Complex analysis

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Contents

Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations)**.** Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a, f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Definition 1.1.2. *Wirtinger derivatives* are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a. The function f is *holomorphic* if it is holomorphic in every point of Ω. We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with $\mathcal{C}^1$-smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,dw \wedge d\overline{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\mathbb{\Delta}(z,\varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\mathbb{\Delta}(z,\varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w-z}\,dw$ on Ω_ε. As $d\omega = \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w-z}\,dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w-z}\,dw = \oint_{\partial\Omega} \frac{f(w)}{w-z}\,dw - \oint_{\partial\mathbb{\Delta}(z,\varepsilon)} \frac{f(w)}{w-z}\,dw.$$

In the limit, we have

$$\lim_{\varepsilon\to 0} \oint\limits_{\partial\mathbb{\Delta}(z,\varepsilon)} \frac{f(w)}{w-z}\,dw = \lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{f\left(z+\varepsilon e^{it}\right)}{\varepsilon e^{it}}\cdot \varepsilon i e^{it}\,dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon\to 0}\iint\limits_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega\setminus\{z\}} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i}\oint\limits_{\partial\Omega}\frac{f(w)}{w-z}\,dw - f(z) = -\frac{1}{2\pi i}\iint\limits_{\Omega}\frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,dw\wedge d\overline{w}.$$ □

Theorem 1.1.5 (Power series expansion)**.** Let $\Omega\subseteq\mathbb{C}$ be an open subset, $f\in\mathcal{O}(\Omega)$ and $a\in\Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\mathbb{\Delta}(a,r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i}\oint\limits_{\partial\mathbb{\Delta}(a,\rho)}\frac{f(w)}{(w-z)^{k+1}}\,dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k\cdot(z-a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k\to\infty}\sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity)**.** Let $\Omega\subseteq\mathbb{C}$ be a domain and $f\in\mathcal{O}(\Omega)$ a holomorphic function. Let $A\subseteq\Omega$ be a subset such that $f(z)=0$ for all $z\in A$. If A has an accumulation point in Ω, then $f(z)=0$ for all $z\in\Omega$.

Proof. Let $a\in\Omega$ be an accumulation point of A. By continuity, we have $f(a)=0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k(z-a)^k,$$

where we assume $c_{k_0}\neq 0$. But now $g(z)=\frac{f(z)}{(z-a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a)=0$, which is a contradiction. It follows that $c_k=0$ for all $k\in\mathbb{N}_0$. It follows that the set $\operatorname{Int}\{z\in\Omega \mid f(z)=0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to Ω. □

Theorem 1.1.7 (Open mapping)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If f is not constant, it is an open map.

Proof. We first prove the following lemma:

Lemma. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\Delta(a,r)} \subseteq \Omega$. If

$$|f(a)| < \min_{\partial\Delta(a,r)} |f|,$$

then f has a zero in $\Delta(a,r)$.

Proof (lemma). Assume otherwise. From the inequality it follows that f has no zeroes on the boundary either. By continuity, f has no zero on an open set V with $\Delta(a,r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint_{\partial\Delta(a,r)} \frac{g(z)}{z-a}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g\left(a + r \cdot e^{it}\right)}{e^{it}} \cdot rie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} g\left(a + re^{it}\right)\, dt.$$

We can therefore get a bound on $|g(a)|$ as

$$|g(a)| \le \max_{\partial\Delta(a,r)} |g|,$$

but as the condition on f can be rewritten as

$$|g(a)| > \max_{\partial\Delta(a,r)} |g|,$$

we have reached a contradiction. □

Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\Delta(z_0, \rho) \subseteq U$ and z_0 is the only pre-image of w_0 in $\Delta(z_0, 2\rho)$.[1]

Since $\partial\Delta(z_0, \rho)$ is a compact set and

$$|f(z) - w_0| > 0$$

for all $z \in \partial\Delta(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a $w \in \Delta(w_0, \varepsilon)$. As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \ge \varepsilon$$

on the boundary and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

by the above lemma, $f(z_0) - w$ has a root on $\Delta(z, \rho)$. □

Theorem 1.1.8 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function f is constant.

[1] If such a disk does not exist, f is constant by the identity theorem.

Proof. Suppose that f is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As f is an open map, it also attains the value $(1+\varepsilon) \cdot f(z)$, which is a contradiction as the modulus then equals $(1+\varepsilon) \cdot |f(z)| > |f(z)|$. $\square$

Theorem 1.1.9 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

Proof. As $\overline{\Omega}$ is compact, f attains a global maximum on this set. If the maximum is attained in the interior, f is constant, therefore it is also attained on the boundary. $\square$

Definition 1.1.10. A function $f \colon \Omega \setminus \{a\} \to \mathbb{C}$ is *locally bounded* near a if there exists an open neighbourhood $U \subseteq \Omega$ of a such that $f|_{U \setminus \{a\}}$ is bounded.

Theorem 1.1.11 (Riemann removable singularity theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$. If f is locally bounded near a, then there exists a unique function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus \{a\}} = f$.

Proof. Define the function $F \colon \Omega \to \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \displaystyle\frac{1}{2\pi i} \oint\limits_{\partial\mathbb{\Delta}(a,\rho)} \frac{f(w)}{w-a}\, dw & z = a. \end{cases}$$

It remains to check that F is complex differentiable at a. Indeed, for $z \in \mathbb{\Delta}(a,\rho)$ we have

$$\begin{aligned} \lim_{z \to a} \frac{F(z)-F(a)}{z-a} &= \lim_{z \to a} \frac{1}{z-a} \oint\limits_{\partial\mathbb{\Delta}(a,\rho)} \left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a}\right) dw \\ &= \lim_{z \to a} \frac{1}{2\pi i} \cdot \frac{1}{z-a} \cdot \oint\limits_{\partial\mathbb{\Delta}(a,\rho)} f(w) \cdot \frac{z-a}{(w-z)(w-a)}\, dw \\ &= \frac{1}{2\pi i} \oint\limits_{\partial\mathbb{\Delta}(a,\rho)} \frac{f(w)}{(w-a)^2}\, dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem. $\square$

Theorem 1.1.12 (Schwarz lemma)**.** Let $f \colon \mathbb{\Delta} \to \mathbb{\Delta}$ be a holomorphic function with $f(0) = 0$. Then, $|f'(0)| \leq 1$ and the inequality $|f(z)| \leq |z|$ holds for all $z \in \mathbb{\Delta}$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ holds for any $z \neq 0$, then $f(z) = \beta z$ for some $\beta \in \partial\mathbb{\Delta}$.

Proof. We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for g on the domain $\mathbb{\Delta}(\rho)$. We get

$$\sup_{z \in \mathbb{\Delta}(\rho)} |g(z)| \leq \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as $\rho \to 1$, it follows that

$$\sup_{z \in \mathbb{\Delta}} |g(z)| \leq 1.$$

It immediately follows that $|f'(0)| = |g(0)| \leq 1$. Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z| \, .$$

Suppose we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. As then $|g(z_0)| = 1$, it follows that g is constant, therefore $f(z) = \beta z$ for some $\beta \in \partial \mathbb{\Delta}$. If we have $|f'(0)| = 0$, the same argument works for $z_0 = 0$. $\square$

1.2 The $\overline{\partial}$ equation

Lemma 1.2.1. Let $g \in \mathcal{C}^\infty(\mathbb{C})$ be a function with compact support. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Let

$$f(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{g(w)}{w-z} \, dw \wedge d\overline{w}.$$

As

$$dw \wedge d\overline{w} = -2ri \, dr \wedge d\varphi$$

holds for polar coordinates centered at z, we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{rg\left(z + re^{i\varphi}\right)}{re^{i\varphi}} \, dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some R such that $g|_{\mathbb{C} \setminus \Delta(z,R)} = 0$. We get

$$f(z) = -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi,$$

which obviously converges. The function f is therefore well defined. As we are integrating a smooth function on a compact set, the function f is smooth as well.

For $u = re^{i\varphi}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}}(z) &= -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} \frac{\partial}{\partial \overline{z}} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial}{\partial \overline{z}} g(u+z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial g}{\partial \overline{u}}(u+z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w-z} \, dw \wedge d\overline{w}.
\end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint\limits_{\partial \Delta(z,R)} \frac{g(w)}{w-z} \, dw + \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w-z} \, dw \wedge d\overline{w}.$$

by the choice of R, we get

$$\frac{\partial f}{\partial \overline{z}}(z) = g(z).$$

□

Lemma 1.2.2. Given bounded domain $U \subset V \subset \mathbb{R}^n$ such that $\partial U \cap \partial V = \emptyset$, there exists a smooth function $\chi \colon \mathbb{R}^n \to [0,1]$ such that $\chi|_U = 1$ and $\operatorname{supp} \chi \subseteq V$.

Proof. There is a unit partition on the sets V and $\mathbb{R}^n \setminus \overline{U}$. □

Lemma 1.2.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $h_j \colon \Omega \to \mathbb{C}$ be holomorphic functions. If the sequence $(h_j)_{j\in\mathbb{N}}$ converges uniformly on compact sets, the limit is also holomorphic on Ω.

Theorem 1.2.4 (Dolbeault lemma)**.** Let $g \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ for some $R \in (0, \infty]$. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Define discs X_j as follows:

i) If $R = \infty$, set $X_j = \mathbb{\Delta}(j)$.

ii) If $R < \infty$, set $X_j = \mathbb{\Delta}\left(R - \frac{1}{j}\right)$ (for large enough j).

Applying the above lemma, define functions χ_j with $\chi_j|_{X_j} = 1$ and $\operatorname{supp} \chi_j \subseteq X_{j+1}$ and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \mathbb{\Delta}(R), \\ 0 & z \notin \mathbb{\Delta}(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function $f_j \in \mathcal{C}^\infty(\mathbb{C})$ with

$$\frac{\partial f_j}{\partial \overline{z}} = g_j.$$

We inductively construct a new sequence $\widetilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$ such that

$$\frac{\partial \widetilde{f}_j}{\partial \overline{z}} = g$$

on X_j and

$$\left\| \widetilde{f}_j - \widetilde{f}_{j-1} \right\|_{X_{j-2}} \leq 2^{-j}.$$

Set $\widetilde{f}_1 = f_1$. Observe the function $F = f_{j+1} - \widetilde{f}_j$ on X_j. By construction, we have $\frac{\partial F}{\partial \overline{z}} = 0$ on X_j. It follows that F can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on X_j. As power series converge uniformly on compact sets, there exists some polynomial $p \in \mathbb{C}[z]$ such that

$$\|F - p\|_{X_{j-1}} \leq 2^{-j}.$$

Now just set $\widetilde{f}_{j+1} = f_{j+1} - p$.

Let $z \in \mathbb{\Delta}(R)$ be arbitrary. By construction, it is contained in some X_{j_0}, therefore, $\widetilde{f}_j$ is defined for $j \geq j_0$. As $\left(\widetilde{f}_j(z)\right)_{j \geq j_0}$ is a Cauchy sequence, we can define

$$f(z) = \lim_{j\to\infty} \widetilde{f}_j(z).$$

But as

$$f - \widetilde{f}_j = \sum_{k=j}^{\infty} \left(\widetilde{f}_{j+1} - \widetilde{f}_j \right)$$

is a sum of holomorphic functions that converges uniformly, the function $f - \widetilde{f}_j$ is a holomorphic function. Therefore, f is smooth and satisfies $\frac{\partial f}{\partial \overline{z}} = g$. $\square$

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