

Complex analysis

Luka Horjak (luka1.horjak@gmail.com)

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Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f: \Omega \rightarrow \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations). Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a , f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Definition 1.1.2. Wirtinger derivatives are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f: \Omega \rightarrow \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a . The function f is *holomorphic* if it is holomorphic in every point of Ω . We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with \mathcal{C}^1 -smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\Delta(z, \varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\Delta(z, \varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w - z} dw$ on Ω_ε . As $d\omega = \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \oint_{\partial\Omega} \frac{f(w)}{w - z} dw - \oint_{\partial\Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw.$$

In the limit, we have

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial \Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{f(z + \varepsilon e^{it})}{\varepsilon e^{it}} \cdot \varepsilon i e^{it} dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega \setminus \{z\}} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f(w)}{w - z} dw - f(z) = -\frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}. \quad \square$$

Theorem 1.1.5 (Power series expansion). Let $\Omega \subseteq \mathbb{C}$ be an open subset, $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\Delta(a, r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \oint_{\partial \Delta(a, \rho)} \frac{f(w)}{(w - a)^{k+1}} dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z - a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity). Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a holomorphic function. Let $A \subseteq \Omega$ be a subset such that $f(z) = 0$ for all $z \in A$. If A has an accumulation point in Ω , then $f(z) = 0$ for all $z \in \Omega$.

Proof. Let $a \in \Omega$ be an accumulation point of A . By continuity, we have $f(a) = 0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k (z - a)^k,$$

where we assume $c_{k_0} \neq 0$. But now $g(z) = \frac{f(z)}{(z - a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a) = 0$, which is a contradiction. It follows that $c_k = 0$ for all $k \in \mathbb{N}_0$. It follows that the set $\text{Int} \{z \in \Omega \mid f(z) = 0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to Ω . \square

Lemma 1.1.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\Delta(a, r)} \subseteq \Omega$. If

$$|f(a)| < \min_{\partial\Delta(a, r)} |f|,$$

then f has a zero in $\Delta(a, r)$.

Proof. Assume otherwise. From the inequality it follows that f has no zeroes on the boundary either. By continuity, f has no zero on an open set V with $\Delta(a, r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint_{\partial\Delta(a, r)} \frac{g(z)}{z - a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(a + r \cdot e^{it})}{r e^{it}} \cdot r i e^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} g(a + r e^{it}) dt.$$

We can therefore get a bound on $|g(a)|$ as

$$|g(a)| \leq \max_{\partial\Delta(a, r)} |g|,$$

but as the condition on f can be rewritten as

$$|g(a)| > \max_{\partial\Delta(a, r)} |g|,$$

we have reached a contradiction. □

Theorem 1.1.8 (Open mapping). Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If f is not constant, it is an open map.

Proof. Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\Delta(z_0, \rho) \subseteq U$ and z_0 is the only pre-image of w_0 in $\Delta(z_0, 2\rho)$.¹

Since $\partial\Delta(z_0, \rho)$ is a compact set and

$$|f(z) - w_0| > 0$$

for all $z \in \partial\Delta(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a $w \in \Delta(w_0, \varepsilon)$. As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$

on the boundary and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

by the above lemma, $f(z) - w$ has a root on $\Delta(z, \rho)$. □

Theorem 1.1.9 (Maximum principle). Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function f is constant.

¹ If such a disk does not exist, f is constant by the identity theorem.

Proof. Suppose that f is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As f is an open map, it also attains the value $(1 + \varepsilon) \cdot f(z)$, which is a contradiction as the modulus then equals $(1 + \varepsilon) \cdot |f(z)| > |f(z)|$. \square

Theorem 1.1.10 (Maximum principle). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

Proof. As $\overline{\Omega}$ is compact, f attains a global maximum on this set. If the maximum is attained in the interior, f is constant, therefore it is also attained on the boundary. \square

Definition 1.1.11. A function $f: \Omega \setminus \{a\} \rightarrow \mathbb{C}$ is *locally bounded* near a if there exists an open neighbourhood $U \subseteq \Omega$ of a such that $f|_{U \setminus \{a\}}$ is bounded.

Theorem 1.1.12 (Riemann removable singularity theorem). Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$. If f is locally bounded near a , then there exists a unique function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus \{a\}} = f$.

Proof. Define the function $F: \Omega \rightarrow \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{w-a} dw & z = a. \end{cases}$$

It remains to check that F is complex differentiable at a . Indeed, for $z \in \Delta(a, \rho)$ we have

$$\begin{aligned} \lim_{z \rightarrow a} \frac{F(z) - F(a)}{z - a} &= \lim_{z \rightarrow a} \frac{1}{z - a} \oint_{\partial\Delta(a,\rho)} \left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a} \right) dw \\ &= \lim_{z \rightarrow a} \frac{1}{2\pi i} \cdot \frac{1}{z - a} \cdot \oint_{\partial\Delta(a,\rho)} f(w) \cdot \frac{z - a}{(w - z)(w - a)} dw \\ &= \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{(w - a)^2} dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem. \square

Theorem 1.1.13 (Schwarz lemma). Let $f: \Delta \rightarrow \Delta$ be a holomorphic function with $f(0) = 0$. Then, $|f'(0)| \leq 1$ and the inequality $|f(z)| \leq |z|$ holds for all $z \in \Delta$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ holds for any $z \neq 0$, then $f(z) = \beta z$ for some $\beta \in \partial\Delta$.

Proof. We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for g on the domain $\Delta(\rho)$. We get

$$\sup_{z \in \Delta(\rho)} |g(z)| \leq \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as $\rho \rightarrow 1$, it follows that

$$\sup_{z \in \mathbb{A}} |g(z)| \leq 1.$$

It immediately follows that $|f'(0)| = |g(0)| \leq 1$. Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z|.$$

Suppose we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. As then $|g(z_0)| = 1$, it follows that g is constant, therefore $f(z) = \beta z$ for some $\beta \in \partial \mathbb{A}$. If we have $|f'(0)| = 0$, the same argument works for $z_0 = 0$. \square

1.2 The $\bar{\partial}$ equation

Lemma 1.2.1. Let $g \in \mathcal{C}^\infty(\mathbb{C})$ be a function with compact support. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \bar{z}} = g$.

Proof. Let

$$f(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(w)}{w - z} dw \wedge d\bar{w}.$$

As

$$dw \wedge d\bar{w} = -2ri dr \wedge d\varphi$$

holds for polar coordinates centered at z , we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{rg(z + re^{i\varphi})}{re^{i\varphi}} dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some R such that $g|_{\mathbb{C} \setminus \Delta(z, R)} = 0$. We get

$$f(z) = -\frac{1}{\pi} \iint_{\Delta(z, R)} g(z + re^{i\varphi}) e^{-i\varphi} dr \wedge d\varphi,$$

which obviously converges. The function f is therefore well defined. As we are integrating a smooth function on a compact set, the function f is smooth as well.

For $u = re^{i\varphi}$, we have

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}}(z) &= -\frac{1}{\pi} \iint_{\Delta(z, R)} \frac{\partial}{\partial \bar{z}} g(z + re^{i\varphi}) e^{-i\varphi} dr \wedge d\varphi \\ &= \frac{1}{2\pi i} \iint_{\Delta(0, R)} \frac{\partial}{\partial \bar{z}} g(u + z) \frac{1}{u} du \wedge d\bar{u} \\ &= \frac{1}{2\pi i} \iint_{\Delta(0, R)} \frac{\partial g}{\partial \bar{u}}(u + z) \frac{1}{u} du \wedge d\bar{u} \\ &= \frac{1}{2\pi i} \iint_{\Delta(z, R)} \frac{\partial g}{\partial \bar{w}}(w) \frac{1}{w - z} dw \wedge d\bar{w}. \end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint_{\partial \Delta(z, R)} \frac{g(w)}{w - z} dw + \frac{1}{2\pi i} \iint_{\Delta(z, R)} \frac{\partial g}{\partial \bar{w}}(w) \frac{1}{w - z} dw \wedge d\bar{w}.$$

by the choice of R , we get

$$\frac{\partial f}{\partial \bar{z}}(z) = g(z). \quad \square$$

Lemma 1.2.2. Given bounded domain $U \subset V \subset \mathbb{R}^n$ such that $\partial U \cap \partial V = \emptyset$, there exists a smooth function $\chi: \mathbb{R}^n \rightarrow [0, 1]$ such that $\chi|_U = 1$ and $\text{supp } \chi \subseteq V$.

Proof. There is a partition of unity on the sets V and $\mathbb{R}^n \setminus \bar{U}$. \square

Lemma 1.2.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $h_j: \Omega \rightarrow \mathbb{C}$ be holomorphic functions. If the sequence $(h_j)_{j \in \mathbb{N}}$ converges uniformly on compact sets, the limit is also holomorphic on Ω .

Proof. Apply Morera's theorem.² □

Theorem 1.2.4 (Dolbeault lemma). Let $g \in \mathcal{C}^\infty(\Delta(R))$ for some $R \in (0, \infty]$. Then there exists a function $f \in \mathcal{C}^\infty(\Delta(R))$ such that $\frac{\partial f}{\partial \bar{z}} = g$.

Proof. Define disks X_j as follows:

- i) If $R = \infty$, set $X_j = \Delta(j)$.
- ii) If $R < \infty$, set $X_j = \Delta\left(R - \frac{1}{j}\right)$ (for large enough j).

Applying the above lemma, define functions χ_j with $\chi_j|_{X_j} = 1$ and $\text{supp } \chi_j \subseteq X_{j+1}$ and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \Delta(R), \\ 0 & z \notin \Delta(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function $f_j \in \mathcal{C}^\infty(\mathbb{C})$ with

$$\frac{\partial f_j}{\partial \bar{z}} = g_j.$$

We inductively construct a new sequence $\tilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$ such that

$$\frac{\partial \tilde{f}_j}{\partial \bar{z}} = g$$

on X_j and

$$\|\tilde{f}_j - \tilde{f}_{j-1}\|_{X_{j-2}} \leq 2^{-j}.$$

Set $\tilde{f}_1 = f_1$. Observe the function $F = f_{j+1} - \tilde{f}_j$ on X_j . By construction, we have $\frac{\partial F}{\partial \bar{z}} = 0$ on X_j . It follows that F can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on X_j . As power series converge uniformly on compact sets, there exists some polynomial $p \in \mathbb{C}[z]$ such that

$$\|F - p\|_{X_{j-1}} \leq 2^{-j}.$$

Now just set $\tilde{f}_{j+1} = f_{j+1} - p$.

Let $z \in \Delta(R)$ be arbitrary. By construction, it is contained in some X_{j_0} , therefore, \tilde{f}_j is defined for $j \geq j_0$. As $(\tilde{f}_j(z))_{j \geq j_0}$ is a Cauchy sequence, we can define

$$f(z) = \lim_{j \rightarrow \infty} \tilde{f}_j(z).$$

² Analysis 2b, theorem 3.4.6.

But as

$$f - \tilde{f}_j = \sum_{k=j}^{\infty} (\tilde{f}_{k+1} - \tilde{f}_k)$$

is a sum of holomorphic functions that converges uniformly, the function $f - \tilde{f}_j$ is a holomorphic function. Therefore, f is smooth and satisfies $\frac{\partial f}{\partial \bar{z}} = g$. \square

1.3 Meromorphic functions

Definition 1.3.1. Let $\Omega \subset \mathbb{C}$ be an open subset. We call a function f *meromorphic* of Ω if there exists $A \subset \Omega$ such that $f \in \mathcal{O}(\Omega \setminus A)$, A has no accumulation points in Ω and for all $a \in A$ there exists some $k \in \mathbb{N}$ such that

$$\lim_{z \rightarrow a} f(z) \cdot (z - a)^k \neq 0$$

exists. We call A the set of *poles* of the function f . We denote the set of meromorphic functions on Ω with $\mathcal{M}(\Omega)$.

Theorem 1.3.2. Let $0 \leq r < R \leq \infty$. Suppose that $f \in \mathcal{O}(D_{R,r}(a))$ is a holomorphic function, where

$$D_{R,r}(a) = \{z \in \mathbb{C} \mid r < |z - a| < R\}.$$

Then there exists a uniquely determined *Laurent series*

$$\sum_{k \in \mathbb{Z}} c_k (z - a)^k$$

that converges to f uniformly and absolutely on compact subsets of $D_{R,r}(a)$. We have

$$c_k = \frac{1}{2\pi i} \oint_{\partial \Delta(a, \rho)} \frac{f(w)}{(w - a)^k} dw$$

for $r < \rho < R$.

Definition 1.3.3. Let

$$\sum_{k \in \mathbb{Z}} c_k (z - a)^k$$

be a Laurent series. The series

$$\sum_{k=-\infty}^{-1} c_k (z - a)^k$$

is called the *principle part*.

Lemma 1.3.4. Let $f \in \mathcal{O}(\Omega \setminus \{a\})$ be a holomorphic function. Then f is meromorphic on Ω if and only if f has a finite principle part in a .

Proof. Suppose that f is meromorphic on Ω . If a is a removable singularity, f is holomorphic in a , therefore the principle part is trivial. Otherwise, set $m \in \mathbb{N}$ such that

$$\lim_{z \rightarrow a} (z - a)^m f(z) \neq 0$$

exists and set $g(z) = (z - a)^m f(z)$. As g is bounded near a , we can extend it to Ω by the Riemann removable singularity theorem. The power series of g corresponds to a finite Laurent series of f .

The converse is obvious. □

Theorem 1.3.5. If $f \in \mathcal{M}(\mathbb{C})$ is a meromorphic function, there exist entire functions g and h such that $f = \frac{g}{h}$.

Definition 1.3.6. Let $\Omega \subseteq \mathbb{C}$ be an open set. An *additive Cousin problem* on Ω is an open cover $\{U_j\}_{j \in J}$ of Ω and functions $f_j \in \mathcal{M}(U_j)$ such that $f_j - f_k|_{U_j \cap U_k}$ is holomorphic for all $j, k \in J$. A function $f \in \mathcal{M}(\Omega)$ is a solution to the additive Cousin problem if $f|_{U_j} - f_j$ is holomorphic for all $j \in J$.

Definition 1.3.7. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A *generalized additive Cousin problem* is an open cover $\{U_j\}_{j \in J}$ of Ω and functions $f_{j,k} \in \mathcal{O}(U_j \cap U_k)$ for each $(j, k) \in J^2$, such that

- i) $f_{j,k} = -f_{k,j}$ on $U_j \cap U_k$ for all $(j, k) \in J^2$ and
- ii) $f_{j,k} + f_{k,\ell} + f_{\ell,j} = 0$ on $U_j \cap U_k \cap U_\ell$ for all $(j, k, \ell) \in J^3$.

A solution to the generalized additive Cousin problem is given by functions $f_j \in \mathcal{O}(U_j)$ for each $j \in J$ such that $f_{j,k} = f_j - f_k$ for each $(j, k) \in J^2$.

Lemma 1.3.8 (Partition of unity). Let $\Omega \subseteq \mathbb{C}$ be an open set and $\{U_j\}_{j \in J}$ be an open cover of Ω . Then there exists a partition of unity subordinate to $\{U_j\}_{j \in J}$.

Lemma 1.3.9. Given a generalized additive Cousin problem on $\Omega \subseteq \mathbb{C}$, there exist functions $g_j \in \mathcal{C}^\infty(U_j)$ such that $f_{j,k} = g_j - g_k$ for all $(j, k) \in J^2$.

Proof. Let $\{(V_a, \chi_a)\}_{a \in A}$ be a partition of unity, subordinate to $\{U_j\}_{j \in J}$. For all $a \in A$ choose a $j(a) \in J$ such that $V_a \subseteq U_{j(a)}$. For all $k \in J$, define

$$g_k = - \sum_{a \in A} \chi_a \cdot f_{j(a),k}.$$

This is of course a smooth function on U_k . Now note that

$$g_k - g_\ell = \sum_{a \in A} \chi_a \cdot (-f_{j(a),k} + f_{j(a),\ell}) = \sum_{a \in A} \chi_a \cdot f_{k,\ell} = f_{k,\ell}. \quad \square$$

Proposition 1.3.10. The generalized additive Cousin problem is solvable for $\Omega = \Delta(r)$ and $\Omega = \mathbb{C}$.

Proof. Let $f_{j,k} = g_j - g_k$ for $g_j \in \mathcal{C}^\infty(U_j)$. Note that

$$\frac{\partial g_j}{\partial \bar{z}} = \frac{\partial g_k}{\partial \bar{z}},$$

therefore,

$$h|_{U_j} = \frac{\partial g_j}{\partial \bar{z}}$$

induces a smooth function $h: \Omega \rightarrow \mathbb{C}$. By the Dolbeault lemma, there exists a function $g \in \mathcal{C}^\infty(\Omega)$ such that $\frac{\partial g}{\partial \bar{z}} = h$. It is clear that $f_j = g_j - g$ solves the generalized additive Cousin problem. \square

Proposition 1.3.11. The additive Cousin problem is solvable for $\Omega = \Delta(r)$ and $\Omega = \mathbb{C}$.

Proof. An additive Cousin problem induces a generalized additive Cousin problem for functions $f_{j,k} = f_j - f_k$. Let g_j be a solution to the generalized problem. As $f_j - f_k = f_{j,k} = g_j - g_k$ on $U_j \cap U_k$, we can define a function $f \in \mathcal{M}(\Omega)$ with $f|_{U_j} = f_j - g_j$. This function is of course well defined. As $f|_{U_j} - f_j = g_j \in \mathcal{O}(U_j)$, this function indeed solves the additive Cousin problem. \square

October 18, 2023

Theorem 1.3.12 (Mittag-Leffler). Let $(a_k)_{k \in \mathbb{N}}$ be a sequence without repetition and accumulation points. Let

$$f_k(z) = \sum_{\ell=-m_k}^{-1} c_{k,\ell} (z - a_k)^\ell$$

be finite principal parts. Then there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ with poles in $(a_k)_{k \in \mathbb{N}}$ such that f has principle part f_k in a_k for each $k \in \mathbb{N}$.

Proof. For each a_k choose a disk U_k containing no other a_k . Also set $U_0 = \mathbb{C} \setminus \{a_k \mid k \in \mathbb{N}\}$ and $f_0 = 0$. As $\{U_k \mid k \in \mathbb{N}_0\}$ is an open cover of \mathbb{C} , there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ that solves the corresponding additive Cousin problem. It is easy to see that the principle parts of f at a_k are precisely f_k . \square

1.4 Sequences of holomorphic functions

Definition 1.4.1. A family of functions \mathcal{F} from Ω to \mathbb{C} is *locally bounded*, if for all $p \in \Omega$ there exist a $\rho > 0$ and $M > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{z \in \Delta(p, \rho)} |f(z)| < M.$$

Lemma 1.4.2. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ a locally bounded family of functions. Then for all $p \in \Omega$ there exists a $\rho > 0$ such that \mathcal{F} is equi-continuous on $\Omega \cap \Delta(p, \rho)$.

Proof. Fix $p \in \Omega$ and choose $r > 0$ such that $D = \overline{\Delta(p, 2r)} \subseteq \Omega$. For any $z, w \in D$ and $f \in \mathcal{F}$ we have

$$f(z) - f(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - w} d\xi = \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi.$$

Note that the family \mathcal{F} is bounded on every compact. Therefore, we can write

$$\sup_{f \in \mathcal{F}} \sup_{z \in \partial D} |f(z)| < M.$$

Now, for $z, w \in \Delta(p, r)$ we have

$$|f(z) - f(w)| = \left| \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi \right| \leq |z - w| \cdot \frac{2M}{r}. \quad \square$$

Theorem 1.4.3 (Arzelà-Ascoli). Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be an infinite family such that the following conditions hold:

- i) \mathcal{F} is point-wise bounded.
- ii) \mathcal{F} is locally equi-continuous.

Then there \mathcal{F} contains a sequence that converges uniformly on compacts of Ω .

Proof. Choose a dense countable subset $A \subseteq \Omega$ and enumerate it as a sequence $(a_k)_{k \in \mathbb{N}}$. Pick any sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ with pairwise distinct terms. As $|f_n(a_1)| < M$ for all n , we can choose a subsequence $(f_{1,n})_{n \in \mathbb{N}}$ such that $f_{1,n}(a_1)$ converges by Bolzano-Weierstraß.

Similarly, for every $k \in \mathbb{N}$ there exists a subsequence $(f_{k,n})_n$ of $(f_{k-1,n})_n$ such that $(f_{k,n}(a_k))_n$ converges. Now define $F_n = f_{n,n}$. Observe that (F_n) converges at every point in A .

Fix a $p \in \Omega$. By local equi-continuity, there exists a $\rho > 0$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta < \rho$ and $|F_n(z) - F_n(w)| < \frac{\varepsilon}{3}$ for all $z, w \in \Delta(p, \rho)$ such that $|z - w| < \delta$. Choose an element $a \in A \cap \Delta(p, \delta)$.³ Then, we have

$$|F_n(z) - F_m(z)| \leq |F_n(z) - F_n(a)| + |F_n(a) - F_m(a)| + |F_m(a) - F_m(z)| < 3 \cdot \frac{\varepsilon}{3}.$$

It follows that (F_n) is locally uniformly convergent, therefore it converges uniformly on compact sets. \square

³ By compactness of $\overline{\Delta(p, \rho)}$ we can choose a from a finite set.

Theorem 1.4.4 (Montel). Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f_n: \Omega \rightarrow \mathbb{C}$ be a locally bounded sequence of holomorphic functions. Then $(f_n)_n$ contains a subsequence that converges uniformly on compacts.

Proof. As the sequence is locally bounded, it is locally equi-continuous by lemma 1.4.2. By Arzelà-Ascoli, there exists a convergent subsequence. \square

Definition 1.4.5. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family of functions $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is *normal* if every sequence in \mathcal{F} contains a subsequence that converges uniformly on compacts.

Theorem 1.4.6 (Montel). Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is normal if and only if it is locally bounded.

Proof. The proof is obvious and need not be mentioned. \square

Theorem 1.4.7 (Vitali). Let $\Omega \subseteq \mathbb{C}$ be a domain and $(f_n)_n \subseteq \mathcal{O}(\Omega)$ a locally bounded sequence of holomorphic functions. The following statements are equivalent:

- i) The sequence $(f_n)_n$ converges uniformly on compact subsets of Ω .
- ii) For a point $p \in \Omega$ the sequence $(f_n^{(k)}(p))_n$ converges for all $k \in \mathbb{N}_0$.
- iii) The set

$$A = \left\{ z \in \Omega \mid \lim_{n \rightarrow \infty} f_n(z) \text{ converges} \right\}$$

has an accumulation point in Ω .

Proof. Suppose that the sequence converges uniformly on compact subsets. Given a $p \in \Omega$, choose a $\delta > 0$ such that $D = \overline{\Delta}(p, \delta) \subseteq \Omega$. Note that

$$|g^{(k)}(p)| \leq \frac{k!}{\delta^k} \cdot \|g\|_D$$

holds for all holomorphic functions g . As $\|f - f_n\|$ converges to 0, the derivatives of f_n converge.

Suppose that the sequences of derivatives converge at a point $p \in \Omega$ and choose a $\delta > 0$ such that $D = \overline{\Delta}(p, \delta) \subseteq \Omega$. As the sequence is locally bounded, there exists a constant M such that $\|f_n\|_D \leq M$ holds for all $n \in \mathbb{N}$. We can now develop power series

$$f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z-p)^k.$$

They converge uniformly on compact subsets of $\Delta(p, \delta)$. Note that

$$a_{k,n} = \frac{f_n^{(k)}(p)}{k!}.$$

As derivatives converge, we can define the limit

$$a_k = \lim_{n \rightarrow \infty} a_{k,n}.$$

Now define the formal power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - p)^k.$$

The Cauchy bounds give us the inequality

$$|a_{k,n}| = \frac{|f^{(k)}(p)|}{k!} \leq \frac{M}{\delta^k},$$

therefore

$$\limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} \leq \limsup_{k \rightarrow \infty} \frac{\sqrt[k]{M}}{\delta} = \frac{1}{\delta}.$$

We conclude that the radius of convergence is at least δ . Consider some $\rho \in (0, \delta)$ and $z \in \mathbb{A}(p, \rho)$. We have

$$\begin{aligned} |f_n(z) - f(z)| &\leq \left| \sum_{k=0}^m (a_{k,n} - a_k) \cdot (p - z)^k \right| + \left| \sum_{k=m+1}^{\infty} (a_{k,n} - a_k) \cdot (p - z)^k \right| \\ &\leq \sum_{k=0}^m |a_{k,n} - a_k| \rho^k + \sum_{k=m+1}^{\infty} 2M \cdot \frac{\rho^k}{\delta^k} \\ &= \sum_{k=0}^m |a_{k,n} - a_k| \rho^k + 2M \cdot \left(\frac{\rho}{\delta}\right)^{m+1} \cdot \frac{\delta}{\delta - \rho} \\ &= 2 \cdot \frac{\varepsilon}{2} \end{aligned}$$

for large enough m and n . It follows that p is an accumulation point of A .

Suppose now that A has an accumulation point in Ω . By Montel's theorem there exists a subsequence $(f_{n_m})_m$ that converges uniformly on compact subsets of Ω to a limit function f . Note that all such subsequences have the same limit by the identity principle.

Assume that the sequence $(f_n)_n$ does not converge uniformly on a compact subset $K \subseteq \Omega$. We can therefore construct another subsequence $(g_n)_n$ of $(f_n)_n$ such that

$$\|g_n - f\|_K > \varepsilon$$

for all $n \in \mathbb{N}$. But note that $(g_n)_n$ also has a convergent subsequence by Montel's theorem, which is of course a contradiction, as it cannot converge to f . \square

2 Theorems about holomorphic functions

2.1 Riemann mapping theorem

Definition 2.1.1. A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if every closed path in Ω is homotopic to a constant path in Ω .

Lemma 2.1.2. Let $\Omega \subset \mathbb{C}$ be a domain and $a \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Then there exists a holomorphic injection $f: \Omega \rightarrow \mathbb{A}$ such that $f(a) = 0$.

Proof. Fix a point $p \in \mathbb{C} \setminus \Omega$. By our assumption, there exists a function $v \in \mathcal{O}^*(\Omega)$ such that $v(z)^2 = z - p$. Note that v is injective. Similarly, we have $v(\Omega) \cap -v(\Omega) = \emptyset$. Now choose a point $b \in -v(\Omega)$. As v is not constant, it is an open map. Therefore, there exists some $r > 0$ such that $\mathbb{A}(b, r) \cap v(\Omega) = \emptyset$. The Möbius transformation

$$h(w) = r \cdot \left(\frac{1}{w - b} - \frac{1}{v(a) - b} \right)$$

thus maps $v(\Omega)$ into \mathbb{A} . The map f is therefore given as $f = h \circ v$. \square

Definition 2.1.3. An *expansion* is a map $\kappa: \Omega \rightarrow \mathbb{A}$ where $0 \in \Omega \subset \mathbb{A}$ such that $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ holds for all $z \neq 0$.

Lemma 2.1.4. Let $\Omega \subset \mathbb{A}$ be a domain with $0 \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Choose $c \in \mathbb{A}$ such that $c^2 \notin \Omega$. For all $a \in \mathbb{A}$, let

$$g_a = \frac{z - a}{\bar{a}z - 1}$$

and choose $v \in \mathcal{O}(\Omega)$ such that $v(z)^2 = g_{c^2}(z)$ and $v(0) = c$. Then the map $\kappa = g_c \circ v$ is an expansion and

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = \text{id}_\Omega.$$

Proof. Note that v is indeed well-defined. Also note that

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = g_{c^2} \circ (z \mapsto z^2) \circ v = g_{c^2} \circ g_{c^2} = \text{id}.$$

We of course have $\kappa(0) = 0$. Denote $\psi_c = g_{c^2} \circ (z \mapsto z^2) \circ g_c$. It remains to check that $|\kappa(z)| > |z|$, which is equivalent to $|\psi_c(z)| < |z|$ for $z \neq 0$ as $\psi_c \circ \kappa = \text{id}$. Note that $\psi_c: \mathbb{A} \rightarrow \mathbb{A}$ is holomorphic. As it is not a rotation (it is not injective), the conclusion follows from the Schwarz lemma. \square

Lemma 2.1.5 (Hurwitz). Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f_n: \Omega \rightarrow \mathbb{C}$ be holomorphic functions. Suppose that the sequence $(f_n)_n$ converges uniformly on compacts of Ω to a non-constant function $f: \Omega \rightarrow \mathbb{C}$. Then for all points $p \in \Omega$ there exists a sequence $(p_n)_n \subseteq \Omega$ with limit p such that $f_n(p_n) = f(p)$ for all $n > N$.

Proof. Let $w = f(p)$. There exists a disk $\mathbb{A}(p, \delta)$ such that $f(z) \neq w$ for all points $z \in \mathbb{A}(p, \delta) \setminus \{p\}$. Note that we have

$$\min_{z \in \partial \mathbb{A}(p, \delta)} |f(z) - w| > |f(p) - w| = 0.$$

As $(f_n)_n$ converges uniformly on $\overline{\Delta(p, \delta)}$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\min_{z \in \partial \Delta(p, \delta)} |f_n(z) - w| > |f_n(p) - w|.$$

By lemma 1.1.7, $f_n(z) - w$ has a root $p_n \in \Delta(p, \delta)$. For any convergent subsequence $(p_{n_k})_k$ with limit q we have

$$f(p) = \lim_{k \rightarrow \infty} f_{n_k}(p_{n_k}) = f(q),$$

therefore $p = q$. □

Corollary 2.1.5.1. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n: \Omega \rightarrow \mathbb{C}$ be holomorphic functions such that $(f_n)_n$ converges uniformly on compacts of Ω to $f: \Omega \rightarrow \mathbb{C}$. If all the f_n are nowhere vanishing and $f \neq 0$, then f is nowhere vanishing.

Proof. The proof is obvious and need not be mentioned. □

Theorem 2.1.6 (Hurwitz). Let $\Omega, \Omega' \subseteq \mathbb{C}$ be domains and $f_n: \Omega \rightarrow \Omega'$ be holomorphic functions that converge uniformly on compacts of Ω to $f: \Omega \rightarrow \Omega'$. Assume that f is not constant.

- i) If $f_n: \Omega \rightarrow \Omega'$ is injective, f is also injective.
- ii) We have $f(\Omega) \subseteq \Omega'$.

Proof.

- i) Let $p \in \Omega$ and observe the functions $g_n(z) = f_n(z) - f_n(p)$. This is a sequence of nowhere vanishing functions. As f is not constant, $f(z) - f(p)$ is nowhere vanishing as well. It follows that f is injective.
- ii) Suppose otherwise and apply the Hurwitz lemma for a point p with $f(p) \notin \Omega'$. □

Theorem 2.1.7 (Riemann mapping). For a proper domain $\Omega \subset \mathbb{C}$ the following are equivalent:

- i) Ω is simply connected.
- ii) Ω admits a logarithm for any $f \in \mathcal{O}^*(\Omega)$.
- iii) Ω admits a square root for any $f \in \mathcal{O}^*(\Omega)$.
- iv) Ω is biholomorphic to Δ .

Proof. Note that if Ω is biholomorphic to Δ , it is of course simply connected. Suppose that Ω is simply connected. Then

$$F(z) = a + \int_{z_0}^z \frac{f'(z)}{f(z)} dz$$

defines a logarithm for any $f \in \mathcal{O}^*(\Omega)$. Given a logarithm of a function, we can of course construct a square root with $\sqrt{f} = e^{\frac{1}{2} \ln f}$. It remains to check that all domains admitting square roots are biholomorphic to Δ .

By lemma 2.1.2 we can assume that $\Omega \subseteq \mathbb{A}$ and $0 \in \Omega$. Now define the family of functions

$$\mathcal{F} = \{f: \Omega \rightarrow \mathbb{A} \mid f \in \mathcal{O}(\Omega) \wedge f(0) = 0 \wedge f \text{ is injective}\}.$$

If \mathcal{F} has no biholomorphic map, it is infinite. Note that \mathcal{F} is bounded, so it is normal by Montel.

Choose a point $p \in \Omega$ with $p \neq 0$. We claim that if $h \in \mathcal{F}$ and

$$|h(p)| = \sup_{f \in \mathcal{F}} |f(p)|,$$

we have $h(\Omega) = \mathbb{A}$. Indeed, if that were not the case, we'd reach a contradiction with the expansion κ of Ω as

$$|\kappa(h(p))| > |h(p)|$$

and $\kappa \circ h \in \mathcal{F}$.

Let

$$M = \sup_{f \in \mathcal{F}} |f(p)|$$

and find a sequence $(f_n)_n \subseteq \mathcal{F}$ with

$$\lim_{n \rightarrow \infty} |f_n(p)| = M.$$

As \mathcal{F} is a normal family, there exists a convergent subsequence. The limit is not constant as $f(p) \neq 0$. By Hurwitz, f is injective and $f(\Omega) \subseteq \mathbb{A}$. By the above claim, we have $f(\Omega) = \mathbb{A}$. \square

2.2 Bloch's theorem

Lemma 2.2.1. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $f: \overline{\Omega} \rightarrow \mathbb{C}$ a continuous map such that $f|_{\Omega}$ is an open map. Let $a \in \Omega$ be a point such that

$$s = \min_{z \in \partial\Omega} |f(z) - f(a)| > 0.$$

Then $f(\Omega)$ contains the disk $\Delta(f(a), s)$.

Proof. By compactness, there exists a $w_0 \in \partial f(\Omega)$ such that $d(\partial f(\Omega), f(a)) = |w_0 - f(a)|$. Let $(z_k)_k \subseteq \Omega$ be a sequence, convergent to z_0 , such that

$$\lim_{k \rightarrow \infty} f(z_k) = w_0.$$

Of course $f(z_0) = w_0$. Note that, as $f|_{\Omega}$ is open, we have $z_0 \in \partial\Omega$. But then

$$d(\partial f(\Omega), f(a)) = |f(z_0) - f(a)| \geq s. \quad \square$$

Lemma 2.2.2. Let f be a non-constant function, holomorphic in a neighbourhood of $\overline{\Delta(a, r)}$. Assume that

$$\sup_{z \in \overline{\Delta(a, r)}} |f'(z)| \leq 2 |f'(a)|.$$

Then $\Delta(f(a), R) \subseteq f(\Delta(a, r))$, where

$$R = (3 - 2\sqrt{2}) \cdot r \cdot |f'(a)|.$$

Proof. Without loss of generality assume that $a = f(a) = 0$. Define

$$A(z) = f(z) - f'(0)z = \int_0^1 (f'(tz) - f'(0)) z \, dt.$$

Note that

$$f'(v) - f'(0) = \frac{1}{2\pi i} \oint_{\partial\Delta(a, r)} f'(\xi) \cdot \left(\frac{1}{\xi - v} - \frac{1}{\xi} \right) d\xi,$$

therefore

$$|f'(v) - f'(0)| \leq \frac{1}{2\pi} \cdot |v| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r \cdot (r - |v|)} \cdot 2\pi r = |v| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r - |v|}.$$

It follows that

$$\begin{aligned} |A(z)| &\leq \int_0^1 |z| \cdot |f'(tz) - f'(0)| \, dt \\ &\leq |z| \cdot \int_0^1 |tz| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r - |tz|} \, dt \\ &\leq |z|^2 \cdot \|f'\|_{\Delta(a, r)} \cdot \int_0^1 t \cdot \frac{1}{r - |z|} \, dt \\ &= |z|^2 \cdot 2 \frac{|f'(0)|}{r - |z|}. \end{aligned}$$

Now, using the triangle inequality, we get

$$|f(z)| \geq |z| \cdot |f'(0)| - |A(z)|.$$

Let $|z| = \rho \in (0, r)$. We get

$$|f(z)| \geq \rho \cdot |f'(0)| - |A(z)| \geq \rho \cdot |f'(0)| - \frac{\rho^2}{r - \rho} \cdot |f'(0)| \geq |f'(0)| \cdot \left(\rho - \frac{\rho^2}{r - \rho} \right).$$

Note that there exists a ρ_0 such that

$$\rho_0 - \frac{\rho_0^2}{r - \rho_0} = r \cdot (3 - 2\sqrt{2}).$$

Therefore, we get

$$|f(z)| \geq |f'(0)| \cdot r \cdot (3 - 2\sqrt{2}).$$

Now just apply the previous lemma to the disk $\Delta(0, \rho_0)$. □

Theorem 2.2.3 (Bloch). Let f be a function, holomorphic in a neighbourhood of $\bar{\Delta}$, with $f'(0) = 1$. Then $f(\Delta)$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.

Proof. Define $h(z) = |f'(z)| (1 - |z|) \geq 0$. Note that $h \not\equiv 0$ as f is not constant. Therefore h attains a maximum in a point $p \in \bar{\Delta}$. In particular, as $h|_{\partial\Delta} = 0$, we have $p \in \Delta$. Observe $\Omega = \Delta(p, t)$ for $t = \frac{1}{2} \cdot (1 - |p|)$. For all $z \in \Omega$, we have $1 - |z| \geq t$ and

$$|f'(z)| \cdot (1 - |z|) \leq |f'(p)| \cdot (1 - |p|) = |f'(p)| \cdot 2t \leq |f'(p)| \cdot 2 \cdot (1 - |z|).$$

Now, applying lemma 2.2.2, we have $\Delta(f(p), R) \subseteq f(\Delta)$ with

$$R = (3 - 2\sqrt{2}) \cdot \frac{1}{2} \cdot (1 - |p|) \cdot |f'(p)| \geq \frac{3}{2} - \sqrt{2}$$

by choice of p . □

Remark 2.2.3.1. Let

$$\mathcal{F} = \left\{ f \text{ holomorphic on a neighbourhood of } \bar{\Delta} \mid f'(0) = 1 \right\}.$$

For $f \in \mathcal{F}$, denote by L_f the supremum of radii of disks contained in $f(\Delta)$, and by B_f the supremum of radii of disks contained in $f(\Delta)$ that is a biholomorphic image of some subdomain of Δ . We then define the *Landau's constant*

$$L = \inf_{f \in \mathcal{F}} L_f$$

and the *Bloch's constant*

$$B = \inf_{f \in \mathcal{F}} B_f.$$

The current known bounds for the constants are

$$0.5 < L < 0.544 \quad \text{and} \quad \frac{\sqrt{3}}{4} + 10^{-14} < B \leq \sqrt{\frac{\sqrt{3}-1}{2}} \cdot \frac{\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

Corollary 2.2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{O}(\Omega)$ a function and $p \in \Omega$. Let $r = d(p, \partial\Omega)$. Then $f(\Omega)$ contains a disk of radius

$$\left(\frac{3}{2} - \sqrt{2} \right) \cdot r \cdot |f'(p)|.$$

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Proof. The proof is obvious and need not be mentioned. \square

Remark 2.2.3.3. Liouville's theorem follows from this corollary.

Lemma 2.2.4. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $1, -1 \notin f(\Omega)$. Then there exists a function $F \in \mathcal{O}(\Omega)$ such that $f = \cos(F)$.

Proof. Note that, as Ω is simply connected, we can define

$$F(z) = \frac{1}{i} \cdot \ln \left(f(z) + \sqrt{f(z)^2 - 1} \right). \quad \square$$

Theorem 2.2.5. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let $f \in \mathcal{O}(\Omega)$. Suppose that $0, 1 \notin f(\Omega)$. Then the following statements are true:

i) There exists a function $g \in \mathcal{O}(\Omega)$ such that

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g))).$$

ii) If any $g \in \mathcal{O}(\Omega)$ satisfies the above equality, then $g(\Omega)$ contains no disk of radius 1.

Proof.

i) Apply the previous lemma twice.

ii) Define

$$A = \left\{ m \pm \frac{i}{\pi} \ln \left(n + \sqrt{n^2 - 1} \right) \mid m \in \mathbb{Z} \wedge n \in \mathbb{N} \right\}.$$

We claim that $g(\Omega) \cap A = \emptyset$. Indeed, for $a \in A$ we have

$$f(a) = \frac{1}{2} (1 + \cos(\pm \pi \cdot n)) \in \{0, 1\}.$$

Now note that

$$\begin{aligned} \ln \left(n + 1 + \sqrt{n^2 + 2n} \right) - \ln \left(n + \sqrt{n^2 - 1} \right) &= \ln \left(\frac{n + 1 + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 - 1}} \right) \\ &\leq \ln \left(\frac{2n + 2}{n} \right) \\ &\leq \ln(4) \\ &< \pi. \end{aligned}$$

It's straightforward to check that every disk of radius 1 intersects A . \square

Theorem 2.2.6 (Picard's little theorem). Every non-constant entire function omits at most one complex value.

Proof. Without loss of generality assume that f omits 0 and 1. Applying the above theorem, we can write

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g))).$$

Recall that $g(\mathbb{C})$ contains no disk of radius 1. If g is not constant, $g(\mathbb{C})$ contains arbitrarily large disks by corollary 2.2.3.2, which is a contradiction. \square

Corollary 2.2.6.1. Suppose that $f \in \mathcal{M}(\mathbb{C})$ is a non-constant function. Then f omits at most 2 values.

Proof. Suppose that f omits distinct values a, b and c . Then

$$g(z) = \frac{1}{f(z) - a}$$

is an entire function that omits values $\frac{1}{b-a}$ and $\frac{1}{c-a}$, therefore it is constant. \square

Theorem 2.2.7. Let $f \in \mathcal{O}(\mathbb{C})$ be an entire function. Then either $f \circ f$ has a fixed point of $f(z) = z + c$.

Proof. If $f \circ f$ has no fixed point, the same holds for f . We can therefore define an entire holomorphic function g with

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}.$$

Note that g omits both 0 and 1, therefore it is constant. But then

$$f(f(z)) - z = \lambda(f(z) - z)$$

for some $\lambda \notin \{0, 1\}$ by Picard's little theorem. Taking the derivative, we get

$$f'(f(z)) \cdot f'(z) - 1 = \lambda(f'(z) - 1),$$

or equivalently

$$f'(z) \cdot (f'(f(z)) - \lambda) = 1 - \lambda \neq 0.$$

Note that $f' \circ f$ omits both λ and 0, therefore it is constant. But then f' is constant as well. The only option is $f'(z) = 1$. \square

Lemma 2.2.8. For all $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos(\pi v) = w$ and $|v| \leq 1 + |w|$.

Proof. Let $v = \alpha + i\beta$ and note that

$$|w|^2 = \cos(\pi\alpha)^2 + \sinh(\pi\beta)^2 \geq \pi^2\beta^2.$$

Observe that we can choose some α such that $|\alpha| \leq 1$, therefore

$$1 + |w| \geq 1 + \pi \cdot |\beta| \geq |\alpha| + |\beta| \geq |v|. \quad \square$$

Theorem 2.2.9. Let f be a function, holomorphic on a neighbourhood of $\overline{\Delta}$, such that $0, 1 \notin f(\Omega)$. There exists a function g , holomorphic on a neighbourhood of $\overline{\Delta}$, such that

i) the equality

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g)))$$

holds with $|g(0)| \leq 3 + 2|f(0)|$, and

ii) the inequality

$$|g(z)| \leq |g(0)| + \frac{\theta}{\gamma(1 - \theta)}$$

holds for all $|z| \leq \theta$.

Proof. Again, apply lemma 2.2.4 and let

$$2f - 1 = \cos(\pi \cdot F).$$

Using the above lemma, we can transform F such that $|F(0)| \leq 1 + |2f(0) - 1|$. Applying lemma 2.2.4 again, we define g such that

$$F = \cos(\pi g).$$

Again, using the above lemma, set $|g(0)| \leq 1 + |F(0)|$. We therefore have

$$|g(0)| \leq 1 + |F(0)| \leq 2 + |2f(0) - 1| \leq 3 + 2|f(0)|.$$

Recall that $g(\mathbb{A})$ does not contain a disk of radius 1. Let $z \in \overline{\mathbb{A}(\theta)}$. Then, by Bloch's theorem, $g(\mathbb{A})$ contains a disk of radius $R = \gamma \cdot |g'(z)| \cdot (1 - \theta)$. Therefore, we must have

$$|g'(z)| < \frac{1}{\gamma(1 - \theta)}.$$

It follows that

$$|g(z)| = \left| g(0) + \int_0^z g'(\xi) d\xi \right| \leq |g(0)| + \int_0^z |g'(\xi)| d\xi \leq |g(0)| + |z| \cdot \frac{1}{\gamma(1 - \theta)}. \quad \square$$

Definition 2.2.10. For $r \geq 0$, let

$$S(r) = \left\{ f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{A}} \mid 0, 1 \notin f(\overline{\mathbb{A}}) \wedge |f(0)| \leq r \right\}.$$

For $\theta \in (0, 1)$ and $r > 0$, let

$$L(\theta, r) = \exp \left(\pi \cdot \exp \left(3 + 2r + \frac{\theta}{\gamma(1 - \theta)} \right) \right),$$

where γ is any constant such that Bloch's theorem holds, e.g. $\gamma = \frac{3}{2} - \sqrt{2}$.

Theorem 2.2.11 (Schottky). Let $f \in S(r)$. Then for all $z \in \mathbb{A}$ such that $|z| < \theta$ we have

$$|f(z)| \leq L(\theta, r).$$

Proof. Let g be a holomorphic function as in the previous theorem. Note that $|\cos(w)| \leq e^{|w|}$. We must therefore also have

$$\frac{1}{2} \cdot |1 + \cos(w)| \leq e^{|w|}.$$

Using this inequality, we get

$$|f(z)| \leq \exp(\pi \cdot \exp(\pi \cdot |g(z)|)) \leq L(\theta, r). \quad \square$$

2.3 The great Picard theorem

Lemma 2.3.1. Let $\Omega \subseteq \mathbb{C}$ be a domain, $\omega \in \Omega$ and $r \in (0, \infty)$. Let

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

and $\mathcal{F}_{\omega,r} \subseteq \mathcal{F}$ a subfamily with $|f(\omega)| \leq r$ for all $f \in \mathcal{F}_{\omega,r}$.

- i) There exists some $t > 0$ such that $\mathcal{F}_{\omega,r}|_{\Delta(\omega,t)}$ is bounded.
- ii) The family $\mathcal{F}_{\omega,1}$ is locally bounded in Ω .

Proof.

- i) Choose a $t > 0$ such that $\overline{\Delta(\omega, 2t)} \subseteq \Omega$ and set $\varphi(z) = 2tz + \omega$. By Schottky's theorem, we have

$$|f \circ \varphi(z)| \leq L\left(\frac{1}{2}, r\right)$$

for $|z| < \frac{1}{2}$, or equivalently

$$\sup_{v \in \Delta(\omega,t)} |f(v)| \leq L\left(\frac{1}{2}, r\right).$$

The family $\mathcal{F}_{\omega,r}$ is therefore bounded.

- ii) Let

$$\mathcal{U} = \{u \in \Omega \mid \mathcal{F}_{\omega,1} \text{ is bounded in a neighbourhood of } u\}.$$

Note that $\omega \in \mathcal{U}$, therefore the set is non-empty. Also observe that \mathcal{U} is open. Suppose that $\mathcal{U} \neq \Omega$ and let $v \in \partial\mathcal{U} \cap \Omega$. Then there exists a sequence $(f_n)_n \subseteq \mathcal{F}_{\omega,1}$ such that

$$\lim_{n \rightarrow \infty} |f_n(v)| = \infty.$$

Define $g_n = \frac{1}{f_n}$. These functions are holomorphic and omit both 0 and 1 by definition, therefore $g_n \in \mathcal{F}$. Applying the item i) for the sequence $(g_n)_n$ at point v , the sequence is bounded in a neighbourhood of v . By Montel's theorem, there exists a subsequence $(g_{n_k})_k$ that converges to a function g uniformly on compacts of $\Delta(v, s)$. By corollary 2.1.5.1, the function g is constant. But then

$$\lim_{k \rightarrow \infty} |f_{n_k}(z)| = \infty$$

for all $z \in \Delta(v, s)$, which is not possible as v is a boundary point. It follows that $\mathcal{U} = \Omega$. \square

Definition 2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n: \Omega \rightarrow \mathbb{C}$ a sequence of functions. We say that f_n converges to ∞ if

$$\lim_{n \rightarrow \infty} \|f_n\|_K = \infty$$

for every compact $K \subset \Omega$.

Theorem 2.3.3 (Montel – sharp). Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

Then \mathcal{F} is normal in Ω where we also allow convergence to ∞ .

Proof. Let $\Omega \subseteq \mathbb{C}$ be a domain and $p \in \Omega$. Consider the family $\mathcal{F}_{p,1}$. Let $(f_n)_n \subseteq \mathcal{F}$ be a sequence. If there exists a subsequence $(f_{n_k})_k \subseteq \mathcal{F}_{p,1}$, we can apply the above lemma. By the classical Montel's theorem, this subsequence has a convergent subsequence.

Suppose now that no such subsequence exists, that is $(f_n)_n$ has only finitely many terms in $\mathcal{F}_{p,1}$. But then there exists a subsequence $\left(\frac{1}{f_{n_k}}\right)_k \subseteq \mathcal{F}_{p,1}$. As before, this sequence has a convergent subsequence with limit g . If g is nowhere-vanishing, then $\frac{1}{g}$ is the limit of a subsequence of $(f_n)_n$. Otherwise, by corollary 2.1.5.1, we have $g = 0$ and therefore $(f_n)_n$ converges to ∞ . \square

Definition 2.3.4. Let $\Omega \subseteq \mathbb{C}$ be an open set and $p \in \Omega$. A function $f \in \mathcal{O}(\Omega \setminus \{p\})$ has an *essential singularity* in p if the limit

$$\lim_{z \rightarrow p} f(z)$$

does not exist and

$$\lim_{z \rightarrow p} |f(z)| \neq \infty.$$

Theorem 2.3.5 (Picard's great theorem). Let $\Omega \subseteq \mathbb{C}$ be an open set $p \in \Omega$ a point and $f \in \mathcal{O}(\Omega \setminus \{p\})$ a function. If f has an essential singularity at p , then f assumes every complex number as a value infinitely many times with at most one exception.

Proof. Without loss of generality assume that $p = 0$ and consider $\Omega = \Delta(\varepsilon)$. Suppose that f omits two values on $\Delta(\varepsilon)$, without loss of generality 0 and 1.

We now claim that f or $\frac{1}{f}$ is bounded in a neighbourhood of 0. Define the sequence of holomorphic functions $(f_n)_n$ with $f_n(z) = f\left(\frac{z}{n}\right)$. This sequence also omits 0 and 1, therefore either $(f_n)_n$ or $\left(\frac{1}{f_n}\right)_n$ has a convergent subsequence that converges uniformly on compacts by the sharp version of Montel's theorem. Denote the subsequence by $(g_{n_k})_k$ and set $g = f$ or $g = \frac{1}{f}$ accordingly.

Observe that there exists a constant M such that

$$\|g_{n_k}\|_{\partial\Delta(\frac{\varepsilon}{2})} \leq M$$

holds for all $k \in \mathbb{N}$. This is equivalent to

$$|g(z)| \leq M$$

for $|z| = \frac{1}{n_k} \cdot \frac{\varepsilon}{2}$. By the maximum principle, we have

$$|g(z)| \leq M$$

for all z such that

$$\frac{\varepsilon}{2} \cdot \frac{1}{n_k} \leq |z| \leq \frac{\varepsilon}{2}.$$

But as $(n_k)_k$ diverges, the inequality $|g(z)| \leq M$ holds for all z such that $|z| \leq \frac{\varepsilon}{2}$, therefore f or $\frac{1}{f}$ is bounded near 0.

Observe that f is not bounded in a neighbourhood of 0, as otherwise 0 is a removable singularity, which is not possible. Similarly, if $\frac{1}{f}$ is bounded, then f has either a removable singularity or a pole at 0, which is again a contradiction. \square

3 Infinite products

3.1 Definition and convergence

Definition 3.1.1. Let $(a_k)_k$ be a sequence of complex numbers. The sequence

$$n \mapsto \prod_{k=1}^n a_k$$

is called the *sequence of partial products* with factors a_k . We denote

$$p_{m,n} = \prod_{k=m}^n a_k.$$

We say that the infinite product is *convergent* if there exists an index $m \in \mathbb{N}$ such that the limit

$$\hat{a}_m = \lim_{n \rightarrow \infty} p_{m,n}$$

exists and is non-zero. We then define

$$\prod_{k=1}^{\infty} a_k = p_{1,m-1} \cdot \hat{a}_m.$$

as the limit of the infinite product.

Remark 3.1.1.1. The limit is uniquely defined.

Remark 3.1.1.2. An infinite product is convergent if and only if the product of all its non-zero factors has a non-zero limit and only finitely many factors are non-zero.

Lemma 3.1.2. Let $(a_k)_k \subseteq \mathbb{R}_{\geq 0}$ be a sequence such that

$$\sum_{k=1}^{\infty} (1 - a_k) = \infty.$$

Then

$$\lim_{n \rightarrow \infty} \prod_{k=p}^n a_k = 0$$

for all $p \in \mathbb{N}$. In particular, the infinite product is divergent.

Proof. Observe that

$$0 \leq \prod_{k=p}^n a_k \leq \prod_{k=p}^n e^{a_k - 1},$$

which converges to 0. □

Definition 3.1.3. Let $X \subseteq \mathbb{C}$ be a set.

i) A series

$$\sum_{k=1}^{\infty} g_k$$

of continuous functions $g_k \in \mathcal{C}(X)$ is *normally convergent* if for every compact $K \subseteq X$ the series

$$\sum_{k=1}^{\infty} \|g_k\|_K$$

converges.

ii) A product

$$\prod_{k=1}^{\infty} f_k$$

of continuous functions $f_k = 1 + g_k \in \mathcal{C}(X)$ is *normally convergent* if the series

$$\sum_{k=1}^{\infty} g_k$$

is normally convergent.

Definition 3.1.4. Let $X \subseteq \mathbb{C}$ be a set and $f_k \in \mathcal{C}(X)$ be continuous functions. Denote

$$p_{m,n} = \prod_{k=m}^n f_k.$$

We say that the infinite product

$$\prod_{k=1}^{\infty} f_k$$

converges *uniformly* on a set $L \subseteq X$ if there exists an index $m \in \mathbb{N}$ such that $f_k|_L$ has no zeroes for $k \geq m$ and

$$\lim_{n \rightarrow \infty} p_{m,n} = \hat{f}_m$$

exists, is uniform on L and has no zeroes on L . We define

$$\prod_{k=1}^{\infty} f_k = p_{1,m-1} \cdot \hat{f}_m$$

on L .

Theorem 3.1.5 (Reordering of infinite products). Let

$$\prod_{k=1}^{\infty} f_k$$

be a normally convergent product in $X \subseteq \mathbb{C}$. Then there exists a functions $f: X \rightarrow \mathbb{C}$ such that for all bijections $\tau: \mathbb{N} \rightarrow \mathbb{N}$ the product

$$\prod_{k=1}^{\infty} f_{\tau(k)}$$

converges to f uniformly on compacts of X . In particular, the infinite product converges uniformly on compacts.

Proof. Recall that, for $w \in \mathbb{D}$, we can define

$$\log(1+w) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} w^k.$$

Then,

$$|\log(1+w)| \leq |w| \cdot \sum_{k=0}^{\infty} |w|^k = \frac{|w|}{1-|w|}.$$

In particular, if $|w| \leq \frac{1}{2}$, we have

$$|\log(1 + w)| \leq 2|w|.$$

Let $L \subseteq X$ be a compact and write $f_k = 1 + g_k$. For all $k > N$ we have $\|g_k\|_L \leq \frac{1}{2}$, therefore we can write

$$\log f_k = \log(1 + g_k) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} g_k^{\ell}.$$

But then

$$\|\log f_k\|_L \leq 2\|g_k\|_L.$$

It follows that the series

$$\sum_{k=N}^{\infty} \|\log f_k\|_L$$

converges. But then the series

$$h_N = \sum_{k=N}^{\infty} \log f_k$$

converges absolutely, and therefore all reorderings of the series converge as well to the same limit h_N .

Observe that

$$e^{h_N} = \prod_{k=N}^{\infty} e^{\log f_k} = \prod_{k=N}^{\infty} f_k.$$

This product therefore converges uniformly on L , independently of reorderings. We now define

$$f = \prod_{k=1}^{N-1} f_k \cdot e^{h_N}.$$

Note that this holds for all reorderings, as they differ from a suitable one by only finitely many transpositions. \square

3.2 Zeroes of infinite products

Definition 3.2.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{O}(\Omega)$. The *zero set* of f is the set

$$Z(f) = \{z \in \Omega \mid f(z) = 0\}.$$

For all $c \in \Omega$, define the *zero order* of f in c as follows: if

$$f(z) = (z - c)^k \cdot g(z)$$

where $g(c) \neq 0$ is a holomorphic function, then $\text{ord}_c(f) = k$.

Remark 3.2.1.1. For non-zero $f \in \mathcal{O}(\Omega)$, the set $Z(f)$ is discrete in Ω .

Remark 3.2.1.2. We have

$$\text{ord}_c \left(\prod_{k=1}^n f_k \right) = \sum_{k=1}^n \text{ord}_c(f_k).$$

Lemma 3.2.2. Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^{\infty} f_k$$

be a normally convergent product in Ω , where $f_k \in \mathcal{O}(\Omega)$ are non-zero holomorphic functions. Then f is a non-zero function with

$$Z(f) = \bigcup_{k=1}^{\infty} Z(f_k)$$

and

$$\text{ord}_c(f) = \sum_{k=1}^{\infty} \text{ord}_c(f_k).$$

Proof. Recall that normally convergent products converge uniformly on compacts of Ω . In particular, f is a holomorphic function.

Pick a point $c \in \Omega$. By definition of convergence, there exists some $m \in \mathbb{N}$ such that $\hat{f}_m(c) \neq 0$. As \hat{f}_m is holomorphic as well, we have

$$f(c) = (p_{1,m-1} \cdot \hat{f}_m)(c),$$

but then

$$\text{ord}_c(f) = \sum_{k=1}^{m-1} \text{ord}_c(f_k) = \sum_{k=1}^{\infty} \text{ord}_c(f_k). \quad \square$$

Lemma 3.2.3. Let $\Omega \subseteq \mathbb{C}$ be a domain. If

$$f = \prod_{k=1}^{\infty} f_k$$

is a normally convergent product in Ω , where $f_k \in \mathcal{O}(\Omega)$ are holomorphic functions, then the sequence $(\hat{f}_n)_n$ converges to 1 uniformly on compacts.

Proof. Choose $m \in \mathbb{N}$ such that $\hat{f}_m \neq 0$. Then the set $Z(\hat{f}_m)$ has no accumulation points in Ω . We can therefore write

$$\hat{f}_n = \frac{\hat{f}_m}{p_{m,n-1}}$$

on $\Omega \setminus Z(\hat{f}_m)$. As $p_{m,n-1}$ converges to \hat{f}_m on compacts of Ω ,

$$\lim_{n \rightarrow \infty} \hat{f}_n = 1$$

uniformly on compacts of $\Omega \setminus Z(\hat{f}_m)$. For any compact set $K \subseteq \Omega$, taking m large enough, we have $Z(\hat{f}_m) \cap K = \emptyset$. The conclusion follows. \square

Definition 3.2.4. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. The meromorphic function $\frac{f'}{f}$ is called the *logarithmic derivative* of f .

Remark 3.2.4.1. For holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(\Omega)$ we have

$$\left(\prod_{k=1}^n f_k \right)' \cdot \left(\prod_{k=1}^n f_k \right)^{-1} = \sum_{k=1}^n \frac{f'_k}{f_k}.$$

Definition 3.2.5. Let $g_k \in \mathcal{M}(\Omega)$ be meromorphic functions. The series

$$\sum_{k=1}^{\infty} g_k$$

is *normally convergent* in Ω if for every compact $L \subseteq \Omega$ there exists some $m \in \mathbb{N}$ such that

$$\sum_{k=m}^{\infty} \|g_k\|_L$$

converges.

Theorem 3.2.6 (Logarithmic differentiation). Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^{\infty} f_k$$

be a normally convergent product in Ω , where $f_k \in \mathcal{O}(\Omega)$ are non-zero functions. Then

$$\sum_{k=1}^{\infty} \frac{f'_k}{f_k}$$

is normally convergent in Ω and

$$\sum_{k=1}^{\infty} \frac{f'_k}{f_k} = \frac{f'}{f}.$$

Proof. As \hat{f}_n converges to 1 uniformly on compacts, the sequence $(f'_n)_n$ converges to 0 uniformly on compacts by Cauchy estimates. Then for any compact L , $\frac{\hat{f}'_n}{\hat{f}_n}$ converges to 0 as \hat{f}_n has no zeroes in L for n large enough. It follows that

$$\lim_{n \rightarrow \infty} \frac{f'}{f} - \sum_{k=1}^n \frac{f'_k}{f_k} = \lim_{n \rightarrow \infty} \frac{\hat{f}'_{n+1}}{\hat{f}_{n+1}} = 0.$$

Write $f_k = 1 + g_k$ and fix a compact set $L \subseteq \Omega$. Choose an index m such that we have $Z(\hat{f}_m) \cap L = \emptyset$ and

$$\min_{z \in L} |f_k(z)| \geq \frac{1}{2}.$$

Choose $\varepsilon > 0$ such that

$$L_\varepsilon = \{z \in \mathbb{C} \mid d(z, L) \leq \varepsilon\} \subseteq \Omega.$$

By the Cauchy estimates, we have $\|g'_k\|_L \leq \frac{1}{\varepsilon} \|g_k\|_L$. But then

$$\sum_{k=m}^{\infty} \left\| \frac{f'_k}{f_k} \right\|_L = \sum_{k=m}^{\infty} \left\| \frac{g'_k}{f_k} \right\|_L \leq 2 \cdot \sum_{k=m}^{\infty} \|g'_k\|_L \leq \frac{2}{\varepsilon} \cdot \sum_{k=m}^{\infty} \|g_k\|_L,$$

which is convergent by our assumptions. \square

Lemma 3.2.7. Let g be meromorphic on \mathbb{C} with poles in \mathbb{Z} with principal parts $\frac{1}{z-m}$. Moreover, assume that g is an odd function that satisfies

$$2g(2z) = g(z) + g\left(z + \frac{1}{2}\right).$$

Then $g(z) = \pi \cdot \cot(\pi z)$.

Proof. Simple calculations show that $\pi \cdot \cot(\pi z)$ is indeed a solution of the functional equation. Define $h(z) = g(z) - \pi \cdot \cot(\pi z)$. This another solution of the functional equation, and an odd function. In particular, $h(0) = 0$. Observe that the principal parts of h are 0, therefore $h \in \mathcal{O}(\mathbb{C})$ is an entire function.

Suppose that h is not constant. In particular, there exists some $c \in \partial\mathbb{A}(2)$ such that

$$|h(z)| < |h(c)|$$

for all $z \in \mathbb{A}(2)$. As $\frac{c}{2}, \frac{c+1}{2} \in \mathbb{A}(2)$, we can write

$$2|h(c)| = \left| h\left(\frac{c}{2}\right) + h\left(\frac{c+1}{2}\right) \right| \leq \left| h\left(\frac{c}{2}\right) \right| + \left| h\left(\frac{c+1}{2}\right) \right| < 2|h(c)|,$$

which is a contradiction. It follows that $h = 0$. \square

Corollary 3.2.7.1. We have

$$\pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

Proof. Note that

$$\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2} = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k} \right),$$

therefore the series has poles in \mathbb{Z} with principal parts $\frac{1}{z-m}$. It is also an odd function. A calculation shows that, for

$$r_n(z) = \frac{1}{z} + \sum_{k=1}^n \frac{2z}{z^2 - k^2},$$

we have

$$r_n(z) + r_n\left(z + \frac{1}{2}\right) = 2r_{2n}(2z) + \frac{2}{2z + 2n + 1}.$$

Taking $n \rightarrow \infty$, the conclusion follows. \square

Theorem 3.2.8. We have

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Proof. The above product is obviously normally convergent, therefore we can take its logarithmic derivative. A simple calculation shows that it is equal to $\pi \cot(\pi z)$. As logarithmic derivatives are equal only for scalar multiples, we only have to check equality in one point. \square

3.3 The Euler gamma function

Lemma 3.3.1. The infinite product

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-\frac{z}{k}}$$

is normally convergent in \mathbb{C} .

Proof. Write

$$\begin{aligned} |1 - (1 - \omega)e^{\omega}| &= |1 - e^{\omega} + \omega e^{\omega}| \\ &= \left| -\sum_{k=1}^{\infty} \frac{\omega^k}{k!} + \sum_{k=0}^{\infty} \frac{\omega^{k+1}}{k!} \right| \\ &= \left| \omega^2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \omega^{k-1} \right| \\ &\leq |\omega|^2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \\ &= |\omega|^2 \end{aligned}$$

for $|\omega| \leq 1$. But then the sum

$$\sum_{k=\lceil |z| \rceil}^{\infty} \left| 1 - \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right| \leq \sum_{k=\lceil |z| \rceil}^{\infty} \left| \frac{z^2}{k^2} \right|$$

converges normally. The infinite product must then converge normally in \mathbb{C} as well. \square

Lemma 3.3.2. Let

$$H(z) = z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Then $H(1) = e^{-\gamma}$, where γ is the *Euler-Mascheroni constant*, that is

$$\gamma = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \log(n).$$

Proof. First note that

$$\prod_{k=1}^n \left(1 + \frac{1}{k}\right) = \prod_{k=1}^n \frac{k+1}{k} = n+1.$$

We therefore have

$$\prod_{k=1}^n \left(1 + \frac{1}{k}\right) e^{-\frac{1}{k}} = \exp \left(\log(n+1) - \sum_{k=1}^n \frac{1}{k} \right),$$

therefore

$$H(1) = \lim_{n \rightarrow \infty} \exp \left(\log(n+1) - \sum_{k=1}^n \frac{1}{k} \right) = e^{-\gamma}. \quad \square$$

Lemma 3.3.3. Let $\Delta(z) = e^{\gamma z} H(z)$.

i) We have $\Delta(1) = 1$ and $\Delta(z) = z\Delta(z+1)$.

ii) We have $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$.

Proof. Note that $\Delta(1) = 1$ by the previous lemma. Rewrite the partial products as

$$z \cdot \prod_{k=1}^n \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} = \frac{z}{n!} \cdot \prod_{k=1}^n (z+k) \cdot \exp\left(-z \sum_{k=1}^n \frac{1}{k}\right).$$

We therefore have

$$\begin{aligned} \Delta(z) &= \lim_{n \rightarrow \infty} \frac{e^{\gamma z}}{n!} \cdot \prod_{k=0}^n (z+k) \cdot \exp\left(-z \sum_{k=1}^n \frac{1}{k}\right) \\ &= \lim_{n \rightarrow \infty} \frac{e^{\gamma z}}{n! \cdot n^z} \cdot \prod_{k=0}^n (z+k) \cdot \exp\left(z \log(n) - z \sum_{k=1}^n \frac{1}{k}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n! \cdot n^z} \cdot \prod_{k=0}^n (z+k). \end{aligned}$$

We can now calculate

$$z \cdot \Delta(z+1) = \lim_{n \rightarrow \infty} z \cdot \frac{1}{n! \cdot n^{z+1}} \cdot \prod_{k=1}^{n+1} (z+k) = \Delta(z) \cdot \lim_{n \rightarrow \infty} \frac{z+n+1}{n} = \Delta(z).$$

It remains to check the equality $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$. We have

$$\begin{aligned} \pi \cdot \Delta(z)\Delta(1-z) &= \pi \cdot \Delta(z) \cdot \frac{\Delta(-z)}{-z} \\ &= \pi e^{\gamma z} \cdot z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \cdot e^{-\gamma z} \cdot \frac{-z}{-z} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{\frac{z}{k}} \\ &= \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \\ &= \sin(\pi z). \end{aligned}$$

□

Definition 3.3.4. The *Euler gamma function* is defined as

$$\Gamma(z) = \frac{1}{\Delta(z)}.$$

Theorem 3.3.5. The Γ function satisfies the following properties:

1. The function Γ is meromorphic with simple poles in $-\mathbb{N}_0$.
2. We have $\Gamma(1) = 1$.
3. The function Γ satisfies $\Gamma(z+1) = z\Gamma(z)$.
4. The function Γ satisfies

$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

5. We have

$$\Gamma(z) = \lim_{n \rightarrow \infty} n! \cdot n^z \cdot \left(\prod_{k=0}^n (z+k) \right)^{-1}.$$

Proof. The proof is obvious and need not be mentioned. \square

Theorem 3.3.6. Let F be holomorphic in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ and assume $F(z+1) = z \cdot F(z)$. Furthermore, assume that F is bounded on the strip $1 \leq \operatorname{Re}(z) < 2$ and $F(1) = 1$. Then $F = \Gamma$.

3.4 Weierstraß factors

Definition 3.4.1. The *Weierstraß factors* are functions

$$E_n(z) = (1 - z) \cdot \exp \left(\sum_{\ell=1}^n \frac{z^\ell}{\ell} \right).$$

Lemma 3.4.2. The Weierstraß factors satisfy the following:

i) For $n \geq 1$ we have

$$E'_n(z) = -z^n \cdot \exp \left(\sum_{\ell=1}^n \frac{z^\ell}{\ell} \right).$$

ii) For $n \geq 0$ we have

$$E_n(z) = 1 + \sum_{k=n+1}^{\infty} a_k z^k,$$

where

$$\sum_{k=n+1}^{\infty} |a_k| = 1.$$

iii) For $n \geq 0$ and $|z| \leq 1$ we have

$$|E_n(z) - 1| \leq |z|^{n+1}.$$

Proof.

i) Evident.

ii) Observing the derivative, we see that $a_1 = a_2 = \dots = a_n = 0$, and $a_k \leq 0$ for $k > n$.
But then

$$\sum_{k=n+1}^{\infty} |a_k| = - \sum_{k=n+1}^{\infty} a_k = 1 - E_n(1) = 1.$$

iii) We have

$$|E_n(z) - 1| = \left| \sum_{k=n+1}^{\infty} a_k z^k \right| \leq \sum_{k=n+1}^{\infty} |a_k| \cdot |z|^k \leq |z|^{n+1}. \quad \square$$

Lemma 3.4.3. Let $(a_k)_k \subset \mathbb{C}^*$ be a sequence of complex numbers with no accumulation point and let $(p_k)_k \subseteq \mathbb{N}_0$ be non-negative integers with

$$\sum_{k=1}^{\infty} \left| \frac{r}{a_k} \right|^{p_k+1}$$

converges for every $r > 0$. Then the *Weierstraß product*

$$\prod_{k=1}^{\infty} E_{p_k} \left(\frac{z}{a_k} \right)$$

converges normally on \mathbb{C} .

Proof. Note that $|a_k| > |z|$ for all but finitely many k . Now just apply the previous lemma. \square

Theorem 3.4.4 (Weierstraß factorization theorem). For any sequence $(a_k)_k \subset \mathbb{C}$ with no accumulation point there exists a Weierstraß product

$$z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{p_k} \left(\frac{z}{a_k} \right)$$

that converges normally on \mathbb{C} .

Proof. Set $p_k = k - 1$. For any $r > 0$ choose $m \in \mathbb{N}_0$ such that $|a_k| > 2r$ for all $k \geq m$. We then have

$$\sum_{k=m}^{\infty} \left| \frac{r}{a_k} \right|^{p_k+1} \leq \sum_{k=m}^{\infty} \frac{1}{2^k} \leq 2. \quad \square$$

Theorem 3.4.5 (Weierstraß product theorem). Let $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$ be a holomorphic function. Then there exists a function $g \in \mathcal{O}(\mathbb{C})$ such that

$$f = e^g \cdot z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{k-1} \left(\frac{z}{a_k} \right),$$

where a_k are zeroes of f on $\mathbb{C} \setminus \{0\}$, counted with multiplicities, and $q = \text{ord}_0(f)$.

Proof. The proof is obvious and need not be mentioned. \square

Lemma 3.4.6. Let $\Omega \subset \mathbb{C}$ be an open subset, $(a_k)_k \subset \Omega$ a sequence with no accumulation point in Ω and $A = \{a_k \mid k \in \mathbb{N}\}$. Let $(b_k)_k \subset \mathbb{C} \setminus \Omega$ and $(p_k)_k \subseteq \mathbb{N}$ be sequences such that the series

$$\sum_{k=1}^{\infty} |r(a_k - b_k)|^{p_k+1}$$

converges for all $r > 0$ and denote $B = \{b_k \mid k \in \mathbb{N}\}$. Then the infinite product

$$\prod_{k=1}^{\infty} E_{p_k} \left(\frac{a_k - b_k}{z - b_k} \right)$$

converges normally on $\mathbb{C} \setminus \overline{B}$.

Proof. Let $L \subseteq \mathbb{C} \setminus \overline{B}$ be a compact set and let $\ell = d(L, \overline{B}) > 0$. We then have $|z - b_k| \geq \ell$ for all $z \in L$ and $k \in \mathbb{N}$.

We can now bound

$$\left\| \frac{a_k - b_k}{z - b_k} \right\|_L \leq \frac{|a_k - b_k|}{\ell}.$$

By the assumption of convergence for $r = \frac{1}{\ell}$, we must have

$$|r \cdot (a_k - b_k)| < 1$$

for all $k \geq n(L)$, but then

$$\sum_{k=n(L)}^{\infty} \left\| E_{p_k} \left(\frac{a_k - b_k}{z - b_k} \right) - 1 \right\|_L \leq \sum_{k=n(L)}^{\infty} \left\| \frac{a_k - b_k}{z - b_k} \right\|_L^{p_k+1} \leq \sum_{k=n(L)}^{\infty} |r \cdot (a_k - b_k)|^{p_k+1},$$

which converges. \square

Remark 3.4.6.1. The Weierstraß factor $E_{p_k} \left(\frac{a_k - b_k}{z - b_k} \right)$ is zero if and only if $z = a_k$.

Lemma 3.4.7. Let $A \subset \mathbb{C}$ be a discrete set and define $A' = \overline{A} \setminus A$. Suppose that $A' \neq \emptyset$ and let

$$A_1 = \{z \in A \mid |z| \cdot d(z, A') \geq 1\}$$

and $A_2 = A \setminus A_1$. Now let

$$A_2(\varepsilon) = \{z \in A_2 \mid d(z, A') \geq \varepsilon\}.$$

Then A_1 is a closed set and $A_2(\varepsilon)$ is finite for any $\varepsilon > 0$.

Proof. Assume A_1 has an accumulation point a and let $(a_k)_k \subseteq A$ be a sequence, converging to a . But then

$$\lim_{k \rightarrow \infty} |a_k| \cdot d(a_k, A') = 0,$$

which is a contradiction.

Note that, for all $z \in A_2(\varepsilon)$, we have $|z| < \frac{1}{\varepsilon}$. If the set is infinite, it has an accumulation point, which is impossible as $d(z, A') \geq \varepsilon$. \square

Remark 3.4.7.1. If $A \subset \mathbb{C}$ is a discrete set, then A' is a closed set in \mathbb{C} .

Theorem 3.4.8 (Weierstraß product theorem). Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k)_k \subset \Omega$ be a sequence without accumulation points in Ω and denote $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$. Then there exists a Weierstraß product for $(a_k)_k$ that converges normally in $\mathbb{C} \setminus A'$. This product has zeros precisely in $(a_k)_k$, counted with multiplicities.

Proof. Assume that $\Omega \neq \mathbb{C}$ and $A' \neq \emptyset$.⁴ Write $A = A_1 \cup A_2$ as in the above lemma. Recall that A_1 has no accumulation points, therefore we can apply theorem 3.4.5 for A_1 . It remains to construct a Weierstraß product for A_2 .

Observe that $A' = A'_2$. As this is a closed space, for all $a_k \in A_2$ there exists some $b_k \in A'_2$ such that

$$|a_k - b_k| = d(a_k, A'_2).$$

Observe that

$$\lim_{\substack{k \rightarrow \infty \\ a_k \in A_2}} |a_k - b_k| = 0,$$

as the sets $A_2(\varepsilon)$ are finite. Now set $p_k = k$ and apply lemma 3.4.6. \square

⁴ Otherwise just apply theorem 3.4.5.

Corollary 3.4.8.1 (Blaschke products). Let $(a_k)_k \subset \mathbb{A} \setminus \{0\}$ be a sequence without accumulation points in \mathbb{A} . If the series

$$\sum_{k=1}^{\infty} (1 - |a_k|)$$

converges, then the product

$$\prod_{k=1}^{\infty} E_0 \left(\frac{a_k - \frac{1}{\bar{a}_k}}{z - \frac{1}{\bar{a}_k}} \right)$$

converges normally in \mathbb{A} and has zeros precisely in $(a_k)_k$, counted with multiplicities.

Proof. Note that

$$|a_k - b_k| = \left| a_k - \frac{1}{\bar{a}_k} \right| = \left| \frac{1}{\bar{a}_k} \right| \cdot \left| |a_k|^2 - 1 \right| = \left| \frac{1}{\bar{a}_k} \right| \cdot (1 - |a_k|)(1 + |a_k|) \leq \frac{2}{m} \cdot (1 - |a_k|),$$

where

$$m = \min \{ |a_k| \mid k \in \mathbb{N} \}.$$

It follows that the series

$$\sum_{k=1}^{\infty} r \cdot |a_k - b_k|$$

converges, therefore we can apply lemma 3.4.6. □

Theorem 3.4.9. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega) \setminus \{0\}$. Then we can write

$$f = g \cdot \prod_{k=1}^{\infty} f_k,$$

where $g \in \mathcal{O}^*(\Omega)$ and f_k are Weierstraß factors.

Proof. The proof is obvious and need not be mentioned. □

Theorem 3.4.10. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{M}(\Omega)$. Then we can write $f = \frac{g}{h}$, where $g, h \in \mathcal{O}(\Omega)$.

Proof. Define h as the Weierstraß product of the poles of f . □

Remark 3.4.10.1. Let $\Omega \subseteq \mathbb{C}$ be a domain. Then $\mathcal{O}(\Omega)$ is not a factorial ring,⁵ but $\gcd(f, g) \in \mathcal{O}(\Omega)$ exists.

Definition 3.4.11. Let Ω be an open subset and $\{a_k\}_k$ be a sequence without accumulation points and without repetitions. Let

$$q_k(z) = \sum_{m=1}^{\infty} c_{k,m} (z - a_k)^{-m}$$

be a principal part in a_k for each k .

⁵ “Kolobar z enolično faktorizacijo.”

If there exist functions $g_k \in \mathcal{O}(\Omega)$ for each k such that

$$\sum_{k=1}^{\infty} q_k - g_k$$

converges normally in Ω , we call it the *Mittag-Leffler series* for the distribution of principal parts (a_k, q_k) .

Remark 3.4.11.1. We adopt the following convention: If $0 \in \{a_k\}_k$, then $a_1 = 0$.

Theorem 3.4.12 (Mittag-Leffler for \mathbb{C}). For every distribution of principal parts in \mathbb{C} there exists a corresponding Mittag-Leffler series.

Proof. Let g_k be the Taylor series of q_k about 0 in the disk $\Delta(|a_k|)$ such that the inequality $\|q_k - g_k\|_{\overline{\Delta(\frac{1}{2}|a_k|)}} < 2^{-k}$ holds for each $k \geq 2$. Note that

$$\lim_{k \rightarrow \infty} |a_k| = \infty$$

as the points don't accumulate. For each $r > 0$ we can therefore find an integer n such that $r < \frac{1}{2}|a_k|$ for all $k \geq n$. Then

$$\sum_{k=n}^{\infty} \|q_k - g_k\|_{\overline{\Delta(r)}} \leq 1. \quad \square$$

Remark 3.4.12.1. The above series $f \in \mathcal{O}(\mathbb{C} \setminus \{a_1, a_2, \dots\})$ with principal parts q_k in a_k for each $k \in \mathbb{N}$. If the principal part are finite, then $f \in \mathcal{M}$.

Lemma 3.4.13. Let $a \in \mathbb{C}$, $q \in \mathcal{O}(\Omega \setminus \{a\})$ be a principal part and $b \in \mathbb{C} \setminus \{a\}$. Then q has a Laurent series expansion about b in the annulus $\{z \in \mathbb{C} \mid |z - b| > |a - b|\}$ of the form

$$q(z) = \sum_{m=1}^{\infty} c_m (z - b)^{-m}$$

that converges uniformly for $|z - b| \geq r > |a - b|$.

Proof. Choose a path γ_r that goes around the circle centered at b of radius r . We claim that

$$c_m = \frac{1}{2\pi} \int_{\gamma_r} \frac{q(z)}{(z - b)^{-m+1}} dz$$

for $m \in \mathbb{Z}$ suffice. We can estimate

$$|c_m| \leq \frac{1}{2\pi} \cdot 2\pi \frac{\|q\|_{\gamma_r}}{r^{-m}} = \frac{\|q\|_{\gamma_r}}{r^{-m}}.$$

We know that $q(z)$ is of the form

$$q(z) = \sum_{m=1}^{\infty} d_m (z - a)^{-m}$$

for some $d_m \in \mathbb{C}$ when developed into a Laurent series around a . It is trivial to show that

$$\lim_{|z| \rightarrow \infty} q(z) = 0.$$

Thus, $\|q\|_{\gamma_r}$ approaches zero as r goes to infinity. If $m \leq 0$, then

$$\lim_{r \rightarrow \infty} \|q\|_{\gamma_r} r^m = 0.$$

Therefore, $c_m = 0$ for $m \leq 0$ and

$$q(z) = \sum_{m=1}^{\infty} c_m (z - b)^{-m}$$

is indeed a power series in $z - b$ which converges uniformly for $|z - b|^{-1} \leq r$. \square

Definition 3.4.14. The partial sums of

$$q_\ell(z) = \sum_{m=1}^{\ell} c_m (z - b)^{-m}$$

are called the ℓ -th *Laurent terms* of q about b .

Lemma 3.4.15. Let $(a_k, q_k)_k$ be a distribution of principal parts in an open set $\Omega \subseteq \mathbb{C}$, $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$.⁶ Assume there exists a sequence $(b_k)_k \subseteq A'$ with

$$\lim_{k \rightarrow \infty} |a_k - b_k| = 0.$$

Let $q_{k,\ell}$ be the ℓ -th Laurent term of q_k about b_k . Then there exists a sequence $(\ell_k)_k \subseteq \mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty} (q_k - q_{k,\ell_k})$$

is a Mittag-Leffler series for $(a_k, q_k)_k$.

Proof. For a principal part q_k the Laurent series converges uniformly on $|z - b_k| > r$ for any $r > |a_k - b_k|$ by the previous lemma. Thus, we can choose ℓ_k large enough such that

$$|q_k(z) - q_{k,\ell_k}(z)| < 2^{-k}$$

for all z such that $|z - b_k| \geq 2|a_k - b_k|$.

For any compact set $L \subseteq \mathbb{C} \setminus A'$, the distance to A' is strictly positive. Since

$$\lim_{k \rightarrow \infty} |a_k - b_k| = 0,$$

the point b_k must lie outside L for large enough k . Thus, there exists some $n(L) \in \mathbb{N}$ such that

$$L \subseteq \bigcap_{k \geq n(L)} \{z \in \mathbb{C} \mid |z - b_k| \geq 2|a_k - b_k|\}.$$

Therefore, we can use the previous estimate on L , to get

$$\sum_{k \geq n(L)} \|q_k - q_{k,\ell_k}\|_L \leq \sum_{k \geq n(L)} 2^{-k} \leq 2. \quad \square$$

⁶ The closure is taken in \mathbb{C} .

Theorem 3.4.16 (Mittag-Leffler for open subsets). Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k, q_k)_k$ be a distribution of principal parts in Ω and $A = \{a_k \mid k \in \mathbb{N}\}$. Then there exists a Mittag-Leffler series for (a_k, q_k) that converges normally in $\mathbb{C} \setminus A'$ where $A' = \overline{A} \setminus A$.

Proof. By lemma 3.4.7, $(A_1)'$ is empty and $(A_2)' = A'$. If A' is empty, then

$$\lim_{k \rightarrow \infty} |a_k| = \infty$$

and we can apply theorem 3.4.12. Similarly, we can assume $\Omega \neq \mathbb{C}$. Again by the lemma, $A_2(\epsilon)$ is finite. Hence, there exist $(b_k)_k \in A'$ such that

$$\lim_{k \rightarrow \infty} |a_k - b_k| = 0.$$

We can apply lemma 3.4.15 to obtain a Mittag-Leffler series. Now we apply theorem 3.4.12 for \mathbb{C} to A_1 . Sum up these two series to get the series from the statement. \square

Theorem 3.4.17 (Mittag-Leffler osculation theorem). Let $\Omega \subseteq \mathbb{C}$ be an open subset and $(a_k)_k \subseteq \Omega$ be a sequence without accumulation points and without repetition. Furthermore, let

$$f_k(z) = \sum_{\ell=-\infty}^{n(k)} c_{k,\ell} (z - a_k)^\ell,$$

where $n(k) \in \mathbb{N}_0$, be normally convergent on $\mathbb{C} \setminus A$, where A is the set of a_k . Then there exists a function $f \in \mathcal{O}(\Omega \setminus A)$ such that $\text{ord}_{a_k}(f - f_k) > n(k)$ for all $k \in \mathbb{N}$.

Proof. By the Weierstraß product theorem, there exists a function $h \in \mathcal{O}(\Omega)$ such that $\text{ord}_{a_k}(h) > n(k)$ and h has no zeroes on $\mathbb{C} \setminus A$. Then $(a_k, \frac{f_k}{h})_k$ is a distribution of principal parts. By theorem 3.4.16, there exists a $g \in \mathcal{O}(\Omega \setminus A)$ with these principal parts.

Now define $f = g \cdot h$. Then

$$f - f_k = g \cdot h - f_k = h \cdot \left(g - \frac{f_k}{h} \right),$$

which vanishes to order larger than $n(k)$ in a_k . \square

Corollary 3.4.17.1. For every sequence $(a_k)_k \subseteq \Omega$ without accumulation points and without repetition and every sequence $(c_k)_k \subseteq \mathbb{C}$ there exists a function $f \in \mathcal{O}(\Omega)$ such that $f(a_k) = c_k$ for each $k \in \mathbb{N}$.

3.5 Ring structure of holomorphic functions

Definition 3.5.1. Let $\Omega \subseteq \mathbb{C}$ be an open set. A *divisor* of a meromorphic function $f \in \mathcal{M}^*(\Omega)$ is the function $(f): \Omega \rightarrow \mathbb{Z}$, given by

$$(f)(z) = \begin{cases} n & f \text{ has a zero of order } n \text{ in } z \\ -n & f \text{ has a pole of order } n \text{ in } z \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.5.1.1. The divisor of a product is the sum of divisors, i.e. $(f \cdot g) = (f) + (g)$.

Definition 3.5.2. Let $S \subseteq \mathcal{O}(\Omega)$ be a subset that contains a non-zero holomorphic function.

Define

$$d(z) = \min_{f \in S \setminus \{0\}} (f)(z) \in \mathbb{N}_0.$$

By Weierstraß product theorem there exists a function $g \in \mathcal{O}(\Omega)$ such that $(g) = d$. We define $\gcd(S) = g$.⁷

Lemma 3.5.3 (Wedderburn). Let $\Omega \subseteq \mathbb{C}$ be a domain and $f, g \in \mathcal{O}(\Omega)$ be functions with $\gcd(f, g) = 1$. Then there exist functions $a, b \in \mathcal{O}(\Omega)$ so that $af + bg = 1$. Moreover, we can choose a to be nowhere vanishing.

Proof. If $g = 0$ and $f \neq 0$ then f cannot vanish by assumption on \gcd , therefore $a = \frac{1}{f}$ and $b = 1$ suffice. Therefore, we can assume both f, g are nonzero. Note that $Z(f) \cap Z(g)$ is empty, since $(z - p)$ divides $\gcd(f, g)$ for any $p \in Z(f) \cap Z(g)$. The set $Z(f) \cup Z(g)$ is thus discrete. Further, for each zero p of g , there exists a disk of radius ε and a holomorphic function $f_p \in \mathcal{O}(\Delta(p, \varepsilon))$ such that

$$f = e^{f_p}.$$

By the Mittag-Leffler osculation theorem there exists a function $h \in \mathcal{O}(\Omega)$ such that $\text{ord}_p(h - f_p) > \text{ord}_p(g)$.

Here we stop for a short observation. Developing into the power series, we get that $e^{w^n} - 1 = w^n + O(w^{2n})$. Then,

$$\text{ord}_p(f - e^h) = \text{ord}_p(e^h \cdot (e^{f_p - h} - 1)) = \text{ord}_p((e^{f_p - h} - 1)) = \text{ord}_p(f_p - h) > \text{ord}_p(g).$$

Define $k = \frac{f - e^h}{g} \in \mathcal{O}(\Omega)$. We claim that $a = e^{-h}$ and $b = -ke^{-h}$ satisfy the conditions. Clearly, a doesn't vanish, and

$$af + bg = e^{-h}f - ke^{-h}g = e^{-h}(f - kg) = e^{-h} \left(f - \frac{f - e^h}{g}g \right) = e^{-h}e^h = 1. \quad \square$$

Corollary 3.5.3.1. For holomorphic functions $f_j \in \mathcal{O}(\Omega)$, where $j \leq n$, we can write $f = \gcd(f_1, f_2, \dots, f_n)$ as

$$f = \sum_{j=1}^n a_j f_j$$

⁷ There are of course multiple possible functions that satisfy this condition, but their quotients are invertible.

Proof. We proceed by induction. The base case is just Wedderburn's lemma. Now let $\hat{f} = \gcd(f_2, f_3, \dots, f_n)$, which can be written as

$$\hat{f} = \sum_{j=2}^n \hat{a}_j f_j$$

by the induction hypothesis. Then $\frac{f_1}{\hat{f}}, \frac{\hat{f}}{\hat{f}} \in \mathcal{O}(\Omega)$ are holomorphic functions with gcd equal to 1. We can therefore apply Wedderburn's lemma to get functions a and b such that

$$a \frac{f_1}{\hat{f}} + b \frac{\hat{f}}{\hat{f}} = 1.$$

The conclusion follows □

Theorem 3.5.4. Let $I \triangleleft \mathcal{O}(\Omega)$ be the ideal generated by holomorphic functions f_1, f_2, \dots, f_n on Ω . Then there exists a holomorphic function f such that $I = (f)$.

Proof. Take $f = \gcd(f_j)$. This function is an element of I by the previous corollary. Since $f \mid f_j$, this implies that $I = (f)$. □

Definition 3.5.5. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal.

- i) We call I *closed* if for every sequence $(f_n)_n \subseteq I$ that converges uniformly on compacts of Ω to some function f , we also have $f \in I$.
- ii) We call $p \in \Omega$ a *zero* of I if $f(p) = 0$ for every $f \in I$.

Lemma 3.5.6. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal. Let $p \in \Omega$ be a point that is not a zero of I . Let $f, g \in \mathcal{O}(\Omega)$ be functions such that $f(z) \neq 0$ for all $z \neq p$. If $fg \in I$, then $g \in I$.

Proof. Since p is not a zero of I , then there exists a function $h \in I$ such that $h(p) \neq 0$. Let $n = \text{ord}_p(f)$. If $n = 0$, then f is a unit, so $g \in I$. Otherwise, we have

$$\frac{f(z)}{z-p} g = -\frac{1}{h(p)} \cdot \left(\frac{h-h(p)}{z-p} fg - \frac{fg}{z-p} h \right) \in I$$

since $\frac{f}{z-p}$ is holomorphic.

We can iterate this process to find $\frac{f}{(z-p)^n} g \in I$. Since $\frac{f}{(z-p)^n}$ is a unit, g must be an element of I . □

Theorem 3.5.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal. If I has no zeroes and is closed, then $I = \mathcal{O}(\Omega)$.

Proof. Let f be an arbitrary nonzero element of I . By the Weierstraß product theorem, we can write

$$f = \prod_{k=1}^{\infty} f_k,$$

where each f_k has exactly one zero in Ω , and the tails

$$\hat{f}_n = \prod_{k=n}^{\infty} f_k$$

converge to 1 uniformly on compacts of Ω . As $f = \widehat{f}_1 = f_1 \widehat{f}_2$, we can apply the previous lemma to find $\widehat{f}_2 \in I$. Inductively, $\widehat{f}_n \in I$ and since the ideal I is assumed to be closed, we have

$$1 = \lim_{k \rightarrow \infty} \widehat{f}_k \in I. \quad \square$$

4 Approximation of holomorphic functions

4.1 Runge's little theorem

Lemma 4.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $K \subseteq \Omega$ a non-empty compact. Then there exist finitely many horizontal or vertical line segments $\sigma_1, \dots, \sigma_n$ of equal length in $\Omega \setminus K$ such that for all $f \in \mathcal{O}(\Omega)$ we have

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\sigma_k} \frac{f(\xi)}{\xi - z} d\xi$$

for all $z \in K$.

Proof. Define

$$\delta = \begin{cases} 1, & \Omega = \mathbb{C}, \\ d(K, \partial\Omega), & \Omega \neq \mathbb{C}. \end{cases}$$

Let Q be a grid of squares, parallel to the coordinate axes, with side length $d < \frac{\delta}{\sqrt{2}}$. As K is compact, it only intersects finitely many of them. Now just choose the boundary of the union of those squares. It is clear that all those segments are subsets of $\Omega \setminus K$.

Let Q_k denote the above squares. Then,

$$\frac{1}{2\pi i} \sum_{k=1}^m \oint_{\partial Q_k} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\sigma_k} \frac{f(\xi)}{\xi - z} d\xi.$$

If $z \in \text{Int } Q_\ell$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q_\ell} \frac{f(\xi)}{\xi - z} d\xi$$

and

$$\frac{1}{2\pi i} \int_{\partial Q_k} \frac{f(\xi)}{\xi - z} d\xi = 0$$

for all $k \neq \ell$, therefore the lemma holds for all such z . As both sides of the equations are continuous functions, they must agree on the whole set K . \square

Lemma 4.1.2. Let σ be a compact line segment and $K \subseteq \mathbb{C}$ be a compact set such that $K \cap \sigma = \emptyset$. Let $h \in \mathcal{C}(\sigma)$ be a function. Then for all $\varepsilon > 0$ there exist points $c_1, \dots, c_m \in \mathbb{C}$ and $w_1, \dots, w_m \in \sigma$ such that

$$\left\| \int_{\sigma} \frac{h(\xi)}{\xi - z} d\xi - \sum_{k=1}^m \frac{c_k}{z - w_k} \right\|_K < \varepsilon.$$

Proof. Let ℓ be the length of σ and define the function $v: \sigma \times K \rightarrow \mathbb{C}$ with

$$v(\xi, z) = \frac{h(\xi)}{\xi - z}.$$

It is clearly continuous, therefore it is uniformly continuous. In particular, there exists a $\delta > 0$ such that

$$|v(\xi, z) - v(\xi', z)| < \frac{\varepsilon}{\ell}$$

for all $z \in \mathbb{C}$ and $|\xi - \xi'| < \delta$.

Let τ_1, \dots, τ_m be the partition of σ into line segments of length $d < \sigma$. Choose points $w_k \in \tau_k$ and set $c_k = -h(w_k)d$. We therefore have

$$\left| \int_{\tau_k} v(\xi, z) d\xi - \frac{c_k}{z - w_k} \right| = \left| \int_{\tau_k} v(\xi, z) d\xi - d \cdot v(w_k, z) \right| < d \cdot \frac{\varepsilon}{\ell}.$$

Summing up, we get the desired inequality. \square

Lemma 4.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open set and $K \subseteq \Omega$ be a non-empty compact. Then there exist finitely many line segments $\sigma_1, \dots, \sigma_n$ in $\Omega \setminus K$ such that for any holomorphic function $f \in \mathcal{O}(\Omega)$ and $\varepsilon > 0$ there exists a rational function q of the form

$$q(z) = \sum_{k=1}^m \frac{c_k}{z - w_k},$$

where $c_k \in \mathbb{C}$ and $w_k \in \sigma$, such that

$$\|f - q\|_K < \varepsilon.$$

Proof. By lemma 4.1.1 there exist line segments σ_k such that

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\sigma_k} \frac{f(\xi)}{\xi - z} d\xi$$

for all $z \in K$. Now just apply lemma 4.1.2 to each line segment separately. \square

Lemma 4.1.4 (Shifting poles). Let $K \subseteq \mathbb{C}$ be a compact set, Z a connected component of $\mathbb{C} \setminus K$ and $a, b \in Z$. Then, for all $\varepsilon > 0$ there exists a polynomial q such that

$$\left\| \frac{1}{z - a} - q\left(\frac{1}{z - b}\right) \right\|_K < \varepsilon.$$

If Z is the unbounded component, we can approximate $\frac{1}{z-a}$ by $q(z)$ instead.

Proof. Let L_ω be the family of all functions that are holomorphic in a neighbourhood of K that can be approximated uniformly on K by polynomials in $\frac{1}{z-\omega}$. Note that, if $\frac{1}{z-p} \in L_q$, then $L_p \subseteq L_q$.

Consider the set

$$S = \left\{ s \in Z \mid \frac{1}{z-s} \in L_b \right\}.$$

We claim that $S = Z$. First note that $b \in S$, therefore $S \neq \emptyset$. Take any point $p \in S$ and $\delta > 0$ such that $\Delta(p, \delta) \subseteq Z$. For any $s \in \Delta(p, \delta)$, we can write

$$\frac{1}{z-s} = \frac{1}{z-p} \cdot \frac{1}{1 - \frac{s-p}{z-p}} = \frac{1}{z-p} \cdot \sum_{k=0}^{\infty} \left(\frac{s-p}{z-p} \right)^k,$$

which converges as $|s-p| < |z-p|$ by choice of δ . We conclude that $\Delta(p, \delta) \subseteq S$. In particular, S is an open set. Take any point $p \in \partial S \cap Z$. Suppose that $\Delta(p, 3\delta) \subseteq Z$ and

choose a point $p' \in \Delta(p, \delta) \cap S$. Then, $\Delta(p', 2\delta) \subseteq S$ and therefore $\Delta(p, \delta) \subseteq S$, which is a contradiction and therefore proves our claim.

Now suppose that Z is the unbounded component and take a point $d \in Z$ such that $K \subseteq \Delta(0, |d|)$. Then all functions $\left(\frac{1}{z-d}\right)^n$ can be approximated uniformly on K by Taylor's polynomials about 0. \square

Definition 4.1.5. For any set $P \subseteq \mathbb{C}$ denote by $\mathbb{C}_P[z]$ the family of rational functions with poles in P .

Theorem 4.1.6 (Runge's approximation theorem). Let $K \subseteq \mathbb{C}$ be a compact. If P intersects every bounded connected component of $\mathbb{C} \setminus K$, then for every function f , holomorphic in a neighbourhood of K , and every $\varepsilon > 0$ there exists a function $q \in \mathbb{C}_P[z]$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. By lemma 4.1.3, we can find a compact union σ of line segments such that every such f can be approximated by

$$\tilde{q} = \sum_{k=1}^m \frac{c_k}{z - w_k},$$

where $c_k \in \mathbb{C}$ and $w_k \in \sigma$. Suppose then

$$\|f - \tilde{q}\|_K < \frac{\varepsilon}{2}.$$

Let Z_k be the connected component of $\mathbb{C} \setminus K$ that contains w_k .

If Z_k is bounded, then choose $t_k \in P \cap Z_k$. By the pole shifting lemma, we can find a polynomial g_k such that

$$\left\| \frac{c_k}{z - w_k} - g_k(1)z - t_k \right\|_K < \frac{\varepsilon}{2m}.$$

If Z_k is unbounded, we can instead approximate $\frac{c_k}{z - w_k}$ by a polynomial instead. Choosing

$$q = \sum_{k=1}^m g_k,$$

we find

$$\|f - q\|_K \leq \|f - \tilde{q}\|_K + \|\tilde{q} - q\|_K < \frac{\varepsilon}{2} + m \cdot \frac{\varepsilon}{2m} = \varepsilon. \quad \square$$

Theorem 4.1.7 (Runge's approximation theorem). Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $K \subseteq \Omega$ be a compact. If every bounded component of $\mathbb{C} \setminus K$ intersects $\mathbb{C} \setminus \Omega$, then for every function f that is holomorphic on a neighbourhood of K and every $\varepsilon > 0$ there exists a function $q \in \mathcal{O}(\Omega)$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Choose $P = \mathbb{C} \setminus \Omega$ in theorem 4.1.6. \square

Corollary 4.1.7.1 (Runge's little theorem). Let $K \subseteq \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. Then for every holomorphic function on a neighbourhood of K and $\varepsilon > 0$ there exists a polynomial $q \in \mathbb{C}[z]$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Choose $P = \emptyset$ in theorem 4.1.6. □

Definition 4.1.8. Let V be a vector space over \mathbb{C} , equipped with a topology, and let $T: V \rightarrow V$ be a linear map.

i) We call T *cyclic* if there exists some $f \in V$, called a *cyclic vector*, such that

$$\text{span}_{\mathbb{C}} \{T^n(f) \mid n \in \mathbb{N}_0\} = V.$$

ii) We call T *hypercyclic* if there exists some $f \in V$, such that

$$\overline{\{T^n(f) \mid n \in \mathbb{N}_0\}} = V.$$

Theorem 4.1.9 (Birkhoff). Let $\tau: \mathbb{C} \rightarrow \mathbb{C}$ be given by $\tau(z) = z + a$ for some $a \neq 0$. Then the map $T: \mathcal{O}(\mathbb{C}) \rightarrow \mathcal{O}(\mathbb{C})$, given by $T(f) = f \circ \tau$, is hypercyclic.

Proof. Set $K_n = \overline{\Delta(\ell_n \cdot a, n)}$ for a sequence $(\ell_n)_n \subseteq \mathbb{N}$, such that all K_n are pairwise disjoint and $K_n \cap \Delta(0, n) = \emptyset$ for all $n \in \mathbb{N}$. Choose a sequence $(\varepsilon_n)_n \subseteq \mathbb{R}^+$, converging to 0. Furthermore, let $(p_n)_n$ be a sequence of all polynomials $(\mathbb{Q} \oplus i\mathbb{Q})[z]$.

We first construct a sequence of holomorphic functions $(f_n)_n$, such that the following conditions hold:

i) For all $n \in \mathbb{N}$, we have

$$\left\| \sum_{k=1}^n f_k - \tau^{-\ell_n} \circ p_n \right\|_{K_n} < \frac{\varepsilon_n}{2^n}.$$

ii) For all $m < n$, we have

$$\|f_n\|_{K_m} < \frac{\varepsilon_m}{2^n}.$$

iii) For all $n \in \mathbb{N}$, we have

$$\|f_n\|_{\Delta(0, n)} < \frac{1}{2^n}.$$

Choose ℓ_n inductively such that $K_n \cap K_m = \emptyset$ for all $m < n$. Then the set

$$\mathbb{C} \setminus \left(\bigcup_{m=1}^n K_m \cup \Delta(0, n) \right)$$

is obviously connected. We can now just apply Runge's little theorem by choosing the function 0 on $\Delta(0, n)$ and K_m for $m < n$, and

$$\tau^{-\ell_n} \circ p_n - \sum_{k=1}^{n-1} f_k$$

on K_n .

Note that the series

$$f = \sum_{n=1}^{\infty} f_n$$

is uniformly convergent by the third condition. We now compute

$$\begin{aligned} \|f \circ \tau^{\ell_n} - p_n\|_{\tau^{-\ell_n}(K_n)} &= \|f - \tau^{-\ell_n} \circ p_n\|_{K_n} \\ &\leq \sum_{k>n} \|f_k\|_{K_n} + \left\| \sum_{k=1}^n f_k - \tau^{-\ell_n} \circ p_n \right\|_{K_n} \\ &< \varepsilon_n. \end{aligned}$$

It follows that we can approximate every rational polynomial with iterations $T^n(f)$. \square

Lemma 4.1.10. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $K \subseteq \Omega$ a compact subset.

- i) For every component Z of $\Omega \setminus K$ we have $\Omega \cap \partial Z \subseteq K$. If Z is relatively compact in Ω , then $\|f\|_Z \leq \|f\|_K$ for all $f \in \mathcal{O}(\Omega)$.
- ii) Every component Z_0 of $\mathbb{C} \setminus K$ that is contained in Ω is a component of $\Omega \setminus K$. If Z_0 is bounded, then $Z_0 \subseteq \Omega$ is relatively compact.

Proof. Let p be a point in $\Omega \cap \partial Z$ that is not in K . Then there exists a disk D , centered at p , that is contained in $\Omega \setminus K$. Note that every point of D is in the same connected components as p , which implies $p \in \text{Int } Z$, a contradiction.

Suppose now that Z is relatively compact. As $\partial Z \subseteq K$, we must have

$$\|f\|_Z = \|f\|_{\partial Z} \leq \|f\|_K$$

by the maximum modulus principle.

For the second item, let Z_0 be a component of $\mathbb{C} \setminus K$ contained in Ω . There must exist a connected component $Z_1 \subseteq \Omega \setminus K$ that contains Z_0 . Since Z_1 is also a connected domain in $\mathbb{C} \setminus K$, it follows that $Z_0 = Z_1$.

Finally, suppose that Z_0 is bounded. The closure of Z_0 in \mathbb{C} is contained in Ω by the first item. But then it is compact as it is bounded and closed in Ω . \square

Theorem 4.1.11. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $K \subseteq \Omega$ a compact subset. The following statements are equivalent:

- i) The set $\Omega \setminus K$ has no relatively compact components.
- ii) Every bounded component of $\mathbb{C} \setminus K$ intersects $\mathbb{C} \setminus \Omega$.
- iii) Every function, holomorphic in a neighbourhood of K , can be approximated uniformly on K by rational functions without poles in Ω .
- iv) Every function, holomorphic in a neighbourhood of K , can be approximated uniformly by functions, holomorphic in Ω .
- v) For every point $p \in \Omega \setminus K$ there exists a function $f \in \mathcal{O}(\Omega)$ such that $|f(p)| > \|f\|_K$.
- vi) We have $\widehat{K}_{\mathcal{O}(\Omega)} = K$.

Proof. As usual, we will prove a sequence of implications.

Suppose first that $\Omega \setminus K$ has no relatively compact components. If Z_0 is a bounded component of $\mathbb{C} \setminus K$ that is contained in Ω , then it is also a component of $\Omega \setminus K$ and relatively compact in Ω by the previous lemma. This is a clear contradiction, therefore **i)** implies **ii)**.

From theorem 4.1.6 we immediately get that statement **ii)** implies **iii)**. Trivially, we have that **iii)** implies **iv)**.

Suppose now that **iv)** holds and that the set $\Omega \setminus K$ has a relatively compact component Z . Choose a point $a \in Z$ and consider the function $f(z) = \frac{1}{z-a}$. By our assumption, there exists a function $g \in \mathcal{O}(\Omega)$ such that

$$\left\| \frac{1}{z-a} - g \right\|_K < \frac{1}{\|z-a\|_K}.$$

Then $\|1 - (z-a)g\|_K < 1$. By the previous lemma, we conclude

$$\|1 - (z-a)g\|_Z \leq \|1 - (z-a)g\|_K < 1,$$

but this is not the case for $z = a$. This implies **i)**.

As **v)** and **vi)** are equivalent by definition, it remains to check that **v)** is equivalent to **i)**.

Suppose that **v)** holds and that Z is a relatively compact component of $\Omega \setminus K$. As before, we have that $\|f\|_Z \leq \|f\|_K$ for all functions $f \in \mathcal{O}(\Omega)$. In particular, this holds for f from the assumption of statement **v)**, which is again a contradiction.

Finally, suppose that $\Omega \setminus K$ has no relatively compact components and let $p \in \Omega \setminus K$ be a point. Consider the set $\Omega' = \Omega \setminus (K \cup \{p\})$, which has the same connected components as $\Omega \setminus K$, and are therefore still not relatively compact.⁸ From the previous arguments, it follows that **iv)** holds for Ω' . In particular, let g be a holomorphic function, defined on a neighbourhood of $K \cup \{p\}$, with $g|_K = 0$ and $g(p) = 1$. Thus there exist a function $h \in \mathcal{O}(\Omega)$ such that

$$\|g - h\|_{K \cup p} < \frac{1}{2}.$$

Then $\|h\|_K < \frac{1}{2} < |h(p)|$, which is exactly the statement **v)**. \square

Lemma 4.1.12. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $K \subseteq \Omega$ a compact subset. Then there exists a compact set K' with $K \subseteq K' \subseteq \Omega$, such that every bounded component of $\mathbb{C} \setminus K'$ intersects $\mathbb{C} \setminus \Omega$ and contains a compact component of $\mathbb{C} \setminus \Omega$.

Proof. For the case $\Omega = \mathbb{C}$, we can take $K' = \overline{\mathbb{D}(0, r)}$, where r is large enough so that $K \subseteq K'$. Suppose then that $\Omega \neq \mathbb{C}$. Choose $0 < \rho < d(K, \partial\Omega) = d(K, \mathbb{C} \setminus \Omega)$ and let

$$M = \{z \in \Omega \mid d(z, \mathbb{C} \setminus \Omega) \geq \rho\}.$$

Choose $r > 0$ such that $K \subseteq \overline{\mathbb{D}(0, r)}$ and set $K' = M \cap \overline{\mathbb{D}(0, r)}$, which is clearly compact, contained inside Ω , and contains K . It remains to prove the statement about the bounded components.

⁸ We did remove a point from one of the components, but this property remains unchanged.

Let Z be a bounded component of $\mathbb{C} \setminus K'$. Note that $\mathbb{C} \setminus K' = (\mathbb{C} \setminus M) \cup (\mathbb{C} \setminus \overline{\Delta(0, r)})$. As Z is bounded, it must then be contained in $\mathbb{C} \setminus M$, as $\mathbb{C} \setminus \overline{\Delta(0, r)}$ is connected and unbounded.

We can rewrite

$$\mathbb{C} \setminus M = \bigcup_{w \in \mathbb{C} \setminus \Omega} \Delta(w, \rho)$$

by the definition of M . For each such disk, we either have $\Delta(w, \rho) \subseteq Z$ or $\Delta(w, \rho) \cap Z = \emptyset$. Thus we can write

$$Z = \bigcup_{w \in Z \setminus \Omega} \Delta(w, \rho).$$

In particular, Z intersects $\mathbb{C} \setminus \Omega$. Let S be the component of $\mathbb{C} \setminus \Omega$ that Z intersects. As $\mathbb{C} \setminus \Omega \subseteq \mathbb{C} \setminus K'$, S is also connected in $\mathbb{C} \setminus K'$. As Z is a maximal connected subset, we conclude that $S \subseteq Z$, as desired. \square

Definition 4.1.13. Let $\Omega \subseteq \mathbb{C}$ be an open subset. We call every bounded component of $\mathbb{C} \setminus \Omega$ a *hole* of Ω .

Remark 4.1.13.1. If K is the Cantor set, then $\Omega = \mathbb{C} \setminus K$ has uncountably many holes.

Theorem 4.1.14 (Runge's theorem for rational approximations). Let $\Omega \subseteq \mathbb{C}$ be an open set and $P \subseteq \mathbb{C} \setminus \Omega$ a set such that \overline{P} intersects every hole of Ω . Then $\mathbb{C}_P[z]$ is dense in $\mathcal{O}(\Omega)$.

Proof. Let $K \subseteq \Omega$ be compact subset. Construct K' as in the previous lemma. Then \overline{P} intersects every bounded component of $\mathbb{C} \setminus K'$. We can thus apply theorem 4.1.6 for the compact K' . \square

Corollary 4.1.14.1 (Runge's theorem for polynomial approximation). If Ω has no holes then $\mathbb{C}[z]$ is dense in $\mathcal{O}(\Omega)$.

Proof. Take $P = \emptyset$. \square

Theorem 4.1.15 (Mittag-Leffler for open subsets). Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k, q_k)_k$ be a distribution of principal parts in Ω and $A = \{a_k \mid k \in \mathbb{N}\}$. Then there exists a Mittag-Leffler series for (a_k, q_k) that converges normally in $\mathbb{C} \setminus A'$ where $A' = \overline{A} \setminus A$.

Proof. This is the same theorem as 3.4.16. We present another proof using Runge's approximation theorem.

First note that $A' \subseteq \partial\Omega$. Let

$$K_j = \overline{\left\{ z \in \Omega \mid |z| \leq j \wedge d(z, \partial\Omega) \geq \frac{1}{j} \right\}},$$

which form an exhaustion by compacts. Let K'_j be the compact furnished by the previous lemma for K_j . Choose a subsequence $(L_j)_j$ of $(K'_j)_j$ that is an exhaustion by compacts, which can be done by the properties of exhaustion, i.e. for every K'_j there exists some $k \in \mathbb{N}$ so that $K'_j \subseteq K_k \subseteq K'_k$.

We may assume $K_1 \cap \overline{A} = \emptyset$. Let $A_n = A \cap (L_{n+1} \setminus L_n)$. For each $a \in A_n$ the principal part of q is holomorphic in L_n . By Runge's theorem, there exists a function $g \in \mathcal{O}(\Omega)$ such that

$$\|q - g\|_{L_n} < \frac{1}{2^n \cdot |A_n|}.$$

Then

$$\sum_{n=1}^{\infty} \|q_n - g_n\|_{L_n} \leq \sum_{n=1}^{\infty} 2^{-n} = 1,$$

thus the series converges uniformly on compacts, as each compact subset is contained in all but finitely many L_n . \square

Definition 4.1.16. A pair of sets (Ω, Ω') is a *Runge pair* if for every function $f \in \mathcal{O}(\Omega)$, $\varepsilon > 0$ and compact $K \subseteq \Omega$ there exists a function $F \in \mathcal{O}(\Omega')$ such that

$$\|f - F\|_K < \varepsilon.$$

Theorem 4.1.17. Let $\Omega \subseteq \Omega' \subseteq \mathbb{C}$ be open sets. The following statements are equivalent:

- i) The set $\Omega' \setminus \Omega$ has no compact components.
- ii) The set $\mathbb{C}_P[z]$ with $P = \mathbb{C} \setminus \Omega'$ is dense in $\mathcal{O}(\Omega)$.
- iii) The pair (Ω, Ω') is a Runge pair.
- iv) The set $\Omega' \setminus \Omega$ contains no non-empty open compact set.

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