

Complex analysis

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Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f: \Omega \rightarrow \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations). Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a , f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Definition 1.1.2. Wirtinger derivatives are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f: \Omega \rightarrow \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a . The function f is *holomorphic* if it is holomorphic in every point of Ω . We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with \mathcal{C}^1 -smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\Delta(z, \varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\Delta(z, \varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w - z} dw$ on Ω_ε . As $d\omega = \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \oint_{\partial\Omega} \frac{f(w)}{w - z} dw - \oint_{\partial\Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw.$$

In the limit, we have

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial \Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{f(z + \varepsilon e^{it})}{\varepsilon e^{it}} \cdot \varepsilon i e^{it} dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega \setminus \{z\}} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f(w)}{w - z} dw - f(z) = -\frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}. \quad \square$$

Theorem 1.1.5 (Power series expansion). Let $\Omega \subseteq \mathbb{C}$ be an open subset, $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\Delta(a, r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \oint_{\partial \Delta(a, \rho)} \frac{f(w)}{(w - a)^{k+1}} dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z - a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity). Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a holomorphic function. Let $A \subseteq \Omega$ be a subset such that $f(z) = 0$ for all $z \in A$. If A has an accumulation point in Ω , then $f(z) = 0$ for all $z \in \Omega$.

Proof. Let $a \in \Omega$ be an accumulation point of A . By continuity, we have $f(a) = 0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k (z - a)^k,$$

where we assume $c_{k_0} \neq 0$. But now $g(z) = \frac{f(z)}{(z - a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a) = 0$, which is a contradiction. It follows that $c_k = 0$ for all $k \in \mathbb{N}_0$. It follows that the set $\text{Int} \{z \in \Omega \mid f(z) = 0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to Ω . \square

Lemma 1.1.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\Delta(a, r)} \subseteq \Omega$. If

$$|f(a)| < \min_{\partial\Delta(a, r)} |f|,$$

then f has a zero in $\Delta(a, r)$.

Proof. Assume otherwise. From the inequality it follows that f has no zeroes on the boundary either. By continuity, f has no zero on an open set V with $\Delta(a, r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint_{\partial\Delta(a, r)} \frac{g(z)}{z - a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(a + r \cdot e^{it})}{r e^{it}} \cdot r i e^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} g(a + r e^{it}) dt.$$

We can therefore get a bound on $|g(a)|$ as

$$|g(a)| \leq \max_{\partial\Delta(a, r)} |g|,$$

but as the condition on f can be rewritten as

$$|g(a)| > \max_{\partial\Delta(a, r)} |g|,$$

we have reached a contradiction. □

Theorem 1.1.8 (Open mapping). Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If f is not constant, it is an open map.

Proof. Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\Delta(z_0, \rho) \subseteq U$ and z_0 is the only pre-image of w_0 in $\Delta(z_0, 2\rho)$.¹

Since $\partial\Delta(z_0, \rho)$ is a compact set and

$$|f(z) - w_0| > 0$$

for all $z \in \partial\Delta(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a $w \in \Delta(w_0, \varepsilon)$. As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$

on the boundary and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

by the above lemma, $f(z_0) - w$ has a root on $\Delta(z, \rho)$. □

Theorem 1.1.9 (Maximum principle). Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function f is constant.

¹ If such a disk does not exist, f is constant by the identity theorem.

Proof. Suppose that f is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As f is an open map, it also attains the value $(1 + \varepsilon) \cdot f(z)$, which is a contradiction as the modulus then equals $(1 + \varepsilon) \cdot |f(z)| > |f(z)|$. \square

Theorem 1.1.10 (Maximum principle). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

Proof. As $\overline{\Omega}$ is compact, f attains a global maximum on this set. If the maximum is attained in the interior, f is constant, therefore it is also attained on the boundary. \square

Definition 1.1.11. A function $f: \Omega \setminus \{a\} \rightarrow \mathbb{C}$ is *locally bounded* near a if there exists an open neighbourhood $U \subseteq \Omega$ of a such that $f|_{U \setminus \{a\}}$ is bounded.

Theorem 1.1.12 (Riemann removable singularity theorem). Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$. If f is locally bounded near a , then there exists a unique function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus \{a\}} = f$.

Proof. Define the function $F: \Omega \rightarrow \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{w-a} dw & z = a. \end{cases}$$

It remains to check that F is complex differentiable at a . Indeed, for $z \in \Delta(a, \rho)$ we have

$$\begin{aligned} \lim_{z \rightarrow a} \frac{F(z) - F(a)}{z - a} &= \lim_{z \rightarrow a} \frac{1}{z - a} \oint_{\partial\Delta(a,\rho)} \left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a} \right) dw \\ &= \lim_{z \rightarrow a} \frac{1}{2\pi i} \cdot \frac{1}{z - a} \cdot \oint_{\partial\Delta(a,\rho)} f(w) \cdot \frac{z - a}{(w - z)(w - a)} dw \\ &= \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{(w - a)^2} dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem. \square

Theorem 1.1.13 (Schwarz lemma). Let $f: \Delta \rightarrow \Delta$ be a holomorphic function with $f(0) = 0$. Then, $|f'(0)| \leq 1$ and the inequality $|f(z)| \leq |z|$ holds for all $z \in \Delta$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ holds for any $z \neq 0$, then $f(z) = \beta z$ for some $\beta \in \partial\Delta$.

Proof. We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for g on the domain $\Delta(\rho)$. We get

$$\sup_{z \in \Delta(\rho)} |g(z)| \leq \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as $\rho \rightarrow 1$, it follows that

$$\sup_{z \in \mathbb{A}} |g(z)| \leq 1.$$

It immediately follows that $|f'(0)| = |g(0)| \leq 1$. Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z|.$$

Suppose we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. As then $|g(z_0)| = 1$, it follows that g is constant, therefore $f(z) = \beta z$ for some $\beta \in \partial \mathbb{A}$. If we have $|f'(0)| = 0$, the same argument works for $z_0 = 0$. \square

1.2 The $\bar{\partial}$ equation

Lemma 1.2.1. Let $g \in \mathcal{C}^\infty(\mathbb{C})$ be a function with compact support. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \bar{z}} = g$.

Proof. Let

$$f(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(w)}{w - z} dw \wedge d\bar{w}.$$

As

$$dw \wedge d\bar{w} = -2ri dr \wedge d\varphi$$

holds for polar coordinates centered at z , we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{rg(z + re^{i\varphi})}{re^{i\varphi}} dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some R such that $g|_{\mathbb{C} \setminus \Delta(z, R)} = 0$. We get

$$f(z) = -\frac{1}{\pi} \iint_{\Delta(z, R)} g(z + re^{i\varphi}) e^{-i\varphi} dr \wedge d\varphi,$$

which obviously converges. The function f is therefore well defined. As we are integrating a smooth function on a compact set, the function f is smooth as well.

For $u = re^{i\varphi}$, we have

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}}(z) &= -\frac{1}{\pi} \iint_{\Delta(z, R)} \frac{\partial}{\partial \bar{z}} g(z + re^{i\varphi}) e^{-i\varphi} dr \wedge d\varphi \\ &= \frac{1}{2\pi i} \iint_{\Delta(0, R)} \frac{\partial}{\partial \bar{z}} g(u + z) \frac{1}{u} du \wedge d\bar{u} \\ &= \frac{1}{2\pi i} \iint_{\Delta(0, R)} \frac{\partial g}{\partial \bar{u}}(u + z) \frac{1}{u} du \wedge d\bar{u} \\ &= \frac{1}{2\pi i} \iint_{\Delta(z, R)} \frac{\partial g}{\partial \bar{w}}(w) \frac{1}{w - z} dw \wedge d\bar{w}. \end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint_{\partial \Delta(z, R)} \frac{g(w)}{w - z} dw + \frac{1}{2\pi i} \iint_{\Delta(z, R)} \frac{\partial g}{\partial \bar{w}}(w) \frac{1}{w - z} dw \wedge d\bar{w}.$$

by the choice of R , we get

$$\frac{\partial f}{\partial \bar{z}}(z) = g(z). \quad \square$$

Lemma 1.2.2. Given bounded domain $U \subset V \subset \mathbb{R}^n$ such that $\partial U \cap \partial V = \emptyset$, there exists a smooth function $\chi: \mathbb{R}^n \rightarrow [0, 1]$ such that $\chi|_U = 1$ and $\text{supp } \chi \subseteq V$.

Proof. There is a partition of unity on the sets V and $\mathbb{R}^n \setminus \bar{U}$. \square

Lemma 1.2.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $h_j: \Omega \rightarrow \mathbb{C}$ be holomorphic functions. If the sequence $(h_j)_{j \in \mathbb{N}}$ converges uniformly on compact sets, the limit is also holomorphic on Ω .

Proof. Apply Morera's theorem.² □

Theorem 1.2.4 (Dolbeault lemma). Let $g \in \mathcal{C}^\infty(\Delta(R))$ for some $R \in (0, \infty]$. Then there exists a function $f \in \mathcal{C}^\infty(\Delta(R))$ such that $\frac{\partial f}{\partial \bar{z}} = g$.

Proof. Define disks X_j as follows:

- i) If $R = \infty$, set $X_j = \Delta(j)$.
- ii) If $R < \infty$, set $X_j = \Delta\left(R - \frac{1}{j}\right)$ (for large enough j).

Applying the above lemma, define functions χ_j with $\chi_j|_{X_j} = 1$ and $\text{supp } \chi_j \subseteq X_{j+1}$ and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \Delta(R), \\ 0 & z \notin \Delta(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function $f_j \in \mathcal{C}^\infty(\mathbb{C})$ with

$$\frac{\partial f_j}{\partial \bar{z}} = g_j.$$

We inductively construct a new sequence $\tilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$ such that

$$\frac{\partial \tilde{f}_j}{\partial \bar{z}} = g$$

on X_j and

$$\|\tilde{f}_j - \tilde{f}_{j-1}\|_{X_{j-2}} \leq 2^{-j}.$$

Set $\tilde{f}_1 = f_1$. Observe the function $F = f_{j+1} - \tilde{f}_j$ on X_j . By construction, we have $\frac{\partial F}{\partial \bar{z}} = 0$ on X_j . It follows that F can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on X_j . As power series converge uniformly on compact sets, there exists some polynomial $p \in \mathbb{C}[z]$ such that

$$\|F - p\|_{X_{j-1}} \leq 2^{-j}.$$

Now just set $\tilde{f}_{j+1} = f_{j+1} - p$.

Let $z \in \Delta(R)$ be arbitrary. By construction, it is contained in some X_{j_0} , therefore, \tilde{f}_j is defined for $j \geq j_0$. As $(\tilde{f}_j(z))_{j \geq j_0}$ is a Cauchy sequence, we can define

$$f(z) = \lim_{j \rightarrow \infty} \tilde{f}_j(z).$$

² Analysis 2b, theorem 3.4.6.

But as

$$f - \tilde{f}_j = \sum_{k=j}^{\infty} (\tilde{f}_{k+1} - \tilde{f}_k)$$

is a sum of holomorphic functions that converges uniformly, the function $f - \tilde{f}_j$ is a holomorphic function. Therefore, f is smooth and satisfies $\frac{\partial f}{\partial \bar{z}} = g$. \square

1.3 Meromorphic functions

Definition 1.3.1. Let $\Omega \subset \mathbb{C}$ be an open subset. We call a function f *meromorphic* of Ω if there exists $A \subset \Omega$ such that $f \in \mathcal{O}(\Omega \setminus A)$, A has no accumulation points in Ω and for all $a \in A$ there exists some $k \in \mathbb{N}$ such that

$$\lim_{z \rightarrow a} f(z) \cdot (z - a)^k \neq 0$$

exists. We call A the set of *poles* of the function f . We denote the set of meromorphic functions on Ω with $\mathcal{M}(\Omega)$.

Theorem 1.3.2. Let $0 \leq r < R \leq \infty$. Suppose that $f \in \mathcal{O}(D_{R,r}(a))$ is a holomorphic function, where

$$D_{R,r}(a) = \{z \in \mathbb{C} \mid r < |z - a| < R\}.$$

Then there exists a uniquely determined *Laurent series*

$$\sum_{k \in \mathbb{Z}} c_k (z - a)^k$$

that converges to f uniformly and absolutely on compact subsets of $D_{R,r}(a)$. We have

$$c_k = \frac{1}{2\pi i} \oint_{\partial \Delta(a, \rho)} \frac{f(w)}{(w - a)^k} dw$$

for $r < \rho < R$.

Definition 1.3.3. Let

$$\sum_{k \in \mathbb{Z}} c_k (z - a)^k$$

be a Laurent series. The series

$$\sum_{k=-\infty}^{-1} c_k (z - a)^k$$

is called the *principle part*.

Lemma 1.3.4. Let $f \in \mathcal{O}(\Omega \setminus \{a\})$ be a holomorphic function. Then f is meromorphic on Ω if and only if f has a finite principle part in a .

Proof. Suppose that f is meromorphic on Ω . If a is a removable singularity, f is holomorphic in a , therefore the principle part is trivial. Otherwise, set $m \in \mathbb{N}$ such that

$$\lim_{z \rightarrow a} (z - a)^m f(z) \neq 0$$

exists and set $g(z) = (z - a)^m f(z)$. As g is bounded near a , we can extend it to Ω by the Riemann removable singularity theorem. The power series of g corresponds to a finite Laurent series of f .

The converse is obvious. □

Theorem 1.3.5. If $f \in \mathcal{M}(\mathbb{C})$ is a meromorphic function, there exist entire functions g and h such that $f = \frac{g}{h}$.

Definition 1.3.6. Let $\Omega \subseteq \mathbb{C}$ be an open set. An *additive Cousin problem* on Ω is an open cover $\{U_j\}_{j \in J}$ of Ω and functions $f_j \in \mathcal{M}(U_j)$ such that $f_j - f_k|_{U_j \cap U_k}$ is holomorphic for all $j, k \in J$. A function $f \in \mathcal{M}(\Omega)$ is a solution to the additive Cousin problem if $f|_{U_j} - f_j$ is holomorphic for all $j \in J$.

Definition 1.3.7. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A *generalized additive Cousin problem* is an open cover $\{U_j\}_{j \in J}$ of Ω and functions $f_{j,k} \in \mathcal{O}(U_j \cap U_k)$ for each $(j, k) \in J^2$, such that

- i) $f_{j,k} = -f_{k,j}$ on $U_j \cap U_k$ for all $(j, k) \in J^2$ and
- ii) $f_{j,k} + f_{k,\ell} + f_{\ell,j} = 0$ on $U_j \cap U_k \cap U_\ell$ for all $(j, k, \ell) \in J^3$.

A solution to the generalized additive Cousin problem is given by functions $f_j \in \mathcal{O}(U_j)$ for each $j \in J$ such that $f_{j,k} = f_j - f_k$ for each $(j, k) \in J^2$.

Lemma 1.3.8 (Partition of unity). Let $\Omega \subseteq \mathbb{C}$ be an open set and $\{U_j\}_{j \in J}$ be an open cover of Ω . Then there exists a partition of unity subordinate to $\{U_j\}_{j \in J}$.

Lemma 1.3.9. Given a generalized additive Cousin problem on $\Omega \subseteq \mathbb{C}$, there exist functions $g_j \in \mathcal{C}^\infty(U_j)$ such that $f_{j,k} = g_j - g_k$ for all $(j, k) \in J^2$.

Proof. Let $\{(V_a, \chi_a)\}_{a \in A}$ be a partition of unity, subordinate to $\{U_j\}_{j \in J}$. For all $a \in A$ choose a $j(a) \in J$ such that $V_a \subseteq U_{j(a)}$. For all $k \in J$, define

$$g_k = - \sum_{a \in A} \chi_a \cdot f_{j(a),k}.$$

This is of course a smooth function on U_k . Now note that

$$g_k - g_\ell = \sum_{a \in A} \chi_a \cdot (-f_{j(a),k} + f_{j(a),\ell}) = \sum_{a \in A} \chi_a \cdot f_{k,\ell} = f_{k,\ell}. \quad \square$$

Proposition 1.3.10. The generalized additive Cousin problem is solvable for $\Omega = \Delta(r)$ and $\Omega = \mathbb{C}$.

Proof. Let $f_{j,k} = g_j - g_k$ for $g_j \in \mathcal{C}^\infty(U_j)$. Note that

$$\frac{\partial g_j}{\partial \bar{z}} = \frac{\partial g_k}{\partial \bar{z}},$$

therefore,

$$h|_{U_j} = \frac{\partial g_j}{\partial \bar{z}}$$

induces a smooth function $h: \Omega \rightarrow \mathbb{C}$. By the Dolbeault lemma, there exists a function $g \in \mathcal{C}^\infty(\Omega)$ such that $\frac{\partial g}{\partial \bar{z}} = h$. It is clear that $f_j = g_j - g$ solves the generalized additive Cousin problem. \square

Proposition 1.3.11. The additive Cousin problem is solvable for $\Omega = \Delta(r)$ and $\Omega = \mathbb{C}$.

Proof. An additive Cousin problem induces a generalized additive Cousin problem for functions $f_{j,k} = f_j - f_k$. Let g_j be a solution to the generalized problem. As $f_j - f_k = f_{j,k} = g_j - g_k$ on $U_j \cap U_k$, we can define a function $f \in \mathcal{M}(\Omega)$ with $f|_{U_j} = f_j - g_j$. This function is of course well defined. As $f|_{U_j} - f_j = g_j \in \mathcal{O}(U_j)$, this function indeed solves the additive Cousin problem. \square

Theorem 1.3.12 (Mittag-Leffler). Let $(a_k)_{k \in \mathbb{N}}$ be a sequence without repetition and accumulation points. Let

$$f_k(z) = \sum_{\ell=-m_k}^{-1} c_{k,\ell} (z - a_k)^\ell$$

be finite principal parts. Then there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ with poles in $(a_k)_{k \in \mathbb{N}}$ such that f has principle part f_k in a_k for each $k \in \mathbb{N}$.

Proof. For each a_k choose a disk U_k containing no other a_k . Also set $U_0 = \mathbb{C} \setminus \{a_k \mid k \in \mathbb{N}\}$ and $f_0 = 0$. As $\{U_k \mid k \in \mathbb{N}_0\}$ is an open cover of \mathbb{C} , there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ that solves the corresponding additive Cousin problem. It is easy to see that the principle parts of f at a_k are precisely f_k . \square

1.4 Sequences of holomorphic functions

Definition 1.4.1. A family of functions \mathcal{F} from Ω to \mathbb{C} is *locally bounded*, if for all $p \in \Omega$ there exist a $\rho > 0$ and $M > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{z \in \Delta(p, \rho)} |f(z)| < M.$$

Lemma 1.4.2. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ a locally bounded family of functions. Then for all $p \in \Omega$ there exists a $\rho > 0$ such that \mathcal{F} is equi-continuous on $\Omega \cap \Delta(p, \rho)$.

Proof. Fix $p \in \Omega$ and choose $r > 0$ such that $D = \overline{\Delta(p, 2r)} \subseteq \Omega$. For any $z, w \in D$ and $f \in \mathcal{F}$ we have

$$f(z) - f(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - w} d\xi = \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi.$$

Note that the family \mathcal{F} is bounded on every compact. Therefore, we can write

$$\sup_{f \in \mathcal{F}} \sup_{z \in \partial D} |f(z)| < M.$$

Now, for $z, w \in \Delta(p, r)$ we have

$$|f(z) - f(w)| = \left| \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi \right| \leq |z - w| \cdot \frac{2M}{r}. \quad \square$$

Theorem 1.4.3 (Arzelà-Ascoli). Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be an infinite family such that the following conditions hold:

- i) \mathcal{F} is point-wise bounded.
- ii) \mathcal{F} is locally equi-continuous.

Then there \mathcal{F} contains a sequence that converges uniformly on compacts of Ω .

Proof. Choose a dense countable subset $A \subseteq \Omega$ and enumerate it as a sequence $(a_k)_{k \in \mathbb{N}}$. Pick any sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ with pairwise distinct terms. As $|f_n(a_1)| < M$ for all n , we can choose a subsequence $(f_{1,n})_{n \in \mathbb{N}}$ such that $f_{1,n}(a_1)$ converges by Bolzano-Weierstrass.

Similarly, for every $k \in \mathbb{N}$ there exists a subsequence $(f_{k,n})_n$ of $(f_{k-1,n})_n$ such that $(f_{k,n}(a_k))_n$ converges. Now define $F_n = f_{n,n}$. Observe that (F_n) converges at every point in A .

Fix a $p \in \Omega$. By local equi-continuity, there exists a $\rho > 0$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta < \rho$ and $|F_n(z) - F_n(w)| < \frac{\varepsilon}{3}$ for all $z, w \in \Delta(p, \rho)$ such that $|z - w| < \delta$. Choose an element $a \in A \cap \Delta(p, \delta)$.³ Then, we have

$$|F_n(z) - F_m(z)| \leq |F_n(z) - F_n(a)| + |F_n(a) - F_m(a)| + |F_m(a) - F_m(z)| < 3 \cdot \frac{\varepsilon}{3}.$$

It follows that (F_n) is locally uniformly convergent, therefore it converges uniformly on compact sets. \square

³ By compactness of $\overline{\Delta(p, \rho)}$ we can choose a from a finite set.

Theorem 1.4.4 (Montel). Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f_n: \Omega \rightarrow \mathbb{C}$ be a locally bounded sequence of holomorphic functions. Then $(f_n)_n$ contains a subsequence that converges uniformly on compacts.

Proof. As the sequence is locally bounded, it is locally equi-continuous by lemma 1.4.2. By Arzelà-Ascoli, there exists a convergent subsequence. \square

Definition 1.4.5. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family of functions $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is *normal* if every sequence in \mathcal{F} contains a subsequence that converges uniformly on compacts.

Theorem 1.4.6 (Montel). Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is normal if and only if it is locally bounded.

Proof. The proof is obvious and need not be mentioned. \square

Theorem 1.4.7 (Vitali). Let $\Omega \subseteq \mathbb{C}$ be a domain and $(f_n)_n \subseteq \mathcal{O}(\Omega)$ a locally bounded sequence of holomorphic functions. The following statements are equivalent:

- i) The sequence $(f_n)_n$ converges uniformly on compact subsets of Ω .
- ii) For a point $p \in \Omega$ the sequence $(f_n^{(k)}(p))_n$ converges for all $k \in \mathbb{N}_0$.
- iii) The set

$$A = \left\{ z \in \Omega \mid \lim_{n \rightarrow \infty} f_n(z) \text{ converges} \right\}$$

has an accumulation point in Ω .

Proof. Suppose that the sequence converges uniformly on compact subsets. Given a $p \in \Omega$, choose a $\delta > 0$ such that $D = \overline{\Delta}(p, \delta) \subseteq \Omega$. Note that

$$|g^{(k)}(p)| \leq \frac{k!}{\delta^k} \cdot \|g\|_D$$

holds for all holomorphic functions g . As $\|f - f_n\|$ converges to 0, the derivatives of f_n converge.

Suppose that the sequences of derivatives converge at a point $p \in \Omega$ and choose a $\delta > 0$ such that $D = \overline{\Delta}(p, \delta) \subseteq \Omega$. As the sequence is locally bounded, there exists a constant M such that $\|f_n\|_D \leq M$ holds for all $n \in \mathbb{N}$. We can now develop power series

$$f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z-p)^k.$$

They converge uniformly on compact subsets of $\Delta(p, \delta)$. Note that

$$a_{k,n} = \frac{f_n^{(k)}(p)}{k!}.$$

As derivatives converge, we can define the limit

$$a_k = \lim_{n \rightarrow \infty} a_{k,n}.$$

Now define the formal power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - p)^k.$$

The Cauchy bounds give us the inequality

$$|a_{k,n}| = \frac{|f^{(k)}(p)|}{k!} \leq \frac{M}{\delta^k},$$

therefore

$$\limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} \leq \limsup_{k \rightarrow \infty} \frac{\sqrt[k]{M}}{\delta} = \frac{1}{\delta}.$$

We conclude that the radius of convergence is at least δ . Consider some $\rho \in (0, \delta)$ and $z \in \mathbb{A}(p, \rho)$. We have

$$\begin{aligned} |f_n(z) - f(z)| &\leq \left| \sum_{k=0}^m (a_{k,n} - a_k) \cdot (p - z)^k \right| + \left| \sum_{k=m+1}^{\infty} (a_{k,n} - a_k) \cdot (p - z)^k \right| \\ &\leq \sum_{k=0}^m |a_{k,n} - a_k| \rho^k + \sum_{k=m+1}^{\infty} 2M \cdot \frac{\rho^k}{\delta^k} \\ &= \sum_{k=0}^m |a_{k,n} - a_k| \rho^k + 2M \cdot \left(\frac{\rho}{\delta}\right)^{m+1} \cdot \frac{\delta}{\delta - \rho} \\ &= 2 \cdot \frac{\varepsilon}{2} \end{aligned}$$

for large enough m and n . It follows that p is an accumulation point of A .

Suppose now that A has an accumulation point in Ω . By Montel's theorem there exists a subsequence $(f_{n_m})_m$ that converges uniformly on compact subsets of Ω to a limit function f . Note that all such subsequences have the same limit by the identity principle.

Assume that the sequence $(f_n)_n$ does not converge uniformly on a compact subset $K \subseteq \Omega$. We can therefore construct another subsequence $(g_n)_n$ of $(f_n)_n$ such that

$$\|g_n - f\|_K > \varepsilon$$

for all $n \in \mathbb{N}$. But note that $(g_n)_n$ also has a convergent subsequence by Montel's theorem, which is of course a contradiction, as it cannot converge to f . \square

2 Theorems about holomorphic functions

2.1 Riemann mapping theorem

Definition 2.1.1. A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if every closed path in Ω is homotopic to a constant path in Ω .

Lemma 2.1.2. Let $\Omega \subset \mathbb{C}$ be a domain and $a \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Then there exists a holomorphic injection $f: \Omega \rightarrow \mathbb{A}$ such that $f(a) = 0$.

Proof. Fix a point $p \in \mathbb{C} \setminus \Omega$. By our assumption, there exists a function $v \in \mathcal{O}^*(\Omega)$ such that $v(z)^2 = z - p$. Note that v is injective. Similarly, we have $v(\Omega) \cap -v(\Omega) = \emptyset$. Now choose a point $b \in -v(\Omega)$. As v is not constant, it is an open map. Therefore, there exists some $r > 0$ such that $\mathbb{A}(b, r) \cap v(\Omega) = \emptyset$. The Möbius transformation

$$h(w) = r \cdot \left(\frac{1}{w - b} - \frac{1}{v(a) - b} \right)$$

thus maps $v(\Omega)$ into \mathbb{A} . The map f is therefore given as $f = h \circ v$. \square

Definition 2.1.3. An *expansion* is a map $\kappa: \Omega \rightarrow \mathbb{A}$ where $0 \in \Omega \subset \mathbb{A}$ such that $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ holds for all $z \neq 0$.

Lemma 2.1.4. Let $\Omega \subset \mathbb{A}$ be a domain with $0 \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Choose $c \in \mathbb{A}$ such that $c^2 \notin \Omega$. For all $a \in \mathbb{A}$, let

$$g_a = \frac{z - a}{\bar{a}z - 1}$$

and choose $v \in \mathcal{O}(\Omega)$ such that $v(z)^2 = g_{c^2}(z)$ and $v(0) = c$. Then the map $\kappa = g_c \circ v$ is an expansion and

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = \text{id}_\Omega.$$

Proof. Note that v is indeed well-defined. Also note that

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = g_{c^2} \circ (z \mapsto z^2) \circ v = g_{c^2} \circ g_{c^2} = \text{id}.$$

We of course have $\kappa(0) = 0$. Denote $\psi_c = g_{c^2} \circ (z \mapsto z^2) \circ g_c$. It remains to check that $|\kappa(z)| > |z|$, which is equivalent to $|\psi_c(z)| < |z|$ for $z \neq 0$ as $\psi_c \circ \kappa = \text{id}$. Note that $\psi_c: \mathbb{A} \rightarrow \mathbb{A}$ is holomorphic. As it is not a rotation (it is not injective), the conclusion follows from the Schwarz lemma. \square

Lemma 2.1.5 (Hurwitz). Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f_n: \Omega \rightarrow \mathbb{C}$ be holomorphic functions. Suppose that the sequence $(f_n)_n$ converges uniformly on compacts of Ω to a non-constant function $f: \Omega \rightarrow \mathbb{C}$. Then for all points $p \in \Omega$ there exists a sequence $(p_n)_n \subseteq \Omega$ with limit p such that $f_n(p_n) = f(p)$ for all $n > N$.

Proof. Let $w = f(p)$. There exists a disk $\mathbb{A}(p, \delta)$ such that $f(z) \neq w$ for all points $z \in \overline{\mathbb{A}(p, \delta)} \setminus \{p\}$. Note that we have

$$\min_{z \in \partial \mathbb{A}(p, \delta)} |f(z) - w| > |f(p) - w| = 0.$$

As $(f_n)_n$ converges uniformly on $\overline{\Delta(p, \delta)}$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\min_{z \in \partial \Delta(p, \delta)} |f_n(z) - w| > |f_n(p) - w|.$$

By lemma 1.1.7, $f_n(z) - w$ has a root $p_n \in \Delta(p, \delta)$. For any convergent subsequence $(p_{n_k})_k$ with limit q we have

$$f(p) = \lim_{k \rightarrow \infty} f_{n_k}(p_{n_k}) = f(q),$$

therefore $p = q$. □

Corollary 2.1.5.1. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n: \Omega \rightarrow \mathbb{C}$ be holomorphic functions such that $(f_n)_n$ converges uniformly on compacts of Ω to $f: \Omega \rightarrow \mathbb{C}$. If all the f_n are nowhere vanishing and $f \neq 0$, then f is nowhere vanishing.

Proof. The proof is obvious and need not be mentioned. □

Theorem 2.1.6 (Hurwitz). Let $\Omega, \Omega' \subseteq \mathbb{C}$ be domains and $f_n: \Omega \rightarrow \Omega'$ be holomorphic functions that converge uniformly on compacts of Ω to $f: \Omega \rightarrow \Omega'$. Assume that f is not constant.

- i) If $f_n: \Omega \rightarrow \Omega'$ is injective, f is also injective.
- ii) We have $f(\Omega) \subseteq \Omega'$.

Proof.

- i) Let $p \in \Omega$ and observe the functions $g_n(z) = f_n(z) - f_n(p)$. This is a sequence of nowhere vanishing functions. As f is not constant, $f(z) - f(p)$ is nowhere vanishing as well. It follows that f is injective.
- ii) Suppose otherwise and apply the Hurwitz lemma for a point p with $f(p) \notin \Omega'$. □

Theorem 2.1.7 (Riemann mapping). For a proper domain $\Omega \subset \mathbb{C}$ the following are equivalent:

- i) Ω is simply connected.
- ii) Ω admits a logarithm for any $f \in \mathcal{O}^*(\Omega)$.
- iii) Ω admits a square root for any $f \in \mathcal{O}^*(\Omega)$.
- iv) Ω is biholomorphic to Δ .

Proof. Note that if Ω is biholomorphic to Δ , it is of course simply connected. Suppose that Ω is simply connected. Then

$$F(z) = a + \int_{z_0}^z \frac{f'(z)}{f(z)} dz$$

defines a logarithm for any $f \in \mathcal{O}^*(\Omega)$. Given a logarithm of a function, we can of course construct a square root with $\sqrt{f} = e^{\frac{1}{2} \ln f}$. It remains to check that all domains admitting square roots are biholomorphic to Δ .

By lemma 2.1.2 we can assume that $\Omega \subseteq \mathbb{A}$ and $0 \in \Omega$. Now define the family of functions

$$\mathcal{F} = \{f: \Omega \rightarrow \mathbb{A} \mid f \in \mathcal{O}(\Omega) \wedge f(0) = 0 \wedge f \text{ is injective}\}.$$

If \mathcal{F} has no biholomorphic map, it is infinite. Note that \mathcal{F} is bounded, so it is normal by Montel.

Choose a point $p \in \Omega$ with $p \neq 0$. We claim that if $h \in \mathcal{F}$ and

$$|h(p)| = \sup_{f \in \mathcal{F}} |f(p)|,$$

we have $h(\Omega) = \mathbb{A}$. Indeed, if that were not the case, we'd reach a contradiction with the expansion κ of Ω as

$$|\kappa(h(p))| > |h(p)|$$

and $\kappa \circ h \in \mathcal{F}$.

Let

$$M = \sup_{f \in \mathcal{F}} |f(p)|$$

and find a sequence $(f_n)_n \subseteq \mathcal{F}$ with

$$\lim_{n \rightarrow \infty} |f_n(p)| = M.$$

As \mathcal{F} is a normal family, there exists a convergent subsequence. The limit is not constant as $f(p) \neq 0$. By Hurwitz, f is injective and $f(\Omega) \subseteq \mathbb{A}$. By the above claim, we have $f(\Omega) = \mathbb{A}$. \square

2.2 Bloch's theorem

Lemma 2.2.1. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $f: \overline{\Omega} \rightarrow \mathbb{C}$ a continuous map such that $f|_{\Omega}$ is an open map. Let $a \in \Omega$ be a point such that

$$s = \min_{z \in \partial\Omega} |f(z) - f(a)| > 0.$$

Then $f(\Omega)$ contains the disk $\Delta(f(a), s)$.

Proof. By compactness, there exists a $w_0 \in \partial f(\Omega)$ such that $d(\partial f(\Omega), f(a)) = |w_0 - f(a)|$. Let $(z_k)_k \subseteq \Omega$ be a sequence, convergent to z_0 , such that

$$\lim_{k \rightarrow \infty} f(z_k) = w_0.$$

Of course $f(z_0) = w_0$. Note that, as $f|_{\Omega}$ is open, we have $z_0 \in \partial\Omega$. But then

$$d(\partial f(\Omega), f(a)) = |f(z_0) - f(a)| \geq s. \quad \square$$

Lemma 2.2.2. Let f be a non-constant function, holomorphic in a neighbourhood of $\overline{\Delta(a, r)}$. Assume that

$$\sup_{z \in \overline{\Delta(a, r)}} |f'(z)| \leq 2 |f'(a)|.$$

Then $\Delta(f(a), R) \subseteq f(\Delta(a, r))$, where

$$R = (3 - 2\sqrt{2}) \cdot r \cdot |f'(a)|.$$

Proof. Without loss of generality assume that $a = f(a) = 0$. Define

$$A(z) = f(z) - f'(0)z = \int_0^1 (f'(tz) - f'(0)) z \, dt.$$

Note that

$$f'(v) - f'(0) = \frac{1}{2\pi i} \oint_{\partial\Delta(a, r)} f'(\xi) \cdot \left(\frac{1}{\xi - v} - \frac{1}{\xi} \right) d\xi,$$

therefore

$$|f'(v) - f'(0)| \leq \frac{1}{2\pi} \cdot |v| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r \cdot (r - |v|)} \cdot 2\pi r = |v| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r - |v|}.$$

It follows that

$$\begin{aligned} |A(z)| &\leq \int_0^1 |z| \cdot |f'(tz) - f'(0)| \, dt \\ &\leq |z| \cdot \int_0^1 |tz| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r - |tz|} \, dt \\ &\leq |z|^2 \cdot \|f'\|_{\Delta(a, r)} \cdot \int_0^1 t \cdot \frac{1}{r - |z|} \\ &= |z|^2 \cdot \frac{|f'(0)|}{r - |z|}. \end{aligned}$$

Now, using the triangle inequality, we get

$$|f(z)| \geq |z| \cdot |f'(0)| - |A(z)|.$$

Let $|z| = \rho \in (0, r)$. We get

$$|f(z)| \geq \rho \cdot |f'(0)| - |A(z)| \geq \rho \cdot |f'(0)| - \frac{\rho^2}{r - \rho} \cdot |f'(0)| \geq |f'(0)| \cdot \left(\rho - \frac{\rho^2}{r - \rho} \right).$$

Note that there exists a ρ_0 such that

$$\rho_0 - \frac{\rho_0^2}{r - \rho_0} = r \cdot (3 - 2\sqrt{2}).$$

Therefore, we get

$$|f(z)| \geq |f'(0)| \cdot r \cdot (3 - 2\sqrt{2}).$$

Now just apply the previous lemma to the disk $\Delta(0, \rho_0)$. □

Theorem 2.2.3 (Bloch). Let f be a function, holomorphic in a neighbourhood of $\bar{\Delta}$, with $f'(0) = 1$. Then $f(\Delta)$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.

Proof. Define $h(z) = |f'(z)| (1 - |z|) \geq 0$. Note that $h \not\equiv 0$ as f is not constant. Therefore h attains a maximum in a point $p \in \bar{\Delta}$. In particular, as $h|_{\partial\Delta} = 0$, we have $p \in \Delta$. Observe $\Omega = \Delta(p, t)$ for $t = \frac{1}{2} \cdot (1 - |p|)$. For all $z \in \Omega$, we have $1 - |z| \geq t$ and

$$|f'(z)| \cdot (1 - |z|) \leq |f'(p)| \cdot (1 - |p|) = |f'(p)| \cdot 2t \leq |f'(p)| \cdot 2 \cdot (1 - |z|).$$

Now, applying lemma 2.2.2, we have $\Delta(f(p), R) \subseteq f(\Delta)$ with

$$R = (3 - 2\sqrt{2}) \cdot \frac{1}{2} \cdot (1 - |p|) \cdot |f'(p)| \geq \frac{3}{2} - \sqrt{2}$$

by choice of p . □

Remark 2.2.3.1. Let

$$\mathcal{F} = \left\{ f \text{ holomorphic on a neighbourhood of } \bar{\Delta} \mid f'(0) = 1 \right\}.$$

For $f \in \mathcal{F}$, denote by L_f the supremum of radii of disks contained in $f(\Delta)$, and by B_f the supremum of radii of disks contained in $f(\Delta)$ that is a biholomorphic image of some subdomain of Δ . We then define the *Landau's constant*

$$L = \inf_{f \in \mathcal{F}} L_f$$

and the *Bloch's constant*

$$B = \inf_{f \in \mathcal{F}} B_f.$$

The current known bounds for the constants are

$$0.5 < L < 0.544 \quad \text{and} \quad \frac{\sqrt{3}}{4} + 10^{-14} < B \leq \sqrt{\frac{\sqrt{3}-1}{2}} \cdot \frac{\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

Corollary 2.2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{O}(\Omega)$ a function and $p \in \Omega$. Let $r = d(p, \partial\Omega)$. Then $f(\Omega)$ contains a disk of radius

$$\left(\frac{3}{2} - \sqrt{2} \right) \cdot r \cdot |f'(p)|.$$

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Proof. The proof is obvious and need not be mentioned. \square

Remark 2.2.3.3. Liouville's theorem follows from this corollary.

Lemma 2.2.4. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $1, -1 \notin f(\Omega)$. Then there exists a function $F \in \mathcal{O}(\Omega)$ such that $f = \cos(F)$.

Proof. Note that, as Ω is simply connected, we can define

$$F(z) = \frac{1}{i} \cdot \ln \left(f(z) + \sqrt{f(z)^2 - 1} \right). \quad \square$$

Theorem 2.2.5. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let $f \in \mathcal{O}(\Omega)$. Suppose that $0, 1 \notin f(\Omega)$. Then the following statements are true:

i) There exists a function $g \in \mathcal{O}(\Omega)$ such that

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g))).$$

ii) If any $g \in \mathcal{O}(\Omega)$ satisfies the above equality, then $g(\Omega)$ contains no disk of radius 1.

Proof.

i) Apply the previous lemma twice.

ii) Define

$$A = \left\{ m \pm \frac{i}{\pi} \ln \left(n + \sqrt{n^2 - 1} \right) \mid m \in \mathbb{Z} \wedge n \in \mathbb{N} \right\}.$$

We claim that $g(\Omega) \cap A = \emptyset$. Indeed, for $a \in A$ we have

$$f(a) = \frac{1}{2} (1 + \cos(\pm \pi \cdot n)) \in \{0, 1\}.$$

Now note that

$$\begin{aligned} \ln \left(n + 1 + \sqrt{n^2 + 2n} \right) - \ln \left(n + \sqrt{n^2 - 1} \right) &= \ln \left(\frac{n + 1 + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 - 1}} \right) \\ &\leq \ln \left(\frac{2n + 2}{n} \right) \\ &\leq \ln(4) \\ &< \pi. \end{aligned}$$

It's straightforward to check that every disk of radius 1 intersects A . \square

Theorem 2.2.6 (Picard's little theorem). Every non-constant entire function omits at most one complex value.

Proof. Without loss of generality assume that f omits 0 and 1. Applying the above theorem, we can write

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g))).$$

Recall that $g(\mathbb{C})$ contains no disk of radius 1. If g is not constant, $g(\mathbb{C})$ contains arbitrarily large disks by corollary 2.2.3.2, which is a contradiction. \square

Corollary 2.2.6.1. Suppose that $f \in \mathcal{M}(\mathbb{C})$ is a non-constant function. Then f omits at most 2 values.

Proof. Suppose that f omits distinct values a, b and c . Then

$$g(z) = \frac{1}{f(z) - a}$$

is an entire function that omits values $\frac{1}{b-a}$ and $\frac{1}{c-a}$, therefore it is constant. \square

Theorem 2.2.7. Let $f \in \mathcal{O}(\mathbb{C})$ be an entire function. Then either $f \circ f$ has a fixed point of $f(z) = z + c$.

Proof. If $f \circ f$ has no fixed point, the same holds for f . We can therefore define an entire holomorphic function g with

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}.$$

Note that g omits both 0 and 1, therefore it is constant. But then

$$f(f(z)) - z = \lambda(f(z) - z)$$

for some $\lambda \notin \{0, 1\}$ by Picard's little theorem. Taking the derivative, we get

$$f'(f(z)) \cdot f'(z) - 1 = \lambda(f'(z) - 1),$$

or equivalently

$$f'(z) \cdot (f'(f(z)) - \lambda) = 1 - \lambda \neq 0.$$

Note that $f' \circ f$ omits both λ and 0, therefore it is constant. But then f' is constant as well. The only option is $f'(z) = 1$. \square

Lemma 2.2.8. For all $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos(\pi v) = w$ and $|v| \leq 1 + |w|$.

Proof. Let $v = \alpha + i\beta$ and note that

$$|w|^2 = \cos(\pi\alpha)^2 + \sinh(\pi\beta)^2 \geq \pi^2\beta^2.$$

Observe that we can choose some α such that $|\alpha| \leq 1$, therefore

$$1 + |w| \geq 1 + \pi \cdot |\beta| \geq |\alpha| + |\beta| \geq |v|. \quad \square$$

Theorem 2.2.9. Let f be a function, holomorphic on a neighbourhood of $\overline{\Delta}$, such that $0, 1 \notin f(\Omega)$. There exists a function g , holomorphic on a neighbourhood of $\overline{\Delta}$, such that

i) the equality

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g)))$$

holds with $|g(0)| \leq 3 + 2|f(0)|$, and

ii) the inequality

$$|g(z)| \leq |g(0)| + \frac{\theta}{\gamma(1 - \theta)}$$

holds for all $|z| \leq \theta$.

Proof. Again, apply lemma 2.2.4 and let

$$2f - 1 = \cos(\pi \cdot F).$$

Using the above lemma, we can transform F such that $|F(0)| \leq 1 + |2f(0) - 1|$. Applying lemma 2.2.4 again, we define g such that

$$F = \cos(\pi g).$$

Again, using the above lemma, set $|g(0)| \leq 1 + |F(0)|$. We therefore have

$$|g(0)| \leq 1 + |F(0)| \leq 2 + |2f(0) - 1| \leq 3 + 2|f(0)|.$$

Recall that $g(\mathbb{A})$ does not contain a disk of radius 1. Let $z \in \overline{\mathbb{A}(\theta)}$. Then, by Bloch's theorem, $g(\mathbb{A})$ contains a disk of radius $R = \gamma \cdot |g'(z)| \cdot (1 - \theta)$. Therefore, we must have

$$|g'(z)| < \frac{1}{\gamma(1 - \theta)}.$$

It follows that

$$|g(z)| = \left| g(0) + \int_0^z g'(\xi) d\xi \right| \leq |g(0)| + \int_0^z |g'(\xi)| d\xi \leq |g(0)| + |z| \cdot \frac{1}{\gamma(1 - \theta)}. \quad \square$$

Definition 2.2.10. For $r \geq 0$, let

$$S(r) = \left\{ f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{A}} \mid 0, 1 \notin f(\overline{\mathbb{A}}) \wedge |f(0)| \leq r \right\}.$$

For $\theta \in (0, 1)$ and $r > 0$, let

$$L(\theta, r) = \exp \left(\pi \cdot \exp \left(3 + 2r + \frac{\theta}{\gamma(1 - \theta)} \right) \right),$$

where γ is any constant such that Bloch's theorem holds, e.g. $\gamma = \frac{3}{2} - \sqrt{2}$.

Theorem 2.2.11 (Schottky). Let $f \in S(r)$. Then for all $z \in \mathbb{A}$ such that $|z| < \theta$ we have

$$|f(z)| \leq L(\theta, r).$$

Proof. Let g be a holomorphic function as in the previous theorem. Note that $|\cos(w)| \leq e^{|w|}$. We must therefore also have

$$\frac{1}{2} \cdot |1 + \cos(w)| \leq e^{|w|}.$$

Using this inequality, we get

$$|f(z)| \leq \exp(\pi \cdot \exp(\pi \cdot |g(z)|)) \leq L(\theta, r). \quad \square$$

2.3 The great Picard theorem

Lemma 2.3.1. Let $\Omega \subseteq \mathbb{C}$ be a domain, $\omega \in \Omega$ and $r \in (0, \infty)$. Let

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

and $\mathcal{F}_{\omega,r} \subseteq \mathcal{F}$ a subfamily with $|f(\omega)| \leq r$ for all $f \in \mathcal{F}_{\omega,r}$.

- i) There exists some $t > 0$ such that $\mathcal{F}_{\omega,r}|_{\Delta(\omega,t)}$ is bounded.
- ii) The family $\mathcal{F}_{\omega,1}$ is locally bounded in Ω .

Proof.

- i) Choose a $t > 0$ such that $\overline{\Delta(\omega, 2t)} \subseteq \Omega$ and set $\varphi(z) = 2tz + \omega$. By Schottky's theorem, we have

$$|f \circ \varphi(z)| \leq L\left(\frac{1}{2}, r\right)$$

for $|z| < \frac{1}{2}$, or equivalently

$$\sup_{v \in \Delta(\omega, t)} |f(v)| \leq L\left(\frac{1}{2}, r\right).$$

The family $\mathcal{F}_{\omega,r}$ is therefore bounded.

- ii) Let

$$\mathcal{U} = \{u \in \Omega \mid \mathcal{F}_{\omega,1} \text{ is bounded in a neighbourhood of } u\}.$$

Note that $\omega \in \mathcal{U}$, therefore the set is non-empty. Also observe that \mathcal{U} is open. Suppose that $\mathcal{U} \neq \Omega$ and let $v \in \partial\mathcal{U} \cap \Omega$. Then there exists a sequence $(f_n)_n \subseteq \mathcal{F}_{\omega,1}$ such that

$$\lim_{n \rightarrow \infty} |f_n(v)| = \infty.$$

Define $g_n = \frac{1}{f_n}$. These functions are holomorphic and omit both 0 and 1 by definition, therefore $g_n \in \mathcal{F}$. Applying the item i) for the sequence $(g_n)_n$ at point v , the sequence is bounded in a neighbourhood of v . By Montel's theorem, there exists a subsequence $(g_{n_k})_k$ that converges to a function g uniformly on compacts of $\Delta(v, s)$. By corollary 2.1.5.1, the function g is constant. But then

$$\lim_{k \rightarrow \infty} |f_{n_k}(z)| = \infty$$

for all $z \in \Delta(v, s)$, which is not possible as v is a boundary point. It follows that $\mathcal{U} = \Omega$. \square

Definition 2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n: \Omega \rightarrow \mathbb{C}$ a sequence of functions. We say that f_n converges to ∞ if

$$\lim_{n \rightarrow \infty} \|f_n\|_K = \infty$$

for every compact $K \subset \Omega$.

Theorem 2.3.3 (Montel – sharp). Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

Then \mathcal{F} is normal in Ω where we also allow convergence to ∞ .

Proof. Let $\Omega \subseteq \mathbb{C}$ be a domain and $p \in \Omega$. Consider the family $\mathcal{F}_{p,1}$. Let $(f_n)_n \subseteq \mathcal{F}$ be a sequence. If there exists a subsequence $(f_{n_k})_k \subseteq \mathcal{F}_{p,1}$, we can apply the above lemma. By the classical Montel's theorem, this subsequence has a convergent subsequence.

Suppose now that no such subsequence exists, that is $(f_n)_n$ has only finitely many terms in $\mathcal{F}_{p,1}$. But then there exists a subsequence $\left(\frac{1}{f_{n_k}}\right)_k \subseteq \mathcal{F}_{p,1}$. As before, this sequence has a convergent subsequence with limit g . If g is nowhere-vanishing, then $\frac{1}{g}$ is the limit of a subsequence of $(f_n)_n$. Otherwise, by corollary 2.1.5.1, we have $g = 0$ and therefore $(f_n)_n$ converges to ∞ . \square

Definition 2.3.4. Let $\Omega \subseteq \mathbb{C}$ be an open set and $p \in \Omega$. A function $f \in \mathcal{O}(\Omega \setminus \{p\})$ has an *essential singularity* in p if the limit

$$\lim_{z \rightarrow p} f(z)$$

does not exist and

$$\lim_{z \rightarrow p} |f(z)| \neq \infty.$$

Theorem 2.3.5 (Picard's great theorem). Let $\Omega \subseteq \mathbb{C}$ be an open set $p \in \Omega$ a point and $f \in \mathcal{O}(\Omega \setminus \{p\})$ a function. If f has an essential singularity at p , then f assumes every complex number as a value infinitely many times with at most one exception.

Proof. Without loss of generality assume that $p = 0$ and consider $\Omega = \Delta(\varepsilon)$. Suppose that f omits two values on $\Delta(\varepsilon)$, without loss of generality 0 and 1.

We now claim that f or $\frac{1}{f}$ is bounded in a neighbourhood of 0. Define the sequence of holomorphic functions $(f_n)_n$ with $f_n(z) = f\left(\frac{z}{n}\right)$. This sequence also omits 0 and 1, therefore either $(f_n)_n$ or $\left(\frac{1}{f_n}\right)_n$ has a convergent subsequence that converges uniformly on compacts by the sharp version of Montel's theorem. Denote the subsequence by $(g_{n_k})_k$ and set $g = f$ or $g = \frac{1}{f}$ accordingly.

Observe that there exists a constant M such that

$$\|g_{n_k}\|_{\partial\Delta(\frac{\varepsilon}{2})} \leq M$$

holds for all $k \in \mathbb{N}$. This is equivalent to

$$|g(z)| \leq M$$

for $|z| = \frac{1}{n_k} \cdot \frac{\varepsilon}{2}$. By the maximum principle, we have

$$|g(z)| \leq M$$

for all z such that

$$\frac{\varepsilon}{2} \cdot \frac{1}{n_k} \leq |z| \leq \frac{\varepsilon}{2}.$$

But as $(n_k)_k$ diverges, the inequality $|g(z)| \leq M$ holds for all z such that $|z| \leq \frac{\varepsilon}{2}$, therefore f or $\frac{1}{f}$ is bounded near 0.

Observe that f is not bounded in a neighbourhood of 0, as otherwise 0 is a removable singularity, which is not possible. Similarly, if $\frac{1}{f}$ is bounded, then f has either a removable singularity or a pole at 0, which is again a contradiction. \square

3 Infinite products

3.1 Definition

Definition 3.1.1. Let $(a_k)_k$ be a sequence of complex numbers. The sequence

$$n \mapsto \prod_{k=1}^n a_k$$

is called the *sequence of partial products* with factors a_k . We denote

$$p_{m,n} = \prod_{k=m}^n a_k.$$

We say that the infinite product is *convergent* if there exists an index $m \in \mathbb{N}$ such that the limit

$$\hat{a}_m = \lim_{n \rightarrow \infty} p_{m,n}$$

exists and is non-zero. We then define

$$\prod_{k=1}^{\infty} a_k = p_{1,m-1} \cdot \hat{a}_m.$$

as the limit of the infinite product.

Remark 3.1.1.1. The limit is uniquely defined.

Remark 3.1.1.2. An infinite product is convergent if and only if the product of all its non-zero factors has a non-zero limit and only finitely many factors are non-zero.

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