A Preliminary Report on

Lid-Driven Cavity Problem

by

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GOVERNING EQUATIONS

THE CONTINUITY EQUATION

Physical principle : **Mass is conserved**
Net mass flow out of control volume through S = Time rate decrease of mass inside control volume

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \, d\mathcal{V} + \iint_{S} \rho \, \mathbf{V} \cdot \mathbf{dS} = 0 \tag{1}$$

Eq. (1) is an *integral form of the continuity equation* derived on the basis of a *finite* control volume *fixed* in space. The fact that the element was *fixed* in *space* leads to the specific integral form given by Eq. (1), which is called the *concervation form.*

$$\frac{D}{Dt}\iiint_{\mathcal{V}} \rho d\mathcal{V} = 0 \tag{2}$$

Eq. (2) is an *integral form of the continuity equation* derived on the basis of a *finite* control volume *moving* with the fluid. The fact that the control volume is *moving* with the fluid leads to the specific integral from given by Eq. (2), which is called the *nonconcervation form.*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{3}$$

Eq. (3) is a *partial differential equation form of the continuity equation* derived on the basis of an *infinitesimally small* element *fixed* in space. The fact that the element was *fixed* in *space* leads to the specific differential form given by Eq. (3), which is called the *concervation form.*

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0 \tag{4}$$

Eq. (4) is a *partial differential equation form of the continuity equation* derived on the basis of an *infinitesimally small* fluid element *moving* with the flow. The fact that the element is *moving* with the fluid leads to the specific integral from given by Eq. (4), which is called the *nonconcervation form.*

All the four equations are the same with some manupulation. The integral form of the equation allows for the presence of discontinuities inside the fixed volume, but the differential form do not. Therefore the integral form of the equations to be considered more fundamental than the differential form.

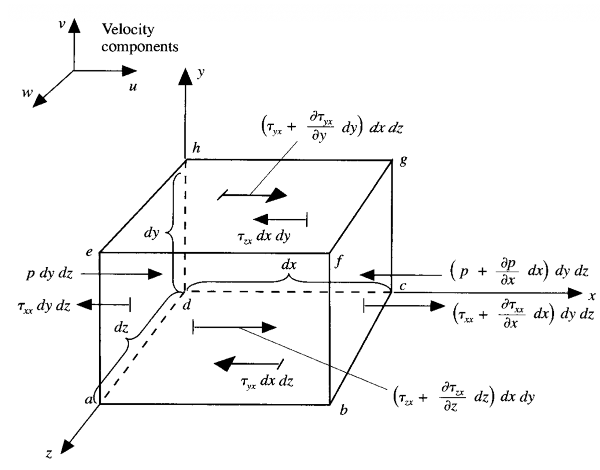
THE MOMENTUM EQUATION

Physical principle : $\mathbf{F} = m\mathbf{a}$
Body Forces : The forces which acts directly on the volumetric mass of the fluid element.

Eg. Gravitational, electric and magnetic forces.

Surface Forces : They act directly on the surface of the fluid element.

Eg. Pressure force, shear force and normal force.



Navier-Stokes equations in x, y and z directions in *nonconservation form* and in scalar form.

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \tag{5}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{6}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \tag{7}$$

Navier-Stokes equations in x, y and z directions in *conservation form.*

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \tag{8}$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{9}$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{V}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \tag{10}$$

where f_x, f_y and f_z are the body force per unit mass. Shear stress in a fluid is proportional to the time rate of stain i.e., velocity gradient, and such fluids are called *newtonial* fluids. Assuming the fluid to be newtonian,

$$\tau_{xx} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial u}{\partial x} \tag{11}$$

$$\tau_{yy} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial v}{\partial y} \tag{12}$$

$$\tau_{zz} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial w}{\partial z} \tag{13}$$

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \tag{14}$$

$$\tau_{xz} = \tau_{zx} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \tag{15}$$

$$\tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \tag{16}$$

where μ is the molecular viscosity coefficient and λ is the second viscosity coefficient.

Bulk Viscosity

In Stokes' hypothese, μ_b is set to be identical to 0. The stress-strain rate relationship (i.e., constitutive relation) for a Newtonian fluid is given as follows:

$$\sigma_{ik} = -P_{thermo}\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + \lambda\frac{\partial u_j}{\partial x_j}\delta_{ik} \tag{17}$$

where, σ_{ik}, is Cauchy's stress tensor, δ_{ik} is the Kronecker delta, u_i is velocity of the fluid, x_i is spatial coordinate, the scalar quantity P_{thermo} is the thermodynamic pressure or hydrostatic pressure, μ is the first coefficient of viscosity, and λ is the second coefficient of viscosity. Eq. (17) can be rearranged as:

$$\sigma_{ik} = -P_{thermo}\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\frac{\partial u_j}{\partial x_j}\delta_{ik}\right) + \mu_b\frac{\partial u_j}{\partial x_j}\delta_{ik} \tag{18}$$

where $\mu_b = \left(\frac{2}{3}\mu + \lambda\right)$ and is called bulk viscosity which represents the irreversible resistance, over and above the reversible resistance, caused by isentropic bulk modulus to change of volume. The bulk viscosity of monatomic gases in the dilute gas limit is zero. By Stokes' hypothesis $\lambda = -\frac{2}{3}\mu$.

The mechanical pressure, P_{mech}, is defined as the negative average of the diagonal terms of stress tensor, as given below:

$$P_{mech} = -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = P_{thermo} - \mu_b\nabla\cdot\overrightarrow{u} \tag{19}$$

which means by Stokes' hypothesis $P_{mech} = P_{thermo}$ i.e., viscous force do not depend upon the rate of expansion or compression at all. But later, if the analysis with or without bulk modulus provides same result, then it could be because of smaller $\nabla\cdot\overrightarrow{u}$ rather than μ_b being 0.

Gas	μ_b/μ	Gas	μ_b/μ
Carbon monoxide	0.548	Dimethylpropane	3.265
Nitrogen	0.769	Water vapour	7.36
n-Pentane	0.896	Hydrogen	28.95
iso-Pentane	1.057	Chlorine	751.88
n-Butane	1.13	Fluorine	2329
iso-Butane	2.00	Carbon dioxide	3828

Table 1: Ratio of bulk viscosity to shear viscosity for common gases at 300 K

Temperature (°C)	Bulk Viscosity	Standard Error
7	4.50E-03	1.20E-05
10	4.03E-03	1.17E-05
15	3.38E-03	1.10E-05
25	2.47E-03	1.08E-05
40	1.84E-03	2.70E-05
50	1.48E-03	2.76E-05

Table 2: Parameter values for water for 6 selected temperatures.

When the atmosphere is majorly consists of gases like CO_2, then it becomes necessary for the bulk modulus in Navier-Stokes equation. The effects of bulk viscosity should be accounted for in the study of high-speed entry into planetary atmospheres. The inclusion of bulk viscosity could significantly increase heat transfer in the hypersonic boundary layer. Bulk viscosity effects cannot be neglected for turbulent flows of fluids with high bulk to shear viscosity ratio. It increases the decay rate of turbulent kinetic energy. During a study of bulk viscosity on how it affects turbulent flows, the results of refined large-eddy simulations (LES) might show dependence on the presence/absence of bulk viscosity, but Reynolds-averaged Navier–Stokes (RANS) simulations might not as they are based on statistical averages.

During a testing of compressible turbulent one-, two-, and three-dimensional Couette flows through DNS simulations for carbon capturing and storage (CCS) compressors, assuming bulk viscosity to be zero does not produce any significant errors, despite the compressors operating at supersonic conditions. However, bulk viscosity effects may become significantly close to the thermodynamic critical point.

Complete Navier-Stokes equation is concervation form:

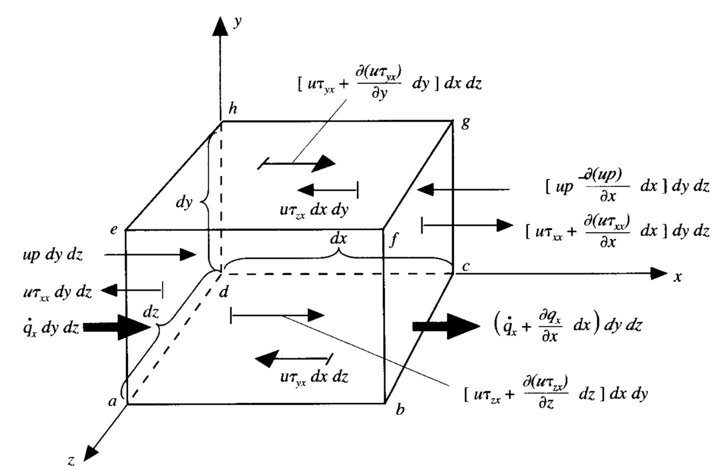
$$\begin{aligned}\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} &= -\frac{\partial p}{\partial x} \\ &+ \frac{\partial}{\partial x}\left(\lambda\nabla\cdot\mathbf{V} + 2\mu\frac{\partial u}{\partial x}\right) + \\ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] &+ pf_x\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} &= -\frac{\partial p}{\partial y} \\ + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left(\lambda\nabla\cdot\mathbf{V} + 2\mu\frac{\partial v}{\partial y}\right) & \\ + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] &+ pf_y\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} &= -\frac{\partial p}{\partial z} \\ + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] & \\ + \frac{\partial}{\partial z}\left(\lambda\nabla\cdot\mathbf{V} + 2\mu\frac{\partial w}{\partial z}\right) &\end{aligned} \tag{22}$$

THE ENERGY EQUATION

Physical principle : **Energy is conserved**
Rate of change of energy inside fluid element = Net flux of heat into element + Rate of work done on element due to body and surface forces



Rate of work done on element due to body and surface forces in a infinitesimally small element

$$\begin{aligned}= \Bigg[&- \left(\frac{\partial(u\rho)}{\partial x} + \frac{\partial(v\rho)}{\partial y} + \frac{\partial(w\rho)}{\partial z}\right) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} \\ &+ \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(u\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} \\ &+ \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\Bigg] dx\,dy\,dz + \rho\mathbf{f}\cdot\mathbf{V}\,dx\,dy\,dz\end{aligned} \tag{23}$$

The heat flux is due to (a) volumetric heating or emission of radiation and (b) heat transfer across the surface due to temperature gradients i.e., thermal conduction. The net flux of heat into element is

$$= \left[\rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dx\, dy\, dz \quad (24)$$

The fluid element has two contributions to its energy, which are:
(1) Internal energy, e (per unit mass).
(2) Kinetic energy per unit mass, $V^2/2$.

Total energy in *nonconcervation* from is

$$= \rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\, dy\, dz \quad (25)$$

The energy equation in *concervation* form

$$\begin{aligned}\rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) &= \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \\ &+ \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \frac{\partial(u\rho)}{\partial x} - \frac{\partial(v\rho)}{\partial y} - \frac{\partial(w\rho)}{\partial z} \\ &+ \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} \\ &+ \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} \\ &+ \frac{\partial(w\tau_{zz})}{\partial z} + \rho\mathbf{f}\cdot\mathbf{V}\end{aligned} \quad (26)$$

The energy equation in *nonconcervation* from by removing Kinetic energy term, applying velocity gradients and viscosity coefficients is

$$\begin{aligned}\rho\frac{De}{Dt} &= \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\ &- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \\ &+ \mu\Bigg[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \\ &\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\Bigg]\end{aligned} \quad (27)$$

Eq. (27) in *concervation* form is written as:

$$\begin{aligned}&\frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e\mathbf{V}) = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\ &- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \mu\Bigg[2\left(\frac{\partial u}{\partial x}\right)^2 + \\ &2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \\ &\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\Bigg]\end{aligned} \quad (28)$$

The energy equation with Kinetic and internal energy in *concervation* for is written as:

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\right] = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \\ &\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \frac{\partial(u\rho)}{\partial x} - \frac{\partial(v\rho)}{\partial y} - \\ &\frac{\partial(w\rho)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \\ &\frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho\mathbf{f}\cdot\mathbf{V}\end{aligned} \quad (29)$$

EQUATION OF STATE

$$p = \rho RT \quad (30)$$

$$\begin{aligned}P = \frac{RT}{V-b} + \frac{A_2 + B_2T + C_2e^{-5.475T/T_c}}{(V-b)^2} + \\ \frac{A_3 + B_3T + C_3e^{-5.475T/T_c}}{(V-b)^3} + \frac{A_4}{(V-b)^4} + \frac{B_5T}{(V-b)^5}\end{aligned} \quad (31)$$

where $A_2, B_2, C_2, A_3, B_3, C_3, A_4$ and B_5 = characteristic constant
P = pressure
R = gas constant
T = temperature in absolute scale
V = specific volume
b = a characteristic constant for a given substance

Parameters	CO_2	H_2O	N_2
T_c	304.17 °R	647.28 °R	126.1K
P_c	72.75 lb./sq. in.	2181.4 lb./sq. in.	33.5 atm.
V_c	0.00216 cu. ft./lb.	0.00314 cu. ft./lb.	9.01×10^{-5} cc./g
b	0.007495	0.0063101	22.1466
R	0.24381	0.59545	82.055
A_2	-8.9273631	-87.7394396	-1592238.2
B_2	0.005262476	0.0312961744	3,221.616
C_2	-150.97587	-2590.5815	-22,359,930
A_3	0.18907819	3.09248249	89,845,367
B_3	-0.0000704617	-0.00082418321	-134,024.05
C_3	0.0831424	113.95968	1,518,821,653
A_4	-0.002112459	-0.0567185967	-2,467,297,480
B_5	1.9565593×10^{-8}	4.2388378×10^{-7}	244,915,183

Table 3: Constants evaluated for the equation
All the constants for the equation of state are given in units of T_c, P_c and V_c of that compound.

FORMS OF GOVERNING EQUATIONS SUITED FOR CFD

The use of conservation form of governing equationa provides a numerical and computer programing convenience in that the continuity, momentum, and energy equations in conservation form can all be expressed by the same generic equation. We are more consерned about the flux of some physical quantity rather than primitive variables such as $p, \rho, \mathbf{V}$.

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = J \quad (32)$$

where U is called the *solution vector*, F, G and H are called *flux terms* and J represents the *source term.* The below equation is done by time marching methord.

$$\frac{\partial U}{\partial t} = J - \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} - \frac{\partial H}{\partial z}$$

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho\left(e + \frac{V^2}{2}\right) \end{Bmatrix} \quad (33)$$

$$F = \begin{Bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho vu - \tau_{xy} \\ \rho wu - \tau_{xz} \\ \rho\left(e + \frac{V^2}{2}\right)u + pu - k\frac{\partial T}{\partial x} - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} \end{Bmatrix} \quad (34)$$

$$G = \begin{Bmatrix} \rho v \\ \rho uv - \tau_{yx} \\ \rho v^2 + p - \tau_{yy} \\ \rho wv - \tau_{yz} \\ \rho\left(e + \frac{V^2}{2}\right)v + pv - k\frac{\partial T}{\partial y} - u\tau_{yx} - v\tau_{yy} - w\tau_{yz} \end{Bmatrix} \quad (35)$$

$$H = \begin{Bmatrix} \rho w \\ \rho u w - \tau_{zx} \\ \rho v w - \tau_{yz} \\ \rho w^2 + p - \tau_{zz} \\ \rho\left(e + \frac{V^2}{2}\right) w + p w - k\frac{\partial T}{\partial z} - u\tau_{zx} - v\tau_{zy} - w\tau_{zz} \end{Bmatrix} \tag{36}$$

$$J = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(u f_x + v f_y + w f_z) + \rho\dot{q} \end{Bmatrix} \tag{37}$$

For an *inviscid* flow, the equations can be reduced as:

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho u\left(e + \frac{V^2}{2}\right) \end{Bmatrix} \tag{38}$$

$$F = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ \rho u\left(e + \frac{V^2}{2}\right) + p u \end{Bmatrix} \tag{39}$$

$$G = \begin{Bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho w v \\ \rho v\left(e + \frac{V^2}{2}\right) + p v \end{Bmatrix} \tag{40}$$

$$H = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho u\left(e + \frac{V^2}{2}\right) \end{Bmatrix} \tag{41}$$

$$J = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(u f_x + v f_y + w f_z) + \rho\dot{q} \end{Bmatrix} \tag{42}$$

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho} \tag{43}$$

From eq.(38) - eq.(43), the solutions obtained are the terms in U vector. These dependent variables are then used to find primitive variables such as ρ, u, v, w, e and p.

For steady state, $\partial U/\partial t = 0$. In such cases the below equation is used (using space marching):

$$\frac{\partial F}{\partial x} = J - \frac{\partial G}{\partial y} - \frac{\partial H}{\partial z}$$

Here F becomes the solution vector.

DISCRETIZATION

Finite Difference Methord

Discretization is the the process by which a closed-form mathematical expression, such as a function or a differential or integral equation involving functions, all of which are viewed as having infinite continuum of values throughout some domain, is approximated by analogous (but different) expressions which prescribe values at only a finite number of discrete points or volumes in the domain. Discretization of partial differential equation is called *finite differences*, and discretization of integral form of the equations is called *finite volumes*.

FINITE DIFFERENCES

If $u_{i,j}$ represents the x component of velocity at point (i,j), then by Taylor series expansion:

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j}\Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{3!} + \dots \tag{1}$$

$$f(x+\Delta x) = \underbrace{f(x)}_{\substack{\text{First guess}\\ \text{(not very good)}}} + \underbrace{\frac{\partial f}{\partial x}\Delta x}_{\substack{\text{Add to capture}\\ \text{slope}}} + \underbrace{\frac{\partial^2 f}{\partial x^2}\frac{(\Delta x)^2}{2}}_{\substack{\text{Add to account}\\ \text{for curvature}}} + \dots$$

From Eq. (1), solving for $\partial u/\partial x$ at (i,j)

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \underbrace{\frac{u_{i+1,j}-u_{i,j}}{\Delta x}}_{\substack{\text{Finite-}\\ \text{difference}\\ \text{representation}}} \underbrace{- \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{\Delta x}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^2}{6} + \dots}_{\text{Truncation error}} \tag{2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j}-u_{i,j}}{\Delta x} + \mathbf{O}(\Delta x) \tag{3}$$

If $\mathbf{O}(\Delta x)$ is removed from Eq. (3), then it is called *first-order-accurate*. $\mathbf{O}(\Delta x)$ is a formal mathematical notation which represents "terms of order Δx". Finite difference in Eq. (3) is called *forward difference* as it uses $u_{i+1,j}$ and $u_{i,j}$. Thus Eq. (3) is called *first-order forward difference*.

$$u_{i-1,j} = u_{i,j} - \left(\frac{\partial u}{\partial x}\right)_{i,j}\Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{(\Delta x)^2}{2!} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{3!} + \dots \tag{4}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j}-u_{i-1,j}}{\Delta x} + \mathbf{O}(\Delta x) \tag{5}$$

Similarly Eq. (4) is called *first-order rearward difference.* Subtracting Eq.(4) from Eq.(1)

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} + \mathbf{O}(\Delta x)^2 \tag{6}$$

Eq.(6) is called *second-order central difference.* For second partial derivatives, we sum Eq.(1) and Eq.(4), we get *second-order central second difference.*

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2} + \mathbf{O}(\Delta x)^2 \tag{7}$$

For forth-order-accurate, we need five known grid points, for this accuracy:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{-u_{i+2,j}+16u_{i+1,j}-30u_{i,j}+16u_{i-1,j}-u_{i-2,j}}{12(\Delta x)^2} + \mathbf{O}(\Delta x)^4 \tag{8}$$

For mixed derivatives, such as $\partial^2 x/\partial x\,\partial y$, we differentiate Eq. (1), Eq. (4) with respect to y, we get

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \left(\frac{\partial u}{\partial y}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j}\Delta x + \left(\frac{\partial^3 u}{\partial x^2\,\partial y}\right)_{i,j}\frac{(\Delta x)^2}{2!} + \left(\frac{\partial^4 u}{\partial x^3\,\partial y}\right)_{i,j}\frac{(\Delta x)^3}{3!} + \dots \tag{9}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \left(\frac{\partial u}{\partial y}\right)_{i,j} - \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j}\Delta x + \left(\frac{\partial^3 u}{\partial x^2\,\partial y}\right)_{i,j}\frac{(\Delta x)^2}{2!} - \left(\frac{\partial^4 u}{\partial x^3\,\partial y}\right)_{i,j}\frac{(\Delta x)^3}{3!} + \dots \tag{10}$$

Subtracting Eq. (10) from Eq. (9), we get

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j} = 2\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j}\Delta x + 2\left(\frac{\partial^4 u}{\partial x^3\,\partial y}\right)_{i,j}\frac{(\Delta x)^3}{3!} + \dots \tag{11}$$

Solving for $(\partial^2 u/\partial x\,\partial y)$

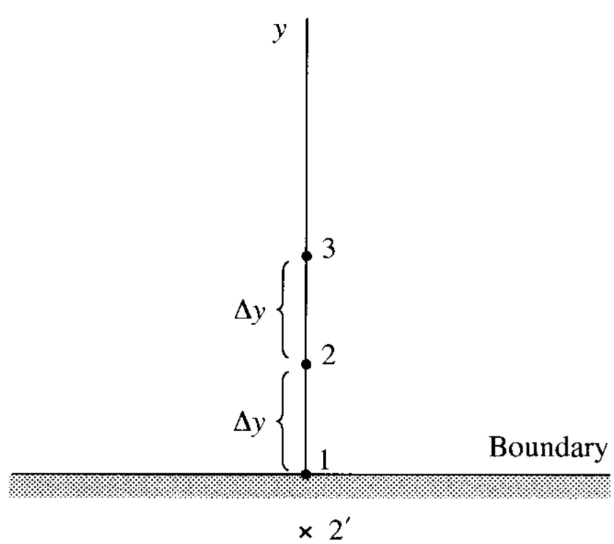
$$\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} = \frac{(\partial u/\partial y)_{i+1,j}-(\partial u/\partial y)_{i-1,j}}{2\Delta x} - \left(\frac{\partial^4 u}{\partial x^3\,\partial y}\right)_{i,j}\frac{(\Delta x)^2}{3!} + \dots$$

Substituting for $(\partial u/\partial y)$ for appropriate grid values, we get

$$\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} = \frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i-1,j-1}}{4\Delta x\,\Delta y} + \mathbf{O}[(\Delta x)^2(\Delta y)^2] \tag{12}$$

Eq. (12) is called *second-order central difference for the mixed derivative.*

For boundary conditions, forward difference can give only upto first-order accuracy. So instead of Taylors series, we use *polynomial approach.*



Taking $u = a + by + cy^2$, at boundary i.e. grid point 1, $y = 0$, which implies $u_1 = a$. At grid point 2, $y = \Delta y$, which implies $u_2 = a + b\Delta y + c(\Delta y)^2$. At grid point 3 where $y = 2\Delta y$, $u_3 = a + b(2\Delta y) + c(2\Delta y)^2$. Solving for b, we get

$$b = \frac{-3u_1 + 4u_2 - u_3}{2\Delta y}$$

differentiating with respect to y,

$$\frac{\partial u}{\partial y} = b + 2cy$$

evaluating at $y = 0$, yields

$$\left(\frac{\partial u}{\partial y}\right)_1 = b$$

which implies

$$\left(\frac{\partial u}{\partial y}\right)_1 = \frac{-3u_1 + 4u_2 - u_3}{2\Delta y}$$

comparing with Taylor series expansion of $u(y)$, the numerator yields $\mathbf{O}(\Delta y)^3$, but since we are dividing by Δy which is in denominator, we get second order accuracy.

$$\left(\frac{\partial u}{\partial y}\right)_1 = \frac{-3u_1 + 4u_2 - u_3}{2\Delta y} + \mathbf{O}(\Delta y)^2 \tag{13}$$

Eq. (13) is the second-order-accurate difference quotient at the boundary and is called *one-sided differences.* These can be applied to internal grid points as well, not limited to boundary.

DIFFERENCE EQUATIONS

When all the partial derivatives in a given partial differential equation are replaced by finite-difference quotients, the resulting *algebraic* equation is called a *difference equation*, which is an algebraic representation of the partial differential equation. In a unsteady, one-dimensional heat conduction with constant thermal diffusivity

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{14}$$

has two independent variables x and t, where the marching variable is t. In CFD it is conventional to write marching variable index in superscript in the finite-difference quotient.

$$\left(\frac{\partial T}{\partial t}\right)_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} - \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \dots \tag{15}$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} - \left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + \dots \tag{16}$$

Eq. (14) can be written as

$$\begin{aligned}\underbrace{\frac{\partial T}{\partial t} - \alpha\frac{\partial^2 T}{\partial x^2}}_{\text{Partial differential equation}} = 0 = \underbrace{\frac{T_i^{n+1} - T_i^n}{\Delta t} - \frac{\alpha\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)}{(\Delta x)^2}}_{\text{Difference equation}} + \\ \underbrace{\left[-\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \alpha\left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + \dots\right]}_{\text{Truncation error}}\end{aligned} \tag{17}$$

Difference equation for Eq.(17) is:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)}{(\Delta x)^2} \tag{18}$$

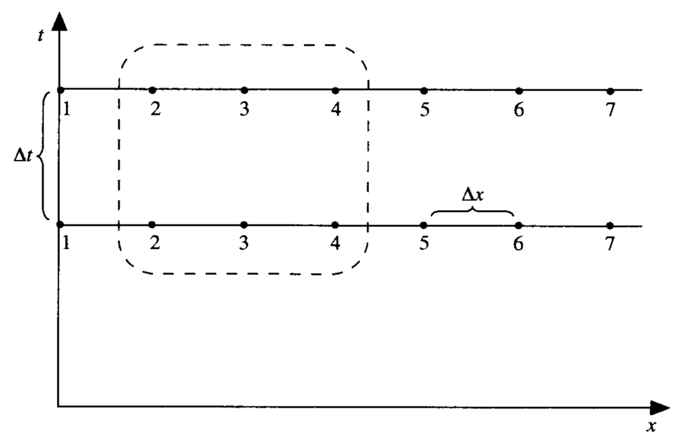
having a truncation error of $\mathbf{O}[\Delta t, (\Delta x)^2]$. If $\Delta x \to 0$ and $\Delta t \to 0$, in this case the finite difference representation of the partial differential equation is said to be *consistent.*

In an explicit approach each difference equation contains only one unknown and therefore can be solved *explicitly* for this unknown in a straightforward manner. Writting the spatial difference on the right-hand side in terms of *average* properties between time levels n and $n+1$ of Eq. (14):

$$\begin{aligned}\frac{T_i^{n+1} - T_i^n}{\Delta t} = \\ \alpha\frac{\frac{1}{2}\left(T_{i+1}^{n+1} + T_{i+1}^n\right) + \frac{1}{2}\left(-2T_i^{n+1} - 2T_i^n\right) + \frac{1}{2}\left(T_{i-1}^{n+1} + T_{i-1}^n\right)}{(\Delta x)^2}\end{aligned} \tag{19}$$

The special type of differencing employed in Eq. (19) is called the *Crank-Nicolson form.* This differencing methord is commonly used to solve problem governed by parabolic equation. In CFD, it is frequently used for finite-difference solutions of the boundary-layer equations. Eq. (19) cannot be solved explicitly, rather the equation must be written at all interior gird points, resulting in a system of algebraic equation for which the unknowns T_i^{n+1} for all i can be solved simultaneously. This is *implicit* approach. *Implicit* approach is the one where the unknowns must be obtained by means of a *simultaneous solution* of the difference equations applied to *all* the grid points arrayed at a given time level. Rearranging Eq. (40) we get:

$$\begin{aligned}\frac{\alpha\Delta t}{2(\Delta x)^2}T_{i-1}^{n+1} - \left[1 + \frac{\alpha\Delta t}{(\Delta)^2}\right]T_i^{n+1} + \frac{\alpha\Delta t}{2(\Delta x)^2}T_{i+1}^{n+1} = \\ -T_i^n - \frac{\alpha\Delta t}{2(\Delta x)^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)\end{aligned} \tag{20}$$



Taking

$$A = \frac{\alpha\Delta t}{2(\Delta x)^2}$$

$$B = 1 + \frac{\alpha\Delta t}{(\Delta)^2}$$

$$K_i = -T_i^n - \frac{\alpha\Delta t}{2(\Delta x)^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)$$

Eq. (20) can be written as

$$AT_{i-1}^{n+1} - BT_i^{n+1} + AT_{i+1}^{n+1} = K_i \tag{21}$$

Since grid point 1, 7 are boundary conditions, T_1 and T_7 are known. At grid points 2, 3 ,4, 5, 6

$$-BT_2 + AT_3 = K_2' \tag{22}$$

where $K_2' = K_2 - AT_1$

$$AT_2 - BT_3 + AT_4 = K_3 \tag{23}$$

$$AT_3 - BT_4 + AT_5 = K_4 \tag{24}$$

$$AT_4 - BT_5 + AT_6 = K_5 \tag{25}$$

$$-BT_6 + AT_7 = K_6' \tag{26}$$

where $K_6' = K_6 - AT_7$. Combining Eq. (22) to Eq. (26) in matrix form

$$\begin{bmatrix} -B & A & 0 & 0 & 0 \\ A & -B & A & 0 & 0 \\ 0 & A & -B & A & 0 \\ 0 & 0 & A & -B & A \\ 0 & 0 & 0 & A & -B \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} K_2' \\ K_3 \\ K_4 \\ K_5 \\ K_6' \end{bmatrix} \tag{27}$$

The coefficient matrix is a *tridiagonal matrix*, defined as having nonzero elements only along the three diagonals.

In Explicit approach for a given Δx, Δt must be less than some limit imposed be stability constraints to maintain stability. This can cause long computer running times for small values of Δt. In implicit approach, since larger Δt can be taken, the truncation error is large. In such case the implicit approach may not be accurate than explicit approach. However, for a time-dependent solution in which the steady state is the desired result, this relative timewise inaccuracy is not important.

STABILITY ANALYSIS AND ERRORS

Discretization error is the difference between the exact analytical solution of the partial differential equation and the exact solution of the corresponding difference equation. *Round-off error* is the numerical error introduced after a repetitive number of calculations in which the computer is constantly rounding the numbers to some significant figure. If

A = analytical solution of partial differential equation

D = exact solution of difference equation

N = numerical solution from a real computer with finite accracy

$$\begin{aligned} &\text{Discretization error} = A - D \\ &\text{Round-off error} = \epsilon = N - D \\ &N = D + \epsilon \end{aligned} \tag{28}$$

The numerical solution N must satisify Eq. (18). Then by Eq. (28)

$$\frac{D_i^{n+1} + \epsilon_i^{n+1} - D_i^{n+1} - \epsilon_i^{n+1}}{\alpha \Delta t} = \frac{D_{i+1}^n + \epsilon_{i+1}^n - 2D_i^n - 2\epsilon_1^n + D_{i-1}^n + \epsilon_{i-1}^n}{(\Delta x)^2} \tag{29}$$

Also the exact solution of difference equation D must also solve Eq. (18).

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} = \frac{\alpha \left(D_{i+1}^n - 2D_i^n + D_{i-1}^n\right)}{(\Delta x)^2} \tag{30}$$

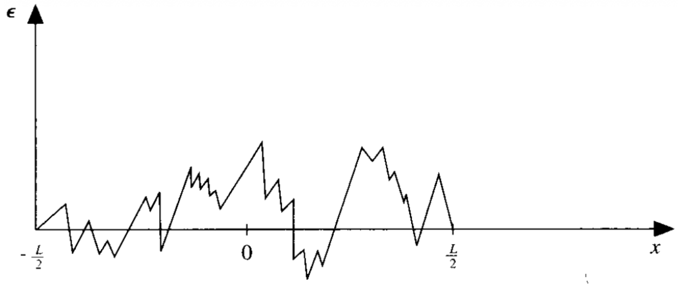
Subtracting Eq. (30) from Eq. (29)

$$\frac{\epsilon_i^{n+1} - \epsilon_i^n}{\Delta t} = \frac{\alpha \left(\epsilon_{i+1}^n - 2\epsilon_i^n + \epsilon_{i-1}^n\right)}{(\Delta x)^2} \tag{31}$$

If the solution is *stable*, the ϵ_i should shrink as solution progress from n to $n+1$, or else it is *unstable*. For the solution to be stable

$$\left| \frac{\epsilon_i^{n+1}}{\epsilon_i^n} \right| \leq 1 \tag{32}$$

At boundary, the error is 0 because of the boundary conditions specified.



Plotting ϵ vs x, the random variation of ϵ with x can be expressed analytically by a Fourier series as:

$$\epsilon(x) = \sum_m A_m e^{ik_m x} = \sum_m A_m (\cos k_m x \, + \, i \sin k_m x) \tag{33}$$

where k_m is called *wave number*. The real part of Eq. (33) represents the error.

$$k_m = \left(\frac{2\pi}{L}\right) m \quad m = 1, 2, 3, \ldots \tag{34}$$

where m is the number of waves fitted into the given interval. In practical senarois, it is not possible to sum it up to infinity to get a countineous variation of ϵ because of the finite grid size. So the longest wavelength which can be fitted can be $\lambda_{max} = L$, which the smallest wavelength which can be fitted in a grid having N grid space is $\lambda_{min} = 2\Delta x$. Where $\Delta x = L/N$, implies $\lambda_{min} = 2L/N$.

$$k_m = \frac{2\pi}{2L/N} = \frac{2\pi}{L}\frac{N}{2}$$

Comparing Eq. (33), highest-order harmonic allowed is $N/2$, also assuming A_m is a function of time.

$$\epsilon(x,t) = \sum_{m=1}^{N/2} A_m(t) e^{ik_m x} \tag{35}$$

It is resonable to assume that the error may grow or diminsh exponentially with time, we write

$$\epsilon(x,t) = \sum_{m=1}^{N/2} e^{at} e^{ik_m x} \tag{36}$$

where a is a constant which may take different values for different m. By taking a particular value of m for Eq. (36) and substituting in Eq. (31) for next step (Δt, Δx)

$$\frac{e^{a(t+\Delta t)} e^{ik_m x} - e^{at} e^{ik_m x}}{\alpha \Delta t} = \frac{e^{at} e^{ik_m (x+\Delta x)} - 2e^{at} e^{ik_m x} + e^{at} e^{ik_m (x-\Delta x)}}{(\Delta x)^2} \tag{37}$$

Dividing Eq. (37) by $e^{at} e^{ik+mx}$ we get

$$\frac{e^{a\Delta t} - 1}{\alpha \Delta t} = \frac{e^{ik_m x} - 2 + e^{-ik_m \Delta x}}{(\Delta x)^2}$$

or

$$e^{a\Delta t} = 1 + \frac{\alpha \Delta t}{(\Delta x)^2} \left(e^{ik_m x} - 2 + e^{-ik_m \Delta x}\right)$$

by cosine relation $\cos(k_m \Delta x) = \frac{e^{ik_m \Delta x} + e^{-ik_m \Delta x}}{2}$

$$e^{a\Delta t} = 1 + \frac{2\alpha \Delta t}{(\Delta x)^2} \left[\cos(k_m \Delta x) - 1\right] + 0i = G$$

where G is called *amplification* factor. The parabolic function only has real terms in G. For stability $|G| \leq 1$. This general methord is called *von Neumann stability methord*, which is used to study stability property of linear difference equation.

by sine relation $\sin^2 \frac{k_m \Delta x}{2} = \frac{1 - \cos(k_m \Delta x)}{2}$

$$e^{a\Delta t} = 1 - \frac{4\alpha \Delta t}{(\Delta x)^2} \sin^2 \frac{k_m \Delta x}{2} \tag{38}$$

$$\frac{\epsilon_i^{n+1}}{\epsilon_i^n} = \frac{e^{a(t+\Delta t)}e^{ik_m x}}{e^{at}e^{ik_m x}} = e^{a\Delta t} \tag{39}$$

from Eq. (32), Eq. (38) and Eq. (39)

$$\left|\frac{\epsilon_i^{n+1}}{\epsilon_i^n}\right| = \left|e^{a\Delta t}\right| = \left|1 - \frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\frac{k_m\Delta x}{2}\right| \leq 1 \tag{40}$$

$$\left|1 - \frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\frac{k_m\Delta x}{2}\right| \equiv |G|$$

1. $$1 - \frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\frac{k_m\Delta x}{2} \leq 1$$

$$\frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\frac{k_m\Delta x}{2} \geq 0 \tag{41}$$

   This condition always holds because $4\alpha\Delta t/(\Delta x)^2$ is always positive.

2. $$1 - \frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\frac{k_m\Delta x}{2} \geq -1$$

   Thus

$$\frac{\alpha\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{42}$$

Eq. (42) gives the *stability requirement* for the solution of the difference equation to be *stable* and applicable for Eq. (14).

In hyperbolic functions such as $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$, using simple forward difference, we get G as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -c\frac{u_{i+1}^n + u_{i-1}^n}{2\Delta x}$$

$$G = e^{a\Delta x} = 1 - ic\frac{\Delta t}{\Delta x}\sin(k_m\Delta x)$$

taking $c\Delta t/\Delta x = C$

$$|G| = \sqrt{1 + (C^2\sin^2(k_m\Delta x))}$$

For any error mode other than the trivial one, $|G|$ will be strictly greater than 1. By von Newmann stability analysis to the above hyperbolic equation, it will be always be unstable no matter what Δt be, and therefore called *unconditionally unstable*. Instead we replace the time derivative with a first-order difference, where $u(t)$ is represented by an average value between grid points $i+1$ and $i-1$. i.e.,

$$u(t) = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)$$

by substituting we get

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - c\frac{\Delta t}{\Delta x}\frac{u_{i+1}^n - u_{i-1}^n}{2}$$

the amplification factor becomes

$$e^{a\Delta t} = \cos(k_m\Delta x) - iC\sin(k_m\Delta x) \tag{43}$$

for stability $|e^{a\Delta t}| \leq 1$, we get

$$C = c\frac{\Delta t}{\Delta x} \leq 1 \tag{44}$$

where C is called the *courant number*. Eq. (44) is called the *Courant-Friedrichs-Lewy* (CFL) condition. It is a important stability criterion for hyperbolic equation. For stability, the numerical domain must include all the analytical domain. It is desirable to have C as close to unity as possible for accuracy. In broader sense we can write the solution itself as a Fourier series similar to Eq. (33).Still $|G| \leq 1$ implies the stability of solution.

GRIDS WITH TRANSFORMATION

In case of different coordinate system, the governing equations must be transformed from (x, y) to (ξ, η) as the new independent variables. When transforming two dimensional independent variables in physical space (x, y, t) in to a new set of independent variables in transformed space ξ, η, τ, where

$$\xi = \xi(x, y, t)$$
$$\eta = \eta(x, y, t)$$
$$\tau = \tau(t)$$

which represents the transformation. By chain rule

$$\left(\frac{\partial}{\partial x}\right)_{y,t} = \left(\frac{\partial}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \xi}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \tau}\right)_{\xi,\eta}\left(\frac{\partial \tau}{\partial x}\right)_{y,t}$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right) \tag{1}$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right) \tag{2}$$

$$\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial t}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial t}\right) + \left(\frac{\partial}{\partial \tau}\right)\left(\frac{\partial \tau}{\partial t}\right) \tag{3}$$

Eq.(1) to Eq.(3) can be used to replace derivatives with respect to x, y and t from the governing equations. The coefficients of the derivative with respect to ξ, η and τ are called *metrics*. It is the metrices that carry all the specific information pertinent to a specific transformation. These metrics can be evaluated by finite-difference quotients, typically central differences. For second derivatives, differentiating Eq. (1) with respect to x

$$A = \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]$$

by product rule

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial \xi}{\partial x}\right)\underbrace{\left(\frac{\partial^2}{\partial x\,\partial \xi}\right)}_{B} + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial \eta}{\partial x}\right)\underbrace{\left(\frac{\partial^2}{\partial x\,\partial \eta}\right)}_{C} \tag{4}$$

Recalling Eq. (1) for B and C

$$B = \frac{\partial^2}{\partial x\,\partial \xi} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi}\right) = \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta\,\partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right) \tag{5}$$

$$C = \frac{\partial^2}{\partial x\,\partial \eta} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \eta}\right) = \left(\frac{\partial^2}{\partial \xi\,\partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right) \tag{6}$$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\left(\frac{\partial^2}{\partial \eta\,\partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right) \tag{7}$$

Similarly, obtaining second derivative of Eq. (2) with respect to y

$$D \equiv \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial D}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$

$$\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial \xi}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial y\,\partial \xi}\right)}_{E} + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial \eta}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial y\,\partial \eta}\right)}_{F} \tag{8}$$

By applying Eq. (2) to E and F

$$E = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \xi}\right) = \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta\,\partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\right) \tag{9}$$

$$F = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \eta}\right) = \left(\frac{\partial^2}{\partial \eta\,\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right) \tag{10}$$

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\left(\frac{\partial^2}{\partial \eta\,\partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right) \tag{11}$$

Similarly obtaining the second partial with respect to x and y.

$$\frac{\partial^2}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial D}{\partial x} = \frac{\partial}{\partial x}\left[\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial}{\partial \xi}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial}{\partial \eta}\right)\right]$$
$$= \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x\,\partial y}\right) + \left(\frac{\partial \xi}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial \xi\,\partial x}\right)}_{B} + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \left(\frac{\partial \eta}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial \eta \partial x}\right)}_{C}$$

Replacing B and C

$$\frac{\partial^2}{\partial x\,\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x\,\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x\,\partial y}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial^2}{\partial \xi\,\partial \eta}\right)\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)\right] \tag{12}$$

METRICS AND JACOBIANS

Let $u = (x, y)$ where $x = (\xi, \eta)$ and $y = (\xi, \eta)$. The total differential of u is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Differentiating u with respect to ξ and η are

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

Solving for $\partial u/\partial x$ and $\partial u/\partial y$ by Cramer's rule

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} \dfrac{\partial u}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}}$$

The denominator determinant is identified as the *jacobian determinant*, denoted by

$$J \equiv \frac{\partial(x,y)}{\partial(\xi,\eta)} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} \tag{13}$$

$$\frac{\partial u}{\partial x} = \frac{1}{J}\left[\left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi}\right)\right] \tag{14}$$

Similarly, $\partial u/\partial y$ can be found. Let us consider for direct transformation $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$. For inverse transformation $x = x(\xi,\eta)$ and $y = y(\xi,\eta)$. The total differential is expressed as

$$d\xi = \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy$$

$$d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy$$

$$dx = \frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta$$

$$dy = \frac{\partial y}{\partial \xi}d\xi + \frac{\partial y}{\partial \eta}d\eta$$

In matrix forms

$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} \quad ; \quad \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}\begin{bmatrix} d\xi \\ d\eta \end{bmatrix}$$

Multiplying second equation by inverse of the 2×2 coefficient matrix, we get

$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^{-1}\begin{bmatrix} dx \\ dy \end{bmatrix}$$

By comparing

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^{-1}$$

By standard rules

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \frac{\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\ -\dfrac{\partial y}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}}{\begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}}$$

The denominator in above matrix and Eq. (13) is same

$$\begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} \equiv J$$

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{J}\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\ -\dfrac{\partial y}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

By comparing

$$\frac{\partial \xi}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \eta} \tag{15}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial \xi} \tag{16}$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial \eta} \tag{17}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial \xi} \tag{18}$$

Substituting Eq. (15) - Eq. (18) in Eq. (1) and (2), we get

$$\frac{\partial}{\partial x} = \frac{1}{J}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi}\right)\right] \tag{19}$$

$$\frac{\partial}{\partial y} = \frac{1}{J}\left[\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial x}{\partial \xi}\right) - \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right)\right] \tag{20}$$

FORM OF THE GOVERNING EQUATIONS SUITED FOR CFD: THE TRANSFORMED VERSION

For an unsteady flow in two spacial dimensions, with no source terms, the equation is

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{21}$$

To transform Eq. (21) in (ξ, η) space

$$\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} = 0$$

Expanding Eq. (21) by chain rule and multiplying by jacobian J

$$\begin{aligned} J\frac{\partial U}{\partial t} + J\left(\frac{\partial F}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + J\left(\frac{\partial F}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right) + \\ J\left(\frac{\partial G}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + J\left(\frac{\partial G}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right) = 0 \end{aligned} \tag{22}$$

For the expansion for the term $JF(\partial \xi/\partial x)$

$$\frac{\partial[JF(\partial \xi/\partial x)]}{\partial \xi} = J\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial F}{\partial \xi} + F\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right)$$

Rearranging

$$J\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial F}{\partial \xi}\right) = \frac{\partial[JF(\partial \xi/\partial x)]}{\partial \xi} - F\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right) \tag{23}$$

Similarly doing for η and G and substituting it in Eq. (22), we get

$$\begin{aligned} J\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi}\left(JF\frac{\partial \xi}{\partial x} + JG\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta}\left(JF\frac{\partial \eta}{\partial x} + JG\frac{\partial \eta}{\partial y}\right) \\ -F\left[\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial x}\right)\right] - G\left[\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial y}\right)\right] = 0 \end{aligned} \tag{24}$$

Using Eq. (15) - Eq. (18) in square brackets in Eq. (24), we get

$$\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial x}\right) = \frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial y}\right) \equiv 0$$

Eq. (24) can be written as

$$\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} = 0 \tag{25}$$

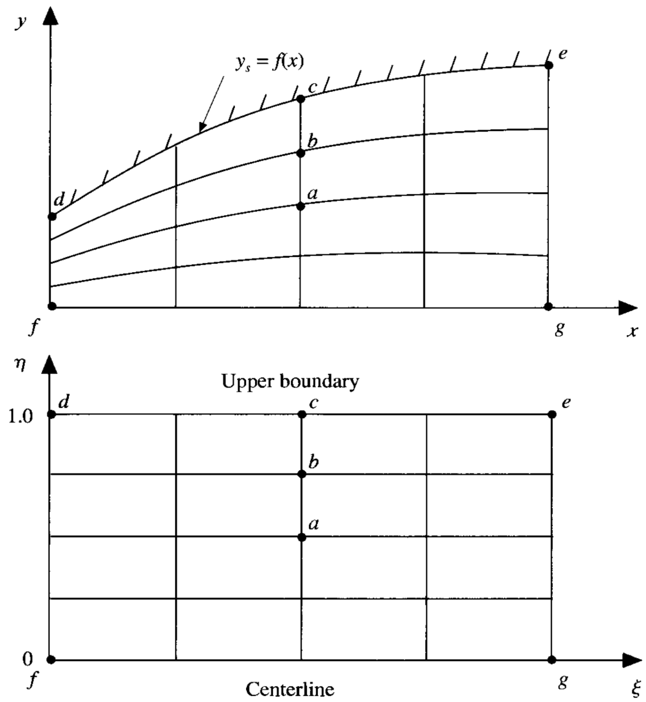
where

$$U_1 = JU \tag{26}$$

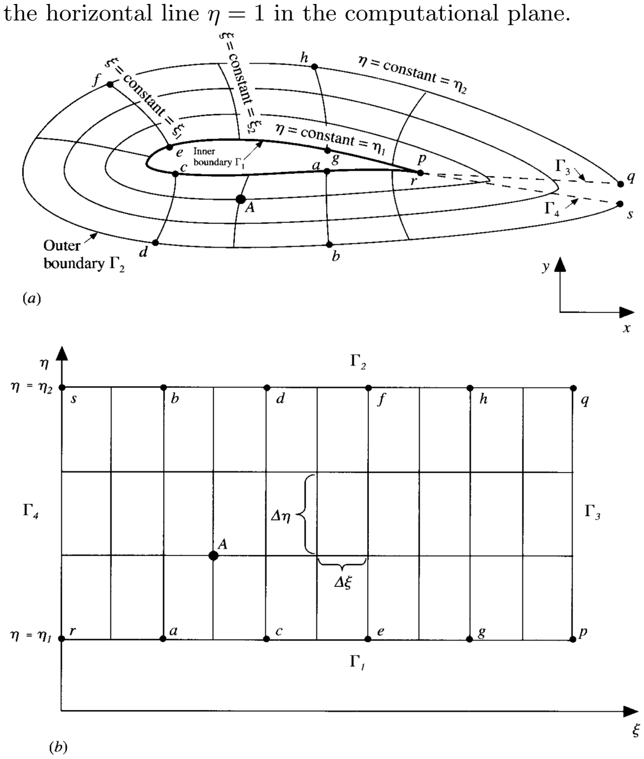
$$F_1 = JF\frac{\partial \xi}{\partial x} + JG\frac{\partial \xi}{\partial y} = F\frac{\partial y}{\partial \eta} - G\frac{\partial x}{\partial \eta} \tag{27}$$

$$G_1 = JF\frac{\partial \eta}{\partial x} + JG\frac{\partial \eta}{\partial y} = -F\frac{\partial y}{\partial \xi} + G\frac{\partial x}{\partial \xi} \tag{28}$$

BOUNDARY-FITTED COORDINATE SYSTEMS: ELLIPTIC GRID GENERATION



Let $y_s = f(x)$ be the ordinate of the upper surface de. Then the transformation will result in a rectangular grid in (ξ, η) space with $\xi = x$ and $\eta = y/y_s$. Then all the points along the curved upper boundary in the physical plane fall, via the transformation along the horizontal line $\eta = 1$ in the computational plane.



The coordinate $\eta = \eta_1 =$ constant denotes the inner boundary of the grid in (a) designated by Γ_1 in both physical and computational planes. Similarly $\eta = \eta_2 =$ constant denotes the outer boundary, designated by Γ_2. Since the grid fits the boundary prefectly, it is called *boundary-fitted coordinate system.* The lines emerging from Γ_1 to Γ_2 have constant ξ. The grid lines of constant η totally encloses, much like elongated circles, such grids are called *O-type grid*, which do not totally closes is called *C-type grid.* The values of (x, y) are known for the known Γ_1 and Γ_2, i.e., these transformations are to be defined by elliptical partial differential equation. For $\eta = \eta_1$ and $\eta = \eta_2$ the boundary conditions are known from Γ_1 and Γ_2. But for left and right of the computational grid boundary, we slice arbitrarily in physical grid with curves rs and pq. But in actual sense, the overlap eachother. We denote those surfaces as Γ_3 and Γ_4 respectively. Since the values of (x, y) are known in Γ_3 and Γ_4, those can be set as boundary conditions in computational plane (left and right edges). Because this transformation is being carrird out via the solution of a system of elliptical partial differential equations, it is called *elliptic gird generation.* At any given point located at (i, j) in both the physical and computational planes, we can write metrices at that point as

$$\left(\frac{\partial \xi}{\partial x}\right)_{i,j} = \frac{\xi_{i+1,j} - \xi_{i-1,j}}{x_{i+1,j} - x_{i-1,j}}$$

ADAPTIVE GRIDS

An adaptive grid is a grid network that automatically clusters grid points in regions of high flow-field gradients; it uses the solution of the flow-field properties to locate the grid points in the physical plane. $\Delta\xi$ and $\Delta\eta$ are uniform in computational $\xi\eta$ plane, whereas Δx and Δy are changing based on location of high flow-field gradient. In adaptive grids

$$\frac{\partial \xi}{\partial t} \equiv \left(\frac{\partial \xi}{\partial t}\right)_{x,y}$$

$$\frac{\partial \eta}{\partial t} \equiv \left(\frac{\partial \eta}{\partial t}\right)_{x,y}$$

These equations as a function of time, the values of ξ and η associated with this fixed (x, y) location will change. Thus the above equations are finite. Time metric $(\partial x/\partial t)_{\xi,\eta}$ can be written as relative change in the x locations of points N and $N+1$ by small time increament. N is for denoting x indexes and M for y indexes.

$$\left(\frac{\partial x}{\partial t}\right)_{\xi,\eta} = \frac{x_N^{t+\Delta t} - x_N^t}{\Delta t} \tag{29}$$

$$\left(\frac{\partial y}{\partial t}\right)_{\xi,\eta} = \frac{y_M^{t+\Delta t} - y_M^t}{\Delta t} \tag{30}$$

By inverse transform as $x = x(\xi, \eta, \tau)$

$$dx = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau} d\xi + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} d\eta + \left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} d\tau$$

If these changes are taking places with respect to time, holding x and y constant.

$$\cancelto{0}{\left(\frac{\partial x}{\partial t}\right)_{x,y}} = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \tag{31}$$

$$\left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta}\cancelto{1}{\left(\frac{\partial \tau}{\partial t}\right)_{x,y}}$$

rewritting

$$-\left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y} \tag{32}$$

Similarly for y coordinate

$$-\left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta} = \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y} \tag{33}$$

Solving for $(\partial\xi/\partial t)_{x,y}$ and $(\partial\eta/\partial t)_{x,y}$ using Cramer's rule

$$\left(\frac{\partial \xi}{\partial t}\right)_{x,y} = \frac{\begin{vmatrix} -\left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} & \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} \\ -\left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta} & \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau} \end{vmatrix}}{\begin{vmatrix} \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau} & \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} \\ \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau} & \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau} \end{vmatrix}}$$

$$\frac{\partial \xi}{\partial t} = \frac{1}{J}\left[-\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial \eta}\right)\right] \tag{34}$$

$$\frac{\partial \eta}{\partial t} = \frac{1}{J}\left[-\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial \xi}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial \xi}\right)\right] \tag{35}$$

The time metrices for Eq. (3) can be calculated from Eq. (33) and Eq. (34). $\partial x/\partial t$ and $\partial y/\partial t$ can be calculated from Eq. (29) and Eq. (30). The spacial metrices in Eq. (33) and Eq. (34) can be replaced by central differences.

$$\frac{\partial x}{\partial \xi} = \frac{x_{i+1,j} - x_{i-1,j}}{2\Delta\xi}$$

$$\frac{\partial x}{\partial \eta} = \frac{x_{i+1,j} - x_{i-1,j}}{2\Delta\eta}$$

$$\frac{\partial y}{\partial \xi} = \frac{y_{i+1,j} - y_{i-1,j}}{2\Delta\xi}$$

$$\frac{\partial y}{\partial \eta} = \frac{y_{i+1,j} - y_{i-1,j}}{2\Delta\eta}$$

LID DRIVEN CAVITY

Finite Difference Methord

Used finite difference method to solve lid driven cavity problem for 2-D incompressible flow using SIMPLE algorithm in 129 by 129 grid.

Steps followed

- Create u, v, p, u^*, v^* and p' stagged grids.
- Setting boundary conditions.
  This step involves setting boundary conditions for u and v stagged grids. It comprises of normal and tangential velocities. Normal velocities of left and right wall i.e., u is set to 0. Similarly normal velocities of top and bottom wall is set to 0.
  For tangential velocities, since the ghost cells are present outside the physical domain, the points outside are set based on average of the both to the boundary layer. At the boundary of top lid, $u = 1$, then at points between boundary

  $$\frac{u[i][y] + u[i][y-1]}{2} = 1$$

  also tangential velocity at other walls, the average velocities of two grid points is 0, i.e., $v[0][j] = -v[1][j]$ and so on.
- Solving the momentum equation
  x-momentum equation

  $$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial\, uv}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

  y-momentum equation

  $$\frac{\partial v}{\partial t} + \frac{\partial\, uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

  Solved the equation for u^{n+1} and v^{n+1} by explicit method, initially guessing the value of $p = 0$ and store it in u^* and v^* grid points.
- Since the guessed pressure is wrong, the velocity obtained from solving the momentum equation is also wrong. So we move to the pressure correction.
  First we set all the elements in p' to 0 before the pressure correction begins. We calculate the *mass source* term by

  $$d = \frac{1}{\Delta x}[(\rho u^*)_{i,j} - (\rho u^*)_{i-1,j}] + \frac{1}{\Delta y}[(\rho v^*)_{i,j} - (\rho v^*)_{i,j-1}]$$

  By *Poisson equation*

  $$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = Q$$

  by expanding we get,

  $$ap'_{i,j} + bp'_{i+1,j} + bp'_{i-1,j} + cp'_{i,j+1} + cp'_{i,j-1} + d = 0$$

  where

  $$a = 2\left[\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2}\right]$$

  $$b = -\frac{\Delta t}{(\Delta x)^2}$$

  $$c = -\frac{\Delta t}{(\Delta y)^2}$$

  solving for p' using iterative loop. At the end of every correction iteration, we apply boundary condition for pressure. We use zero pressure gradient across the wall, because we need to maintain the tangential velocity boundary conditions.
- Correcting flow fields.
  From u^*, v^* and p', we correct the flow fields to u, v and p. We generally under-relax to avoid over shooting of primitive variables.

  $$p^{new}_{i,j} = p^{old}_{i,j} + \alpha_p p'$$

  $$u^{new}_{i,j} = \alpha_u\left[u^*_{i,j} - \frac{\Delta t}{\rho \Delta x}(p'_{i+1,j} + p'_{i,j})\right] + (1-\alpha_u)u^{old}_{i,j}$$

  $$v^{new}_{i,j} = \alpha_v\left[v^*_{i,j} - \frac{\Delta t}{\rho \Delta y}(p'_{i,j+1} + p'_{i,j})\right] + (1-\alpha_v)v^{old}_{i,j}$$

  where α_p, α_u and α_v are the relaxation factors of p, u and v respectively.
- Setting boundary conditions again.

We then repeat steps 3 - 6 iteratively until the solution converges.

Results

The benchmark results are taken from
High-Re Solutions for Incompressible Flow Using the Navier-Stokes Equations and a Multigrid Method* U. GHIA, K. N. GHIA, AND C. T. SHIN *University of Cincinnati, Cincinnati, Ohio 45221*, Received January 15, 1982.

Table 1: Benchmark results for u-velocity along Vertical Line through Geometric Center of Cavity

129-grid		**Re**				
pt. no.	**y**	**100**	**400**	**1000**	**3200**	**5000**
129	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
126	0.9766	0.84123	0.75837	0.65928	0.53236	0.48223
125	0.9688	0.78871	0.68439	0.57492	0.48296	0.46120
124	0.9609	0.73722	0.61756	0.51117	0.46547	0.45992
123	0.9531	0.68717	0.55892	0.46604	0.46101	0.46036
110	0.8516	0.23151	0.29093	0.33304	0.34682	0.33556
95	0.7344	0.00332	0.16256	0.18719	0.19791	0.20087
80	0.6172	-0.13641	0.02135	0.05702	0.07156	0.08183
65	0.5000	-0.20581	-0.11477	-0.06080	-0.04272	-0.03039
59	0.4531	-0.21090	-0.17119	-0.10648	-0.08636	-0.07404
37	0.2813	-0.15662	-0.32726	-0.27805	-0.24427	-0.22855
23	0.1719	-0.10150	-0.24299	-0.38289	-0.34323	-0.33050
14	0.1016	-0.06434	-0.14612	-0.29730	-0.41933	-0.40435
10	0.0703	-0.04775	-0.10338	-0.22220	-0.37827	-0.43643
9	0.0625	-0.04192	-0.09266	-0.20196	-0.35344	-0.42901
8	0.0547	-0.03717	-0.08186	-0.18109	-0.32407	-0.41165
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 2: Results for u-velocity along Vertical Line through Geometric Center of Cavity

129-grid		**Re**				
pt. no.	**y**	**100**	**400**	**1000**	**3200**	**5000**
129	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
126	0.9766	0.843395	0.759556	0.661092	0.514733	0.47296
125	0.9688	0.791507	0.685891	0.577023	0.463983	0.44794
124	0.9609	0.740598	0.619268	0.513321	0.445552	0.445163
123	0.9531	0.691067	0.560716	0.467999	0.440655	0.445559
110	0.8516	0.236286	0.290203	0.332133	0.33339	0.325903
95	0.7344	0.0039853	0.161556	0.185879	0.192303	0.149893
80	0.6172	-0.138734	0.0206828	0.0558917	0.0730172	0.0746594
65	0.5000	-0.208821	-0.115064	-0.0615347	-0.0357403	-0.0305809
59	0.4531	-0.213609	-0.171324	-0.107017	-0.0779771	-0.0714617
37	0.2813	-0.157429	-0.32648	-0.277721	-0.232432	-0.220858
23	0.1719	-0.101639	-0.241837	-0.382141	-0.332014	-0.316744
14	0.1016	-0.0643702	-0.145254	-0.294657	-0.413195	-0.39288
10	0.0703	-0.0466018	-0.102778	-0.219197	-0.385051	-0.411682
9	0.0625	-0.0419528	-0.0921327	-0.199071	-0.363383	-0.400732
8	0.0547	-0.0372008	-0.0814068	-0.178438	-0.336619	-0.381229
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 3: Percentage error (%) of u-velocity results with respect to benchmark values.

129-grid		**Re**				
pt. no.	**y**	**100**	**400**	**1000**	**3200**	**5000**
129	1.00000	0.00	0.00	0.00	0.00	0.00
126	0.9766	0.26	0.16	0.27	3.31	1.92
125	0.9688	0.35	0.22	0.37	3.94	2.88
124	0.9609	0.46	0.28	0.42	4.28	3.21
123	0.9531	0.57	0.32	0.42	4.42	3.22
110	0.8516	2.06	0.25	0.27	3.87	2.88
95	0.7344	20.04	0.62	0.70	2.87	0.97
80	0.6172	1.70	3.12	1.98	2.04	8.78
65	0.5000	1.46	0.26	1.21	16.34	0.63
59	0.4531	1.28	0.08	0.50	9.71	3.48
37	0.2813	0.52	0.24	0.11	4.85	3.37
23	0.1719	0.14	0.47	0.19	3.27	4.16
14	0.1016	0.05	0.59	0.89	1.46	2.84
10	0.0703	2.41	0.58	1.26	1.80	5.67
9	0.0625	0.08	0.57	1.43	2.81	6.59
8	0.0547	0.08	0.55	1.46	3.87	7.39
1	0.0000	0.00	0.00	0.00	0.00	0.00

Table 4: Benchmark results for v-Velocity along Horizontal Line through Geometric Center of Cavity.

129-grid		**Re**				
pt. no.	**x**	**100**	**400**	**1000**	**3200**	**5000**
129	1.0000	0.00000	0.00000	0.00000	0.00000	0.00000
125	0.9688	-0.05906	-0.12146	-0.21388	-0.39017	-0.49774
124	0.9609	-0.07391	-0.15663	-0.27669	-0.47425	-0.55069
123	0.9531	-0.08864	-0.19254	-0.33714	-0.52357	-0.55408
122	0.9453	-0.10313	-0.22847	-0.39188	-0.54053	-0.52876
117	0.9063	-0.16914	-0.23827	-0.51550	-0.44307	-0.41442
111	0.8594	-0.22445	-0.44993	-0.42665	-0.37401	-0.36214
104	0.8047	-0.24533	-0.38598	-0.31966	-0.31184	-0.30018
65	0.5000	0.05454	0.05186	0.02526	0.00999	0.00945
31	0.2344	0.17527	0.30174	0.32235	0.28188	0.27280
30	0.2266	0.17507	0.30203	0.33075	0.29030	0.28066
21	0.1563	0.16077	0.28124	0.37095	0.37119	0.35368
13	0.0938	0.12317	0.22965	0.32627	0.42768	0.42951
11	0.0781	0.10890	0.20920	0.30353	0.41906	0.43648
10	0.0703	0.10091	0.19713	0.29012	0.40917	0.43329
9	0.0625	0.09233	0.18360	0.27485	0.39560	0.42447
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5: Results for v-Velocity along Horizontal Line through Geometric Center of Cavity.

129-grid		**Re**				
pt. no.	**x**	**100**	**400**	**1000**	**3200**	**5000**
129	1.0000	0.00000	0.00000	0.00000	0.00000	0.00000
125	0.9688	-0.0622837	-0.12458	-0.223878	-0.412767	-0.472963
124	0.9609	-0.0778886	-0.160455	-0.288147	-0.489546	-0.523509
123	0.9531	-0.0933204	-0.196988	-0.349204	-0.530057	-0.530188
122	0.9453	-0.108484	-0.233425	-0.40367	-0.539099	-0.511266
117	0.9063	-0.177078	-0.387837	-0.51794	-0.430639	-0.405614
111	0.8594	-0.233508	-0.451022	-0.422439	-0.363346	-0.35363
104	0.8047	-0.253165	-0.383664	-0.316783	-0.300999	-0.291325
65	0.5000	0.0575217	0.0523714	0.0257877	0.013683	0.0106807
31	0.2344	0.179239	0.301425	0.321674	0.27647	0.263477
30	0.2266	0.179032	0.301757	0.330087	0.284575	0.271316
21	0.1563	0.164478	0.281381	0.370792	0.36319	0.343806
13	0.0938	0.126114	0.229785	0.326976	0.412408	0.413328
11	0.0781	0.111534	0.209235	0.30429	0.40126	0.416609
10	0.0703	0.103372	0.197115	0.290814	0.390245	0.411871
9	0.0625	0.0945986	0.183527	0.275419	0.375686	0.402037
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000

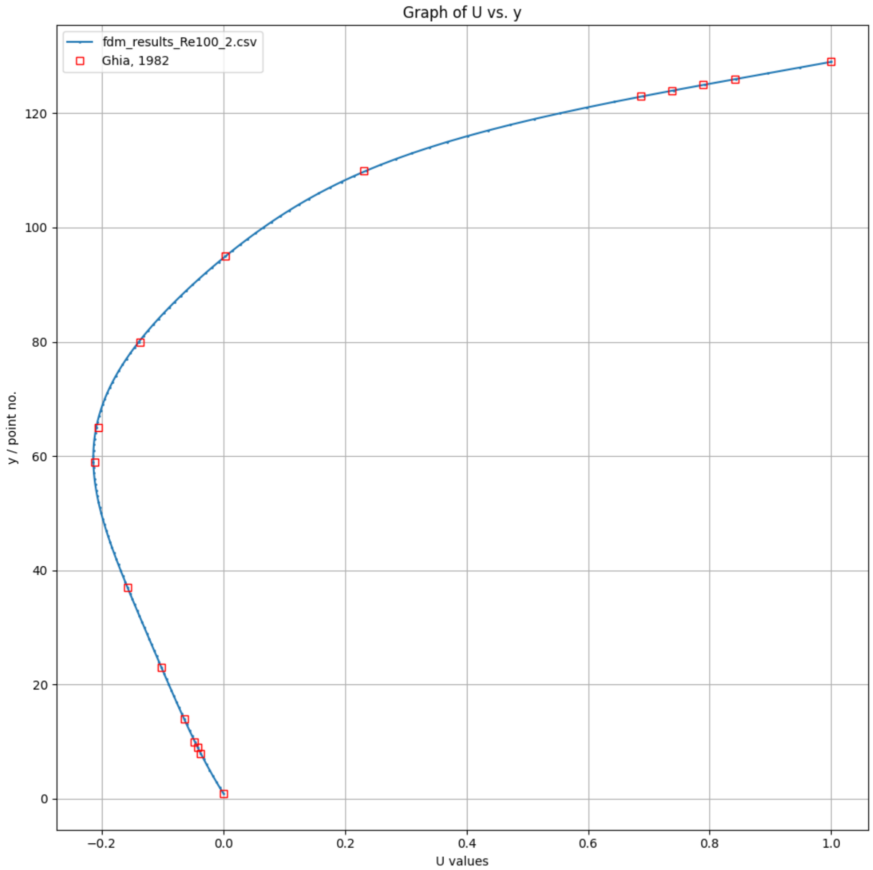
Table 6: Percentage error (%) of v-Velocity results with respect to benchmark values.

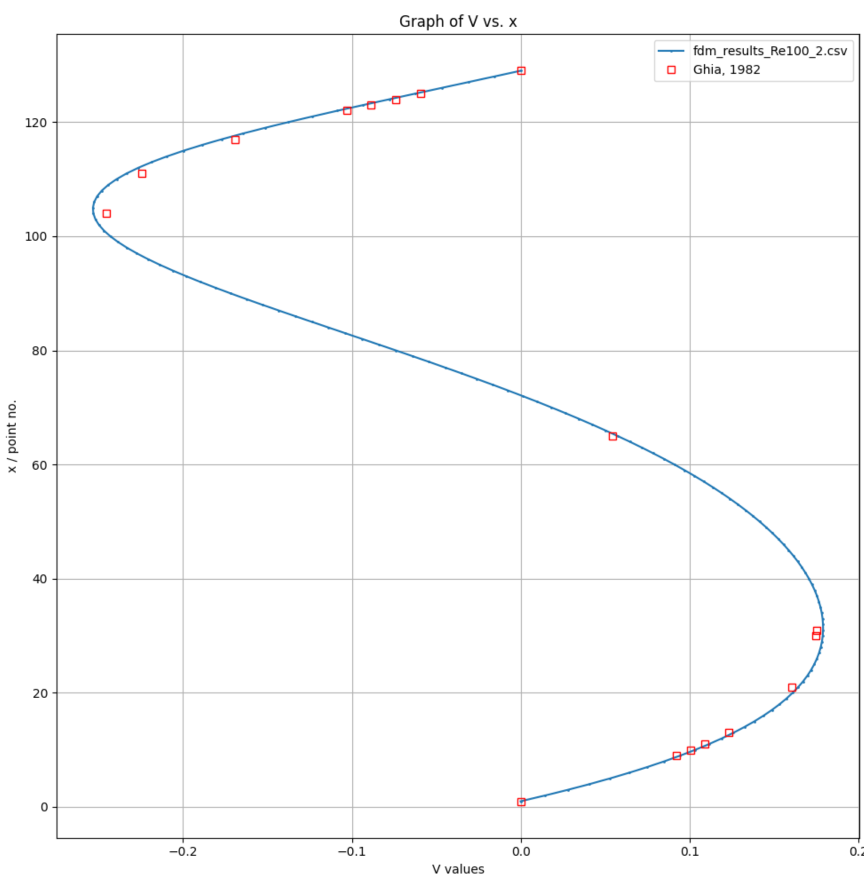
129-grid pt. no.	**x**	**Re**				
		100	**400**	**1000**	**3200**	**5000**
129	1.0000	0.00	0.00	0.00	0.00	0.00
125	0.9688	5.46	2.57	4.67	5.75	4.98
124	0.9609	5.38	2.44	4.14	3.23	4.94
123	0.9531	5.28	2.31	3.58	1.24	4.31
122	0.9453	5.19	2.17	3.01	0.25	3.32
117	0.9063	4.69	62.77	0.47	2.87	2.12
111	0.8594	4.04	0.24	0.92	2.85	2.35
104	0.8047	3.20	0.60	0.90	3.42	2.95
65	0.5000	5.47	0.99	2.09	36.97	13.02
31	0.2344	2.26	0.10	0.21	1.92	3.42
30	0.2266	2.26	0.09	0.20	1.97	3.33
21	0.1563	2.31	0.05	0.02	2.16	2.86
13	0.0938	2.39	0.06	0.22	3.57	3.77
11	0.0781	2.69	0.02	0.25	2.16	4.63
10	0.0703	2.94	0.00	0.24	2.44	4.62
9	0.0625	3.00	0.04	0.21	5.08	5.29
1	0.0000	0.00	0.00	0.00	0.00	0.00

Turn over for Plots

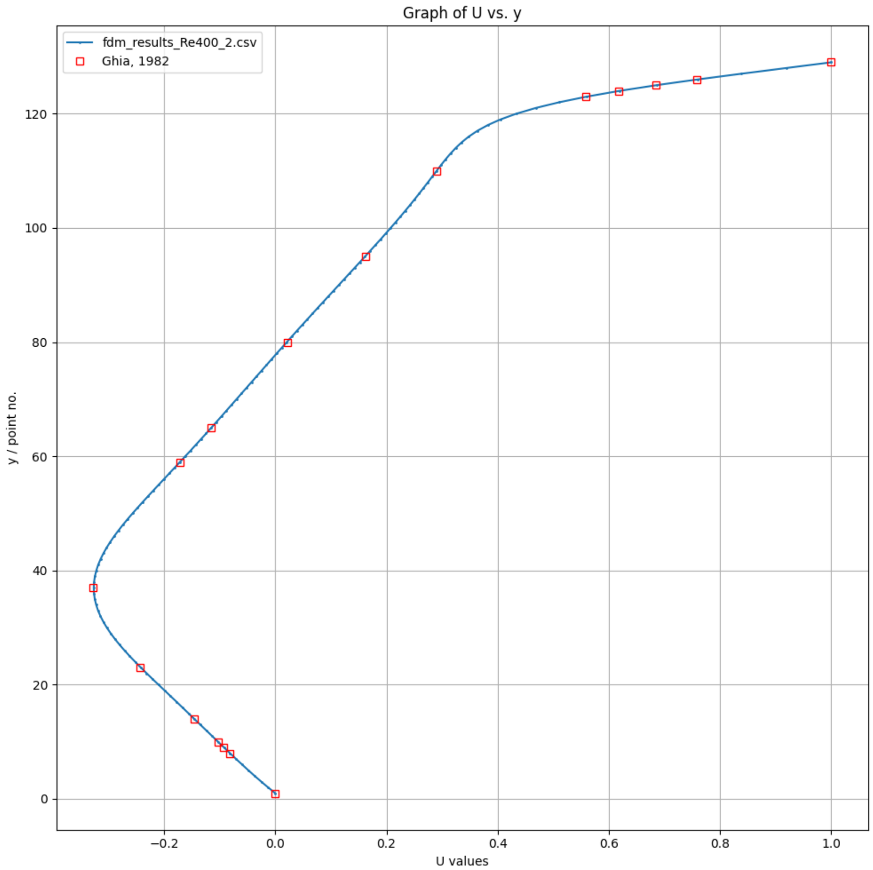
Centerline Plots

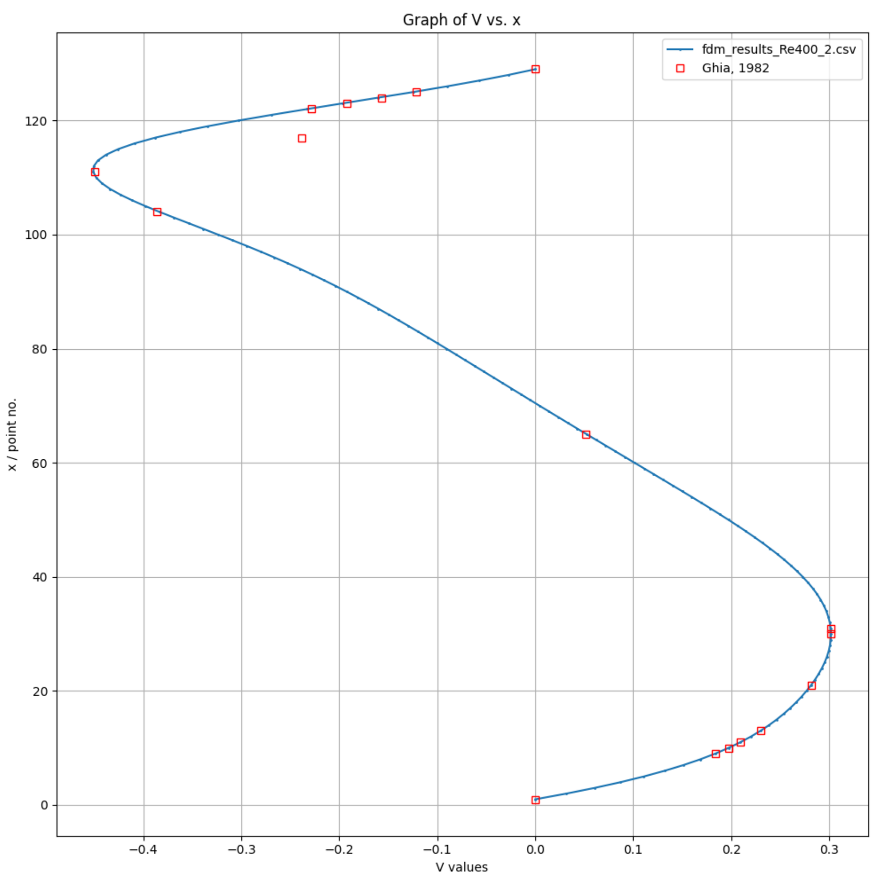
Re = 100



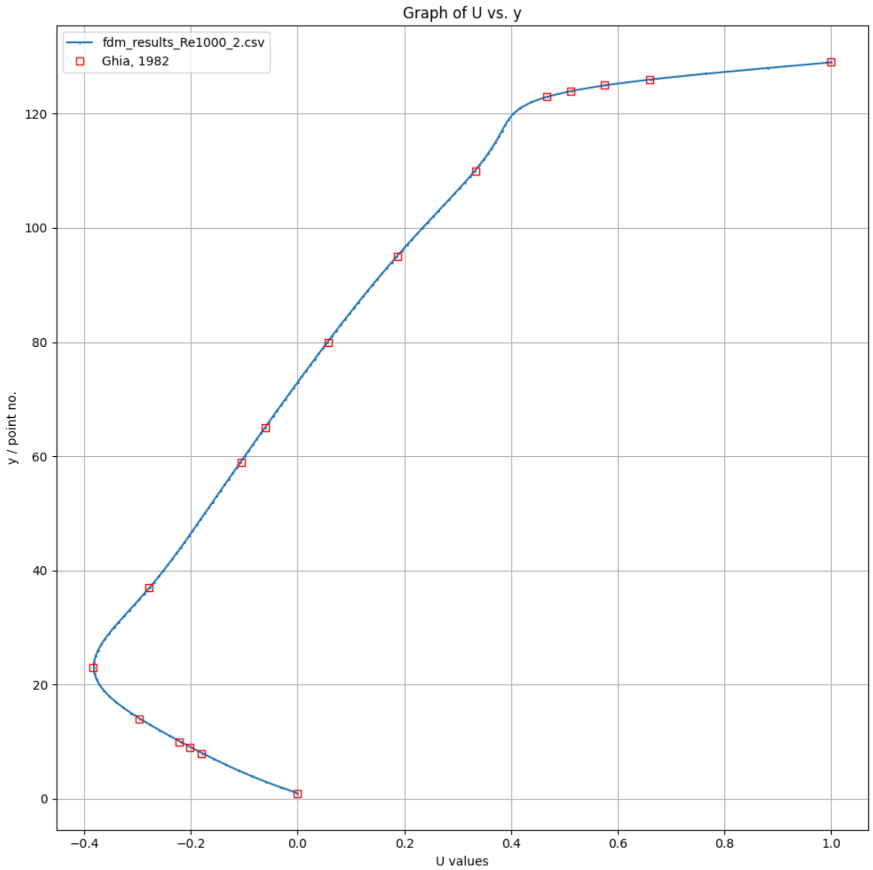


Re = 400

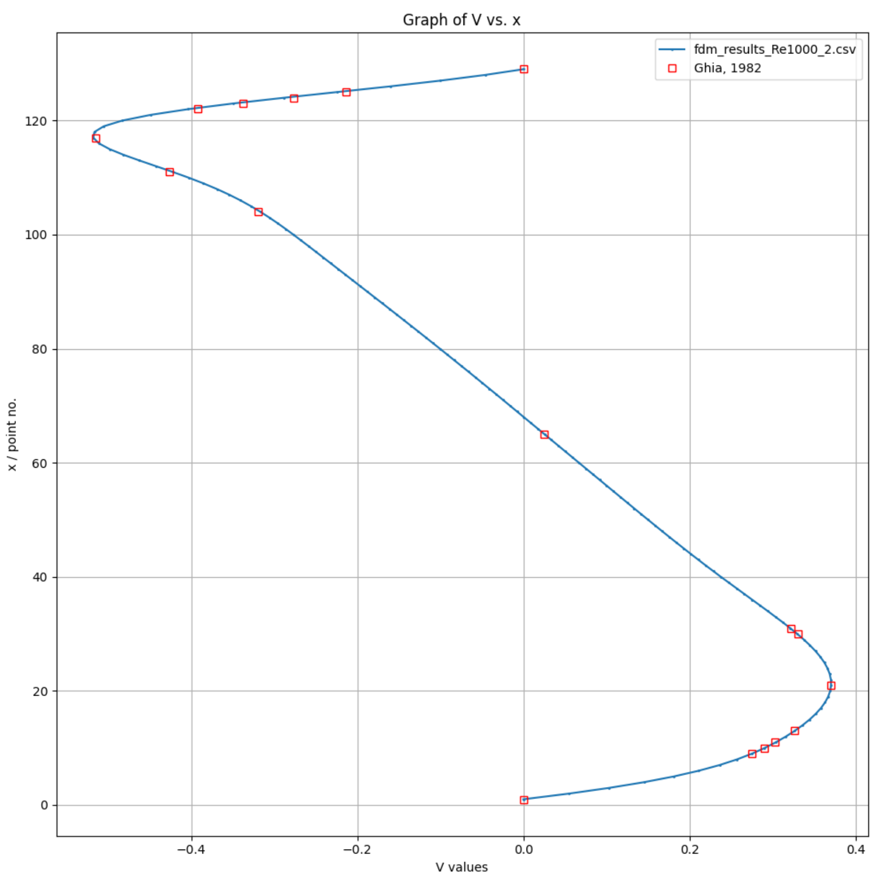

Graph of U vs. y
fdm_results_Re400_2.csv
Ghia, 1982
y / point no.
U values
120
100
80
60
40
20
0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0


Graph of V vs. x
fdm_results_Re400_2.csv
Ghia, 1982
x / point no.
V values
120
100
80
60
40
20
0
−0.4
−0.3
−0.2
−0.1
0.0
0.1
0.2
0.3

Re = 1000

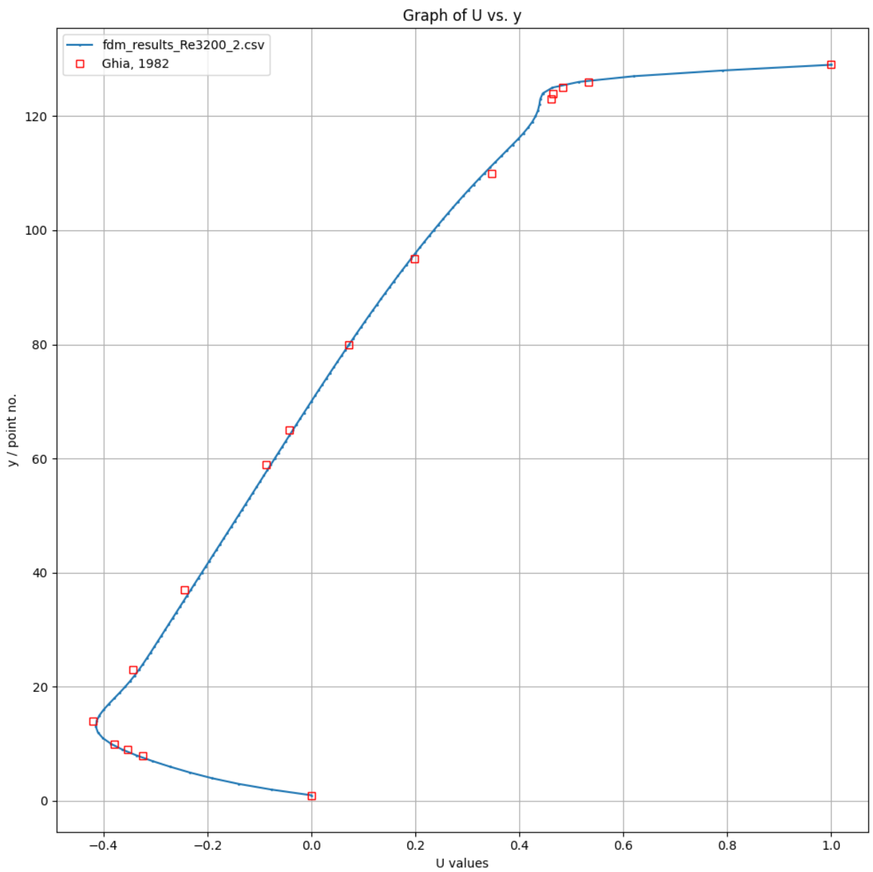


Graph of U vs. y
fdm_results_Re1000_2.csv
Ghia, 1982
y / point no.
U values

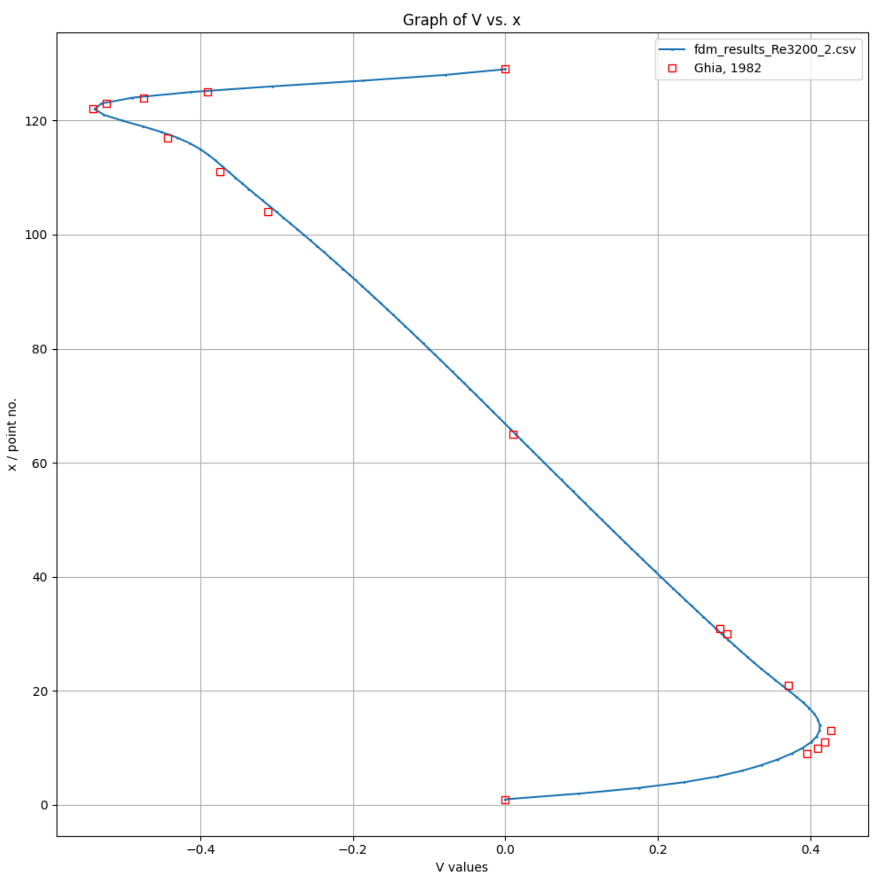


Graph of V vs. x
fdm_results_Re1000_2.csv
Ghia, 1982
x / point no.
V values

Re = 3200

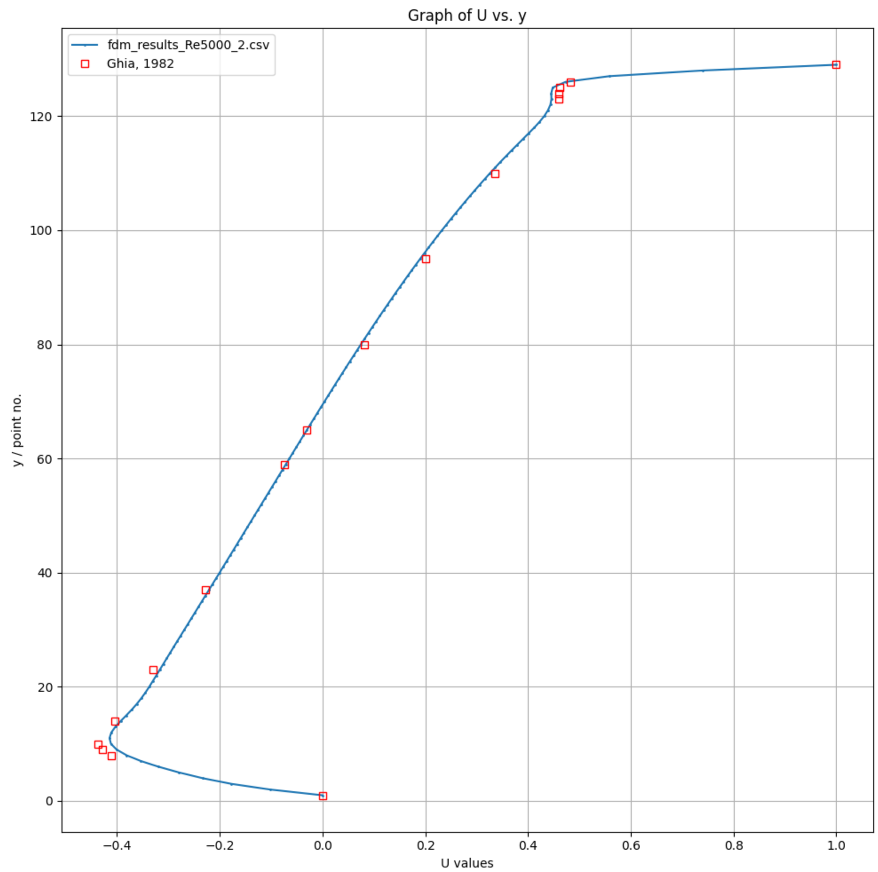


Graph of U vs. y
fdm_results_Re3200_2.csv
Ghia, 1982
y / point no.
U values

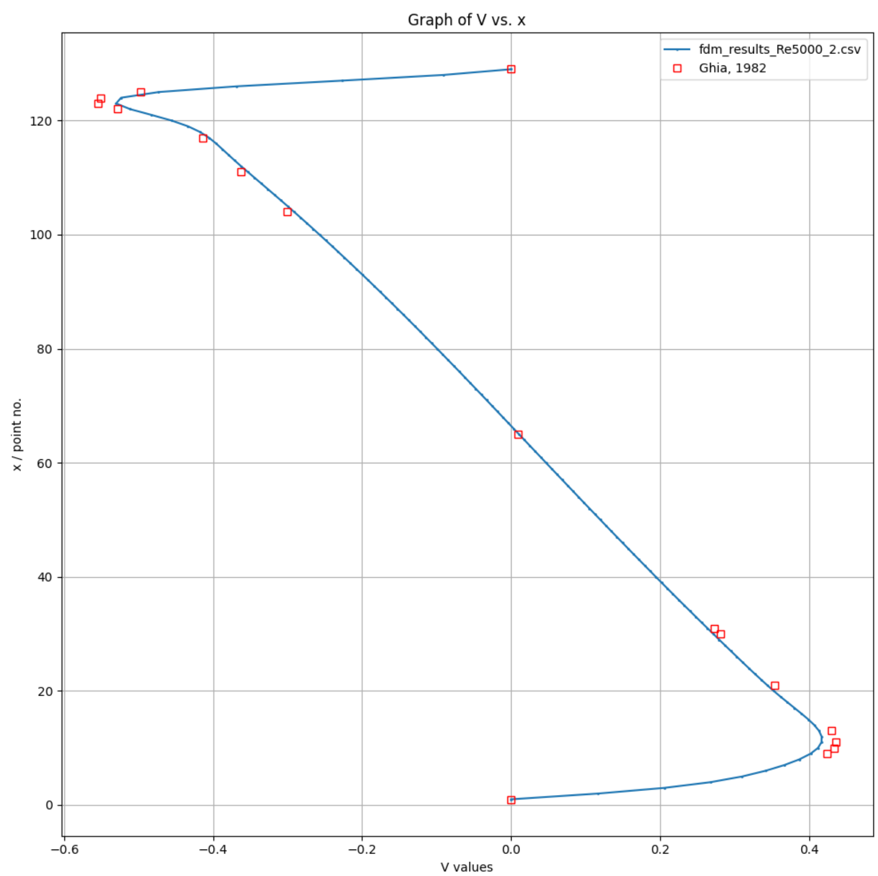


Graph of V vs. x
fdm_results_Re3200_2.csv
Ghia, 1982
x / point no.
V values

Re = 5000



Graph of U vs. y
fdm_results_Re5000_2.csv
Ghia, 1982
y / point no.
U values



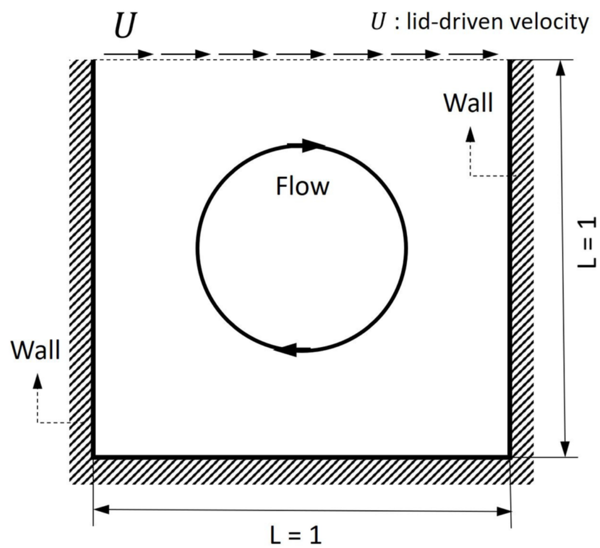
Graph of V vs. x
fdm_results_Re5000_2.csv
Ghia, 1982
x / point no.
V values

LID DRIVEN CAVITY

Finite Volume Methord

Problem Formulation

The steady and transient-state, 2D incompressible flow in a lid-driven cavity is solved by discretizing the Navier-Stokes and continuity equations using the hybrid scheme over a staggered grid of control volumes using the Finite Volume Method. The resulting coupled system of algebraic equations is then solved iteratively using the SIMPLE algorithm, which uses a pressure-correction step to enforce mass conservation and find the final velocity and pressure fields.



Implimentation

- Set Boundary Conditions
  The u velocities are parallel to top and bottom wall, normal to left and right wall. For the bottom wall, $u = 0$, thus the average of grid points in either side of the boundary should be zero. So, the velocity of the grid point outside the physical domain should be the opposite of velocity inside the grid point.

$$\frac{u[i][\text{below bottom wall}] + u[i][\text{above bottom wall}]}{2} = 0$$

$$u[i][\text{below bottom wall}] = -u[i][\text{above bottom wall}]$$

  Similarly the average of grid points in either side of top wall should be 1.

$$u[i][\text{above top wall}] = 2 \times u_{\text{lid}} - u[i][\text{below top wall}]$$

  For the left and right wall, u can be directly assigned to 0. The similar thing can be done to v velocity. v velocity at the boundary is 0. The pressure at every pressure grid point can be guessed initially, 0 would be a good guess.

- Solve momentum equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) = \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vv) = \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) - \frac{\partial p}{\partial y}$$

  To solve these equations, hybrid differencing is used in this case to discretize these equation.

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + a_S \phi_S + a_N \phi_N + a_P^o \phi_P^o$$

  $a_P^o \phi_P^o$ is used in case of transient solver.

$$a_P^o = \frac{\rho_P^o \Delta V}{\Delta t}$$

  ϕ can be u or v as per the equation which is solved

$$a_P = a_W + a_E + a_S + a_N + a_P^o + \Delta F$$

  The coefficients of hybrid differencing scheme for a 2D flow is calculated by

$$a_W = \max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right]$$

$$a_E = \max\left[-F_e, \left(D_e - \frac{F_e}{2}\right), 0\right]$$
$$a_S = \max\left[F_s, \left(D_s + \frac{F_s}{2}\right), 0\right]$$
$$a_N = \max\left[-F_n, \left(D_n - \frac{F_n}{2}\right), 0\right]$$
$$\Delta F = F_e - F_w + F_n - F_s$$

Face	**w**	**e**	**s**	**n**
F	$(\rho u)_w A_w$	$(\rho u)_e A_e$	$(\rho v)_s A_s$	$(\rho v)_n A_n$
D	$\frac{\mu_w}{\delta x_{WP}} A_w$	$\frac{\mu_e}{\delta x_{PE}} A_e$	$\frac{\mu_s}{\delta y_{SP}} A_s$	$\frac{\mu_n}{\delta y_{PN}} A_n$

For 2D case, $A_w = A_e = dy$ and $A_n = A_s = dx$. Since the solver is laminar, $\mu_e = \mu_w = \mu_n = \mu_s = \mu$. The pressure source is also discretized as

$$\frac{\partial p}{\partial x} = (p_{i-1,j} - p_{i,j})A_{i,j}$$

and similarly for y-moment equation. u^* and v^* are calculated and under-relaxed by

$$\frac{a_{i,J} u_{i,J}}{\alpha_u} = \sum a_{nb} u_{nb} + (p_{I-1,J} - p_{I,J})A_{i,J} + a^o_{i,J} u^o_{i,J} + \left[(1-\alpha_u)\frac{a_{i,J}}{\alpha_u}\right] u^{(n-1)}_{i,J}$$
$$\frac{a_{I,j} v_{I,j}}{\alpha_v} = \sum a_{nb} v_{nb} + (p_{I,J-1} - p_{I,J})A_{I,j} + a^o_{I,j} u^o_{I,j} + \left[(1-\alpha_v)\frac{a_{I,j}}{\alpha_v}\right] v^{(n-1)}_{I,j}$$

The solved velocities are stored in u^* and v^* grids.

- Pressure correction

  Pressure correction is obtained by solving

$$a_{I,J} p'_{I,J} = a_{I+1,J} p'_{I+1,J} + a_{I-1,J} p'_{I-1,J} + a_{I,J+1} p'_{I,J+1} + a_{I,J-1} p'_{I,J-1} + b'_{I,J}$$

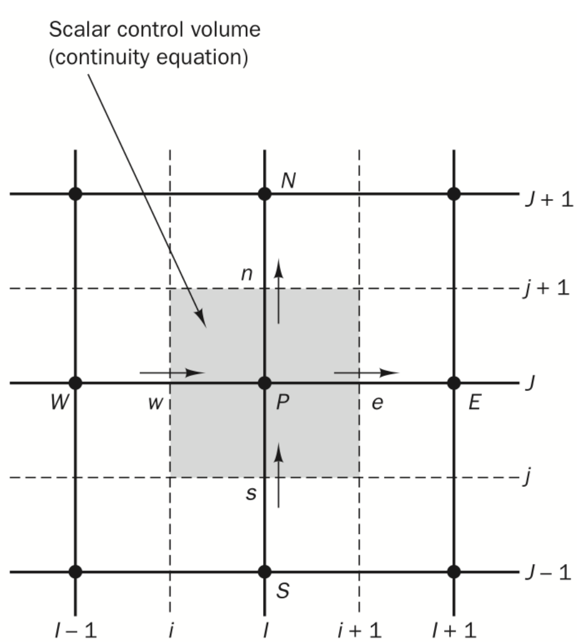
where $a_{I,J} = a_{I+1,J} + a_{I-1,J} + a_{I,J+1} + a_{I,J-1}$ and the coefficients are

$a_{I+1,J}$	$a_{I-1,J}$	$a_{I,J+1}$	$a_{I,J-1}$	$b'_{I,J}$
$(\rho dA)_{i+1,J}$	$(\rho dA)_{i,J}$	$(\rho dA)_{I,j+1}$	$(\rho dA)_{I,j}$	$(\rho u^* A)_{i,J} - (\rho u^* A)_{i+1,J}$ $+(\rho v^* A)_{I,j} - (\rho v^* A)_{I,j+1} + \frac{\rho^o_{I,J} - \rho_{I,J}}{\Delta t}$

also

$$d_{i,J} = \frac{A_{i,J}}{a_{i,J}}$$

At the end of every correction iteration, we apply boundary condition for pressure. We use zero pressure gradient across the wall, because we need to maintain the tangential velocity boundary conditions.



- Correcting flow fields.
  From u^*, v^* and p', we correct the flow fields to u, v and p.

$$p_{i,j}^{new} = p_{i,j}^{old} + \alpha_p p'$$
$$u_{i,J}^{new} = u_{i,J}^* + d_{i,J}(p'_{I-1,J} - p'_{I,J})$$
$$v_{I,j}^{new} = v_{I,j}^* + d_{I,j}(p'_{I,J-1} - p'_{I,J})$$

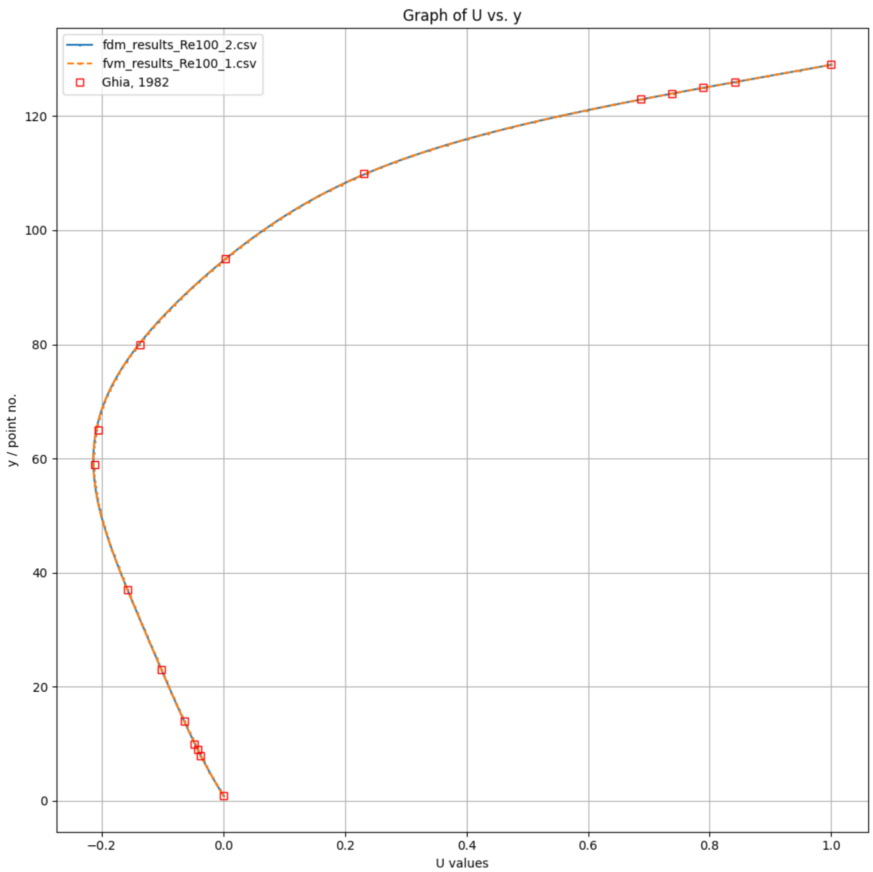
- Setting boundary conditions again.

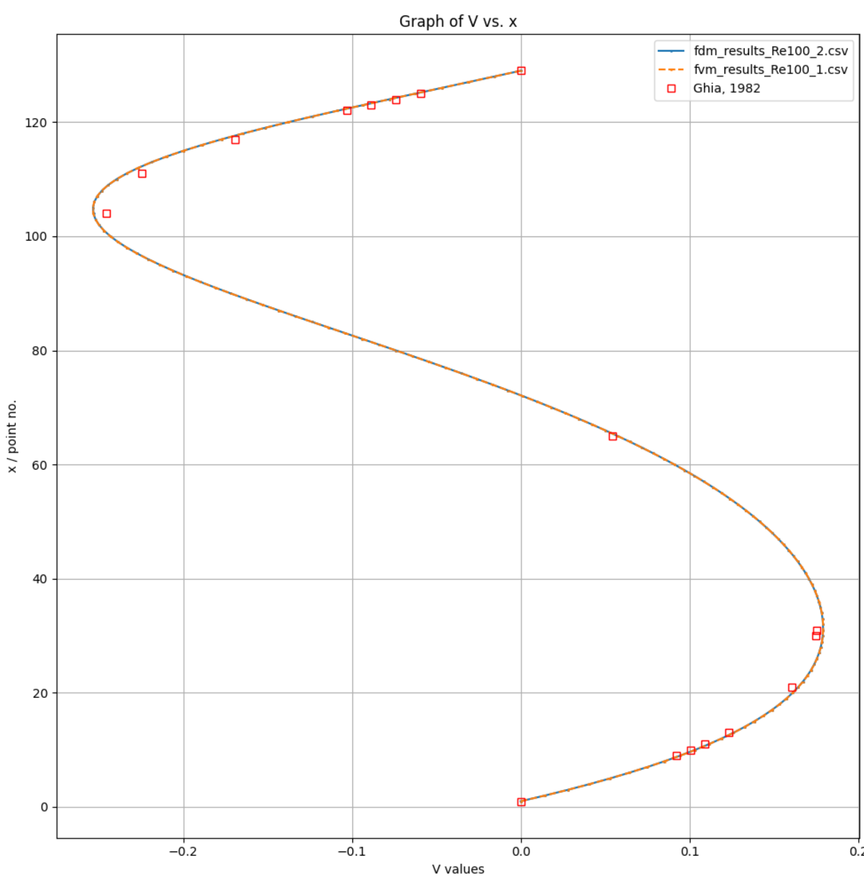
Repeat steps 2 - 5 till convergence.

Turn over for plots

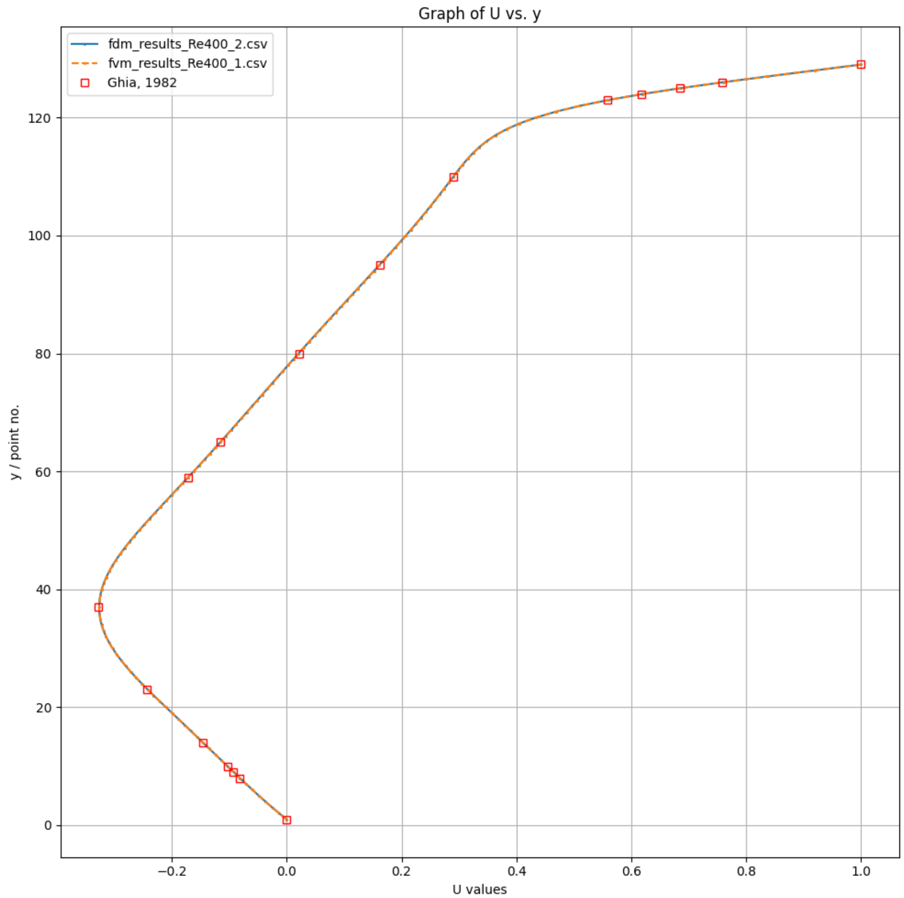
Centerline Plots

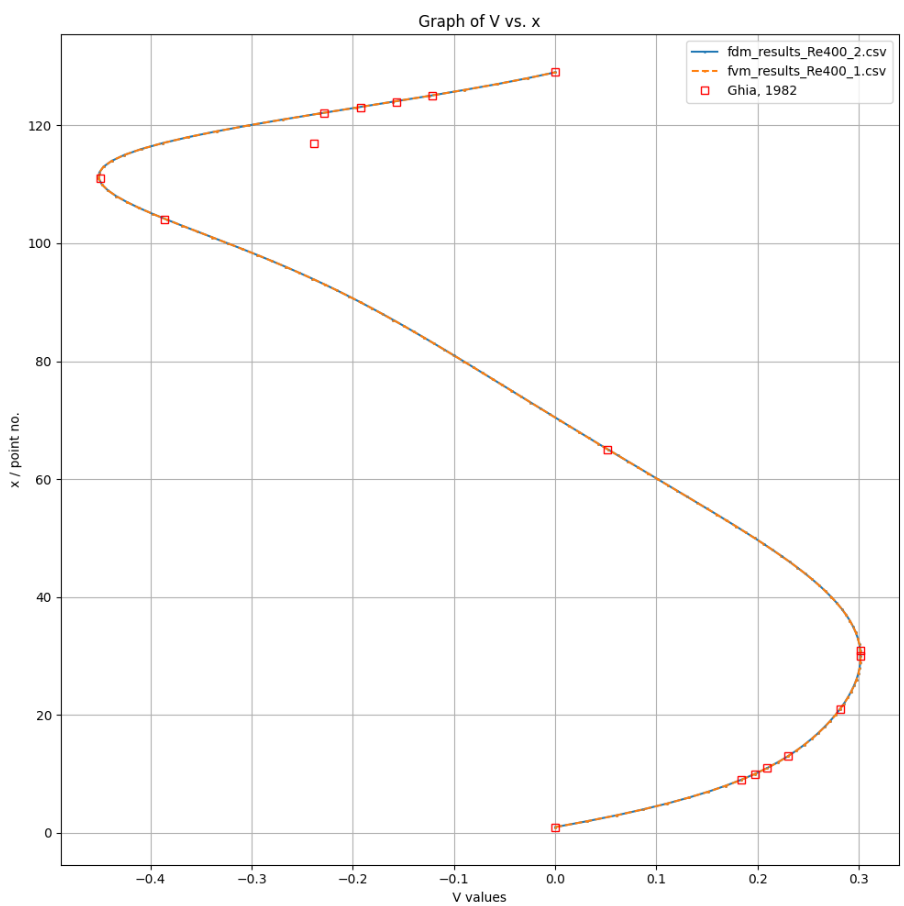
Re = 100



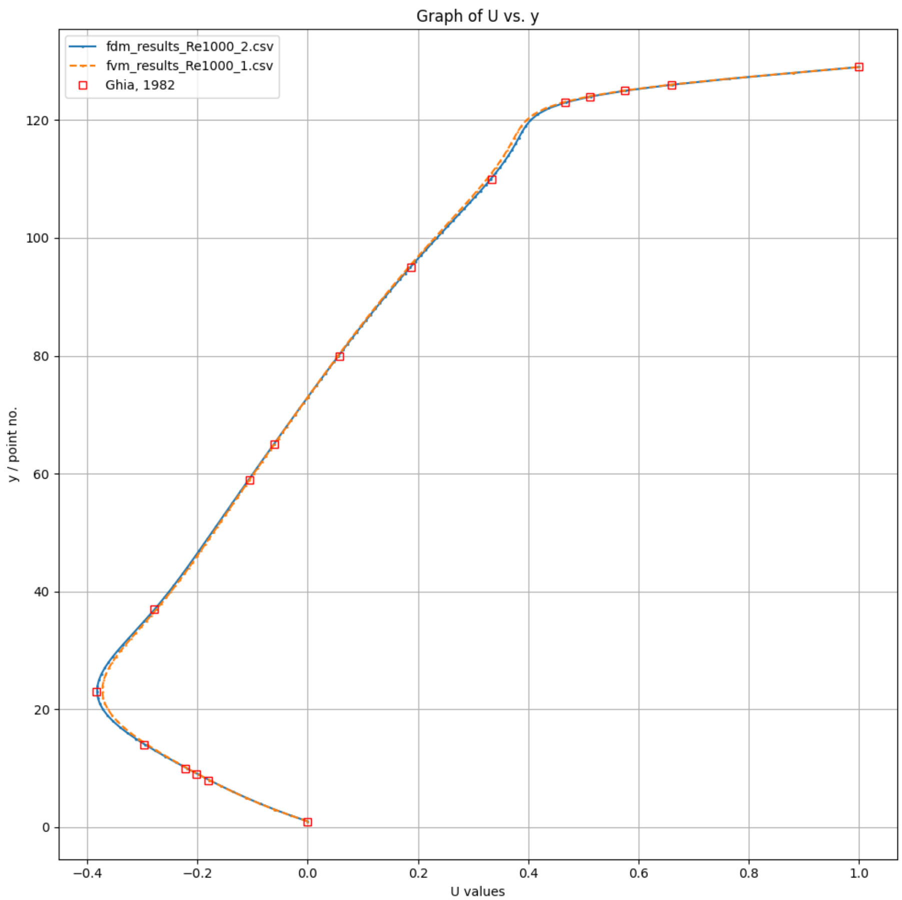


$\mathrm{Re} = 400$

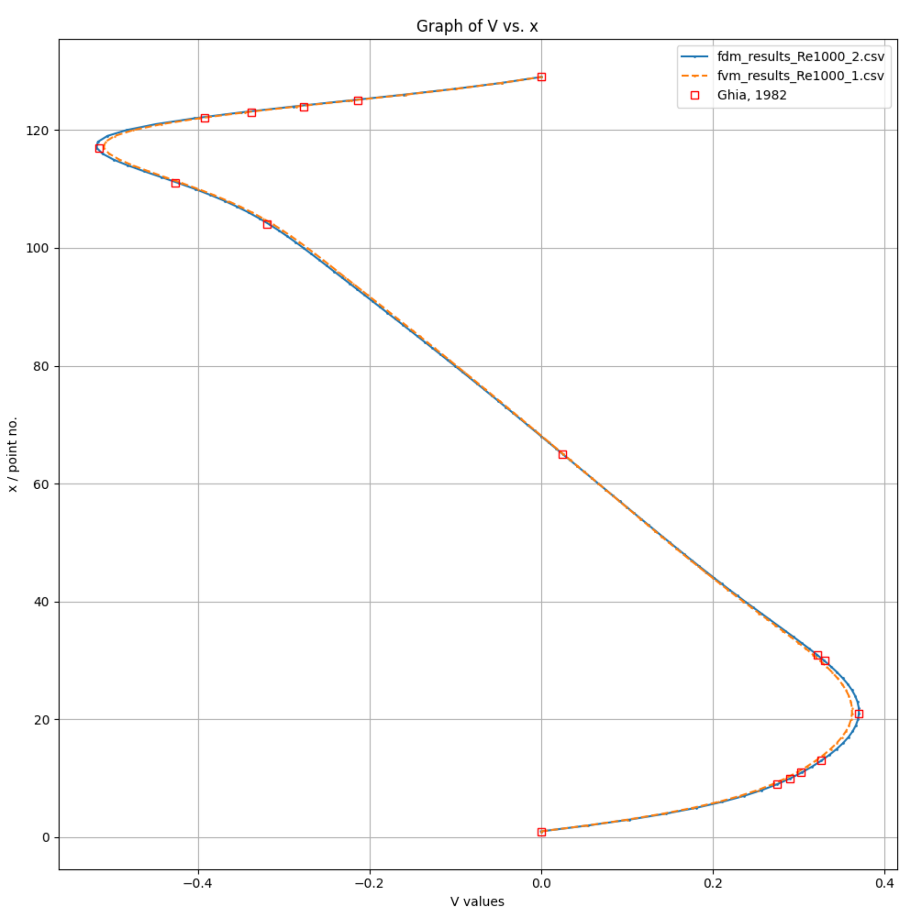




Re = 1000

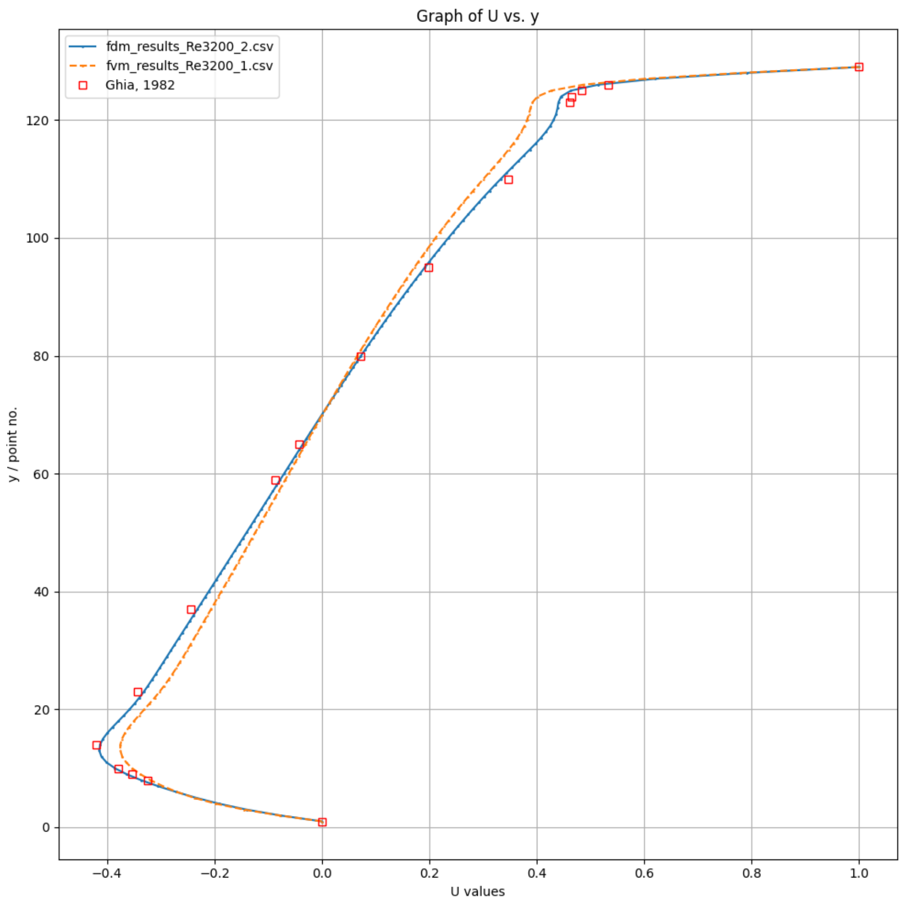


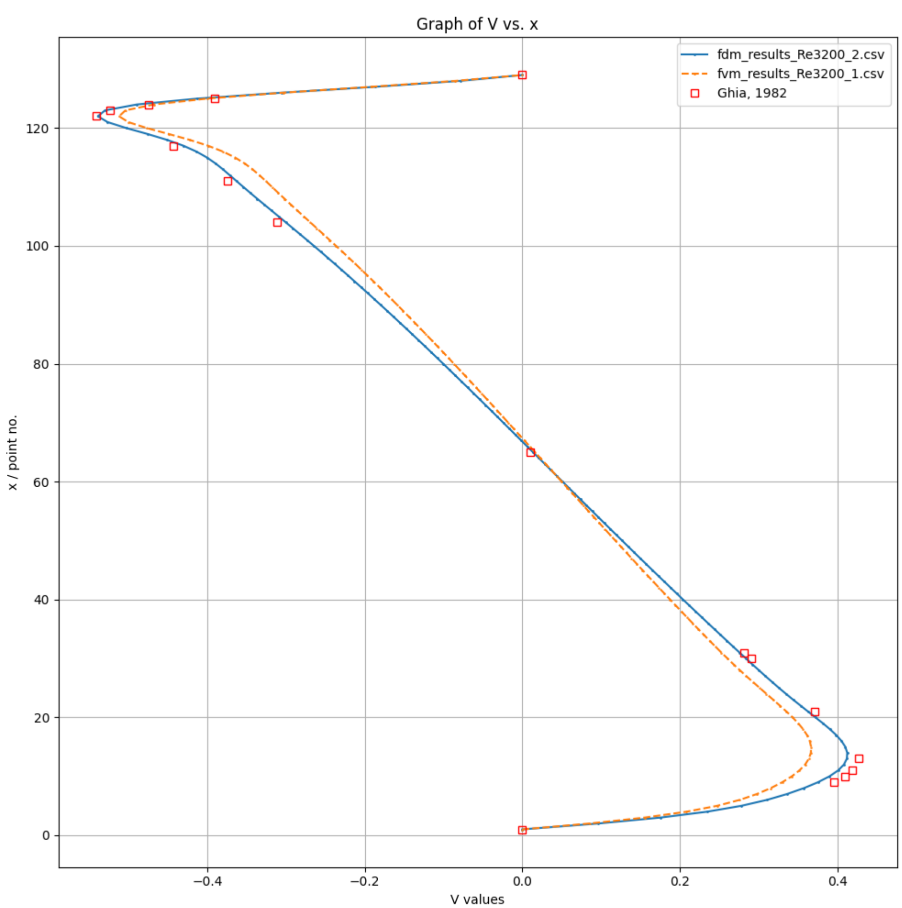
Graph of U vs. y
fdm_results_Re1000_2.csv
fvm_results_Re1000_1.csv
Ghia, 1982
y / point no.
U values



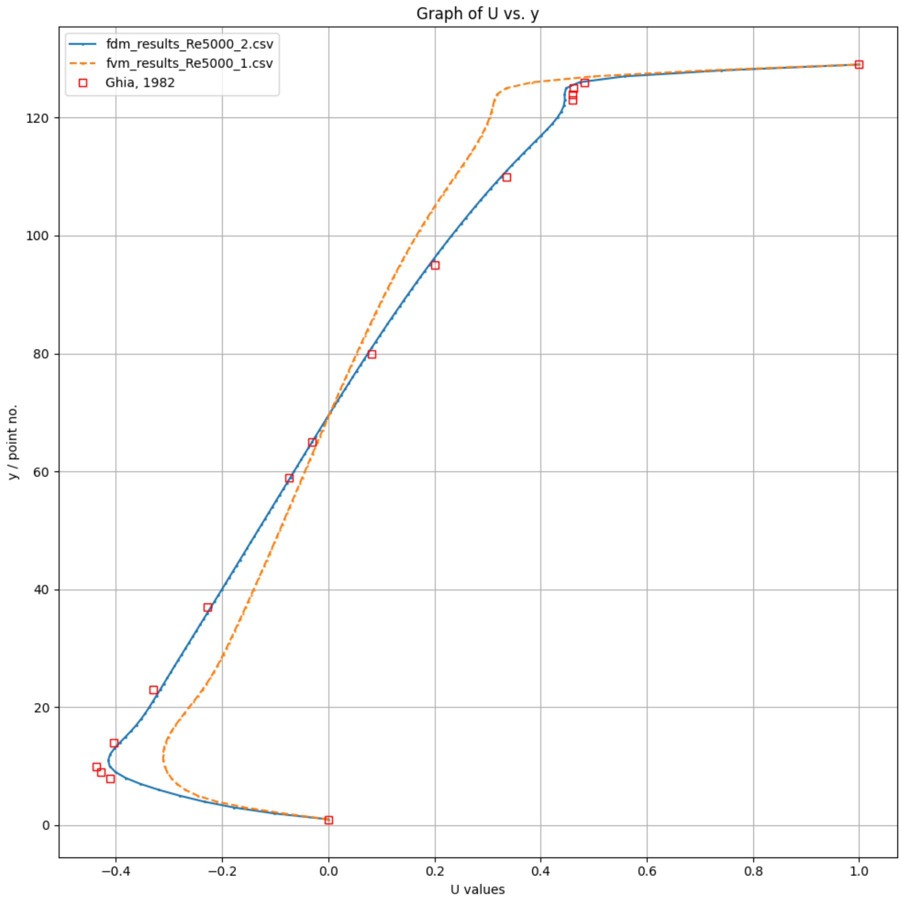
Graph of V vs. x
fdm_results_Re1000_2.csv
fvm_results_Re1000_1.csv
Ghia, 1982
x / point no.
V values

Re = 3200

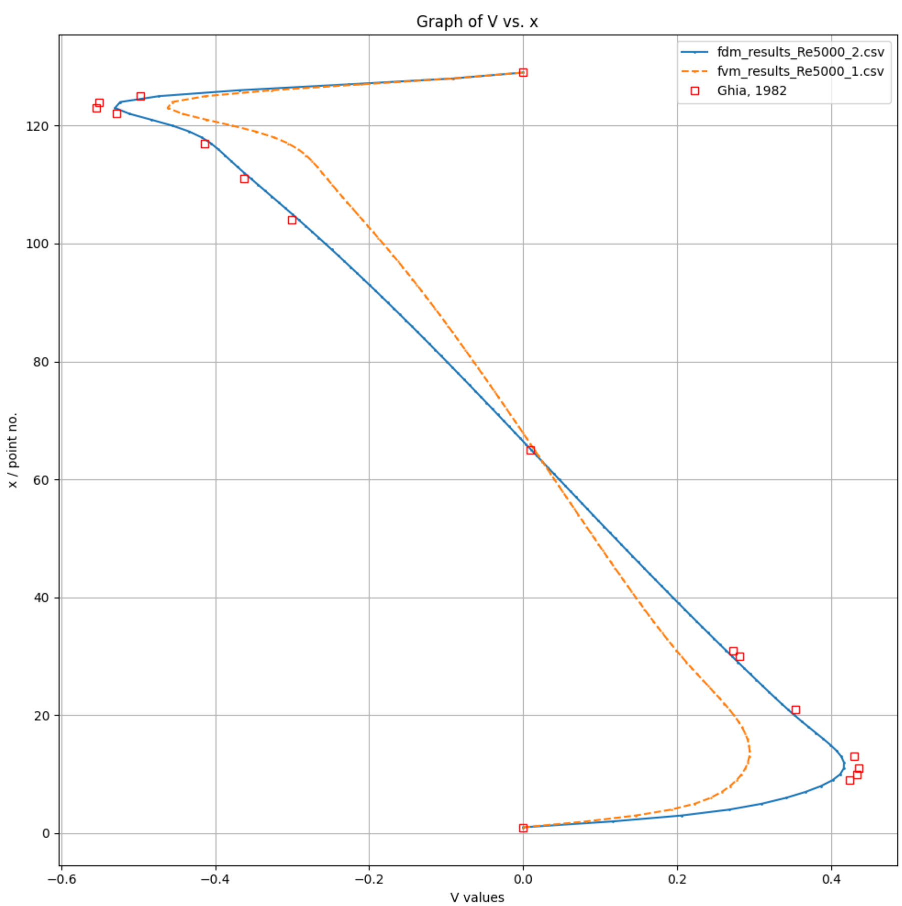

Graph of U vs. y
fdm_results_Re3200_2.csv
fvm_results_Re3200_1.csv
Ghia, 1982
y / point no.
U values
0
20
40
60
80
100
120
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0


Graph of V vs. x
fdm_results_Re3200_2.csv
fvm_results_Re3200_1.csv
Ghia, 1982
x / point no.
V values
0
20
40
60
80
100
120
−0.4
−0.2
0.0
0.2
0.4

Re = 5000



Graph of U vs. y
fdm_results_Re5000_2.csv
fvm_results_Re5000_1.csv
Ghia, 1982
y / point no.
U values



Graph of V vs. x
fdm_results_Re5000_2.csv
fvm_results_Re5000_1.csv
Ghia, 1982
x / point no.
V values

TURBULENCE MODELLING

$k - \epsilon$ model

For varying velocity field $u(t)$ it can be decomposed into a steady mean value U with a fluctuating component $u'(t)$ superimposed on it: $u(t) = U + u'(t)$. This is called *Renyolds decomposition*. It can be applied to other flow properties also, like V, W, P etc.. The largest turbulent eddies interact with and extract energy from the mean flow by a process called *vortex stretching*. Large eddies are dominated by inertia effects and viscous effects are negligible. The *energy cascade* is the one-way trip energy takes from the largest eddies down to the smallest eddies, where it is turned into heat.

The spectral energy ($E(\kappa)$) is the kinetic energy per unit mass per unit wavenumber κ, where $\kappa = 2\pi/\lambda$ and λ is the wavelength of the eddies. The Reynolds number Re_η of the smallest eddies are based on their characteristic velocity υ and characteristic length η is equal to 1, $Re_\eta = \upsilon\eta/\nu = 1$. This means the inertial force is balanced with viscous force. The ratios of small length (η), time (τ) and velocity (υ) scales and large length (ℓ), time (T) and velocity (ϑ) scales are given by

$$\text{Length-scale ratio} \quad \frac{\eta}{\ell} \approx Re_\ell^{-3/4}$$

$$\text{Time-scale ratio} \quad \frac{\tau}{T} \approx Re_\ell^{-1/2}$$

$$\text{Velocity-scale ratio} \quad \frac{\nu}{\vartheta} \approx Re_\ell^{-1/4}$$

Typical values of Re_ℓ might be $10^3 - 10^6$. The behaviour of large eddies are independent of viscosity and are dependent on the velocity scale ϑ and length scale ℓ. The spectral energy content of these eddies should follow: $E(\kappa) \propto \vartheta^2 \ell$. The largest eddies are highly *anisotropic*.

Descriptors of turbulent flow

Time average of mean

The mean Φ of flow property ϕ is defined as

$$\Phi = \frac{1}{\Delta t}\int_0^{\Delta t} \phi(t)\, dt$$

In time-dependent flows the mean of a property at time t is taken to be the average of the instantaneous values of the property over a large number of repeated identical experiments. The time average of the fluctuations ϕ' is by definition 0.

$$\overline{\phi'} = \frac{1}{\Delta t}\int_0^{\Delta t} \phi'(t)\, dt \equiv 0$$

Variance, r.m.s. and turbulence kineric energy

$$\phi_{rms} = \sqrt{\overline{(\phi')^2}} = \left[\frac{1}{\Delta t}\int_0^{\Delta t} (\phi')^2\, dt\right]$$

The total kinetic energy per unit mass k of the turbulence at a given location can be found as

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

Turbulence intensity (T_i) is a percentage that measures how strong the chaotic velocity fluctuations are in a turbulent flow relative to the average speed of the flow.The turbulence intensity T_i is written as

$$T_i = \frac{\left(\frac{2}{3}k\right)^{1/2}}{U_{ref}}$$

Moments of different fluctuating variables

The variance is also called the second moment of the fluctuations. Important details of the structure of the fluctuations are contained in moments constructed from pairs of different variables. If ψ and ϕ are the variables, then

$$\overline{\phi'\psi'} = \frac{1}{\Delta t}\int_0^{\Delta t} \phi'\psi'\, dt$$

If velocity fluctuations in different directions were independent random fluctuations, then the values of the second moments of the velocity components $\overline{u'v'}, \overline{v'w'}$ and $\overline{u'w'}$ would be zero. If they are not independent, then they are not zero. The third and fourth moment are related to skewness (asymmetry) and kurtosis (peakedness)

$$\overline{(\phi')^3} = \frac{1}{\Delta t}\int_0^{\Delta t} (\phi')^3\, dt$$

$$\overline{(\phi')^4} = \frac{1}{\Delta t}\int_0^{\Delta t} (\phi')^4\, dt$$

Correlation function - time and space

Autocorrelation is a statistical tool used to analyze the structure of turbulence by measuring the "memory" of the flow's fluctuations, either in time or in space. The *autocorrelation* function $R_{\phi'\phi'}(\tau)$ is defined as

$$R_{\phi'\phi'}(\tau) = \overline{\phi'(t)\phi'(t+\tau)} = \frac{1}{\Delta t}\int_0^{\Delta t} \phi'(t)\phi'(t+\tau)\, dt$$

Autocorrelation function $R_{\phi'\phi'}(\xi)$ based on two measurements shifter by a certain distance in space

$$R_{\phi'\phi'}(\xi) = \overline{\phi'(\mathbf{x},t)\phi'(\mathbf{x}+\xi,t)} = \frac{1}{\Delta t}\int_t^{t+\Delta t} \phi'(\mathbf{x},t')\phi'(\mathbf{x}+\xi,t')\, dt'$$

When time shift τ is zero, it just corresponds to the variace $\overline{\phi'^2}$ and will have largest possible value. Since the behaviour is chaotic, then $\tau \to \infty$, so the values of autocorrelation functions will decrease to zero.

Probability density function

It is the fraction of time that a fluctuating signal spends between ϕ^* and $\phi^* + d\phi$

$$P(\phi^*)d\phi^* = \text{Prob}(\phi^* < \phi < \phi^* + d\phi^*)$$

$$\overline{\phi} = \int_{-\infty}^{\infty} \phi P(\phi)\, d\phi$$

$$\overline{(\phi')^n} = \int_{-\infty}^{\infty} (\phi')^n P(\phi)\, d\phi$$

for $n = 2$, we obtain variance of ϕ'. For higher values of n we get higher-order moments.

Flat plate boundary layer and pipe flow

Close to the wall the flow is influenced by viscous effects and doesnot depend on free stream parameters. Th mean flow velocity only depend on the distance y from the wall, fluid density ρ and viscosity μ and the wall shear stress τ_w. So

$$U = f(y, \rho, \mu, \tau_w)$$

by dimensional analysis

$$u^+ = \frac{U}{u_\tau} = f\left(\frac{\rho u_\tau y}{\mu}\right) = f(y^+) \tag{1}$$

Eq.(1) is called the *law of the wall*. Where u^+ and y^+ are dimensionless quantity of velocity and distance respectivly. The term $u_\tau = \sqrt{\tau_w/\rho}$ is called friction velocity. Far away from the wall, the velocity is affected by retarding effect of the wall through the value of the wall shear stress, but not through viscosity itself. The appropriate length is the boundary layer thickness δ.

$$U = g(y, \delta, \rho, \tau_w)$$

by dimensional analysis

$$u^+ = \frac{U}{u_\tau} = g\left(\frac{y}{\delta}\right)$$

.
We can use $U_{max} - U$ i.e., velocity deficit in the above equation as it measures how much the fluid is slowed down due to frictional effects.

$$\frac{U_{max} - U}{u_\tau} = g\left(\frac{y}{\delta}\right) \tag{2}$$

Eq. (2) is called *velocity-defect law*.

In *viscous sub-layer* the viscous effects are dominent. The layer is also very this ($y^+ < 5$). We may assume that the shear stress is approximately constant and equal to the wall shear stress τ_w throughout the layer.

$$\tau(y) = \mu\frac{\partial U}{\partial y} \simeq \tau_w$$

solving and applying boundary conditions

$$U = \frac{\tau_w y}{\mu}$$

we get

$$u^+ = y^+$$

because of this linear relationship, the fluid layer adjacent to the wall is also known as the *linear sub-layer*.

Outside the viscous sublayer ($30 < y^+ < 500$) both viscous and turbulent effects are important. The shear stress τ varies slowly with distance from the wall, and within this inner region it is assumed to be constant and equal to the wall shear stress. By further assumption,

$$u^+ = \frac{1}{\kappa}\ln(y^+) + B = \frac{1}{\kappa}\ln(Ey^+) \tag{3}$$

where von Karman's constant $\kappa \approx 0.4$ and the additive constant $B \approx 5.5$ (or $E \approx 9.8$) for smooth walls. Wall roughness causes a decrease in value of B. These constants are valid for smooth wall at high Reynolds number. Eq. (3) is called *log-law*. This region is the *log-law layer*. By experimental measurements, log-law is valid for region $0.02 < y/\delta < 0.2$. For larger values of y, the velocity-defect law provides correct form. In the overlap region, log-law and velocity-defect law must be equal.

$$\frac{U_{max} - U}{u_\tau} = -\frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + A \tag{4}$$

This velocity-defect law is called as *law of the wake*.

The effect of turbulent fluctuations on properties of the mean flow

Due to momentum exchange between the eddies, some additional stresses get introduced called *Renoylds stress*, which are not caused by viscosity, rather by motion of eddies. In the presence of temperature or concentration gradients the eddy motions will also generate turbulent heat or species concentration fluxes.

Reynolds-averaged Navier-Stokes (RANS) equation for incompressible flow

Let there be two fluctuating properties with $\phi = \Phi + \phi'$ and $\psi = \Psi + \psi'$. Their properties over summation, derivatives and integral:

$$\begin{gathered}\overline{\phi'} = \overline{\overline{\phi'}} = 0 \qquad \overline{\Phi} = \Phi \qquad \overline{\frac{\partial \phi}{\partial s}} = \frac{\partial \Phi}{\partial s} \qquad \overline{\int \phi ds} = \int \Phi ds \\ \overline{\phi + \psi} = \Phi + \Psi \qquad \overline{\phi\psi} = \Phi\Psi + \overline{\phi'\psi} \qquad \overline{\phi\Psi} = \Phi\Psi \qquad \overline{\phi'\Psi} = 0\end{gathered} \tag{5}$$

doing the same for divergence and gradient with vector quantity $\mathbf{a} = \mathbf{A} + \mathbf{a}'$ and scalar quantity $\phi = \Phi + \phi'$

$$\begin{aligned}\overline{\nabla \cdot \mathbf{a}} = \nabla \cdot \mathbf{A} \qquad \overline{\nabla \cdot (\phi \mathbf{A})} = \nabla \cdot (\overline{\phi \mathbf{a}}) = \nabla \cdot (\Phi \mathbf{A}) + \nabla \cdot (\overline{\phi' \mathbf{a}'}) \\ , \overline{\nabla \cdot (\nabla \phi)} = \nabla \cdot (\nabla \Phi)\end{aligned} \tag{6}$$

Navier-Stokes equation for velocity vector $\mathbf{u}$ are

$$\nabla \cdot \mathbf{u} = 0 \tag{7}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{u}) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla \cdot (\nabla u) \tag{8}$$

$$\frac{\partial v}{\partial t} + \nabla \cdot (v\mathbf{u}) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla \cdot (\nabla v) \tag{9}$$

$$\frac{\partial w}{\partial t} + \nabla \cdot (w\mathbf{u}) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla \cdot (\nabla w) \tag{10}$$

By writting $\mathbf{u} = \mathbf{U} + \mathbf{u}'$, $u = U + u'$, $v = V + v'$, $w = W + w'$ and $p = P + p'$
By taking time average of the above equations using Eq. (5) and Eq. (6), we get

$$\nabla \cdot \mathbf{U} = 0 \tag{11}$$

$$\frac{\partial U}{\partial t} + \nabla \cdot (U\mathbf{U}) + \nabla \cdot (\overline{u'\mathbf{u}'}) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla \cdot (\nabla U) \tag{12}$$

$$\frac{\partial V}{\partial t} + \nabla \cdot (V\mathbf{U}) + \nabla \cdot (\overline{v'\mathbf{u}'}) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla \cdot (\nabla V) \tag{13}$$

$$\frac{\partial W}{\partial t} + \nabla \cdot (W\mathbf{U}) + \nabla \cdot (\overline{w'\mathbf{u}'}) = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla \cdot (\nabla W) \tag{14}$$

Expressing the above equations as Reynolds stresses

$$\begin{aligned}\frac{\partial U}{\partial t} + \nabla \cdot (U\mathbf{U}) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla \cdot (\nabla U) \\ +\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'^2})}{\partial x} + \frac{\partial(-\rho\overline{u'v'})}{\partial y} + \frac{\partial(-\rho\overline{u'w'})}{\partial z}\right]\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial V}{\partial t} + \nabla \cdot (V\mathbf{U}) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla \cdot (\nabla V) \\ +\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'v'})}{\partial x} + \frac{\partial(-\rho\overline{v'^2})}{\partial y} + \frac{\partial(-\rho\overline{v'w'})}{\partial z}\right]\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial W}{\partial t} + \nabla \cdot (W\mathbf{U}) = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla \cdot (\nabla W) \\ +\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'w'})}{\partial x} + \frac{\partial(-\rho\overline{v'w'})}{\partial y} + \frac{\partial(-\rho\overline{w'^2})}{\partial z}\right]\end{aligned} \tag{17}$$

The extra turbulent stresses are called *Reynolds stresses* and are expressed by

$$\tau_{xx} = -\rho\overline{u'^2} \qquad \tau_{yy} = -\rho\overline{v'^2} \qquad \tau_{zz} = -\rho\overline{w'^2} \tag{18}$$

$$\tau_{xy} = \tau_{yx} = -\rho\overline{u'v'} \qquad \tau_{xz} = \tau_{zx} = -\rho\overline{u'w'} \qquad \tau_{yz} = \tau_{zy} = -\rho\overline{v'w'} \tag{19}$$

The correlations between pairs of different velocity components due to the structure of the vortical eddies ensures that the turbulent shear stresses are non-zero and usually very large compared with the viscous stresses in a turbulent flows. Eq. (15) - Eq. (17) is called the *Reynolds-averages Navier-Stokes equations*. For other time-average transport equation for scalar ϕ is

$$\frac{\partial \Phi}{\partial t} + \nabla \cdot (\Phi \mathbf{U}) = \frac{1}{\rho}\nabla \cdot (\Gamma_\Phi \nabla \Phi) + \left[-\frac{\partial \overline{u'\phi'}}{\partial x} - \frac{\partial \overline{v'\phi'}}{\partial y} - \frac{\partial \overline{w'\phi'}}{\partial z} \right] + S_\Phi \tag{20}$$

For r.m.s. velocity fluctuations are of order 5%, density fluctuations is unimportant upto Mach numbers around 3 to 5. But in free turbulent flows, it can easily go upto 20%. In such cases density fluctuations start to affect the turbulence around Mach number of 1. For compressible flow

Continuity

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla(\bar{\rho}\tilde{\mathbf{U}}) = 0 \tag{21}$$

Reynolds equations

$$\begin{aligned}
1.\quad & \frac{\partial(\bar{\rho}\tilde{U})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{U}\tilde{\mathbf{U}}) = -\frac{\partial \bar{P}}{\partial x} + \nabla \cdot (\mu \nabla \tilde{U}) + \\
& \left[-\frac{\partial(\overline{\rho u'^2})}{\partial x} - \frac{\partial(\overline{\rho u'v'})}{\partial y} - \frac{\partial(\overline{\rho u'w'})}{\partial z} \right] + S_{Mx} \\
2.\quad & \frac{\partial(\bar{\rho}\tilde{V})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{V}\tilde{\mathbf{U}}) = -\frac{\partial \bar{P}}{\partial y} + \nabla \cdot (\mu \nabla \tilde{V}) + \\
& \left[-\frac{\partial(\overline{\rho u'v'})}{\partial x} - \frac{\partial(\overline{\rho v'^2})}{\partial y} - \frac{\partial(\overline{\rho v'w'})}{\partial z} \right] + S_{My} \\
3.\quad & \frac{\partial(\bar{\rho}\tilde{W})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{W}\tilde{\mathbf{U}}) = -\frac{\partial \bar{P}}{\partial z} + \nabla \cdot (\mu \nabla \tilde{W}) + \\
& \left[-\frac{\partial(\overline{\rho u'w'})}{\partial x} - \frac{\partial(\overline{\rho v'w'})}{\partial y} - \frac{\partial(\overline{\rho w'^2})}{\partial z} \right] + S_{Mz}
\end{aligned} \tag{22}$$

Scalar transport equation

$$\begin{aligned}
& \frac{\partial(\bar{\rho}\tilde{\Phi})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{\Phi}\tilde{\mathbf{U}}) = \nabla \cdot (\Gamma_\Phi \nabla \bar{\Phi}) + \\
& \left[-\frac{\partial(\overline{\rho u'\phi'})}{\partial x} - \frac{\partial(\overline{\rho v'\phi'})}{\partial y} - \frac{\partial(\overline{\rho w'\phi'})}{\partial z} \right] + S_\Phi
\end{aligned} \tag{23}$$

Overbar indicates a time-averaged variable and the tilde indicates a density-weighted or Favre-averaged variable.

Eddy viscosity and eddy diffusivity

By Newton's law of viscosity, the viscous stresses are taken to be proportional to the rate of deformation of fluid elements. For an incompressible fluid,

$$\tau_{ij} = \mu s_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{24}$$

Turbulence decays unless there is shear in isothermal incompressible flows. Also turbulent stresses increases as the mean reate of deformation increases. *Boussinesq* proposed that, Reynolds stresses might be proportional to mean rates of deformation.

$$\tau_{ij} = -\rho\overline{u'_i u'_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij} \tag{25}$$

where $k = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$ is the turbulent kinetic energy per unit mass. δ_{ij} is the Kronecker delta, $\delta_{ij} = 1$ for $i = j$ else 0. This ensures the correct result for the normal Reynolds stresses. The turbulent transport of a scalar is taken to be proportional to the gradient of the mean value of the transport quantity.

$$-\rho\overline{u'_i \phi'} = \Gamma_t \frac{\partial \Phi}{\partial x_i}$$

where Γ_t is the turbulent or eddy diffusivity. Turbulent Prandtl/Schmidt number defined as

$$\sigma_t = \frac{\mu_t}{\Gamma_t}$$

In most CFD models, σ_t is around unity.

The $k - \epsilon$ model

The instantaneous kinetic energy $k(t)$ of a turbulent flow is the sum of the mean kinetic energy K and the turbulent kinetic energy k.

$$k(t) = K + k$$

The components of the rate of deformation s_{ij} and the stresses τ_{ij} in tensor form as:

$$s_{ij} = \begin{pmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{pmatrix} \quad \text{and} \quad \tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

also $s_{ij}(t) = S_{ij} + s'_{ij}$

Governing equation for mean flow kinetic energy K

$$\frac{\partial(\rho K)}{\partial t} + \nabla \cdot (\rho K \mathbf{U}) = \nabla \cdot (-P\mathbf{U} + 2\mu \mathbf{U} S_{ij} - \rho \mathbf{U}\overline{u'_i u'_j}) - 2\mu S_{ij} \cdot S_{ij} + \rho\overline{u'_i u'_j} \cdot S_{ij} \tag{26}$$

The two terms that contain the Reynolds stresses $\rho\overline{u'_i u'_j}$ accounts for turbulence effects.

Governing equation for turbulent flow kinetic energy k

$$\begin{aligned}
\frac{\partial(\rho k)}{\partial t} + \mathrm{div}(\rho k \mathbf{U}) = \mathrm{div}(-\overline{p'\mathbf{u}'} + 2\mu\overline{\mathbf{u}' s'_{ij}} - \rho\frac{1}{2}\overline{u'_i \cdot u'_i u'_j}) - \\
2\mu\overline{s'_{ij} \cdot s'_{ij}} - \rho\overline{u'_i u'_j} \cdot S_{ij}
\end{aligned} \tag{27}$$

The changes in turbulent kinetic energymare mainly governed by turbulent interactions. The viscous dissipation term is expressed as

$$-2\mu\overline{s'_{ij} \cdot s'_{ij}} = -2\mu(\overline{s'^2_{11}} + \overline{s'^2_{22}} + \overline{s'^2_{33}} + 2\overline{s'^2_{12}} + 2\overline{s'^2_{13}} + 2\overline{s'^2_{32}})$$

This dissipation of turbulent kinetic energy is caused by work done by the smallest eddies against viscous stresses. The rate of dissipation of turbulent kinetic energy per unit mass ϵ, is given by

$$\epsilon = 2\nu\overline{s'_{ij} \cdot s'_{ij}}$$

The $k - \epsilon$ model equations

Defining velocity scale ϑ and length scale ℓ using k and ϵ as

$$\vartheta = k^{1/2} \qquad \ell = \frac{k^{3/2}}{\epsilon}$$

By applying dimensionless analysis, eddy viscosity is specified as

$$\mu_t = C\rho\vartheta\ell = \rho C_\mu \frac{k^2}{\epsilon} \tag{28}$$

where C_μ is a dimensionless constant. The transport equations for k and ϵ are:

$$\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho k \mathbf{U}) = \nabla \cdot \left[\frac{\mu_t}{\sigma_k} \nabla k \right] + 2\mu_t S_{ij} \cdot S_{ij} - \rho\epsilon \tag{29}$$

$$\frac{\partial(\rho \epsilon)}{\partial t} + \nabla \cdot (\rho \epsilon \mathbf{U}) = \nabla \cdot \left[\frac{\mu_t}{\sigma_\epsilon} \nabla \epsilon \right] + C_{1\epsilon}\frac{\epsilon}{k} 2\mu_t S_{ij} \cdot S_{ij} - C_{2\epsilon}\rho\frac{\epsilon^2}{k} \tag{30}$$

Where $C_\mu = 0.09$, $\sigma_k = 1.00$, $\sigma_\epsilon = 1.30$, $C_{1\epsilon} = 1.44$ and $C_{2\epsilon} = 1.92$. These can be varied, but the constants are arrived after fitting for a wide range of turbulent flows. reynolds stresses can be calculted from Boussinesq relationship.

$$-\rho\overline{u'_i u'_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij} \tag{31}$$

Boundary conditions

The model equations for k and ϵ are elliptical.

- intet : distributions of k and ϵ must by given
- outlet, symmetry axis : $\partial k/\partial n = 0$ and $\partial\epsilon/\partial n = 0$
- free stream : k and ϵ must be given or $\partial k/\partial n = 0$ and $\partial\epsilon/\partial n = 0$

If no information about k and ϵ, approximation for distribution can be obtained from turbulence intensity T_i and a characteristic length L of the equipment.

$$k = \frac{2}{3}(U_{ref}T_i)^2 \qquad \epsilon = C_\mu^{3/4}\frac{k^{3/2}}{\ell} \qquad \ell = 0.07L$$

Natural choice of boundary conditions for turbulent-free free stream would be $k = 0$ and $\epsilon = 0$. At *high Reynolds number*, the log-law is valid. The mean velocity at point y_p with $30 < y_p^+ < 500$ satisfies the log-law. The follows *wall functions* are

$$u^+ = \frac{U}{u_\tau} = \frac{1}{\kappa}\ln(Yy_p^+) \quad k = \frac{u_\tau^2}{\sqrt{C_\mu}} \quad \epsilon = \frac{u_\tau^3}{\kappa y} \tag{32}$$

where $\kappa = 0.41$ (Von Karman's) constant and wall roughness parameter $E = 9.8$ for smooth walls. The wall function for heat transfer is

$$T^+ \equiv -\frac{(T - T_w)C_p\rho u_\tau}{q_w} = \sigma_{T,t}\left(u^+ + P\left[\frac{\sigma_{T,l}}{\sigma_{T,t}}\right]\right) \tag{33}$$

where T_p = temperature at near-wall point y_p; T_w = wall temperature; C_p = fluid specific heat at constant pressure; q_w = wall heat flux; $\sigma_{T,t}$ = turbulent Prandtl number; $\sigma_{T,l} = \mu C_p/\Gamma_T$ = (laminar or molecular) Prandtl number; Γ_T = thermal conductivity. P is the pee-function, a correlation function dependent on the ratio of laminar to turbulent Prandtl numbers.

At *low Reynolds number*, log-law is invalid. The wall functions need to be damped too ensure that viscous stresses take over from turbulent Reynolds stresses at low Reynolds number and in the viscous sub-layer adjacent to solid walls. The modified equations for low Reynolds number $k-\epsilon$ model are

$$\mu_t = \rho C_\mu f_\mu \frac{k^2}{\epsilon} \tag{34}$$

$$\frac{\partial(\rho k)}{\partial t} + \nabla\cdot(\rho k\mathbf{U}) = \nabla\cdot\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\nabla k\right] + 2\mu_t S_{ij}\cdot S_{ij} - \rho\epsilon \tag{35}$$

$$\frac{\partial(\rho\epsilon)}{\partial t} + \nabla\cdot(\rho\epsilon\mathbf{U}) = \nabla\left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\nabla\epsilon\right] + C_{1\epsilon}f_1\frac{\epsilon}{k}2\mu_t S_{ij}\cdot S_{ij} - C_{2\epsilon}f_2\rho\frac{\epsilon^2}{k} \tag{36}$$

here f_μ, f_1 and f_2 are the wall-damping functions, which are themselves functions of the turbulence Reynolds number.

$$Re_t = \frac{\vartheta\ell}{\nu} = \frac{k^2}{\epsilon\nu}$$

$$Re_y = \frac{k^{1/2}y}{\nu}$$

$$f_\mu = \left[1 - e^{-0.0165Re_y}\right]^2\left(1 + \frac{20.5}{Re_t}\right) \tag{37}$$

$$f_1 = \left(1 + \frac{0.05}{f_\mu}\right)^3 \tag{38}$$

$$f_2 = 1 - e^{-Re_t^2} \tag{39}$$

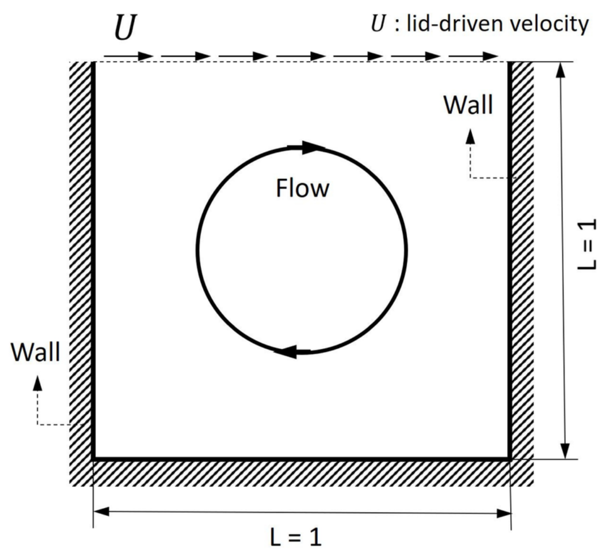
For boundary conditons for ϵ, the rate of dissipation of turbulennt energy rises steeply as the wall is approached and tends to an unknown constant vlaue. Lam and Bremhorst use $\partial\epsilon/\partial y = 0$. Some $k-\epsilon$ model uses modified dissipation rate variable defined as $\tilde{\epsilon} = \epsilon - 2\nu(\partial\sqrt{k}/\partial n)^2$, which sets as $\tilde{\epsilon} = 0$

STANDARD $k - \epsilon$ MODEL

2D Turbulent Lid-Driven Cavity

Problem Formulation

Simulation of steady-state, incompressible turbulent flow within a 2D lid-driven cavity. The fluid motion is described by the Reynolds-Averaged Navier-Stokes (RANS) equations, with turbulent effects modeled using the standard $k - \epsilon$ model and a high and low-Reynolds-number wall function approach. The governing equations are discretized using a finite difference method on a staggered grid and solved iteratively with the SIMPLE algorithm to handle pressure-velocity coupling.



Equations

The countinuity equation for 2D incompressible flow is given by

$$\nabla \cdot \mathbf{U} = 0 \tag{1}$$

The momentum equations are given by

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{U}) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla \cdot (\nabla u) \tag{2}$$

$$\frac{\partial v}{\partial t} + \nabla \cdot (v\mathbf{U}) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla \cdot (\nabla v) \tag{3}$$

The eddy viscosity is calculated as

$$\mu_t = \rho C_\mu f_\mu \frac{k^2}{\epsilon} \tag{4}$$

where C_μ is a dimensionless constant. $C_\mu = 0.09$. The standard $k - \epsilon$ model transport equations for k and ϵ are

$$\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho k \mathbf{U}) = \nabla \cdot \left[\left(\mu + \frac{\mu_t}{\sigma_k}\right) \nabla k\right] + 2\mu_t S_{ij} \cdot S_{ij} - \rho\epsilon \tag{5}$$

$$\frac{\partial(\rho \epsilon)}{\partial t} + \nabla \cdot (\rho \epsilon \mathbf{U}) = \nabla \cdot \left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right) \nabla \epsilon\right] + C_{1\epsilon} f_1 \frac{\epsilon}{k} 2\mu_t S_{ij} \cdot S_{ij} - C_{2\epsilon} f_2 \rho \frac{\epsilon^2}{k} \tag{6}$$

where $\sigma_k = 1.00$, $\sigma_\epsilon = 1.30$, $C_{1\epsilon} = 1.44$ and $C_{2\epsilon} = 1.92$ are adjustable constants. Special *wall functions* are used for ϵ based on the location of ϵ grid point.

The boundary values of ϵ is set based on the location of ϵ grid point from the wall surface. If $y^+ > 5$, then the grid point is in viscous sub layer, then

$$\epsilon = \frac{2k\nu}{y^2} \tag{7}$$

where k is the turbulent kinetic energy at ϵ grid point, y is the normal distance from the wall. If $30 < y^+ < 500$, it satisifies the log-law. If the grid point is in the log-law region, then ϵ at boundary is given by

$$\epsilon = \frac{C_\mu^{3/4} k^{1/2}}{\kappa y} \tag{8}$$

where $\kappa = 0.41$, called Von Karman's constant. y^+ is calculated by

$$y^+ = \frac{C_\mu^{1/4} k^{1/2} y}{\nu}$$

For low Reynolds number, wall damping functions needs to be used in ϵ transport equation and turbulent viscosity equation. f_μ, f_1 and f_2 are inserted to those equation.

$$Re_t = \frac{\vartheta\ell}{\nu} = \frac{k^2}{\epsilon\nu}$$

$$Re_y = \frac{k^{1/2}y}{\nu}$$

$$f_\mu = \left[1 - e^{-0.0165 Re_y}\right]^2 \left(1 + \frac{20.5}{Re_t}\right) \tag{9}$$

$$f_1 = \left(1 + \frac{0.05}{f_\mu}\right)^3 \tag{10}$$

$$f_2 = 1 - e^{-Re_t^2} \tag{11}$$

Implimentation

Step 1 to 5 is similar for FDM and FVM. Step 3 - 5 is same as in laminar steady solver in FVM.

- Set Boundary Conditions
  The u velocities are parallel to top and bottom wall, normal to left and right wall. For the bottom wall, $u = 0$, thus the average of grid points in either side of the boundary should be zero. So, the velocity of the grid point outside the physical domain should be the opposite of velocity inside the grid point.

$$\frac{u[i][\text{below bottom wall}] + u[i][\text{above bottom wall}]}{2} = 0$$

$$u[i][\text{below bottom wall}] = -u[i][\text{above bottom wall}]$$

  Similarly the average of grid points in either side of top wall should be 1.

$$u[i][\text{above top wall}] = 2 \times u_{\text{lid}} - u[i][\text{below top wall}]$$

  For the left and right wall, u can be directly assigned to 0. The similar thing can be done to v velocity. v velocity at the boundary is 0. The pressure at every pressure grid point can be guessed initially, 0 would be a good guess.
  Since there is no turbulent kinetic energy is at the start of the simulation, it can be assigned to a very small value at every grid point, to avoid zero division. Also zero gradient is applied accorss the boundary.
  The value of ϵ depends on the position of grid point from the closest surface. If the gird point is inside viscosity sub-layer, Eq. (7) can be used. If the gird point is in log-law region, Eq.(8) can be used. There is no spesific relation in buffer region, so the critical y^+ is taken as 11.63. Zero gradient is also applied accoss the boundary of ϵ.

- Calculating turbulent viscosity
  Turbulent viscosity is calculated from Eq. (4) for all grid points. The damping functions are used if the Reynolds number is less than or equal to 3200.

- Solving momentum equations
  Eq. (2) and Eq. (3) is solved by discretizing the equations and solving for $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ and storing in a temporary u and v matrices. The dynamic viscosity taken here is the sum of laminar viscosity and turbulent viscosity. Since the turbulent viscosity grid point is not on the u and v grid points, the average of μ_t is taken accross the respective velocity grid point.

- Pressure Correction
  Since the guessed pressure is wrong, the velocity obtained from solving the momentum equation is also wrong. So we move to the pressure correction. First we set all the elements in p' to 0 before the pressure correction begins. We calculate the *mass source* term by

$$d = \frac{1}{\Delta x}[(\rho u^*)_{i,j} - (\rho u^*)_{i-1,j}] + \frac{1}{\Delta y}[(\rho v^*)_{i,j} - (\rho v^*)_{i,j-1}]$$

  By *Poisson equation*

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = Q$$

  by expanding we get,

$$ap'_{i,j} + bp'_{i+1,j} + bp'_{i-1,j} + cp'_{i,j+1} + cp'_{i,j-1} + d = 0$$

  where

$$a = 2\left[\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2}\right]$$

$$b = -\frac{\Delta t}{(\Delta x)^2}$$

$$c = -\frac{\Delta t}{(\Delta y)^2}$$

  solving for p' using iterative loop. At the end of every correction iteration, we apply zero gradient boundary condition for pressure.

- Correcting flow fields.
  From u^*, v^* and p', we correct the flow fields to u, v and p. We generally under-relax to avoid over shooting of variables.

$$p_{i,j}^{n+1} = p_{i,j}^{n} + \alpha_p p'$$

$$u_{i,j}^{n+1} = \alpha_u \left[u_{i,j}^* - \frac{\Delta t}{\rho \Delta x}(p'_{i+1,j} + p'_{i,j}) \right] + (1-\alpha_u) u_{i,j}^n$$

$$v_{i,j}^{n+1} = \alpha_v \left[v_{i,j}^* - \frac{\Delta t}{\rho \Delta y}(p'_{i,j+1} + p'_{i,j}) \right] + (1-\alpha_v) v_{i,j}^n$$

  where α_p, α_u and α_v are the relaxation factors of p, u and v respectively.

- Calculating k and ϵ
  Calculate the dot product of stress $S_{ij} \cdot S_{ij}$

$$S_{ij} \cdot S_{ij} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} \cdot \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = S_{xx}^2 + 2S_{xy}^2 + S_{yy}^2$$

  Substituting the values of the stresses, the production term

$$2\mu_t S_{ij} \cdot S_{ij} = \mu_t \left[2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right]$$

  For finite difference merhord
  Solving for k^{n+1} and ϵ^{n+1} from Eq. (5) and Eq. (6) and store it in a temporary k^* and ϵ^* grid. Also using damping functions for low Reynolds number (Re $\leq$ 3200). k and ϵ are relaxed by

$$k^{n+1} = \alpha_k k^* + (1-\alpha_k) k^n$$

$$\epsilon^{n+1} = \alpha_\epsilon \epsilon^* + (1-\alpha_\epsilon)\epsilon^n$$

  For finite volume merhord
  For steady state solver, the time derivative is zero. By discretizing the turbulent equation in general

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + a_N \phi_N + a_S \phi_S + S_u$$

$$a_P = a_W + a_E + a_N + a_S - S_p$$

  For source term

$$\bar{S}\Delta V = S_u + S_p \phi_P$$

  by solving for k and ϵ

$$S_{uk} = Pk \cdot \Delta V$$

$$S_{u\epsilon} = C_{1\epsilon} f_1 \frac{\epsilon}{k} Pk \Delta V$$

$$S_{pk} = -\rho \frac{\epsilon}{k} \Delta V$$

$$S_{p\epsilon} = C_{2\epsilon} f_2 \frac{\epsilon}{k} \Delta V$$

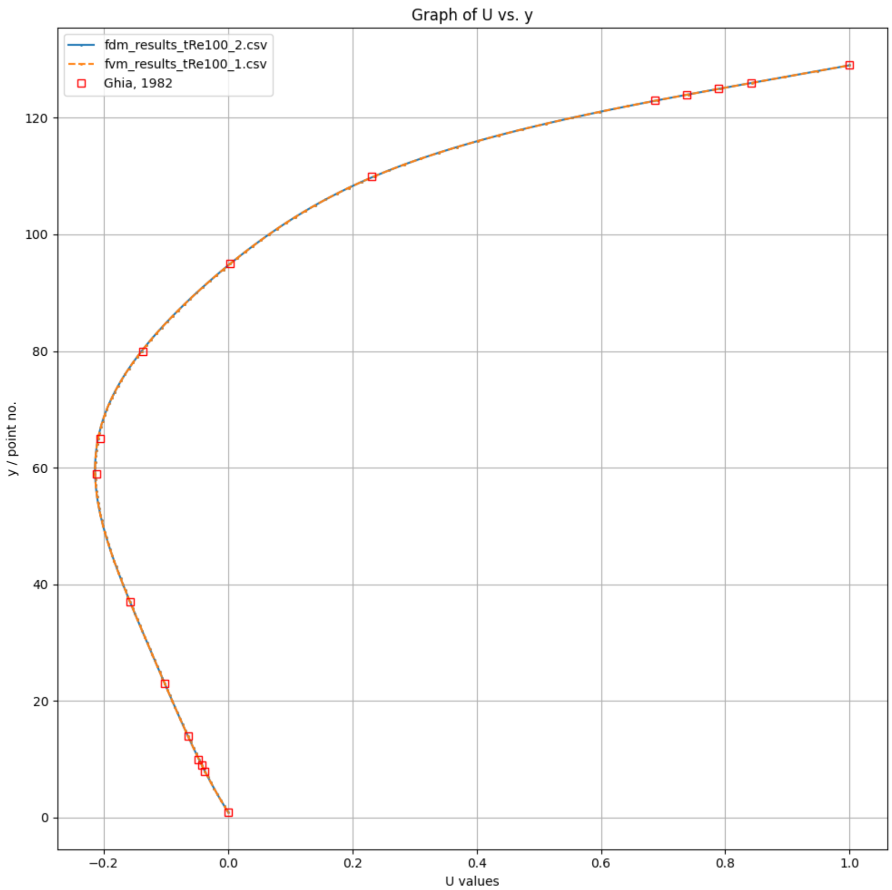
- Set boundary conditions again.

Iterate through Step 2 - 6 till the solutions converges.

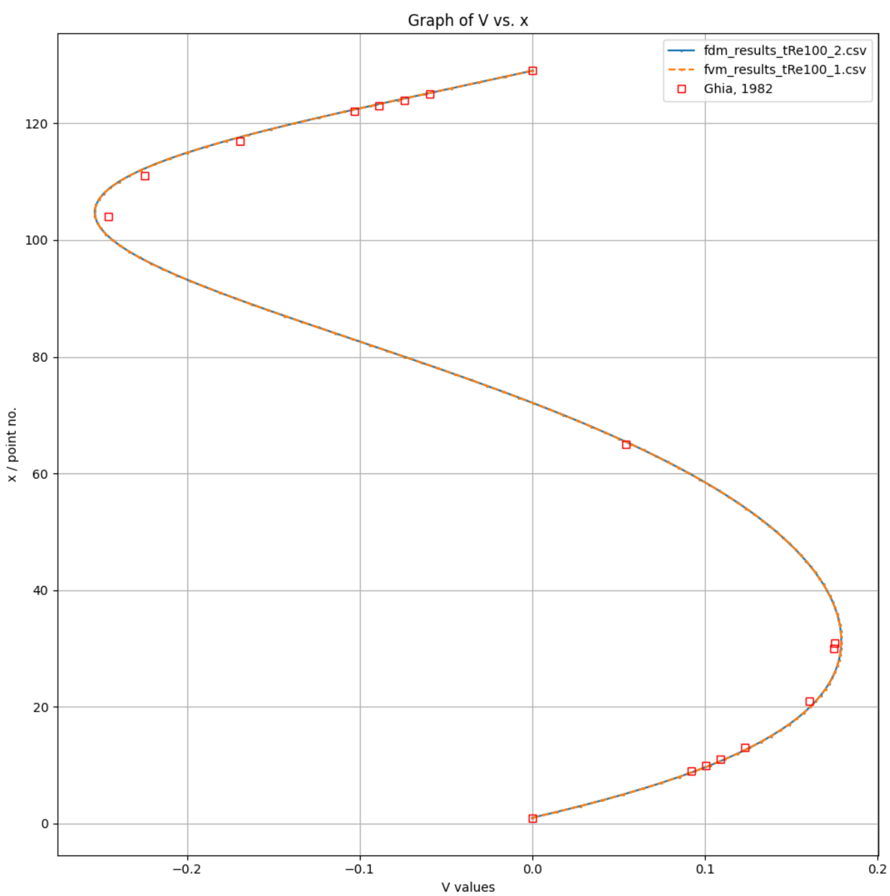
Turn over for plots

Plots

Re = 100

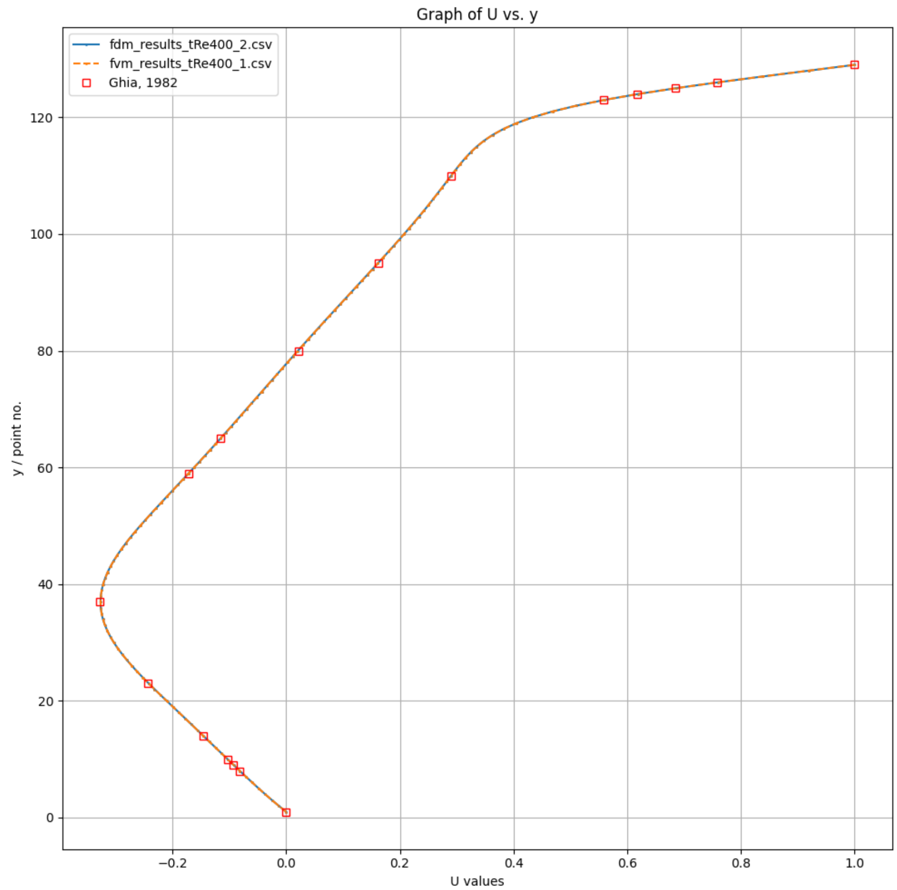


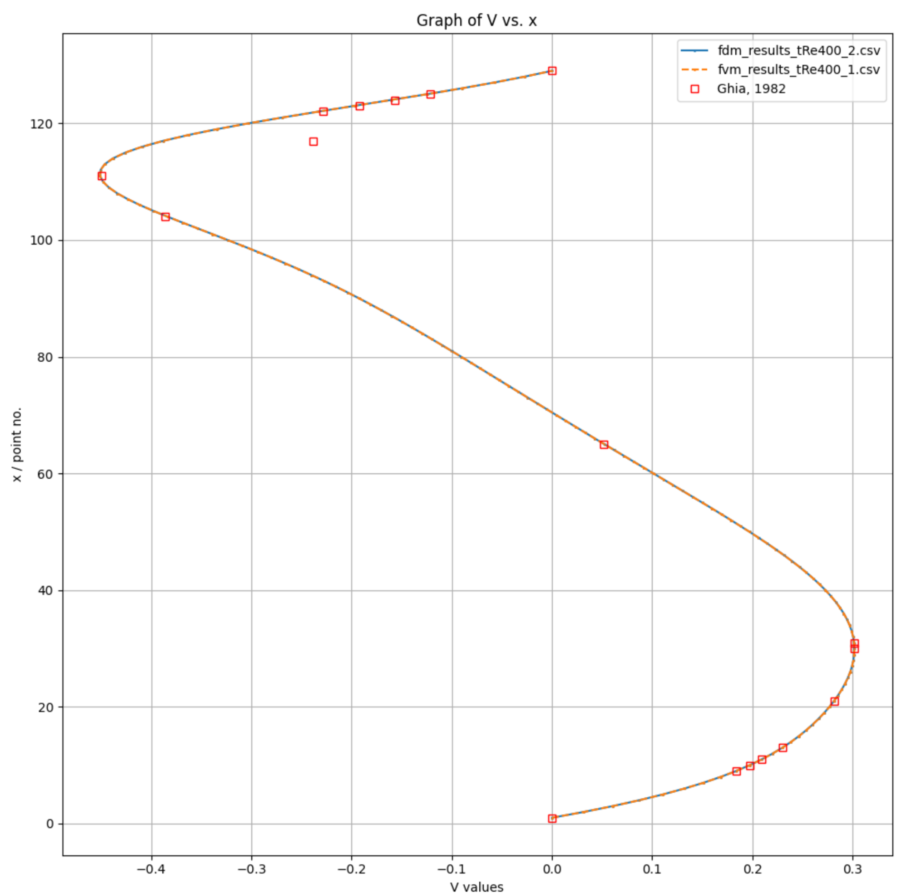
Graph of U vs. y
fdm_results_tRe100_2.csv
fvm_results_tRe100_1.csv
Ghia, 1982
y / point no.
U values



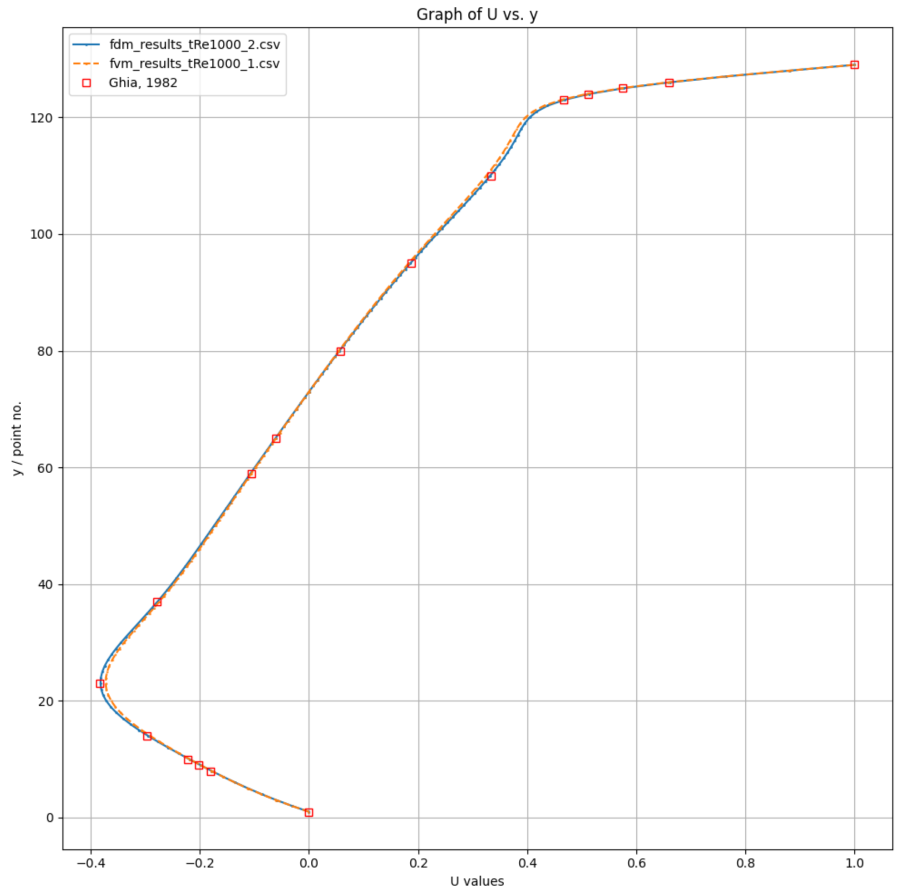
Graph of V vs. x
fdm_results_tRe100_2.csv
fvm_results_tRe100_1.csv
Ghia, 1982
x / point no.
V values

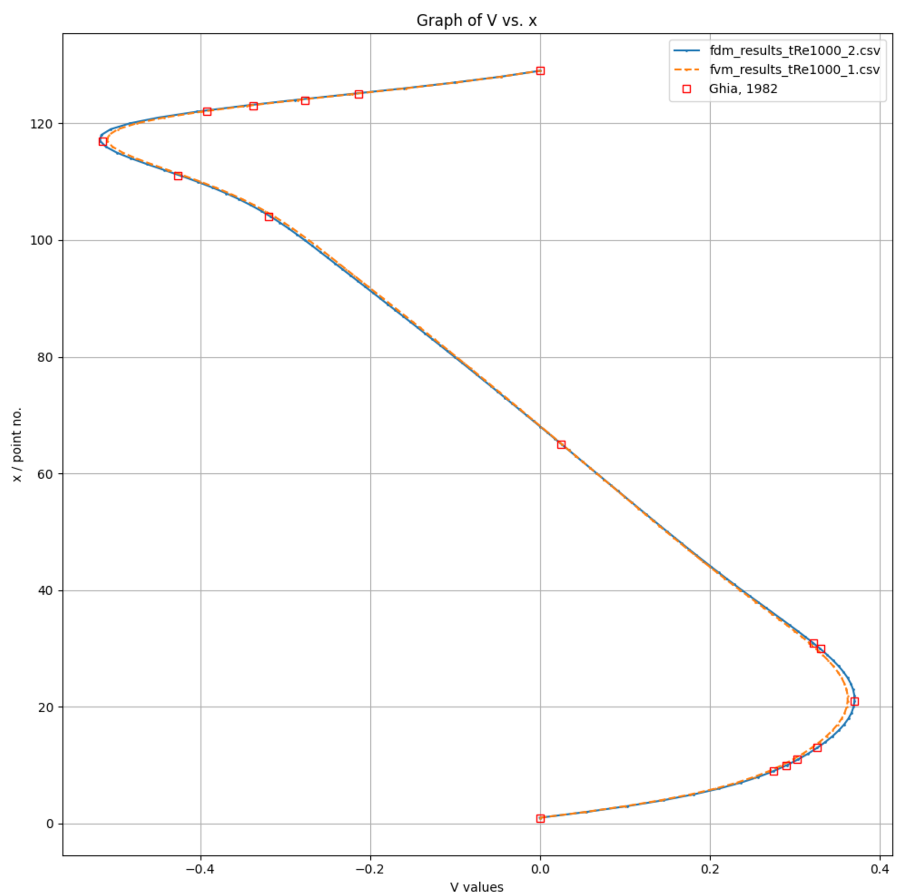
Re = 400


Graph of U vs. y
fdm_results_tRe400_2.csv
fvm_results_tRe400_1.csv
Ghia, 1982
y / point no.
U values
0
20
40
60
80
100
120
−0.2
0.0
0.2
0.4
0.6
0.8
1.0


Graph of V vs. x
fdm_results_tRe400_2.csv
fvm_results_tRe400_1.csv
Ghia, 1982
x / point no.
V values
0
20
40
60
80
100
120
−0.4
−0.3
−0.2
−0.1
0.0
0.1
0.2
0.3

Re = 1000


Graph of U vs. y
fdm_results_tRe1000_2.csv
fvm_results_tRe1000_1.csv
Ghia, 1982
y / point no.
U values
0
20
40
60
80
100
120
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0


Graph of V vs. x
fdm_results_tRe1000_2.csv
fvm_results_tRe1000_1.csv
Ghia, 1982
x / point no.
V values
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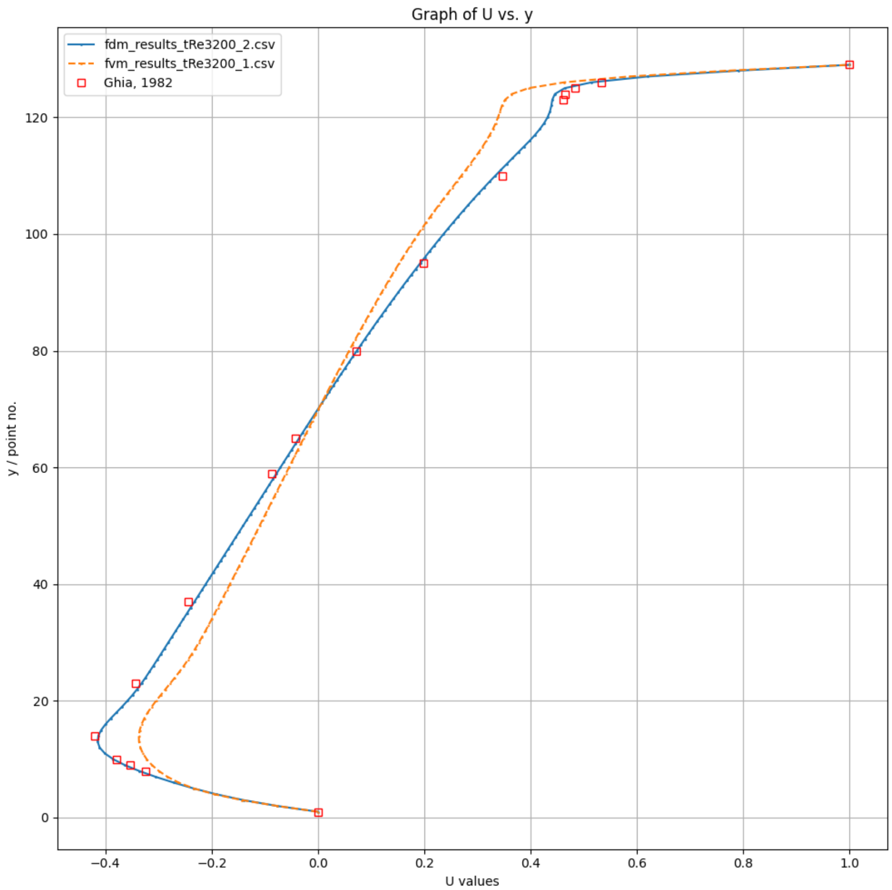
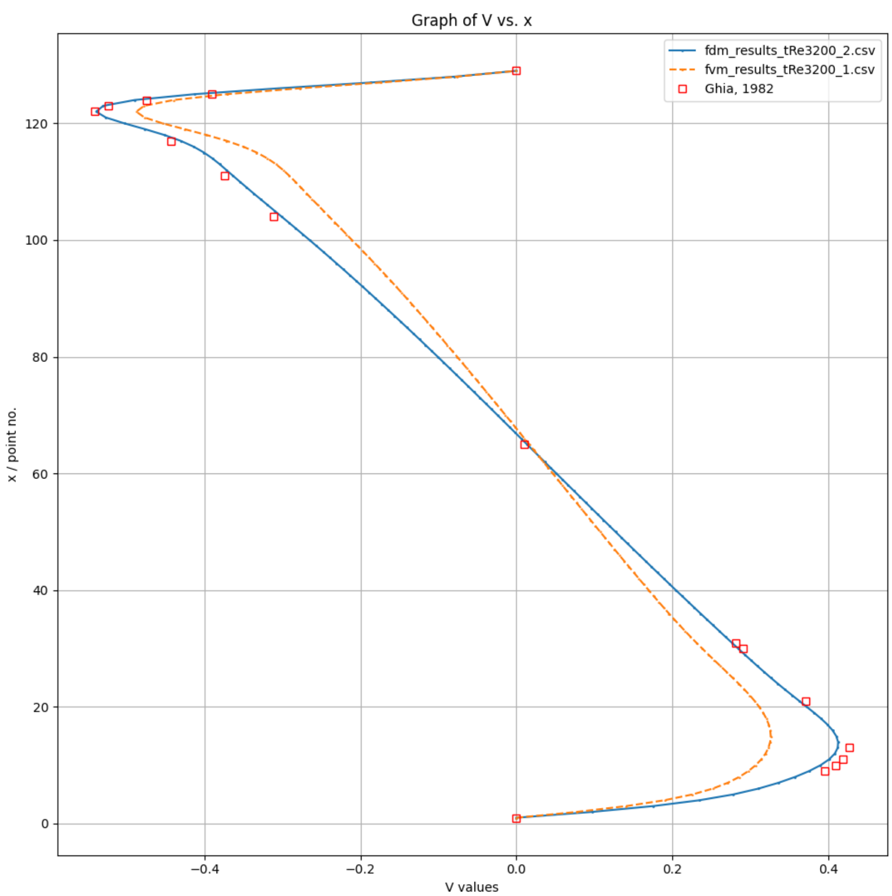
20
40
60
80
100
120
−0.4
−0.2
0.0
0.2
0.4

Re = 3200

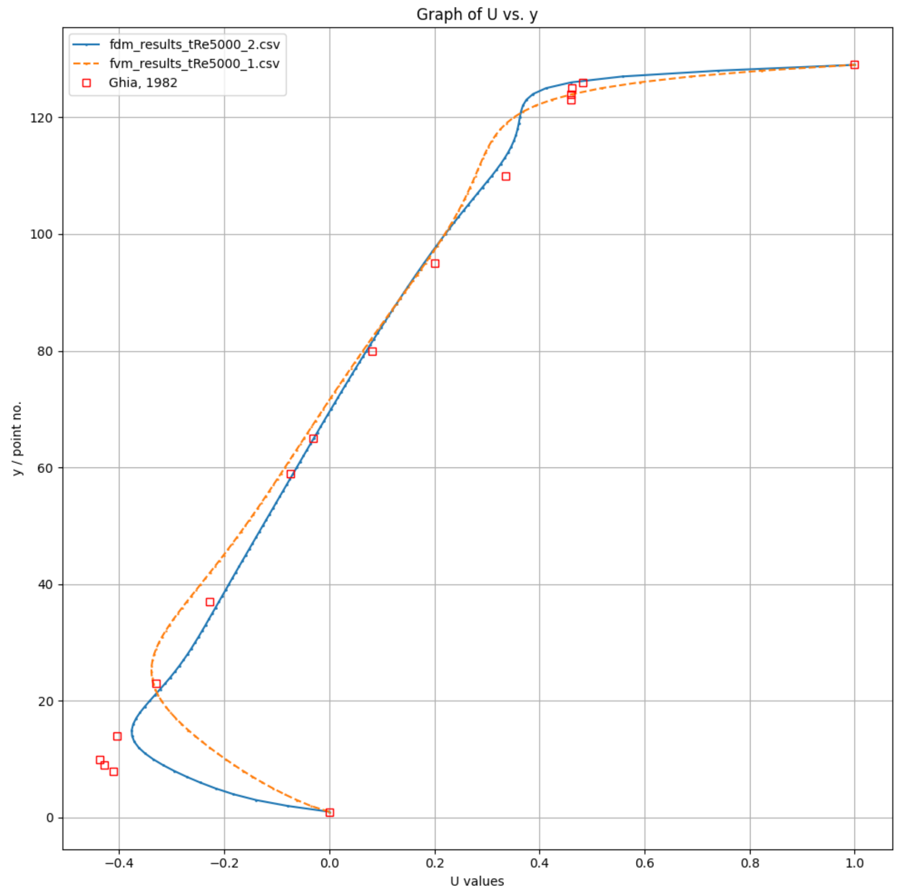


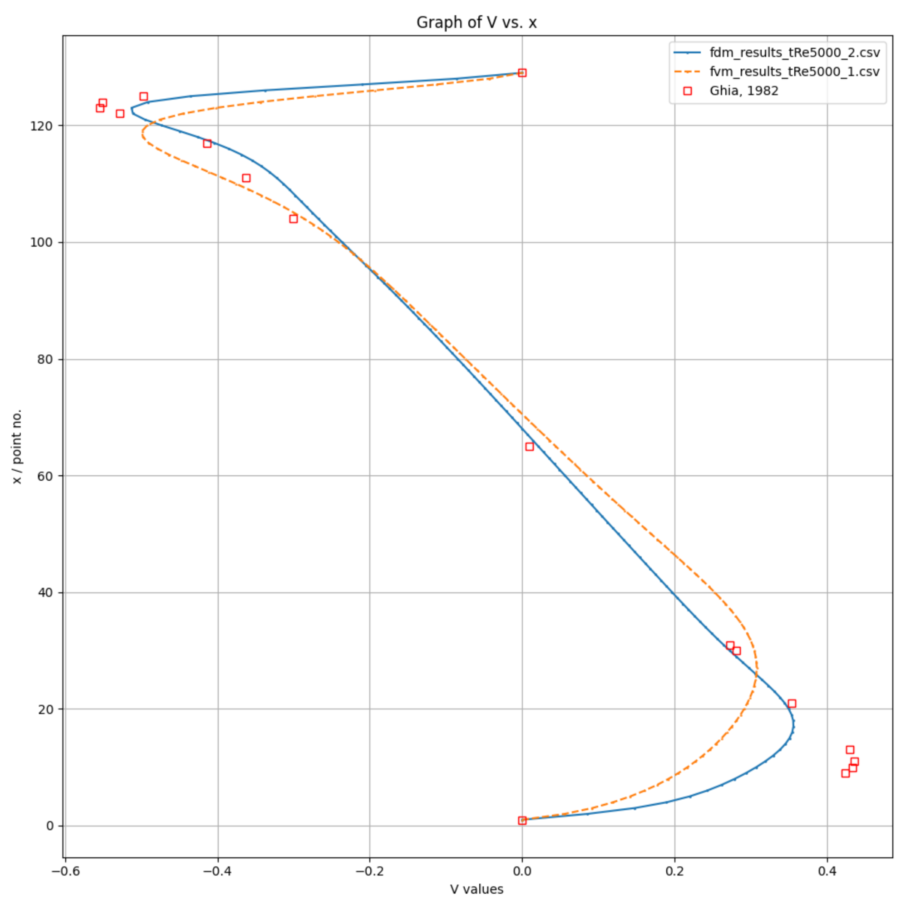
Graph of U vs. y
fdm_results_tRe3200_2.csv
fvm_results_tRe3200_1.csv
Ghia, 1982
y / point no.
U values



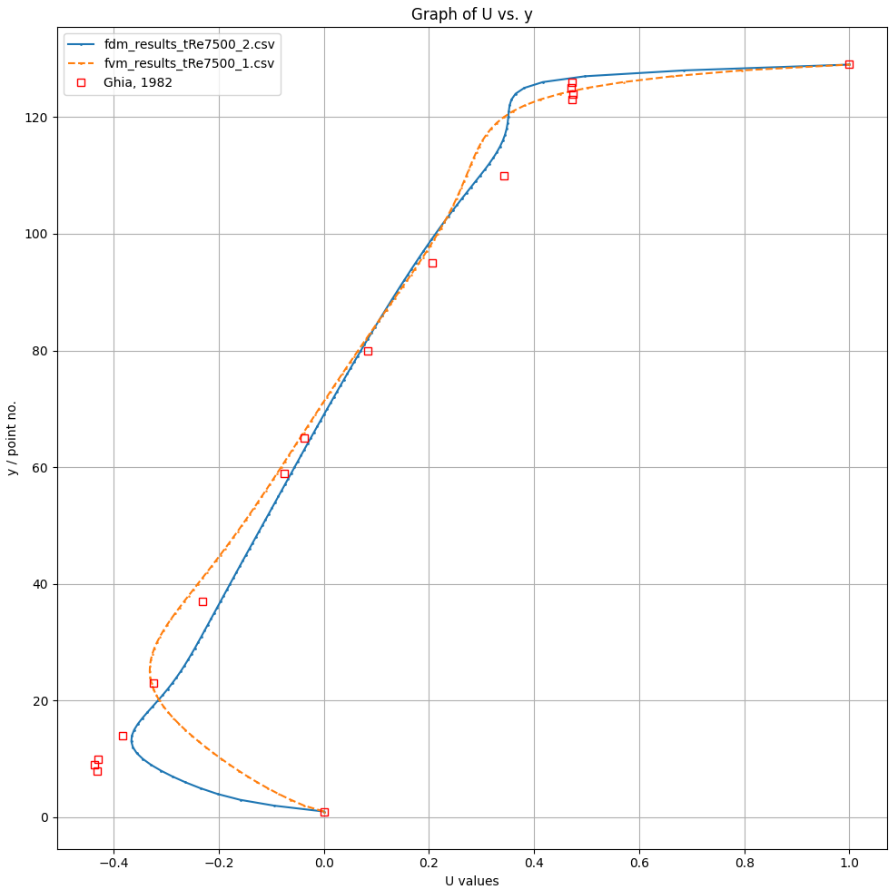
Graph of V vs. x
fdm_results_tRe3200_2.csv
fvm_results_tRe3200_1.csv
Ghia, 1982
x / point no.
V values

Re = 5000

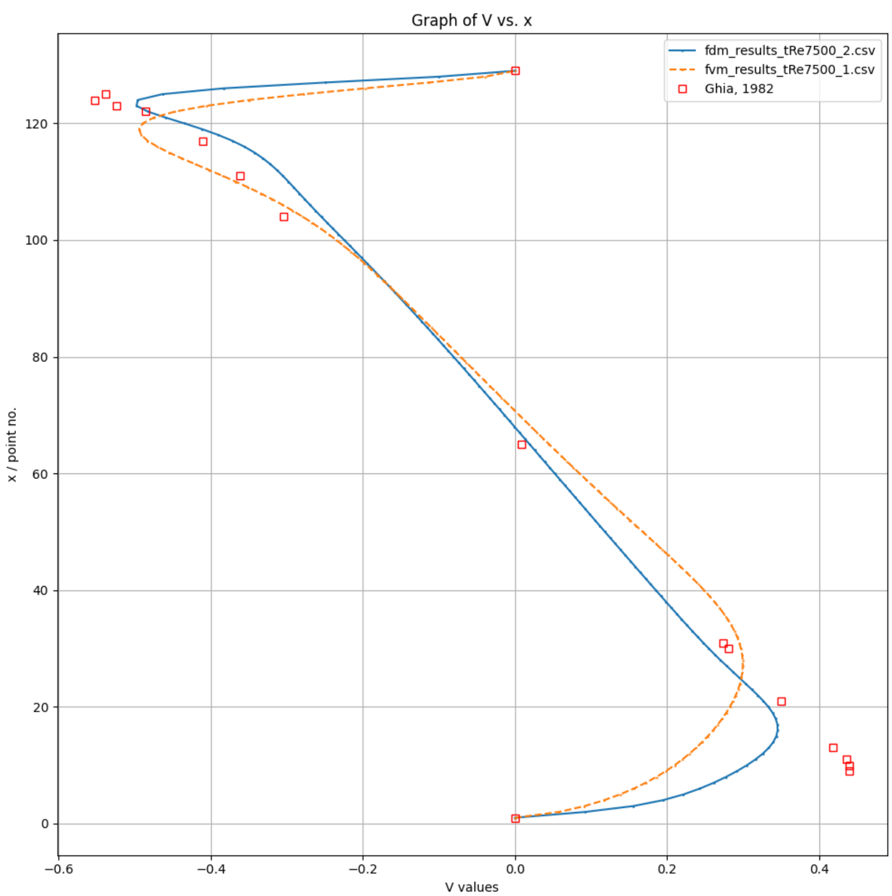

Graph of U vs. y
fdm_results_tRe5000_2.csv
fvm_results_tRe5000_1.csv
Ghia, 1982
y / point no.
U values
0
20
40
60
80
100
120
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0


Graph of V vs. x
fdm_results_tRe5000_2.csv
fvm_results_tRe5000_1.csv
Ghia, 1982
x / point no.
V values
0
20
40
60
80
100
120
−0.6
−0.4
−0.2
0.0
0.2
0.4

Re = 7500

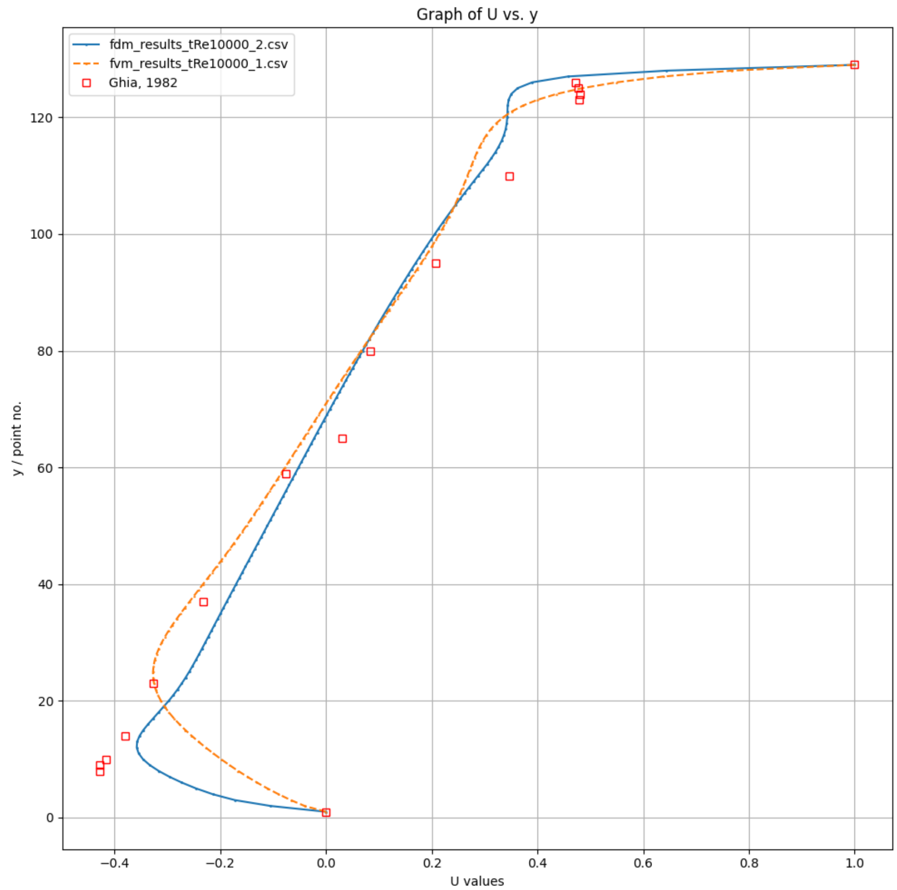


Graph of U vs. y
fdm_results_tRe7500_2.csv
fvm_results_tRe7500_1.csv
Ghia, 1982
y / point no.
U values

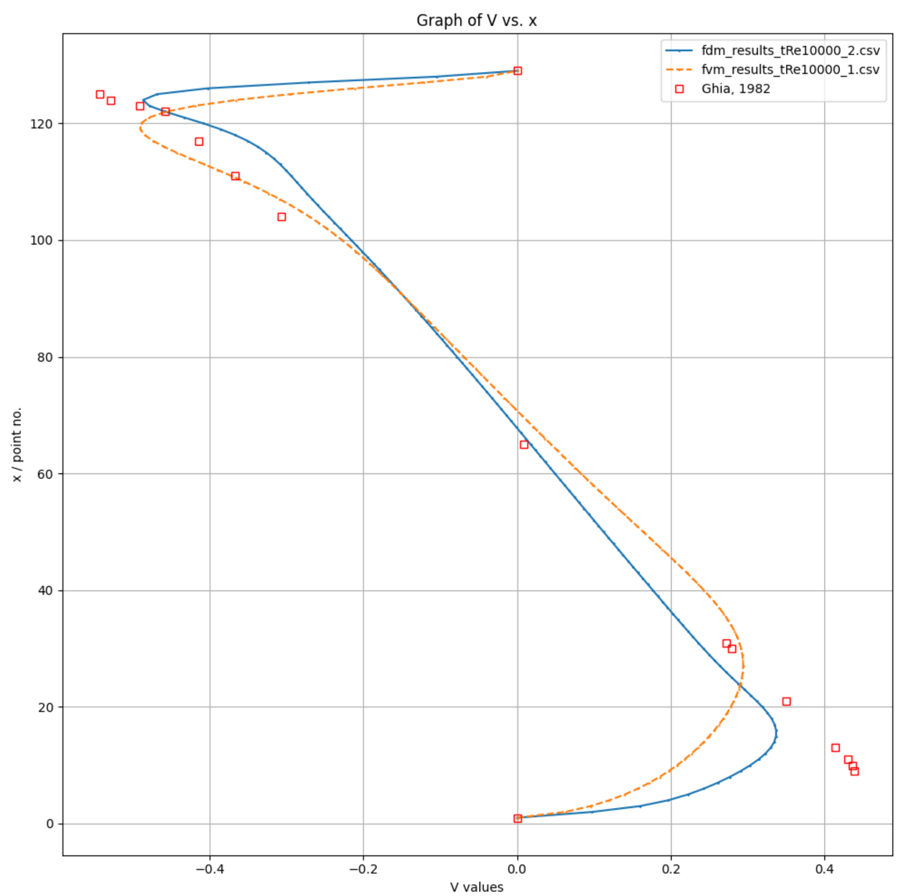


Graph of V vs. x
fdm_results_tRe7500_2.csv
fvm_results_tRe7500_1.csv
Ghia, 1982
x / point no.
V values

Re = 10000



Graph of U vs. y
fdm_results_tRe10000_2.csv
fvm_results_tRe10000_1.csv
Ghia, 1982
y / point no.
U values



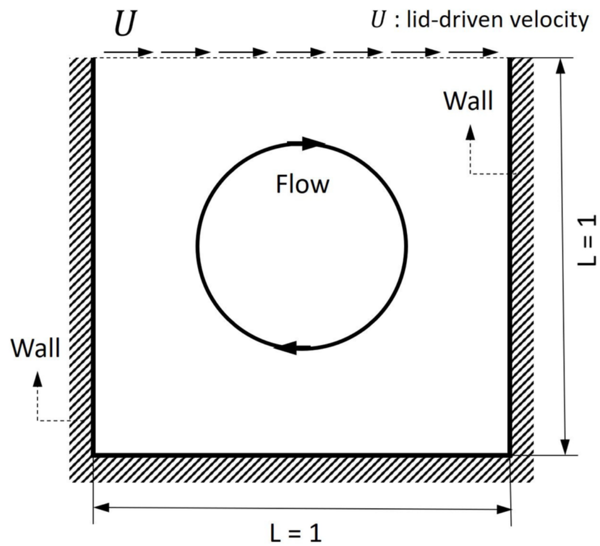
Graph of V vs. x
fdm_results_tRe10000_2.csv
fvm_results_tRe10000_1.csv
Ghia, 1982
x / point no.
V values

SST $k - \omega$ MODEL

2D Turbulent Lid-Driven Cavity

Problem Formulation

This problem solves the steady, incompressible Reynolds-Averaged Navier-Stokes (RANS) equations for a 2D lid-driven cavity. Turbulence is modeled using the SST k-ω model, which adds transport equations for turbulent kinetic energy (k) and specific dissipation rate (ω). The equations are discretized using the Finite Volume Method (FVM) on a staggered grid. A high-resolution TVD scheme is applied to all convective terms for accuracy, while the SIMPLE algorithm is used to couple pressure and velocity. No-slip conditions are set on the three stationary walls, and a constant velocity is applied to the top lid.



Equations

Constants

$\beta_1 = 0.075$; $\sigma_{k1} = 0.85$; $\sigma_{\omega 1} = 0.5$; $\gamma_1 = \dfrac{\beta_1}{\beta^*} - \dfrac{\sigma_{\omega 1}\kappa^2}{\sqrt{\beta^*}}$

$\beta_2 = 0.0828$; $\sigma_{k2} = 1.0$; $\sigma_{\omega 2} = 0.856$; $\gamma_2 = \dfrac{\beta_2}{\beta^*} - \dfrac{\sigma_{\omega 2}\kappa^2}{\sqrt{\beta^*}}$

$\beta^* = 0.09$; $\kappa = 0.41$; $a_1 = 0.31$

The countinuity equation for 2D incompressible flow is given by

$$\nabla \cdot \mathbf{U} = 0 \tag{1}$$

The momentum equations are given by

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{U}) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu_{eff}\nabla \cdot (\nabla u) \tag{2}$$

$$\frac{\partial v}{\partial t} + \nabla \cdot (v\mathbf{U}) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \mu_{eff}\nabla \cdot (\nabla v) \tag{3}$$

The turbulent viscosity is calculated by

$$\mu_t = \frac{a_1 \rho k}{\max(a_1\omega, SF_2)} \tag{4}$$

where $S = \sqrt{2S_{ij} \cdot S_{ij}}$, F_2 is a blending function

$$S_{ij} \cdot S_{ij} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

$$arg_2 = \max\left(2\frac{\sqrt{k}}{\beta^*\omega d}, \frac{500\nu}{d^2\omega}\right)$$

$$F_2 = \tanh(arg_2^2) \tag{5}$$

where d is the perpendicular nearest wall distance.
The transport equations for k and ω are

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial(\rho u_j k)}{\partial x_j} = P - \beta^*\rho\omega k + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_k\mu_t)\frac{\partial k}{\partial x_j}\right] \tag{6}$$

$$\frac{\partial(\rho\omega)}{\partial t} + \frac{\partial(\rho u_j \omega)}{\partial x_j} = \frac{\gamma}{\nu_t} P - \beta\rho\omega^2 + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_\omega \mu_t)\frac{\partial \omega}{\partial x_j}\right] + 2(1-F_1)\frac{\rho\sigma_{\omega 2}}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} \tag{7}$$

where F_1 is a blending function, P is a production term

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}, 10^{-10}\right)$$

$$arg_1 = \min\left[\max\left(\frac{\sqrt{k}}{\beta^* \omega d}, \frac{500\nu}{d^2\omega}\right), \frac{4\rho\sigma_{\omega 2} k}{CD_{k\omega} d^2}\right]$$

$$F_1 = \tanh(arg_1^4) \tag{8}$$

The values of γ, β, σ_k and σ_ω are commonly calculated by

$$\phi = F_1\phi_1 + (1-F_1)\phi_2$$

P is obtained by

$$P = \min(2\mu_t S_{ij}\cdot S_{ij}, 10\beta^*\rho k\omega) \tag{9}$$

Total Variation Diminishing (TVD) Scheme

TVD scheme in a two-dimensional Cartesian grid arrangement is given by

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + S_u^{DC}$$

$$a_P = a_W + a_E + a_S + a_N + (F_e - F_w) + (F_n - F_s)$$

TVD neighbour coefficients

$$a_W = D_w + \max(F_w, 0)$$
$$a_E = D_n + \max(-F_e, 0)$$
$$a_S = D_s + \max(F_s, 0)$$
$$a_N = D_w + \max(-F_n, 0)$$

TVD deferred correction source term

$$\begin{aligned} S_u^{DC} =& \frac{1}{2}F_e[(1-\alpha_e)\psi(r_e^-) - \alpha_e\psi(r_e^+)](\phi_E - \phi_P) + \frac{1}{2}F_w[\alpha_w\psi(r_w^+) - (1-\alpha_w)\psi(r_w^-)](\phi_P - \phi_W) \\ &+ \frac{1}{2}F_n[(1-\alpha_n)\psi(r_n^-) - \alpha_n\psi(r_n^+)](\phi_N - \phi_P) + \frac{1}{2}F_s[\alpha_s\psi(r_s^+) - (1-\alpha_s)\psi(r_s^-)](\phi_P - \phi_S) \end{aligned} \tag{10}$$

$\alpha_j = 1$ for $F_j > 0$, else 0. $\psi(r)$ is the flux limiter function. In this problem, Van Leer flux limiter function is used.

$$\psi(r) = \frac{r + |r|}{1 + r}$$

r is calculated on different faces and for different flow directions by

$$r_e^+ = \frac{\phi_P - \phi_W}{\phi_E - \phi_P} \qquad r_e^- = \frac{\phi_E - \phi_{EE}}{\phi_P - \phi_E}$$

$$r_w^+ = \frac{\phi_W - \phi_{WW}}{\phi_P - \phi_W} \qquad r_w^- = \frac{\phi_P - \phi_E}{\phi_W - \phi_P}$$

$$r_n^+ = \frac{\phi_P - \phi_S}{\phi_N - \phi_P} \qquad r_n^- = \frac{\phi_N - \phi_{NN}}{\phi_P - \phi_N}$$

$$r_s^+ = \frac{\phi_S - \phi_{SS}}{\phi_P - \phi_S} \qquad r_s^- = \frac{\phi_P - \phi_N}{\phi_S - \phi_P}$$

Implimentation

- Set Boundary Conditions
  Setting u and v boundary condition is same as the standard $k-\epsilon$ model. For SST $k-\omega$ model,

$$k_{wall} = 0$$

$$\omega_{wall} = 10\frac{6\nu}{\beta_1(\Delta d_1)^2}$$

- Calculating turbulent viscosity
  Turbulent viscosity is calculated from Eq. (4) for all grid points.

- Solving momentum equations
  Eq. (2) and Eq. (3) is solved by discretizing the equations and solving for $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ and storing in a temporary u and v matrices. The dynamic viscosity taken here is the sum of laminar viscosity and turbulent viscosity. Since the turbulent viscosity grid point is not on the u and v grid points, the average of μ_t is taken accross the respective velocity grid point.

- Pressure correction
  Pressure correction is obtained by solving

$$a_{I,J}p'_{I,J} = a_{I+1,J}p'_{I+1,J} + a_{I-1,J}p'_{I-1,J} + a_{I,J+1}p'_{I,J+1} + a_{I,J-1}p'_{I,J-1} + b'_{I,J}$$

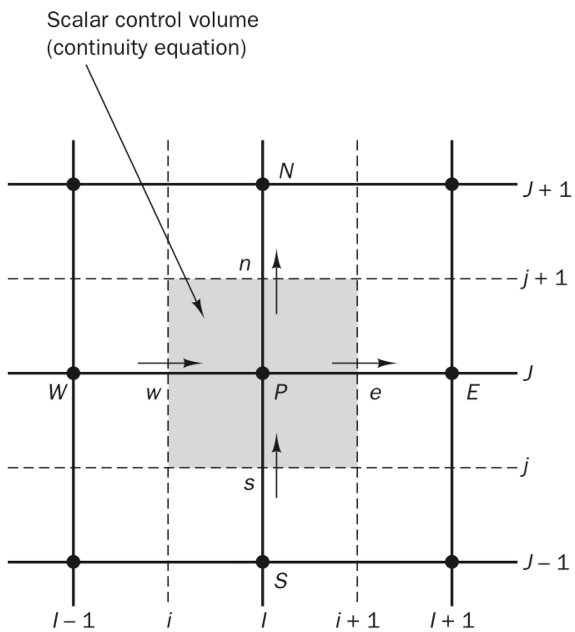
where $a_{I,J} = a_{I+1,J} + a_{I-1,J} + a_{I,J+1} + a_{I,J-1}$ and the coefficients are

$a_{I+1,J}$	$a_{I-1,J}$	$a_{I,J+1}$	$a_{I,J-1}$	$b'_{I,J}$
$(\rho dA)_{i+1,J}$	$(\rho dA)_{i,J}$	$(\rho dA)_{I,j+1}$	$(\rho dA)_{I,j}$	$(\rho u^* A)_{i,J} - (\rho u^* A)_{i+1,J}$ $+(\rho v^* A)_{I,j} - (\rho v^* A)_{I,j+1} + \frac{\rho^o_{I,J} - \rho_{I,J}}{\Delta t}$

also

$$d_{i,J} = \frac{A_{i,J}}{a_{i,J}}$$

At the end of every correction iteration, we apply boundary condition for pressure. We use zero pressure gradient across the wall, because we need to maintain the tangential velocity boundary conditions.



- Correcting flow fields.
  From u^*, v^* and p', we correct the flow fields to u, v and p.

$$p_{i,j}^{new} = p_{i,j}^{old} + \alpha_p p'$$

$$u_{i,J}^{new} = u^*_{i,J} + d_{i,J}(p'_{I-1,J} - p'_{I,J})$$

$$v_{I,j}^{new} = v^*_{I,j} + d_{I,j}(p'_{I,J-1} - p'_{I,J})$$

- Calculating k and ω
  Calculate the dot product of stress $S_{ij} \cdot S_{ij}$

$$S_{ij} \cdot S_{ij} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} \cdot \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = S_{xx}^2 + 2S_{xy}^2 + S_{yy}^2$$

  Substituting the values of the stresses, the production term

$$2\mu_t S_{ij} \cdot S_{ij} = \mu_t \left[2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right]$$

  For steady state solver, the time derivative is zero. By discretizing the turbulent equation in general

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_N\phi_N + a_S\phi_S + S_k^{DC} + S_u$$

$$a_P = a_W + a_E + a_N + a_S + \Delta F - S_p$$

For source term

$$\bar{S}\Delta V = S_u + S_p\phi_P$$

by solving for k and ω

$$S_{uk} = P \cdot \Delta V$$

$$S_{u\omega} = \left[\frac{\gamma}{\nu_t}P + 2(1-F_1)\rho\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}\right]\Delta V$$

$$S_{pk} = -\beta^*\rho\omega\,\Delta V$$

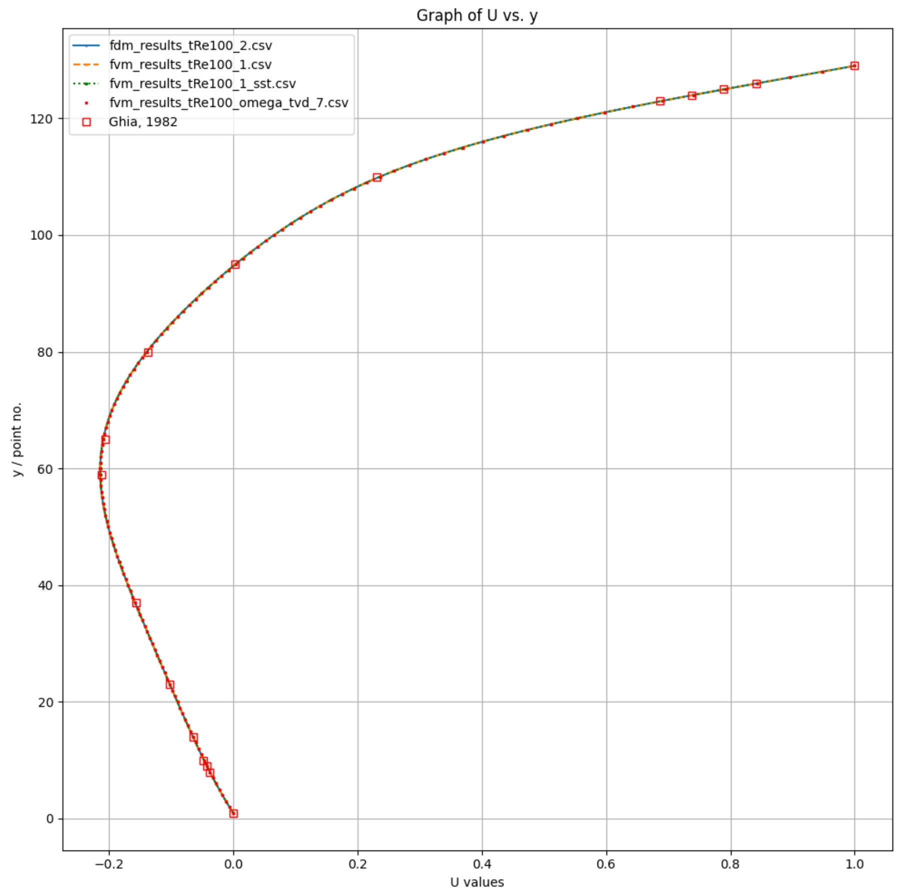
$$S_{p\omega} = -\beta\rho\omega\,\Delta V$$

- Set boundary conditions again.

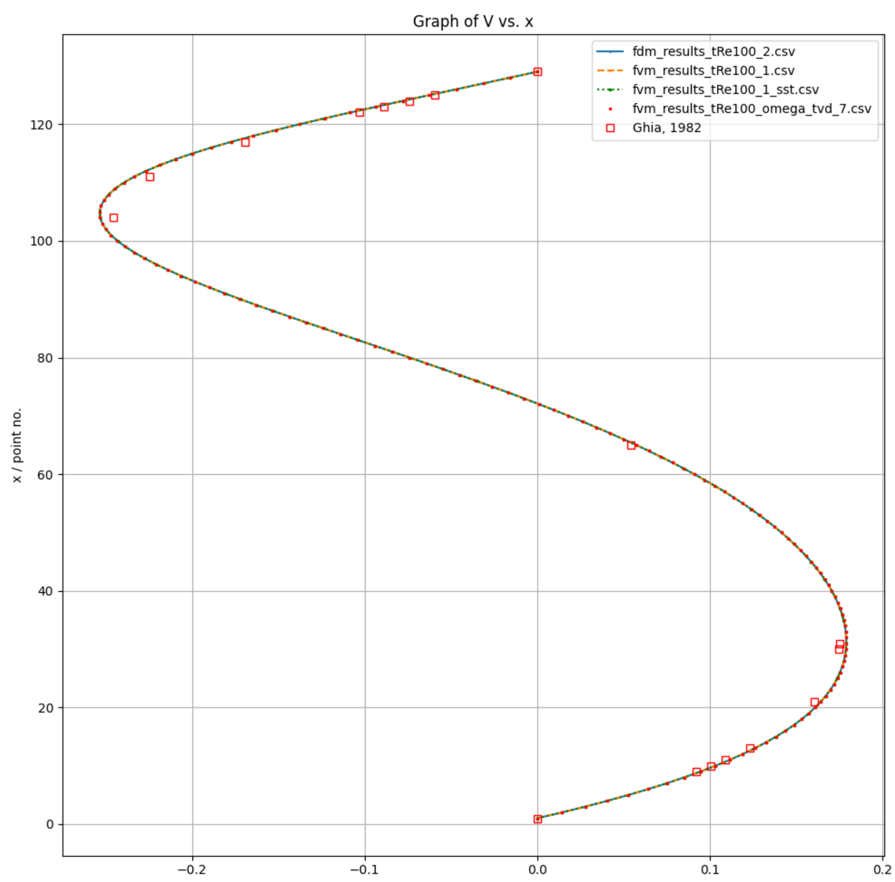
Iterate through Step 2 - 6 till the solutions converges.

Plots

Re = 100

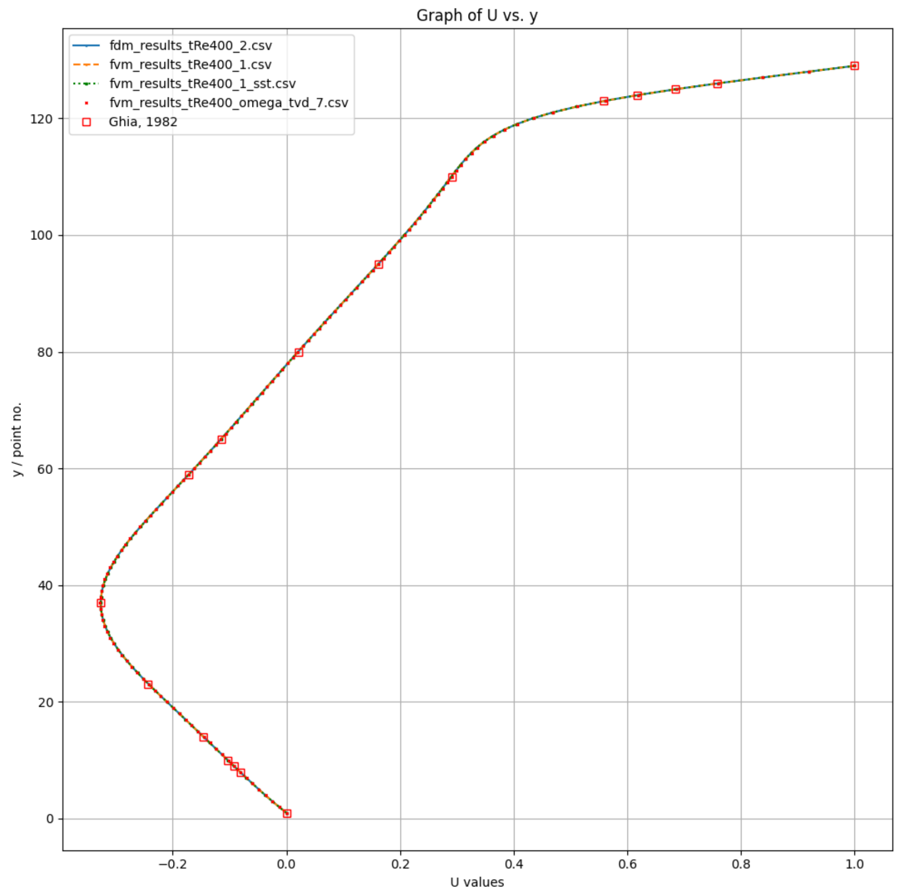


Graph of U vs. y
fdm_results_tRe100_2.csv
fvm_results_tRe100_1.csv
fvm_results_tRe100_1_sst.csv
fvm_results_tRe100_omega_tvd_7.csv
Ghia, 1982
y / point no.
U values

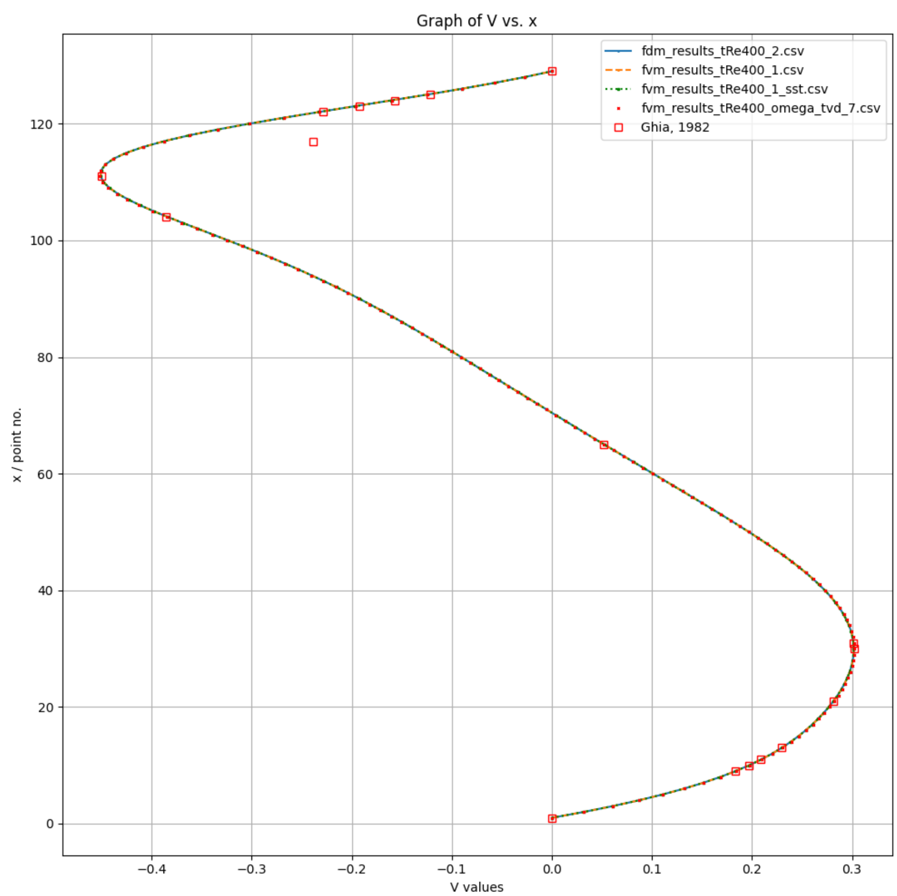


Graph of V vs. x
fdm_results_tRe100_2.csv
fvm_results_tRe100_1.csv
fvm_results_tRe100_1_sst.csv
fvm_results_tRe100_omega_tvd_7.csv
Ghia, 1982
x / point no.
V values

Re = 400

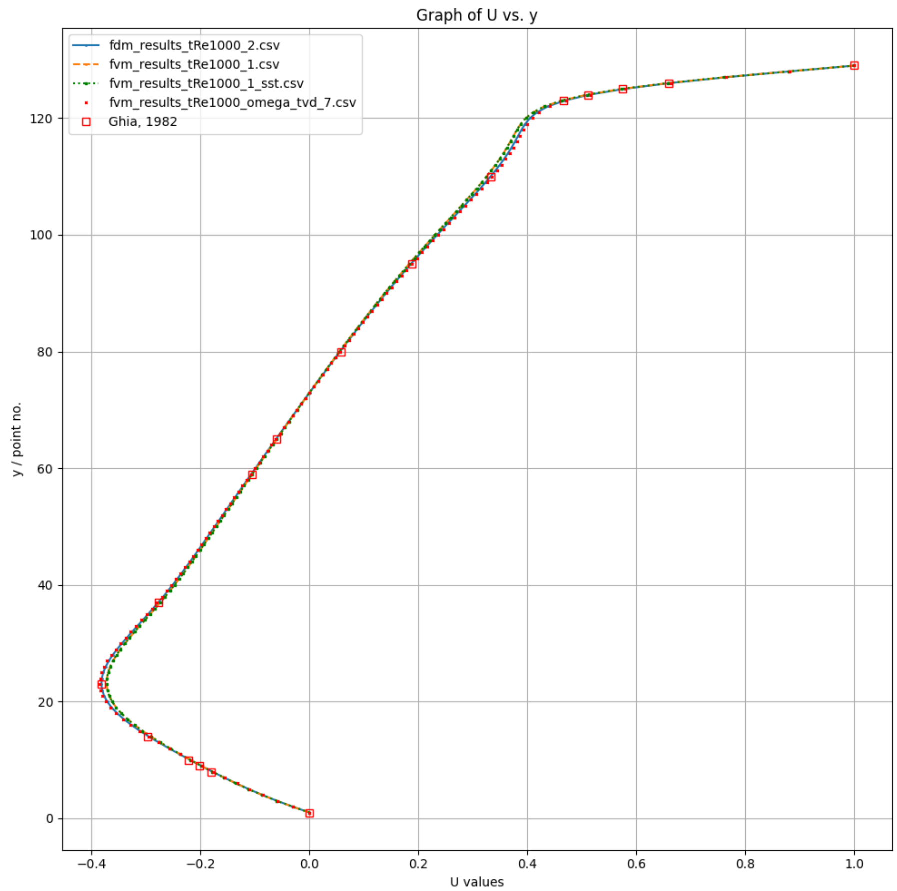


Graph of U vs. y
fdm_results_tRe400_2.csv
fvm_results_tRe400_1.csv
fvm_results_tRe400_1_sst.csv
fvm_results_tRe400_omega_tvd_7.csv
Ghia, 1982
y / point no.
U values

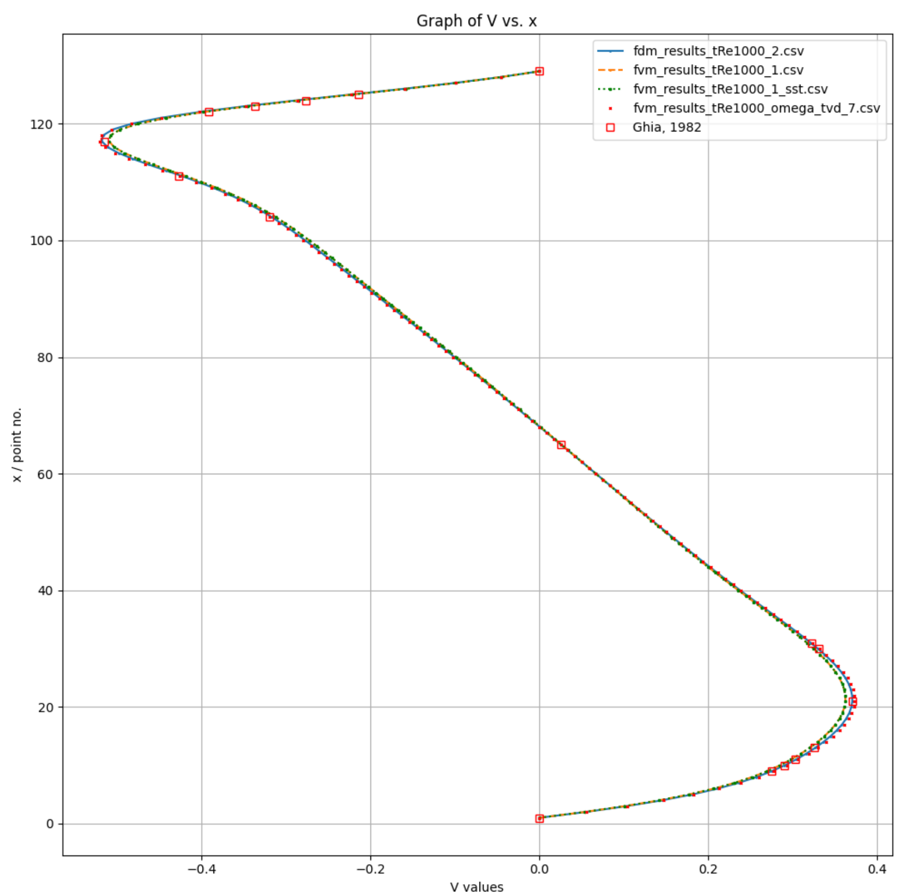


Graph of V vs. x
fdm_results_tRe400_2.csv
fvm_results_tRe400_1.csv
fvm_results_tRe400_1_sst.csv
fvm_results_tRe400_omega_tvd_7.csv
Ghia, 1982
x / point no.
V values

Re = 1000

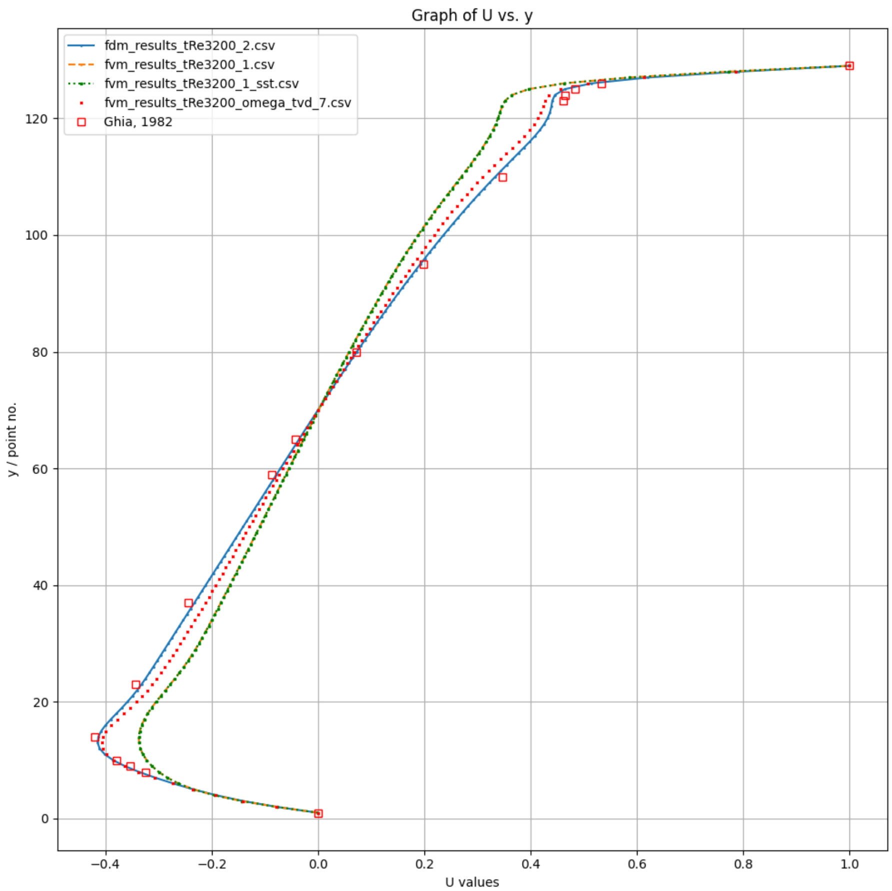


Graph of U vs. y
fdm_results_tRe1000_2.csv
fvm_results_tRe1000_1.csv
fvm_results_tRe1000_1_sst.csv
fvm_results_tRe1000_omega_tvd_7.csv
Ghia, 1982
y / point no.
U values

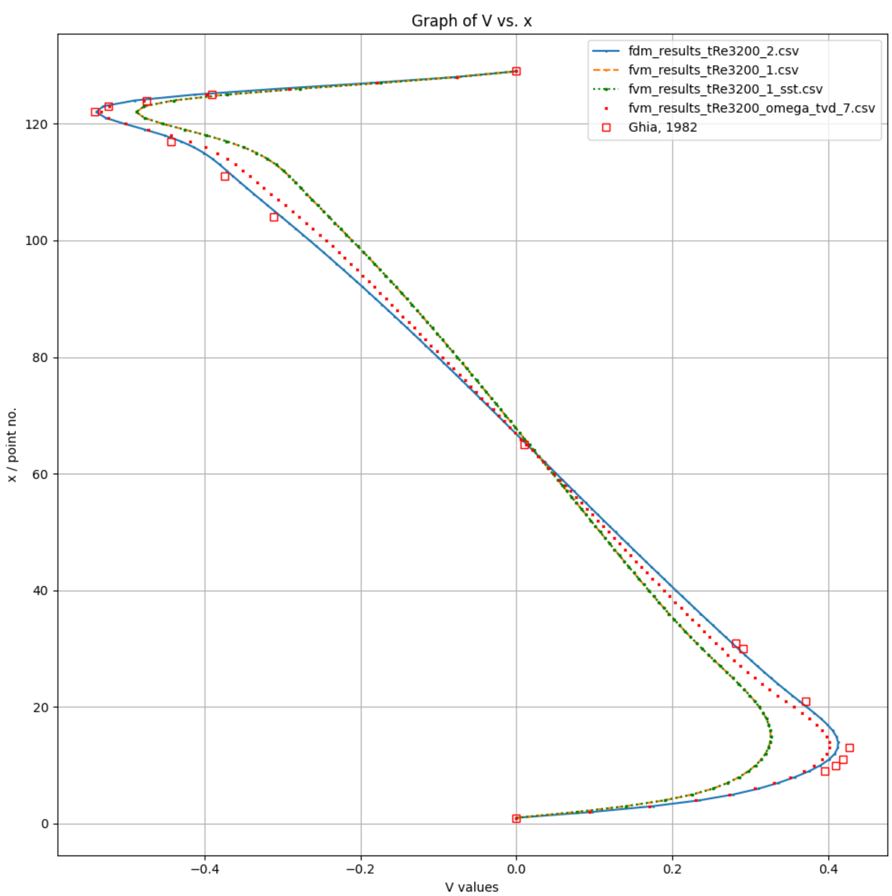


Graph of V vs. x
fdm_results_tRe1000_2.csv
fvm_results_tRe1000_1.csv
fvm_results_tRe1000_1_sst.csv
fvm_results_tRe1000_omega_tvd_7.csv
Ghia, 1982
x / point no.
V values

Re = 3200

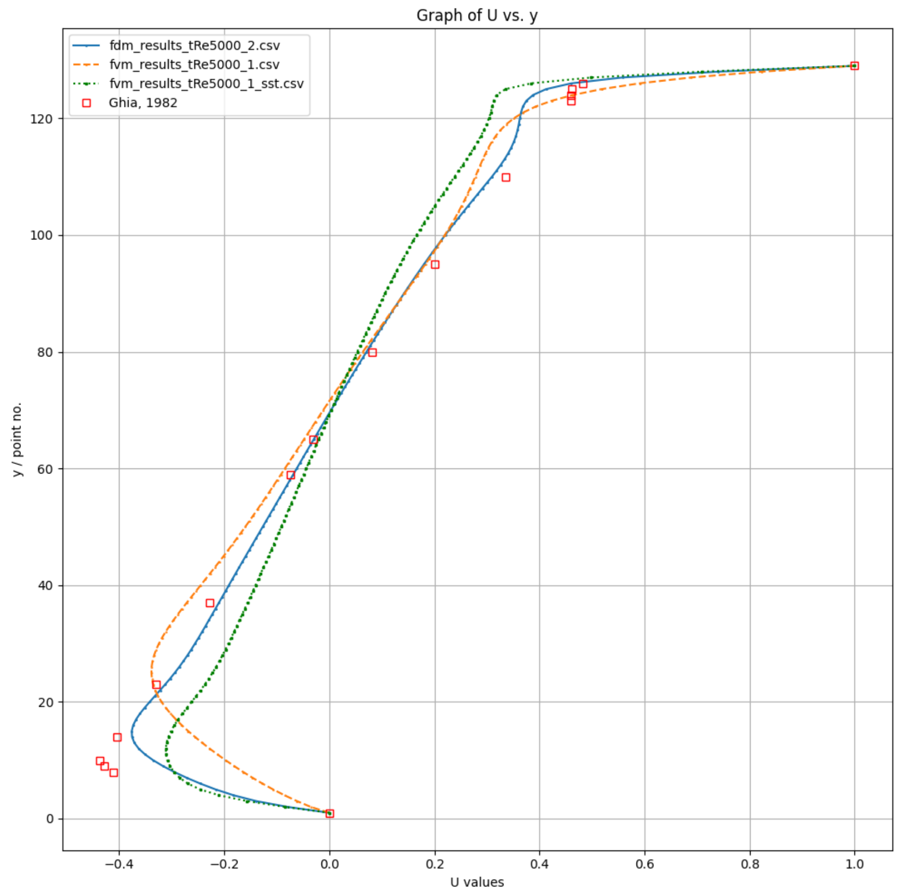


Graph of U vs. y
fdm_results_tRe3200_2.csv
fvm_results_tRe3200_1.csv
fvm_results_tRe3200_1_sst.csv
fvm_results_tRe3200_omega_tvd_7.csv
Ghia, 1982
y / point no.
U values

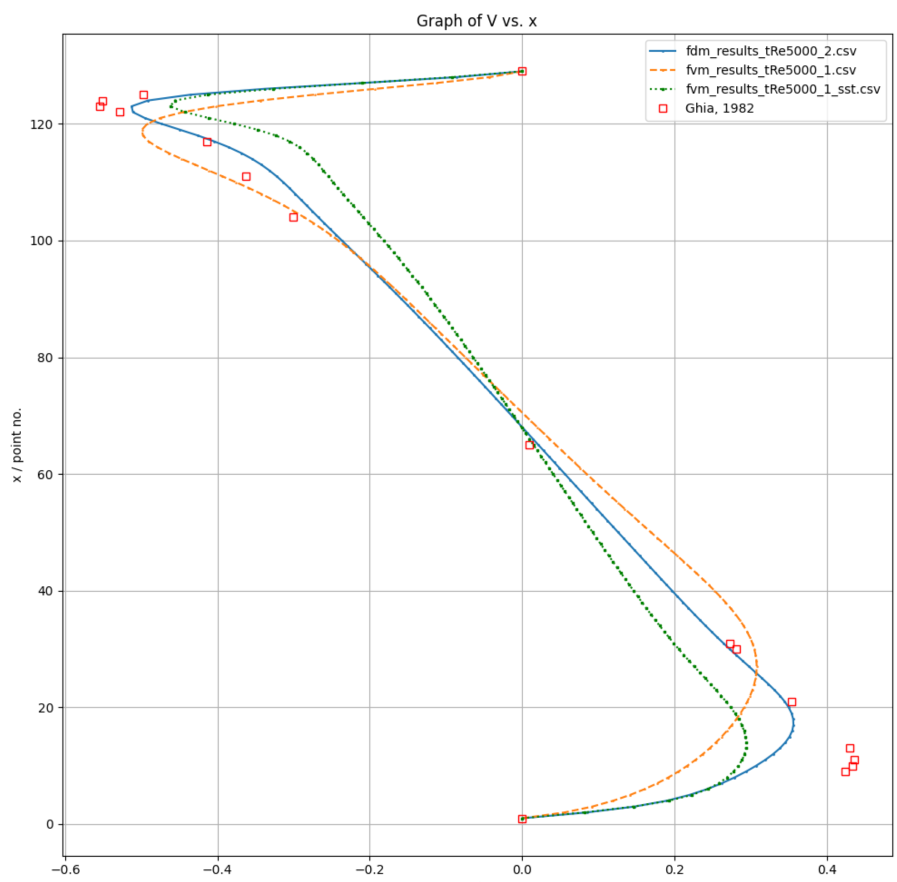


Graph of V vs. x
fdm_results_tRe3200_2.csv
fvm_results_tRe3200_1.csv
fvm_results_tRe3200_1_sst.csv
fvm_results_tRe3200_omega_tvd_7.csv
Ghia, 1982
x / point no.
V values

Re = 5000

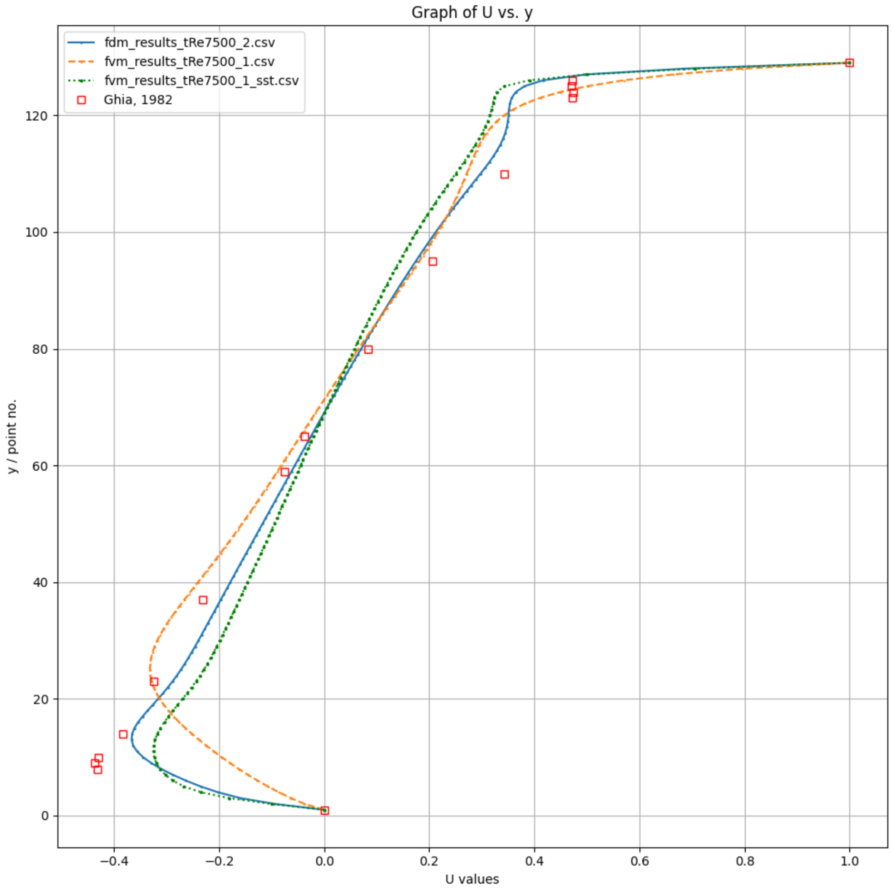


Graph of U vs. y
fdm_results_tRe5000_2.csv
fvm_results_tRe5000_1.csv
fvm_results_tRe5000_1_sst.csv
Ghia, 1982
y / point no.
U values

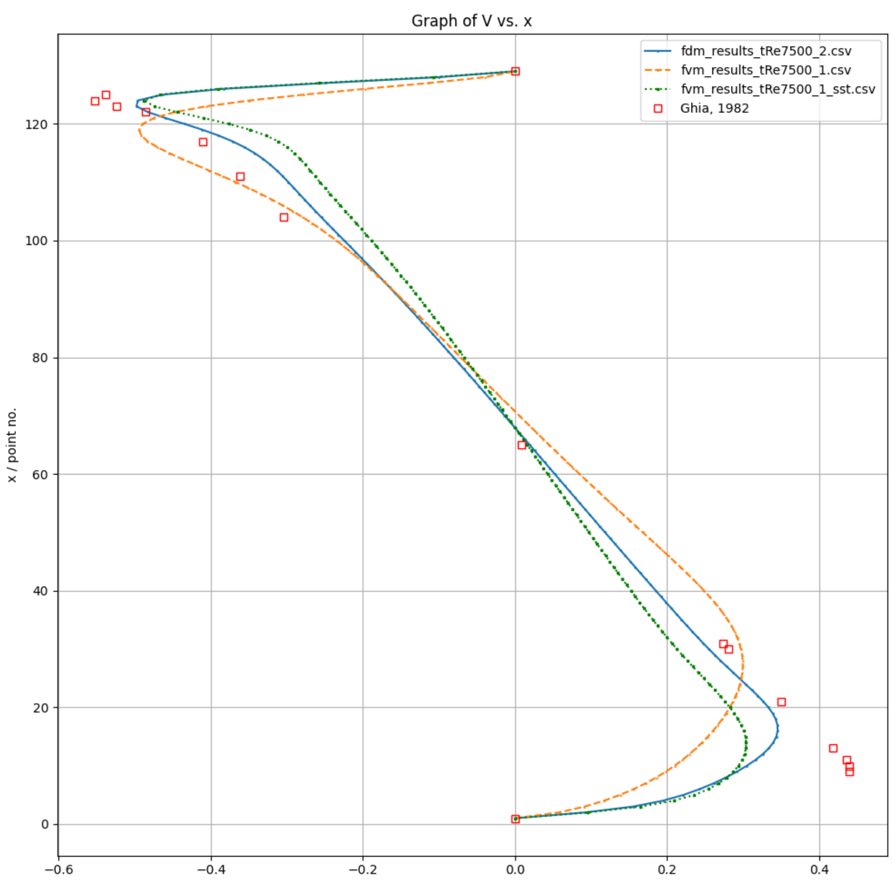


Graph of V vs. x
fdm_results_tRe5000_2.csv
fvm_results_tRe5000_1.csv
fvm_results_tRe5000_1_sst.csv
Ghia, 1982
x / point no.
V values

Re = 7500

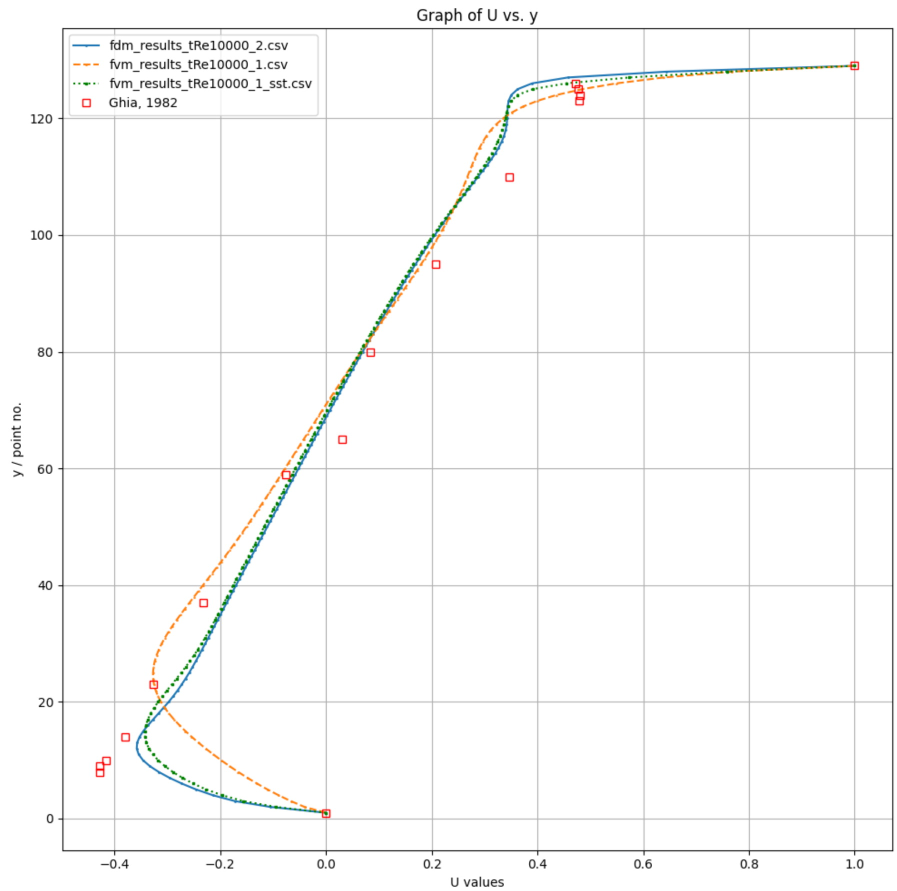


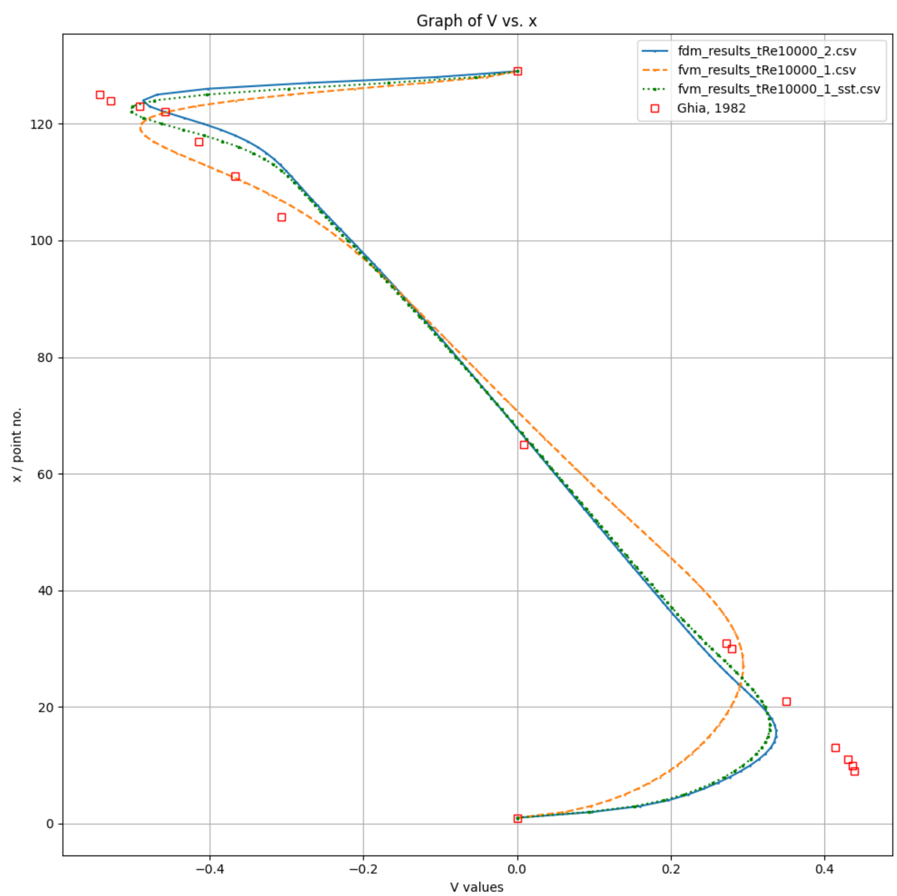
Graph of U vs. y
fdm_results_tRe7500_2.csv
fvm_results_tRe7500_1.csv
fvm_results_tRe7500_1_sst.csv
Ghia, 1982
y / point no.
U values



Graph of V vs. x
fdm_results_tRe7500_2.csv
fvm_results_tRe7500_1.csv
fvm_results_tRe7500_1_sst.csv
Ghia, 1982
x / point no.
V values

Re = 10000


Graph of U vs. y
fdm_results_tRe10000_2.csv
fvm_results_tRe10000_1.csv
fvm_results_tRe10000_1_sst.csv
Ghia, 1982
y / point no.
U values


Graph of V vs. x
fdm_results_tRe10000_2.csv
fvm_results_tRe10000_1.csv
fvm_results_tRe10000_1_sst.csv
Ghia, 1982
x / point no.
V values

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