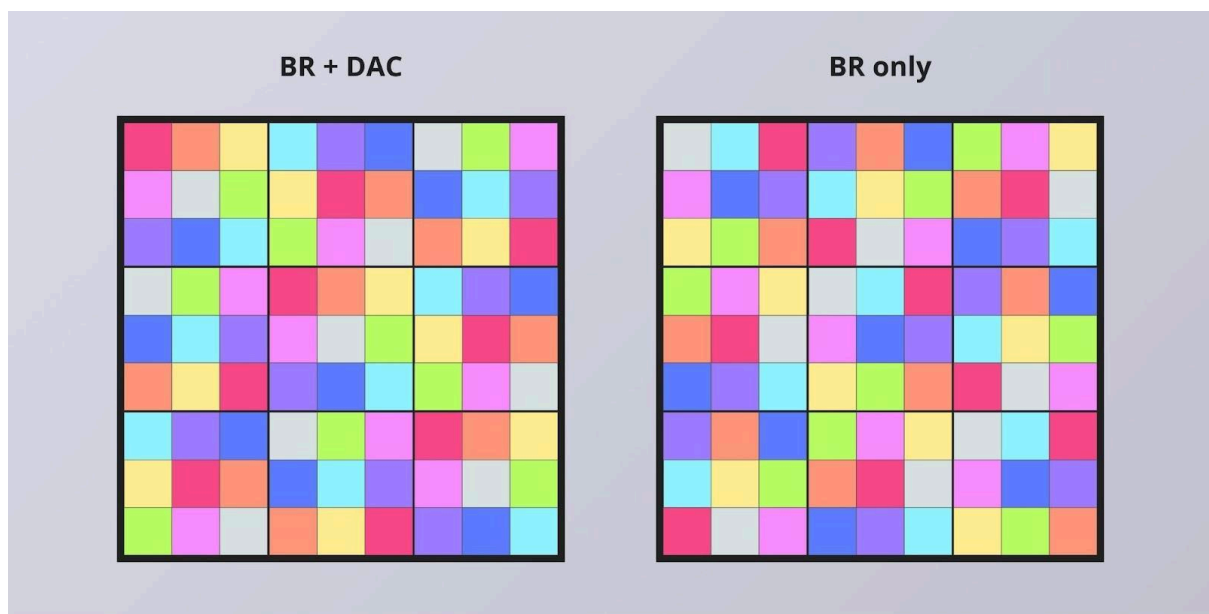
Another characteristic of this pattern is that, in each one of the 3 unique boxes, the same digits are in different horizontal and vertical intra-box positions. For example, if in one box there is a “5” at top-middle, there can’t be a “5” in any of the top and middle positions of the other 2 unique boxes. This applies to all the digits of the 3 unique boxes, and is what allows the Sudoku constraints to not be broken when repeating the 3 unique boxes in each band and stack.



By definition, this pattern can't be present in IBPU configurations (because having exact copies of boxes requires repeating digits in the same intra-box positions).

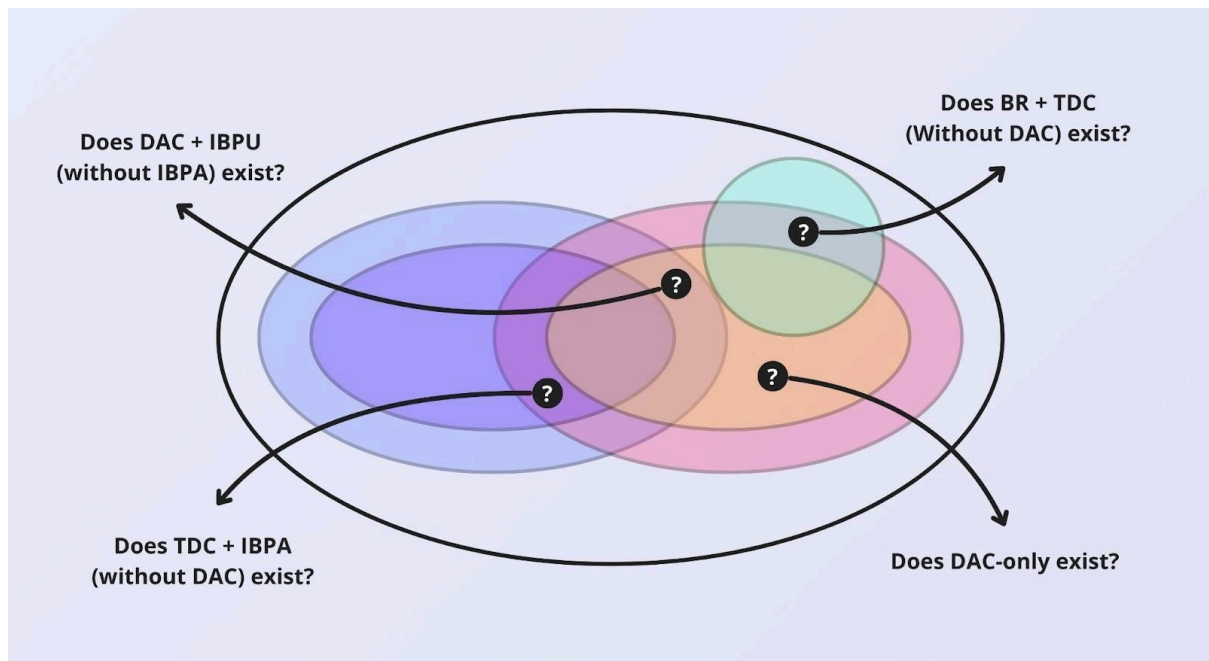


So far, I've only found BR configurations that either satisfy the DAC pattern or fail to satisfy the TDC pattern. I haven't found a BR configuration that satisfies TDC but not DAC. However, I suspect that such a configuration doesn't exist.



I also suspect that DAC configurations either follow the IBPA pattern or the BR pattern; there are no DAC configurations that fail to satisfy both IBPA and BR. But I haven't yet proved this or found a counter proof.

Final diagram of configurations



List of Patterns

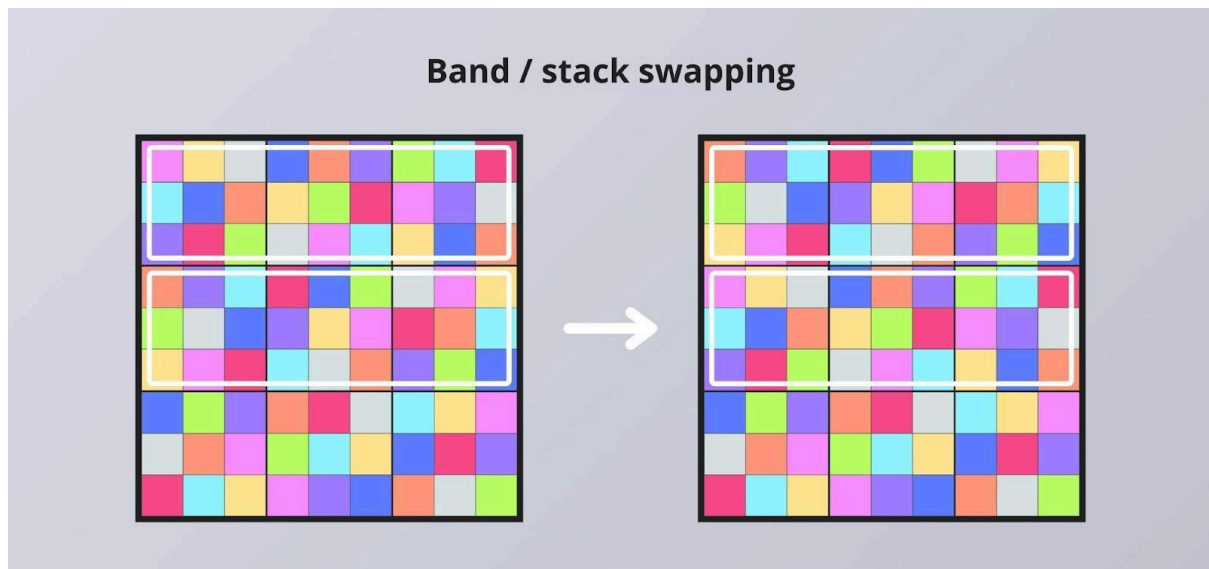
-  Intra-Box Positional Uniqueness - IBPU
-  Triplet Digit Consistency - TDC
-  Intra-Box Positional Alignment - IBPA
-  Digit Adjacency Consistency - DAC
-  Box Repetition - BR

CLASSIFICATION OF TRANSFORMATIONS

Transformations that won't be considered:

- Swapping all instances of one digit with all instances of another digit.
- Rotations or reflections of the Sudoku board.

Band / stack swapping

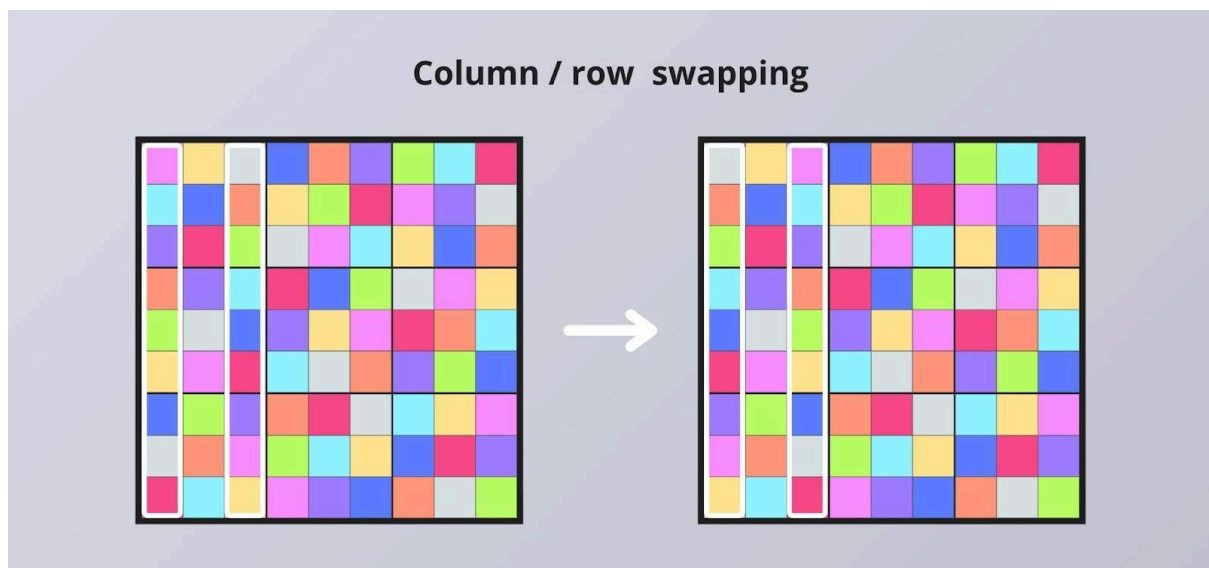


This transformation consists of swapping 2 bands or 2 stacks.

Effects:

- Doesn't break any pattern previously classified (IBPU, IBPA, TDC, DAC, BR).

Row / column swapping within bands / stacks



This transformation consists of swapping 2 columns from the same stack, or 2 rows from the same band.

Effects:

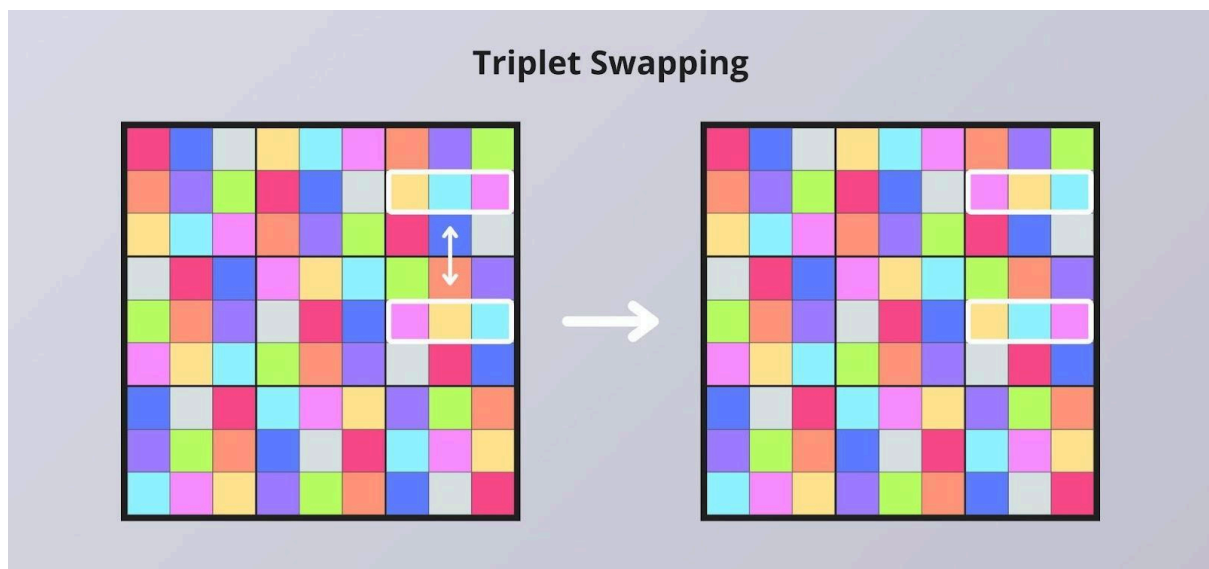
- Breaks IBPA.
- Can break IBPU, but only if IBPA is already broken:

If IBPA isn't broken, this transformation will always swap digits that share the same horizontal or vertical intra-box position. So, for example, a top-left digit will swap with a top-right digit, and there will be no repeated intra-box positions.

But if IBPA is broken, you may swap 2 digits that don't share both horizontal and vertical intra-box position. So, for example, you could swap a "4" at top-left with a "4" at center-right, which would result in a "4" at top-right and a "4" at center-left. But there might already be "4"s at top-right or at center-left, so the intra-box positions might repeat, breaking the IBPU pattern.

- Doesn't break TDC. It changes the intra-box position of the digits in the triplets, but doesn't change the sets of 3 digits.
- Breaks DAC.
- Breaks BR.

Triplet swapping

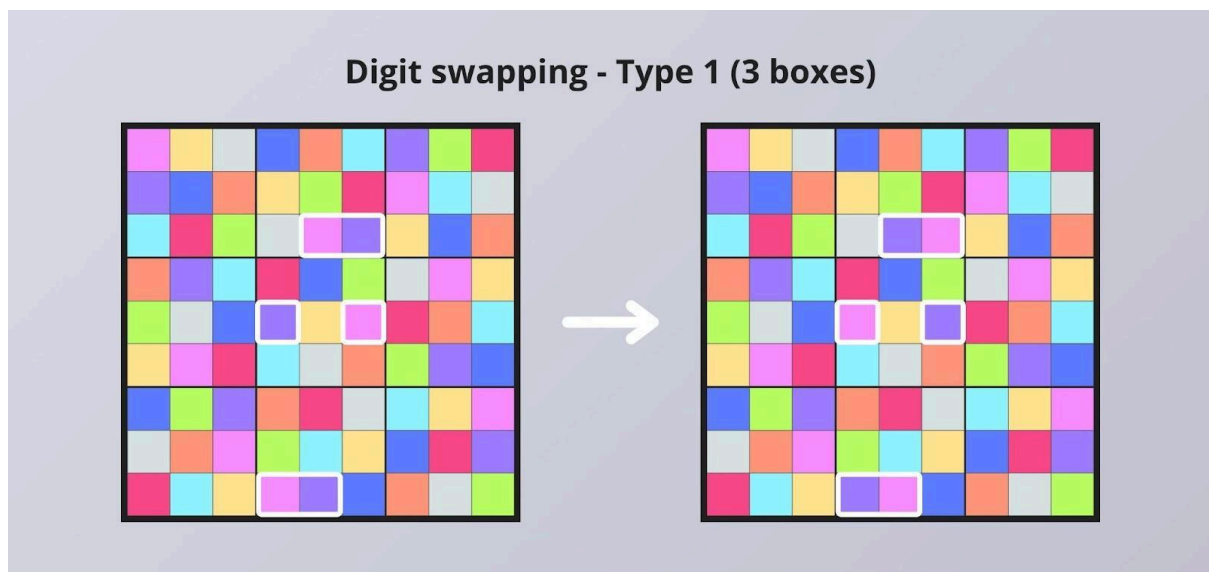


This transformation consists of swapping 2 triplets. The triplets swapped must have the same set of 3 digits, have the same orientation (horizontal or vertical), and be in different boxes along the same band (if triplet orientation is vertical) or stack (if triplet orientation is horizontal).

Effects:

- Breaks IBPA.
- Can break IBPU, but only if IBPA is already broken.
- Breaks TDC.
- Breaks DAC.
- Breaks BR.

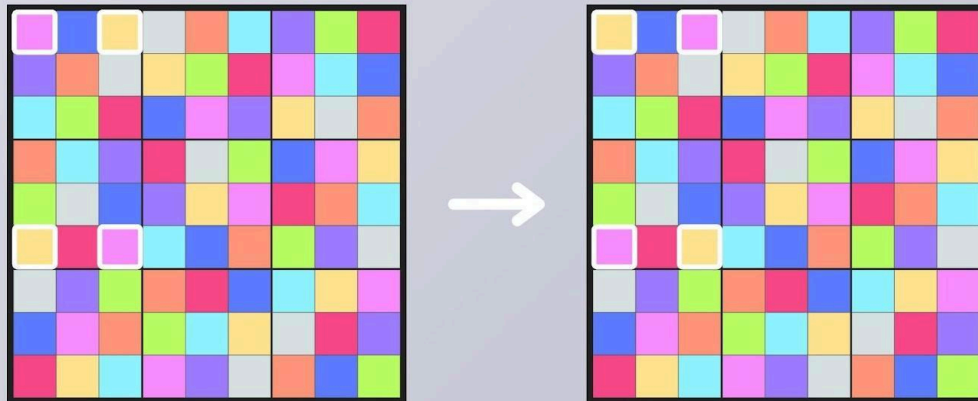
Digit swapping – type 1



This is one type of Digit Swapping. It consists of swapping 2 digits of the same triplet, in the boxes along a band (if the triplet is vertically oriented) or stack (if the triplet is horizontally oriented). The 2 digits swapped must be the same in each box of the band / stack.

It not only works when swapping digits in 3 boxes in a band / stack, but also when swapping digits in 2 boxes:

Digit swapping - Type 1 (2 boxes)

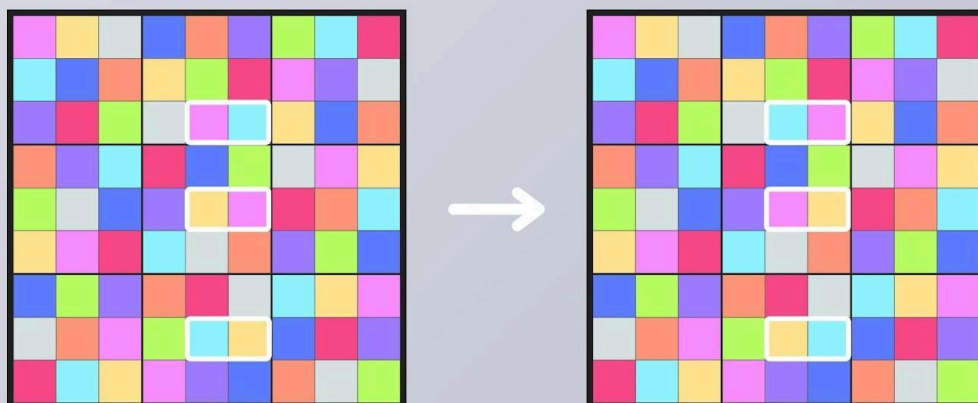


Effects:

- Breaks IBPA.
- Can break IBPU, but only if IBPA is already broken.
- Breaks TDC.
- Breaks DAC.
- Breaks BR.

Digit swapping – type 2

Digit swapping - Type 2

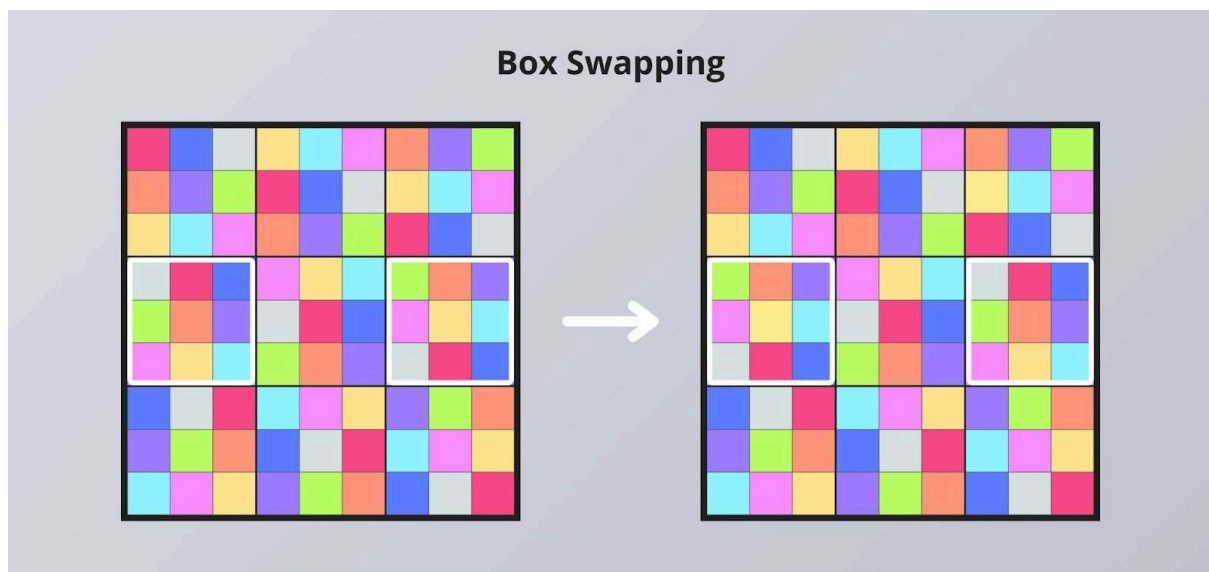


This is the second type of digit swapping. It consists of swapping 2 digits of the same triplet, in the boxes along a band (if the triplet is vertically oriented) or stack (if the triplet is horizontally oriented). The 2 digits swapped must be different in each box of the band / stack, but in total there should be 3 different digits being swapped.

Effects:

- Breaks IBPA.
- Can break IBPU, but only if IBPA is already broken.
- Breaks TDC.
- Breaks DAC.
- Breaks BR.

Box swapping



This transformation consists of swapping 2 boxes within the same band / stack.

Effects:

- Breaks IBPA
- Doesn't break IPBU.
- Doesn't break TDC.
- Doesn't break DAC.
- Can't be applied to BR configurations without breaking Sudoku constraints.

MY CONJECTURE

Every Sudoku configuration can be reached from any other Sudoku configuration by applying a certain sequence of transformations.