Complex analysis

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Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations)**.** Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a, f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Definition 1.1.2. *Wirtinger derivatives* are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a. The function f is *holomorphic* if it is holomorphic in every point of Ω. We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with $\mathcal{C}^1$-smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w - z}\, dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z}\, dw \wedge d\overline{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\mathbb{\Delta}(z, \varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\mathbb{\Delta}(z, \varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w-z}\, dw$ on Ω_ε. As $d\omega = \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w} \wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z}\, dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z}\, d\overline{w} \wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z}\, dw = \oint_{\partial\Omega} \frac{f(w)}{w - z}\, dw - \oint_{\partial\mathbb{\Delta}(z,\varepsilon)} \frac{f(w)}{w - z}\, dw.$$

In the limit, we have

$$\lim_{\varepsilon\to 0} \oint\limits_{\partial\mathbb{\Delta}(z,\varepsilon)} \frac{f(w)}{w-z}\, dw = \lim_{\varepsilon\to 0} \int_0^{2\pi} \frac{f\left(z+\varepsilon e^{it}\right)}{\varepsilon e^{it}} \cdot \varepsilon i e^{it}\, dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon\to 0} \iint\limits_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w}\wedge dw = \iint\limits_{\Omega\setminus\{z\}} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w}\wedge dw = \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w}\wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i} \oint\limits_{\partial\Omega} \frac{f(w)}{w-z}\, dw - f(z) = -\frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, dw \wedge d\overline{w}. \qquad \square$$

Theorem 1.1.5 (Power series expansion)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset, $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\mathbb{\Delta}(a, r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \oint\limits_{\partial\mathbb{\Delta}(a,\rho)} \frac{f(w)}{(w-z)^{k+1}}\, dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k\to\infty} \sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a holomorphic function. Let $A \subseteq \Omega$ be a subset such that $f(z) = 0$ for all $z \in A$. If A has an accumulation point in Ω, then $f(z) = 0$ for all $z \in \Omega$.

Proof. Let $a \in \Omega$ be an accumulation point of A. By continuity, we have $f(a) = 0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k (z-a)^k,$$

where we assume $c_{k_0} \neq 0$. But now $g(z) = \frac{f(z)}{(z-a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a) = 0$, which is a contradiction. It follows that $c_k = 0$ for all $k \in \mathbb{N}_0$. The set $\{z \in \Omega \mid f(z) = 0\}$ is then both open and closed and is therefore equal to Ω. $\square$

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