# Algebraic topology 1

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# Contents

# Introduction

These are my lecture notes on the course Algebraic topology 1 in the year 2023/24. The lecturer that year was prof. dr. Petar Pavešić.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

# 1 Basic homotopy theory

## 1.1 Definition

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**Definition 1.1.1.** Continuous maps $f, g\colon X \to Y$ of topological spaces are *homotopic*, if there is a continuous map $H\colon X \times I \to Y$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such $H$ is called a *homotopy*. We write $H\colon f \simeq g$.

**Remark 1.1.1.1.** If $X$ is a locally compact and Hausdorff space, homotopies coincide with paths in the space $\mathcal{C}(X, Y)$.

**Proposition 1.1.2.** Homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.

*Proof.* The proof is obvious and need not be mentioned. □

**Definition 1.1.3.** We denote the set of equivalence classes of the homotopy relation on $\mathcal{C}(X, Y)$ by $[X, Y]$.

**Remark 1.1.3.1.** If $X$ is a locally compact and Hausdorff space, $[X, Y]$ is the set of path components of $\mathcal{C}(X, Y)$.

**Definition 1.1.4.** With $f\colon (X, A) \to (Y, B)$ we denote maps $f\colon X \to Y$ such that $f(A) \subseteq f(B)$. Similarly, we define $\mathcal{C}((X, A), (Y, B))$ and $[(X, A), (Y, B)]$.

**Definition 1.1.5.** Let $A \subseteq X$ and $f, g\colon X \to Y$ be maps satisfying $f|_A = g|_A$. The map $G\colon X \times I \to Y$ is a *homotopy relative to $A$* if $H\colon f \simeq g$ and $H_t|_A = f|_A$ for all $t \in I$. We write $H\colon f \simeq g$ (rel $A$).

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**Definition 1.1.6.** A map $f\colon X \to Y$ is *null-homotopic* if it is homotopic to a constant.

**Definition 1.1.7.** Let $X$ be a topological space. The *cone* on $X$ is the space

$$CX = {}^{X \times I}\big/_{X \times \{1\}} \,.$$

**Proposition 1.1.8.** A map $f\colon X \to Y$ is null-homotopic if and only if it extends to the cone $CX$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\
CX \arrow[ur, dashed, "F"'] &
\end{tikzcd}$$

*Proof.* In the following diagram, $H$ exists if and only if $F$ exists.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_0"'] & Y \\
X \times I \arrow[ur, dashed, "H"] \arrow[r, "q"'] & CX \arrow[u, dashed, "F"']
\end{tikzcd}$$

□

## 1.2 Homotopy equivalence

**Proposition 1.2.1.** Let $f, f' \colon X \to Y$ and $g, g' \colon Y \to Z$ be maps. If $f \simeq g$ and $f' \simeq g'$, we also have $g \circ f \simeq g' \circ f'$.

*Proof.* Let $H \colon X \times I \to Y$ and $K \colon Y \times I \to Z$ be the two homotopies. It is trivial to check that

$$L(x,t) = K(H(x,t),t)$$

is a homotopy of the compositions. □

**Definition 1.2.2.** The *homotopy category* $\underline{\text{HoTop}}$ is the category with topological spaces as objects and homotopies as morphisms. Operations are induced by the compositions of maps.

**Definition 1.2.3.** The category $\underline{\text{Top}}^2$ has pairs of spaces $(X, A)$ with $A \subseteq X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}^2$.

**Definition 1.2.4.** The category $\underline{\text{Top}}_\bullet$ has pairs $(X, x_0)$ with $x_0 \in X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}_\bullet$.

**Definition 1.2.5.** *Homotopy equivalence* is an isomorphism in the category $\underline{\text{HoTop}}$. If spaces $X$ and $Y$ are homotopy equivalent, we write $X \simeq Y$.

**Remark 1.2.5.1.** A map $f \colon X \to Y$ is a homotopy equivalence if there exists a map $g \colon Y \to X$ such that both $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$ hold. The map $g$ is a *homotopy inverse* of $f$.

**Definition 1.2.6.** A space $X$ is *contractible* if it is homotopy equivalent to a point.

**Remark 1.2.6.1.** Every cone is contractible.

**Proposition 1.2.7.** Let $X$ be a topological space. The following statements are equivalent:

i) The space $X$ is contractible.

ii) The map $\text{id}_X$ is homotopy equivalent to a constant map.

iii) The space $X$ is a retract of $CX$.

*Proof.* The proof is obvious and need not be mentioned. □

**Theorem 1.2.8.** Let $X$ and $Y$ be closed surfaces. If $X \simeq Y$, then $X \approx Y$.

**Definition 1.2.9.** A subspace $A \subseteq X$ is a *deformation retract* of $X$ if it is a retract and the retraction is a homotopy inverse of the inclusion.

**Definition 1.2.10.** A subspace $A \subseteq X$ is a *strong deformation retract* if it is a retract and the retraction is a homotopy inverse of the inclusion relative to $A$.

## 1.3 Extensions of homotopies

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**Definition 1.3.1.** A closed subspace $A \subseteq X$ has the *homotopy extension property* if for every space $Y$, map $f \colon X \to Y$ and homotopy $H \colon A \times I \to Y$ with $H_0 = f|_A$ there exists a homotopy $\overline{H} \colon X \times I \to Y$ such that $\overline{H}_0 = f$ and $\overline{H}\Big|_A = H$.

**Proposition 1.3.2.** A closed subspace $A \subseteq X$ has the homotopy extension property if an only if the space

$$L = A \times I \cup X \times \{0\}$$

is a retract of $X \times I$.

*Proof.* Suppose that $L$ is a retract of $X$. It is easy to see that $\overline{H} = (H \cup f) \circ r$ is the required homotopy extension, where $r \colon X \times I \to L$ is a retraction.

Now suppose that $A$ has the homotopy extension property. Define $i_0 \colon X \hookrightarrow L$ and $H \colon A \times I \hookrightarrow L$ as the inclusions. By the homotopy extension property, there exists a homotopy $\overline{H} \colon X \times I \to L$, which is of course a retraction. □

**Proposition 1.3.3.** Let $A \subseteq X$ be a contractible subspace. If $A$ has the homotopy extension property, then $q \colon X \to X/A$ is a homotopy equivalence.

*Proof.* Let $K \colon A \times I \to A$ be a homotopy equivalence between $\mathrm{id}_A$ and the constant map. Then $K \cup \mathrm{id}_X \colon A \times I \cup X \to X$ is a well defined map. Let $H \colon X \times I \to X$ be its extension.

$$\begin{array}{ccc} X \times I & \xrightarrow{H} & X \\ {\scriptstyle q \times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle q} \\ X/A \times I & \overset{\overline{H}}{\dashrightarrow} & X/A \end{array}$$

The map $q \times \mathrm{id}_I$ is a quotient map because $I$ is locally compact. The induced map $\overline{H}$ is therefore well defined. But now $H_1$ induces a map $h \colon X/A \to X$. Note that, by definition, $h \circ q = H_1 \simeq H_0 = \mathrm{id}_X$, and $q \circ h = \overline{H}_1 \simeq \overline{H}_0 = \mathrm{id}_{X/A}$. □

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**Proposition 1.3.4.** Suppose the map $A \hookrightarrow X$ has the homotopy extension property and $f, g \colon A \to Y$ are homotopic. Then we have $X \cup_f Y \simeq X \cup_g Y$.

**Proposition 1.3.5.** Suppose the map $A \hookrightarrow X$ has the homotopy extension property. If $A \hookrightarrow X$ is a homotopy equivalence, then $A$ is a strong deformation retract of $X$.

**Definition 1.3.6.** Let $f \colon X \to Y$ be a map. The *mapping cylinder* of $f$ is the space

$$M_f = X \times I \sqcup Y \big/ (x,0) \sim f(x) \,.$$

**Remark 1.3.6.1.** The subspace $X \approx X \times \{1\} \subseteq M_f$ has the homotopy extension property.

**Proposition 1.3.7.** Suppose that $f \colon X \to Y$ is a homotopy equivalence. Then both $X$ and $Y$ are strong deformation retracts of $M_f$.

*Proof.* Note that the map $H((x, s), t) = [(x, s(1-t)]$ is a homotopy between id and a retraction $r\colon M_f \to Y$, therefore $Y$ is a strong deformation retract of $M_f$. In particular, $r$ is a homotopy equivalence. Denote by $i\colon X \hookrightarrow M_f$ the inclusion of $X \simeq X \times \{1\}$ in $M_f$. Note that $f = r \circ i$, therefore $i$ is also a homotopy equivalence. It follows that $X$ is also a strong deformation retract of $M_f$. $\square$

# 2 Fundamental group

## 2.1 Definition

**Definition 2.1.1.** Let $\alpha, \beta \colon I \to X$ be two paths with $\alpha(1) = \beta(0)$. The *concatenation product* is the path

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & t \leq \frac{1}{2}, \\ \beta(2t-1), & t \geq \frac{1}{2}. \end{cases}$$

**Definition 2.1.2.** Let $X$ be a topological space and $x_0 \in X$. The set of *loops* with starting point $x_0$ is the set

$$\Omega(X, x_0) = \mathcal{C}((I, \partial I), (X, x_0)).$$

**Theorem 2.1.3.** Concatenation on $\Omega(X, x_0)$ induces a group structure on

$$\Omega(X, x_0) \big/ \simeq \ \operatorname{rel} \partial I \ .$$

**Definition 2.1.4.** The above group is called the *fundamental group* and is denoted by $\pi_1(X, x_0)$.

**Remark 2.1.4.1.** Suppose that $C_{x_0}$ is the path-component of $X$ containing $x_0$. Then we have $\pi_1(X, x_0) = \pi_1(C_{x_0}, x_0)$.

**Definition 2.1.5.** Suppose $x_0, x_1 \in X$ are connected by a path $\gamma \colon (I, 0, 1) \to (X, x_0, x_1)$. Define the map $\operatorname{tr}_\gamma \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ as $\operatorname{tr}_\gamma([\alpha]) = [\overline{\gamma} \cdot \alpha \cdot \gamma]$.

**Proposition 2.1.6.** Suppose $x_0, x_1$ are connected by a path $\gamma \colon (I, 0, 1) \to (X, x_0, x_1)$. Then the map $\operatorname{tr}_\gamma$ is an isomorphism with inverse $(\operatorname{tr}_\gamma)^{-1} = \operatorname{tr}_{\overline{\gamma}}$.

*Proof.* The proof is obvious and need not be mentioned. □

**Proposition 2.1.7.** For paths $\gamma, \delta \colon (I, 0, 1) \to (X, x_0, x_1)$ we have

$$\operatorname{tr}_\delta = \operatorname{tr}_{\overline{\gamma}\delta} \circ \operatorname{tr}_\gamma \ .$$

*Proof.* The proof is obvious and need not be mentioned. □

**Remark 2.1.7.1.** If $\pi_1(X, x_1)$ is commutative, the tr-isomorphisms are independent from the chosen paths. We write $\pi_1(X) = \pi_1(X, x_0)$.

**Proposition 2.1.8.** Let $f \colon (X, x_0) \to (Y, y_0)$ be a map. Then $f$ induces a homomorphism $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

*Proof.* The proof is obvious and need not be mentioned. □

**Proposition 2.1.9.** The fundamental group is a functor $\pi_1 \colon \underline{\text{HoTop}}_\bullet \to \underline{\text{Grp}}$.

**Corollary 2.1.9.1.** The following statements are true:

i) If $f \colon (X, x_0) \to (Y, y_0)$ is a homotopy equivalence relative to $x_0$, the induced map $f_* \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism.



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ii) If $f\colon X \to Y$ is a homotopy equivalence, then the map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

iii) If $X$ is contractible, then $\pi(X, x_0) \cong \{1\}$.

iv) For $x_0 \in X$ and $y_0 \in Y$ we have that

$$((p_X)_*, (p_Y)_*) : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is an isomorphism.

v) If $A$ is a retract of $X$, then $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a monomorphism for all $x_0 \in A$. If $\pi_1(X, x_0)$ is commutative, the group $\pi_1(A, x_0)$ is a direct summand.

*Proof.* Assume that $A$ is a retract of $X$. As $\pi_1\colon \underline{\text{HoTop}}_\bullet \to \underline{\text{Grp}}$ is a functor, we can apply it to the diagram

$$(A, x_0) \xhookrightarrow{i} (X, x_0) \xrightarrow{r} (A, x_0).$$

It follows that $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a monomorphism. $\square$

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**Theorem 2.1.10** (Fundamental group of $S^1$)**.** The fundamental group of $S^1$ is isomorphic to $\mathbb{Z}$.

*Proof.* Let $\alpha\colon (I, 0, 1) \to (S^1, 1, 1)$ be a loop. We can lift the map $\alpha$ to a unique map $\widetilde{\alpha}\colon (I, 0) \to (\mathbb{R}, 0)$, as $\mathbb{R}$ is a covering space of $S^1$. Observe the map $\Phi_0\colon \Omega(S^1, 1) \to \mathbb{Z}$, given by $\Phi_0(\alpha) = \alpha(1)$. As any homotopy $h\colon \alpha \simeq \beta \ (\mathrm{rel}\, \partial I)$ can be lifted to a unique homotopy $H\colon \widetilde{\alpha} \simeq \widetilde{\beta} \ (\mathrm{rel}\, \partial I)$, the map $\Phi_0$ is constant on equivalence classes and therefore induces a map $\Phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$. Note that it is an isomorphism, therefore, $\pi_1(S^1, 1) \cong \mathbb{Z}$. $\square$

**Corollary 2.1.10.1** (Brouwer)**.** Every map $f\colon B^2 \to B^2$ has a fixed point.

*Proof.* If $f$ had no fixed point, we could construct a retraction $r\colon B^2 \to S^1$. This is of course not possible, as $\mathbb{Z}$ is not a subgroup of the trivial group. $\square$

**Proposition 2.1.11.** Let $q\colon I \to S^1$ be the quotient map. Then, the induced map $q^*\colon [(S^1, 1), (X, x_0)] \to \pi_1(X, x_0)$ is an isomorphism of groups.

**Definition 2.1.12.** The *degree* of $\alpha$ is the number $\deg \alpha$, where $\deg\colon [(S^1, 1), (S^1, 1)] \to \mathbb{Z}$ is the above isomorphism.

**Theorem 2.1.13** (Borsuk-Ulam)**.** For any continuous map $f\colon S^1 \to \mathbb{R}$ there exists some $x \in S^1$ such that $f(x) = f(-x)$.

*Proof.* Let $g(x) = f(x) - f(-x)$. Note that $g$ is an odd function and assume that $g(x) \neq 0$ for all $x$. Observe that

$$\frac{g}{|g|}\colon S^1 \to \{-1, 1\}$$

is a well defined, continuous odd function. This is of course impossible, as $S^1$ is connected. $\square$

**Lemma 2.1.14.** For any odd function $f\colon S^1 \to S^1$ we have $2 \nmid \deg(f)$.

*Proof.* Without loss of generality assume $f(1) = 1$. Let $q\colon I \to S^1$ be the quotient map. Note that we can lift the map $f \circ q$ to a map $F\colon I \to \mathbb{R}$ with $F(0) = 0$. Then, $\deg(f) = F(1)$. Note that

$$f \circ q\left(\frac{1}{2}\right) = f(-1) = -1,$$

so $F\left(\frac{1}{2}\right) = k + \frac{1}{2}$. As $f$ is odd, the map

$$G(x) = \begin{cases} F(x), & x \le \frac{1}{2}, \\ F\left(\frac{1}{2}\right) + F\left(x - \frac{1}{2}\right), & x \ge \frac{1}{2} \end{cases}$$

is also a lift of $f \circ q$. By uniqueness, we have $F = G$ and therefore $F(1) = 2k + 1$. □

**Corollary 2.1.14.1.** There are no odd maps $f\colon S^n \to S^1$ for $n > 1$.

*Proof.* Assume that such a map $f$ exists. Note that the inclusion $\iota\colon S^1 \hookrightarrow S^n$ is null-homotopic. But then $f \circ \iota$ should also be null-homotopic, hence have degree 0. □

**Theorem 2.1.15** (Borsuk-Ulam)**.** For any map $f\colon S^2 \to \mathbb{R}^2$ there exists some $x \in S^2$ such that $f(x) = -f(-x)$.

*Proof.* Suppose otherwise and note that $g\colon S^2 \to S^1$, given by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|},$$

is an odd map. □

**Remark 2.1.15.1.** The theorem holds for all maps $f\colon S^n \to \mathbb{R}^n$.

**Theorem 2.1.16** (Stone-Tukey)**.** Let $A$, $B$ and $C$ be bounded measurable subsets of $\mathbb{R}^3$. Then there exists a plane which bisects all of $A$, $B$ and $C$.[1]

**Theorem 2.1.17** (Lusternik-Schnirelmann)**.** Let $A$, $B$ and $C$ form a closed cover of $S^2$. Then, at least one of them contains a pair of antipodal points.

**Theorem 2.1.18** (Fundamental theorem of algebra)**.** Every non-constant polynomial $p \in \mathbb{C}[x]$ has a complex root.

*Proof.* Let $n = \deg p$ and assume $0 \notin p(\mathbb{C})$. Define

$$H(z,t) = \sum_{i=0}^{n} a_i z^i t^{n-i}.$$

Note that $H\colon S^1 \times I \to \mathbb{C} \setminus \{0\}$ is a homotopy between $z^n$ and $p(z)$. Also, the map

$$K(z,t) = p(zt)$$

is a homotopy between $a_0$ and $p(z)$. It follows that $p$ is null-homotopic, so $\deg p = 0$. □

[1] Also called the *Ham Sandwich theorem.*

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## 2.2 Computation of the fundamental group

**Definition 2.2.1.** The *coproduct* of groups $G$ and $H$ is the group

$$G * H = \left\{ \prod_{i=1}^{n} g_i h_i \;\middle|\; n \in \mathbb{N} \land g_i \in G \land h_i \in H \right\}.$$

**Remark 2.2.1.1.** The definition coincides with coproducts in the category $\underline{\mathrm{Grp}}$.

**Theorem 2.2.2** (Seifert-van Kampen)**.** Let $X$ and $Y$ be open subspaces in $X \cup Y$ and $x_0 \in X \cap Y$. Suppose that $X, Y$ and $X \cap Y$ are path-connected. Denote by $i_Z \colon X \cap Y \hookrightarrow Z$ and $j_Z \colon Z \hookrightarrow X \cup Y$ the inclusions for $Z \in \{X, Y\}$. Then

$$\varphi = ((j_X)_*, (j_Y)_*) \colon \pi_1(X, x_0) * \pi_1(Y, x_0) \to \pi_1(X \cup Y, x_0)$$

is an epimorphism with

$$\ker \varphi = \left\langle \left\{ (i_X)_*(\alpha) \cdot (i_Y)_* \left(\alpha^{-1}\right) \;\middle|\; \alpha \in \pi_1(X \cap Y, x_0) \right\} \right\rangle.$$

*Proof.* Let $\alpha \in \pi_1(X \cup Y, x_0)$ be an arbitrary loop. Note that, using the Lebesgue number, we can split $\alpha$ into finitely many paths in $X$ and $Y$. Using the path-connectedness of $X \cap Y$, we can join the path segments into loops starting at $x_0$, thus constructing an element of $\pi_1(X, x_0) * \pi_1(Y, x_0)$ that maps to $\alpha$.

Let

$$N = \left\langle \left\{ (i_X)_*(\alpha) \cdot (i_Y)_* \left(\alpha^{-1}\right) \;\middle|\; \alpha \in \pi_1(X \cap Y, x_0) \right\} \right\rangle.$$

Note that

$$\begin{aligned} \varphi\left((i_X)_*(\alpha) \cdot (i_Y)_* \left(\alpha^{-1}\right)\right) &= (j_X)_*((i_X)_*(\alpha)) \cdot (j_Y)_* \left((i_Y)_* \left(\alpha^{-1}\right)\right) \\ &= (j_Y)_*((i_Y)_*(\alpha)) \cdot (j_Y)_* \left((i_Y)_* (\alpha)\right)^{-1} \\ &= 1, \end{aligned}$$

as $j_X \circ i_X = j_Y \circ i_Y$. It follows that $N \leq \ker \varphi$. It remains to check that $\ker \varphi \leq N$. $\square$

**Definition 2.2.3.** A path-connected topological space $X$ is *simply connected* if any two paths $\alpha, \beta \colon I \to X$, such that $\alpha|_{\partial I} = \beta|_{\partial I}$, are homotopic relative to $\partial I$.

**Remark 2.2.3.1.** Equivalently, $X$ is path-connected with $\pi_1(X) \cong \{1\}$.

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