

Algebraic topology 1

Luka Horjak (luka1.horjak@gmail.com)

October 6, 2023

Contents

Introduction	3
1 Basic homotopy theory	4
1.1 Definition	4
1.2 Homotopy equivalence	5
Index	6

Introduction

These are my lecture notes on the course Algebraic topology 1 in the year 2023/24. The lecturer that year was prof. dr. Petar Pavešić.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Basic homotopy theory

1.1 Definition

Definition 1.1.1. Continuous maps $f, g: X \rightarrow Y$ of topological spaces are *homotopic*, if there is a continuous map $H: X \times I \rightarrow Y$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such H is called a *homotopy*. We write $H: f \simeq g$.

Remark 1.1.1.1. If X is a locally compact and Hausdorff space, homotopies coincide with paths in the space $\mathcal{C}(X, Y)$.

Proposition 1.1.2. Homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.

Proof. The proof is obvious and need not be mentioned. □

Definition 1.1.3. We denote the set of equivalence classes of the homotopy relation on $\mathcal{C}(X, Y)$ by $[X, Y]$.

Remark 1.1.3.1. If X is a locally compact and Hausdorff space, $[X, Y]$ is the set of path components of $\mathcal{C}(X, Y)$.

Definition 1.1.4. With $f: (X, A) \rightarrow (Y, B)$ we denote maps $f: X \rightarrow Y$ such that $f(A) \subseteq B$. Similarly, we define $\mathcal{C}((X, A), (Y, B))$ and $[(X, A), (Y, B)]$.

Definition 1.1.5. Let $A \subseteq X$ and $f, g: X \rightarrow Y$ be maps, such that $f|_A = g|_A$. $G: X \times I \rightarrow Y$ is a *homotopy relative to A* if $H: f \simeq g$ and $H_t|_A = f|_A$ for all $t \in I$.

Definition 1.1.6. A map $f: X \rightarrow Y$ is *null-homotopic* if it is homotopic to a constant.

Definition 1.1.7. Let X be a topological space. The *cone* on X is the space

$$CX = X \times I / X \times \{1\}.$$

Proposition 1.1.8. A map $f: X \rightarrow Y$ is null-homotopic if and only if it extends to the cone CX .

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow i & \nearrow F & \\ CX & & \end{array}$$

Proof. In the following diagram, H exists if and only if F exists.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow i_0 & \nearrow H & \uparrow F \\ X \times I & \xrightarrow{q} & CX \end{array}$$

□

1.2 Homotopy equivalence

Proposition 1.2.1. Let $f, f': X \rightarrow Y$ and $g, g': Y \rightarrow Z$ be maps. If $f \simeq g$ and $f' \simeq g'$, we also have $g \circ f \simeq g' \circ f'$.

Proof. Let $H: X \times I \rightarrow Y$ and $K: Y \times I \rightarrow Z$ be the two homotopies. It is trivial to check that

$$L(x, t) = K(H(x, t), t)$$

is a homotopy of the compositions. □

Definition 1.2.2. The *homotopy category* $\underline{\text{HoTop}}$ is the category with topological spaces as objects and homotopies as morphisms. Operations are induced by the compositions of maps.

Definition 1.2.3. The category $\underline{\text{Top}}^2$ has pairs of spaces (X, A) with $A \subseteq X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}^2$.

Definition 1.2.4. The category $\underline{\text{Top}}_\bullet$ has pairs (X, x_0) with $x_0 \in X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}_\bullet$.

Definition 1.2.5. *Homotopy equivalence* is an isomorphism in the category $\underline{\text{HoTop}}$. If spaces X and Y are homotopy equivalent, we write $X \simeq Y$.

Remark 1.2.5.1. A map $f: X \rightarrow Y$ is a homotopy equivalence if there exists a map $g: Y \rightarrow X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. The map g is a *homotopy inverse* of f .

Definition 1.2.6. A space X is *contractible* if it is homotopy equivalent to a point.

Remark 1.2.6.1. Every cone is contractible.

Proposition 1.2.7. Let X be a topological space. The following statements are equivalent:

- i) The space X is contractible.
- ii) The map id_X is homotopy equivalent to a constant map.
- iii) The space X is a retract of CX .

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.2.8. Let X and Y be closed surfaces. If $X \simeq Y$, then $X \approx Y$.

Definition 1.2.9. A subspace $A \subseteq X$ is a *deformation retract* of X if it is a retract and the retraction is a homotopy inverse of the inclusion.

Definition 1.2.10. A subspace $A \subseteq X$ is a *strong deformation retract* if it is a retract and the retraction is a homotopy inverse of the inclusion relative to A .

Index

C

cone, [4](#)

contractible space, [5](#)

D

deformation retract, [5](#)

H

homotopic, [4](#)

homotopy, [4](#)

category, [5](#)

equivalence, [5](#)

inverse, [5](#)

relative to a subspace, [4](#)

N

null-homotopic, [4](#)

S

strong deformation retract, [5](#)