Functional analysis

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Introduction

These are my lecture notes on the course Functional analysis in the year 2023/24. The lecturer that year was prof. dr. Igor Klep.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Convexity

1.1 Locally convex spaces

Definition 1.1.1. *A topological vector space*[1] V is an $\mathbb{F}$-vector space that is also a topological space, such both addition and scalar multiplication are continuous.

Definition 1.1.2. Let V be an $\mathbb{F}$-vectors pace. A map $p\colon V \to \mathbb{R}$ is a *seminorm* if the following holds:

i) $\forall x \in V\colon p(x) \geq 0$,

ii) $\forall \lambda \in \mathbb{F}, x \in V\colon p(\lambda x) = |\lambda|\, p(x)$,

iii) $\forall x, y \in V\colon p(x+y) \leq p(x) + p(y)$.

Definition 1.1.3. Let V be an $\mathbb{F}$-vector space and $\mathcal{P}$ a family of seminorms on V. We define a topology $\mathcal{T}$ on V with the sets

$$U(x_0, p, \varepsilon) = \{x \in V \mid p(x - x_0) < \varepsilon\}$$

as a subbasis.

Definition 1.1.4. A topological vector space X is a *locally convex space* if its topology is generated by a family of seminorms $\mathcal{P}$ satisfying

$$\bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\} = \{0\}\,.$$

Proposition 1.1.5. A locally convex space X is Hausdorff.

Proof. Let $x, y \in X$ be distinct points. Let $p \in \mathcal{P}$ be a seminorm such that $p(x-y) \neq 0$. Then the sets

$$U = \left\{z \in X \;\middle|\; p(z - x) < \frac{\varepsilon}{2}\right\} \quad \text{and} \quad V = \left\{z \in X \;\middle|\; p(z - y) < \frac{\varepsilon}{2}\right\}$$

split the points x and y. $\square$

Remark 1.1.5.1. The converse is also true.

Definition 1.1.6. A partially ordered set I is *upward directed* if for all $i', i'' \in I$ there exists some $i \in I$ such that $i \geq i'$ and $i \geq i''$.

Definition 1.1.7. A *net* is a pair $((I, \leq), x)$, where $(I, \leq)$ is an upward directed set and $x\colon I \to X$ is a function. We usually write $(x_i)_{i \in I}$.

Remark 1.1.7.1. Let $(X, \mathcal{T})$ be a topological space and $x_0 \in X$. Partially order the set

$$\mathcal{U} = \{U \subseteq X \mid x_0 \in U \land U \text{ is open}\}$$

with reverse inclusion. Then any choice function defines a net $(x_U)_{U \in \mathcal{U}}$.

[1] Also linear topological space.

Definition 1.1.8. Let X be a topological space. A net $(x_i)_{i\in I}$ *converges* so $x \in X$ if for all open sets $U \subseteq X$ with $x \in U$ there exists some index $i_0 \in I$ such that for all $i \geq i_0$ we have $x_i \in U$. We write

$$\lim_{i\in I} x_i = x.$$

Definition 1.1.9. A point $x \in X$ is a *cluster point* of a net $(x_i)_{i\in I}$ if for all open sets $U \subseteq X$ with $x \in U$ and index $i_0 \in I$ there exists some index $i \geq i_0$ such that $x_i \in U$.

Proposition 1.1.10. Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net $(a_i)_{i\in I}$ in A such that

$$\lim_{i\in I} a_i = x.$$

Proof. Suppose a net $(a_i)_{i\in I}$ converges to x. For any neighbourhood U of x and some $i_0 \in I$ we have $a_{i_0} \in U$. Therefore, $U \cap A \neq \emptyset$.

Assume now that $x \in \overline{A}$. Again, define

$$\mathcal{U} = \{U \subseteq X \mid x_0 \in U \wedge U \text{ is open}\}.$$

There is a choice function a such that $a_U \in A$ for all U. The net $(a_U)_{U\in\mathcal{U}}$ then converges to x. ∎

Proposition 1.1.11. Let X and Y be topological spaces and $f\colon X \to Y$ a map. Then, f is continuous in $x_0 \in X$ if and only if

$$\lim_{i\in I} f(x_i) = f(x_0)$$

for all nets $(x_i)_{i\in I}$ that converge to x_0.

Proof. Suppose that f is continuous at x_0. Take an open neighbourhood U of $f(x_0)$. Then there must exist some $i_0 \in I$ such that for all $i \geq i_0$ we have $x_i \in f^{-1}(U)$, therefore $f(x_i) \in U$.

Now suppose f is discontinuous at x_0. Let

$$\mathcal{U} = \{U \subseteq X \mid x_0 \in U \wedge U \text{ is open}\}$$

and $V \subseteq Y$ be an open set such that $f(x_0) \in V$ and x_0 is not an interior point of $f^{-1}(V)$. Now using the discontinuity of f, for all $U \in \mathcal{U}$ choose $x_U \in U$ such that $f(x_U) \notin V$. Trivially the net $(x_V)_{V\in\mathcal{V}}$ converges to x_0, but

$$\lim_{V\in\mathcal{V}} f(x_V) \neq f(x_0).$$ ∎

Proposition 1.1.12. The following statements are true:

i) A net $(x_i)_{i\in I}$ in a locally convex space converges to x_0 if and only if the net $(p(x_i - x_0))_{i\in I}$ converges to 0 for all $p \in \mathcal{P}$.

ii) The topology in a locally convex space X is the coarsest topology in which all the maps $x \mapsto p(x - x_0)$ are continuous for all $x_0 \in X$ and $p \in \mathcal{P}$.

Proof.

i) If $(x_i)_{i\in I}$ converges to x_0, just apply the proposition 1.1.11. Suppose that all the nets $(p(x_i - x_0))_{i\in I}$ converge to 0. Choose an open set from the local basis of x_0. It is given by
$$U = \{x \in X \mid \forall k \leq n \colon p_k(x - x_0) < \varepsilon\}.$$
But as all nets $(p_k(x_i - x_0))_{i\in I}$ converge to 0, there is some index $i_k \in I$ such that for all $i \geq i_k$ we have $p_k(x_i - x_0) < \varepsilon$. Now just take i_0 to be an upper bound of i_k. For all $i \geq i_0$ we then have $x_i \in U$.

ii) Obvious. □

Definition 1.1.13. For all $f \in X^*$ define a seminorm $p_f \colon X \to \mathbb{R}$ as $p_f(x) = |f(x)|$. The family $\mathcal{P} = \{p_f \mid f \in X^*\}$ induces the *weak topology* on X. We denote the weak topology with $\sigma(X, X^*)$.

Remark 1.1.13.1. The space X with the topology $\sigma(X, X^*)$ is a locally convex space by the Hahn-Banach theorem.[2]

Definition 1.1.14. Let X be a normed space. For all $x \in X$ we define a seminorm $p_x \colon X^* \to \mathbb{R}$ as $p_x(f) = |f(x)|$. The family $\mathcal{P} = \{p_x \mid x \in X\}$ induces the *weak-* topology* on X^*. We denote the weak-* topology with $\sigma(X^*, X)$.

Remark 1.1.14.1. The weak topology on X^* is finer than the weak-* topology, as X can be isometrically mapped into X^{**} with the map $x \mapsto (f \mapsto f(x))$.

[2] Introduction to functional analysis, corollary 2.2.5.2.

1.2 Banach-Alaoglu theorem

Theorem 1.2.1 (Banach-Alaoglu)**.** Let X be a normed space. Then the closed unit ball in X^*
$$(X^*)_1 = \{f \in X^* \mid \|f\| \leq 1\}$$
is compact in the weak-* topology on X^*.

Proof. Assign a disk to all $x \in X$ as $D_x = \{z \in \mathbb{F} \mid |z| \leq \|x\|\}$ with the euclidean topology. Define
$$P = \prod_{x \in X} D_x$$
with the product topology. The space P is then compact by Tychonoff's theorem. Now define the map $\Phi \colon (X^*)_1 \to P$ with $\Phi(f) = (f(x))_{x \in X}$. This map is injective.

Let $(f_i)_{i \in I}$ be a net in $(X^*)_1$ that weak-* converges to $f \in X^*$. Equivalently, we have
$$\lim_{i \in I} f_i(x) = f(x)$$
for all $x \in X$. By the definition of the product topology we have
$$\lim_{i \in I} \Phi(f_i) = \Phi(f).$$
Therefore, Φ is continuous. Analogously, $\Phi^{-1} \colon \operatorname{im} \Phi \to (X^*)_1$ is continuous.

Suppose that $(\Phi(f_i))_{i \in I}$ converges to some $p \in P$. By the definition of the product topology this means that $f_i(x)$ converges to p_x for all $x \in X$. Define a map $f \colon X \to \mathbb{F}$ given by $f(x) = p_x$. Then, f is linear and bounded with $\|f\| \leq 1$. Thus $p = \Phi(f) \in \operatorname{im}(\Phi)$, therefore, $\Phi((X^*)_1)$ is closed. As $(X^*)_1$ is homeomorphic to its image which is compact, it is also compact. □

Corollary 1.2.1.1. Every Banach space X is isometrically isomorphic to a closed subspace $\mathcal{C}(K)$ for some compact Hausdorff space K.

Proof. Choose $K = (X^*)_1$ with the weak-* topology. By Banach-Alaoglu, K is compact and Hausdorff. Now define the map $\Delta \colon X \to K$ with $\Delta(x) = (f \mapsto f(x))$. Now observe that
$$\|\Delta(x)\|_\infty = \sup_{g \in K} |\Delta(x)(g)| = \sup_{g \in K} |g(x)| = \|x\|$$
by Hahn-Banach.[3] □

[3] Introduction to functional analysis, corollary 2.2.5.1.

1.3 Minkowski gauge

Definition 1.3.1. Let X be a $\mathbb{F}$-vector space. A set $A \subseteq$ is

i) *balanced*, if for all $x \in A$ and $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$ we have $\alpha x \in A$,

ii) *absorbing*, if for all $x \in X$ there exists some $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$ we have $tx \in A$,

iii) *absorbing in* $a \in A$ if $A - a$ is absorbing.

Theorem 1.3.2. Let X be a $\mathbb{F}$-vector space and $V \subseteq X$ a convex, balanced and absorbing in each of its points. Then there exists a unique seminorm p such that

$$V = \{x \in X \mid p(x) < 1\}.$$

Proof. As A is convex, we can define the *Minkowski gauge*

$$p_V(x) = \inf \{t \geq 0 \mid x \in tV\}.$$

It is of course well defined, as A is absorbing. We can check that

$$\begin{aligned} p_V(\alpha x) &= \inf \left\{ t \geq 0 \;\middle|\; x \in \frac{t}{\alpha} V \right\} \\ &= \inf \left\{ t \geq 0 \;\middle|\; x \in \frac{t}{|\alpha|} V \right\} \\ &= |\alpha| \cdot \inf \left\{ \frac{t}{|\alpha|} \geq 0 \;\middle|\; x \in \frac{t}{|\alpha|} V \right\} \\ &= |\alpha| \, p_V(x) \end{aligned}$$

as A is balanced. Therefore, p_V is homogeneous. As p_V is sublinear,[4] it is a seminorm. It follows that[5]

$$V = \{x \in X \mid p_V(x) < 1\}.$$

Suppose that

$$V = \{x \in X \mid q(x) < 1\}$$

for some seminorm $q \neq p_V$. But then we have $p_V(x) \neq q(x)$ for some $x \in X$, therefore there exists some $t \in \mathbb{R}$ such that $p_V(tx) > 1 > q(tx)$ or $q(tx) > 1 > p_V(tx)$. $\square$

[4] Introduction to functional analysis, proposition 2.3.3.
[5] Introduction to functional analysis, remark 2.3.4.1.

1.4 Applications of the Hahn-Banach theorem

Theorem 1.4.1 (Hahn-Banach)**.** Suppose X is a locally convex space and $A, B \subseteq X$ are disjoint convex sets. If B is compact, there exists a functional $f \in X^*$ that separates A from B – there exist $\alpha, \beta \in \mathbb{R}$ such that for all $a \in A$ and $b \in B$ we have

$$\operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(b).$$

Theorem 1.4.2. Suppose X is a locally convex space and $A \subseteq X$ is a convex space. Then the closure of A is the same as the closure in the weak topology.

Proof. The set $\overline{A}$ is of course a subset of the closure of A in the weak topology. Now choose a point $x \notin \overline{A}$. There exists a functional $f \in X^*$ and numbers $\alpha, \beta \in \mathbb{R}$ such that

$$\operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(x)$$

for all $a \in \overline{A}$. But then

$$\overline{A} \subseteq \{y \in X \mid \operatorname{Re} f(y) < \alpha\} = (\operatorname{Re} f)^{-1}\left((-\infty, \alpha]\right) = C,$$

where C is closed in the weak topology. It follows that the closure of A in the weak topology is a subset of C. As $x \notin C$, we get the desired equality. □

Corollary 1.4.2.1. A convex set is a locally convex space if and only if it is weakly closed.

Proposition 1.4.3. Let X be a topological vector space and $f \colon X \to \mathbb{F}$ a linear functional. The following statements are equivalent:

i) The functional f is continuous.

ii) The functional f is continuous in 0.

iii) The functional f is continuous in some point $x_0 \in X$.

iv) The set $\ker f$ is closed.

v) The function $x \mapsto |f(x)|$ is a continuous seminorm.

If X is a locally compact space and $\mathcal{P}$ is the family of seminorms defining the topology on X, the above conditions are also equivalent to

$$|f(x)| \leq \sum_{k=1}^{r} \alpha_k p_k(x)$$

for some $\alpha_k \in \mathbb{R}^+$ and $p_k \in \mathcal{P}$.

Proof. The proof of the equivalence of the first 5 statements is the same as for normed spaces. Suppose now that

$$|f(x)| \leq \sum_{k=1}^{r} \alpha_k p_k(x).$$

Let $(x_i)_{i \in I}$ be a net in X that converges to 0. Then

$$0 \leq |f(x_i)| \leq \sum_{k=1}^{r} \alpha_k p_k(x_i),$$

which converges to 0. It follows that f is continuous at 0.

Now suppose that f is continuous at 0. The set

$$f^{-1}\left(\mathring{\mathcal{B}}(0,1)\right) = \{x \in X \mid |f(x)| < 1\}$$

contains an open neighbourhood B of the point 0. We can write

$$B = \bigcap_{j=1}^{r} U(0, p_j, \varepsilon).$$

Take $x \in X$. For $\delta > 0$ be such that

$$p_j\left(x \cdot \frac{\varepsilon}{\delta + \sum p_j(x)}\right) = \frac{\varepsilon}{\delta + \sum p_j(x)} \cdot p_j(x) < \varepsilon,$$

therefore,

$$\left| f\left(x \cdot \frac{\varepsilon}{\delta + \sum p_j(x)}\right)\right| < 1,$$

which can be rearranged to

$$|f(x)| < \frac{1}{\varepsilon} \cdot \sum_{j=1}^{r} p_j(x) + \frac{\delta}{\varepsilon}.$$

Taking a limit, we get the desired inequality. □

Theorem 1.4.4 (Riesz-Markov)**.** Let X be a compact Hausdorff space and $\Phi \in \mathcal{C}(X)^*$. Then there exists a unique regular Borel measure μ such that

$$\Phi(f) = \int_X f \, d\mu$$

for all $f \in \mathcal{C}(X)$. Furthermore, we have $\|\Phi\| = \|\mu\| = |\mu|(X)$.

Proposition 1.4.5. Let X be a completely regular space. Endow the space $\mathcal{C}(X)$ with the topology induced by the seminorms $\{p_K \mid K \subseteq X \text{ is compact}\}$. If $L \in \mathcal{C}(X)^*$, then there exists a compact set $K \subseteq X$ and a regular Borel measure on K such that

$$L(f) = \int_K f \, d\mu$$

for all $f \in \mathcal{C}(X)$. Conversely, every such (K, μ) defines a functional $L \in \mathcal{C}(X)^*$.

Proof. Suppose that

$$L(f) = \int_K f \, d\mu$$

for some compact set K and measure μ. Then we have

$$|L(f)| = \left|\int_K f \, d\mu\right| \le \|\mu\| \cdot \sup_K |f| = \|\mu\| \cdot p_K(f),$$

so L is continuous.

Let now $L \in \mathcal{C}(X)^*$. We can therefore write

$$|L(f)| \leq \sum_{k=1}^{r} \alpha_k p_{K_j}(f)$$

for some compact sets K_j. We can simplify the above to

$$|L(f)| \leq \alpha \cdot p_K(f),$$

where

$$K = \bigcup_{j=1}^{r} K_j.$$

Note that if we have $f \in \mathcal{C}(X)$ and $f|_K = 0$, it follows that $L(f) = 0$. Now define $F \colon \mathcal{C} \to \mathbb{F}$ as follows; for any $g \in \mathcal{C}$ choose an extension $\widetilde{g} \in \mathcal{C}(X)$ of g and set

$$F(g) = L\left(\widetilde{g}\right).$$

This map is well defined by the above observation. We can check that F is indeed linear. Note that

$$|F(g)| = |L\left(\widetilde{g}\right)| \leq \alpha \cdot p_K\left(\widetilde{g}\right) = \alpha \cdot \|g\|_{\infty, K},$$

therefore, F is continuous. By the Riesz-Markov theorem there exists a regular Borel measure μ on K such that

$$F(g) = \int_K g \, d\mu.$$

If $f \in \mathcal{C}(X)$, we have $g = f|_K \in \mathcal{C}(K)$, so

$$L(f) = F(g) = \int_K g \, d\mu. \qquad \square$$

1.5 Krein-Milman theorem

Definition 1.5.1. Let X be a vector space and $C \subseteq X$ a convex subset.

i) A non-empty convex subset $F \subseteq C$ is a *face* if for all $t \in (0,1)$ and $x, y \in C$ satisfying $tx + (1-t)x \in F$, we also have $x, y \in F$.

ii) A point $x \in C$ is an *extreme point* if $\{x\} \subseteq C$ is a face. We denote the set of extreme points of C by $\operatorname{ext} C$.

Definition 1.5.2. For a vector space X and $A \subseteq X$ define the *convex hull* of A as

$$\operatorname{co} A = \left\{ \sum_{i=1}^n \alpha_i x_i \;\middle|\; n \in \mathbb{N} \wedge \alpha_j \in \mathbb{R}_{\geq 0} \wedge \sum_{i=1}^n \alpha_i = 1 \wedge x_i \in A \right\}.$$

If X is a topological vector space, define the *closed convex hull* as

$$\overline{\operatorname{co}}\, A = \overline{\operatorname{co} A}.$$

Proposition 1.5.3. The set $\operatorname{co} A$ is the smallest convex set that contains A. The set $\overline{\operatorname{co}}\, A$ is the smallest closed set that contains A.

Proof. The only nontrivial part of the proof is convexity of the set $\overline{\operatorname{co}}\, A$. Let $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ be two nets that converge to x and y, where $x, y \in \overline{\operatorname{co}}\, A$. For any $t \in (0,1)$ we have

$$tx + (1-t)y = \lim_{i \in I} (tx_i + (1-t)y_i) \in \overline{\operatorname{co}}\, A. \qquad \square$$

Lemma 1.5.4. Let X be a topological vector space and $C \subseteq X$ be a non-empty compact convex subset. Then for any $\phi \in X^*$ the set

$$F = \left\{ x \in C \;\middle|\; \operatorname{Re} \phi(x) = \min_C \operatorname{Re} \phi \right\}$$

is a closed face of C.

Proof. As C is a compact set, the set F is obviously non-empty. Also note that, as a preimage of a closed point, F is a closed set. Convexity of F follows from linearity of ϕ. Suppose that $tx + (1-t)y \in F$. As

$$\min_C \operatorname{Re} \phi = \operatorname{Re} \phi(tx + (1-t)y) = t\phi(x) + (1-t)\phi(y) \geq \min_C \operatorname{Re} \phi,$$

it follows that $x, y \in F$. By definition, F is a face. $\square$

Theorem 1.5.5 (Krein-Milman)**.** Let X be a locally convex space and $C \subseteq X$ a non-empty convex compact subset. Then

$$C = \overline{\operatorname{co}}\, (\operatorname{ext} C).$$

Proof. Let $\mathcal{F} = \{\text{closed faces in } C\}$ be a set, ordered with $\supseteq$. As $C \in \mathcal{F}$, this is a non-empty set. As each increasing chain in $\mathcal{F}$ has its intersection[6] as an upper bound, we can apply Zorn's lemma and find a maximal element $F_0 \in \mathcal{F}$.

Suppose there are distinct elements $x, y \in F_0$. By Hahn-Banach, there exists a functional $\phi \in X^*$ such that $\operatorname{Re} \phi(x) < \operatorname{Re} \phi(y)$. Now let

$$F_1 = \left\{ z \in F_0 \;\middle|\; \operatorname{Re} \phi(z) = \min_{F_0} \operatorname{Re} \phi \right\}.$$

As $F_1 \subset F_0$ is a closed face in F_0 by lemma 1.5.4, it is a closed face in C. This is a contradiction, so $|F_0| = 1$. Therefore, $\operatorname{ext} C \neq \emptyset$.

It is clear that $\overline{\operatorname{co}}\,(\operatorname{ext} C) \subseteq C = \overline{\operatorname{co}}\, C$. Suppose that $x \in C \setminus \overline{\operatorname{co}}\,(\operatorname{ext} C)$. By Hahn-Banach, there exists a functional $\psi \in X^*$ such that

$$\operatorname{Re} \psi(x) < \min_{\overline{\operatorname{co}}\,(\operatorname{ext} C)} \operatorname{Re} \psi.$$

Let

$$F = \left\{ z \in C \;\middle|\; \operatorname{Re} \psi(z) = \min_{C} \operatorname{Re} \psi \right\}$$

be a closed face in C. As there exists some $z \in \operatorname{ext} F \subseteq \operatorname{ext} C$, we have

$$\min_C \operatorname{Re} \psi = \operatorname{Re} \psi(z) = \min_{\overline{\operatorname{co}}\,(\operatorname{ext} C)} \operatorname{Re} \psi > \operatorname{Re} \psi(x) \geq \min_C \operatorname{Re} \psi.$$

Such x therefore cannot exist. □

Proposition 1.5.6. The space c_0 is not the dual space of a Banach space.

Proof. Let X be a Banach space. By Banach-Alaoglu, the set $(X^*)_1$ is compact, therefore, $(X^*)_1 = \overline{\operatorname{co}}\,(\operatorname{ext}(X^*)_1)$ by Krein-Milman. In particular, $(X^*)_1$ has extreme points.

Let $x \in (c_0)_1$. There exists some $N \in \mathbb{N}$ such that $|x_n| < \frac{1}{2}$ for all $n > N$. Now define $y, z \in c_0$ with $y_n = z_n = x_n$ for $n \leq N$ and

$$y_n = x_n + \frac{1}{2^n}, \quad z_n = x_n - \frac{1}{2^n}$$

for $n > N$. Clearly, $x = y + z$, therefore, $x \notin \operatorname{ext}(c_0)_1$. It follows that $(c_0)_1$ has no extreme points. □

Theorem 1.5.7 (Milman)**.** Let X be a locally convex space and $K \subseteq K$ a compact space. Suppose that $\overline{\operatorname{co}}\,(K)$ is compact. Then $\operatorname{ext}(\overline{\operatorname{co}}\, K) \subseteq K$.

Proof. Assume that there exists some $x_0 \in \operatorname{ext}(\overline{\operatorname{co}}\, K) \setminus K$. Then there exists a basis neighbourhood V of 0 in X such that $x_0 \notin K + \overline{V}$. Now write

$$K \subseteq \bigcup_{x \in K} (x + V).$$

[6] The intersection is non-empty as C is compact.

As K is compact, we can write

$$K \subseteq \bigcup_{i=1}^{n} (x_i + V)$$

for some points $x_i \in K$. Now form

$$K_i = \overline{\operatorname{co}}\,(K \cap (x_j + V)).$$

Note that K_j is convex and compact as it is a subset of $\overline{\operatorname{co}}\, K$. We also have

$$K_j \subseteq x_j + \overline{V}.$$

Note that

$$K \subseteq \bigcup_{i=1}^{n} K_i.$$

Let

$$\Sigma = \left\{ t \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} t_i = 1 \right\}.$$

Define the map

$$f \colon \Sigma \times \prod_{i=1}^{n} K_i \to X$$

with

$$f(t,k) = \sum_{i=1}^{n} t_i k_i.$$

Note that $C = \operatorname{im} f$. As

$$C \subseteq \operatorname{co}\left(\bigcup_{i=1}^{n} K_i \right),$$

the set C is convex. As it is the image of a compact set, it is also compact. Because $K_j \subseteq C$ for all j, it follows that

$$C = \operatorname{co}\left(\bigcup_{i=1}^{n} K_i \right).$$

It follows that

$$\overline{\operatorname{co}}\, K \subseteq \overline{\operatorname{co}} \left(\bigcup_{i=1}^{n} K_i \right) = \operatorname{co}\left(\bigcup_{i=1}^{n} K_i \right).$$

We can therefore deduce

$$\overline{\operatorname{co}}\, K = \operatorname{co}\left(\bigcup_{i=1}^{n} K_i \right).$$

As x_0 is an element of this set, we can write

$$x_0 = \sum_{i=1}^{n} t_i y_i$$

for $t_i \in [0,1]$ and $y_i \in K_i$. As x_0 is an extreme point, we must have $y_j = x_0$ for some j, therefore $x_0 \in K_j \subseteq x_j + \overline{V} \subseteq K + \overline{V}$, which is a contradiction. □

Remark 1.5.7.1. In finite-dimensional vector spaces, the convex hull of a compact set is compact.

Remark 1.5.7.2. The set $\operatorname{ext} C$ is not always closed.

2 C^*-algebras and continuous functional calculus

2.1 Spectrum

Definition 2.1.1. Let A be a complex algebra with unity 1. Define the set

$$\mathrm{GL}(A) = \{a \in A \mid \exists b \in A \colon ab = ba = 1\}.$$

The *spectrum* of $x \in A$ is the set

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin \mathrm{GL}(A)\}.$$

Proposition 2.1.2. Let A be a complex algebra with unity 1 and $x, y \in A$. Then

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

Proof. By scaling, it is enough to check that $1 \in \sigma_A(xy) \iff 1 \in \sigma_A(yx)$. Suppose that $1 - xy \in \mathrm{GL}(A)$. We can check that $1 - yx$ is invertible with

$$(1 - yx)^{-1} = 1 + y(1 - xy)^{-1}x. \qquad \square$$

2.2 Banach and C^*-algebras

Definition 2.2.1. A *Banach algebra* is a Banach space A that is also an algebra, such that $\|xy\| \le \|x\| \cdot \|y\|$ holds for all $x, y \in A$. If a Banach algebra has an unity, we also demand $\|1\| = 1$.

Definition 2.2.2. An *involution* on a Banach algebra A is a skew-linear map $*\colon A \to A$, satisfying the following for all $x, y \in A$:

i) $(xy)^* = y^* x^*$,

ii) $(x^*)^* = x$,

iii) $\|x^*\| = \|x\|$.

A Banach algebra with involution is called a *Banach *-algebra*.

Definition 2.2.3. A Banach *-algebra A that also satisfies $\|x^* x\| = \|x\|^2$ for all $x \in A$ is called a *C^*-algebra*.

Proposition 2.2.4. Let A be a Banach *-algebra. Then, for all $x \in A$ we have $(x^*)^{-1} = (x^{-1})^*$ and $\sigma_A(x^*) = \overline{\sigma_A(x)}$.

Proposition 2.2.5. Let A be a Banach algebra. The following statements are true:

i) Let $x \in A$. If $\|x\| < 1$, then $1 - x \in \mathrm{GL}(A)$ and
$$(1-x)^{-1} = \sum_{n \in \mathbb{N}_0} x^n.$$

ii) The set $\mathrm{GL}(A)$ is an open subset of A and the map $x \mapsto x^{-1}$ is continuous on $\mathrm{GL}(A)$.

Proof. Let $y \in \mathrm{GL}(A)$. If $\|x - y\| < \frac{1}{\|y^{-1}\|}$, then
$$\left\|1 - xy^{-1}\right\| = \left\|(y-x)y^{-1}\right\| < 1,$$
therefore, xy^{-1} is invertible. It follows that x is also invertible, so $\mathrm{GL}(A)$ is open.

Note that
$$\begin{aligned}
\left\|(xy^{-1})^{-1}\right\| &= \left\|\left(1 - (1 - xy^{-1})\right)^{-1}\right\| \\
&\le \sum_{n \in \mathbb{N}_0} \left\|1 - xy^{-1}\right\|^n \\
&\le \sum_{n \in \mathbb{N}_0} \left\|y^{-1}\right\|^n \cdot \|x - y\|^n \\
&= \frac{1}{1 - \|y^{-1}\| \, \|x - y\|}.
\end{aligned}$$

It follows that
$$\begin{aligned}
\left\|x^{-1} - y^{-1}\right\| &= \left\|x^{-1}(y-x)y^{-1}\right\| \\
&\le \left\|y^{-1}(xy^{-1})^{-1}\right\| \cdot \left\|y^{-1}\right\| \cdot \|x - y\| \\
&\le \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\| \cdot \|x - y\|} \cdot \|x - y\|.
\end{aligned}$$
□

Proposition 2.2.6. Let A be a Banach algebra and $x \in A$. Then $\sigma_A(x)$ is a non-empty compact set.

Proof. Introduction to functional analysis, theorem 6.1.15. $\square$

Theorem 2.2.7 (Gelfald-Mazur)**.** If A is a Banach algebra that is also a division ring, then $A = \mathbb{C}$.

Proof. Let $x \in A$ and $\lambda \in \sigma_A(x)$. As $x - \lambda \cdot 1 \notin \mathrm{GL}(A) = A \setminus \{0\}$, we have $x = \lambda \cdot 1 \in \mathbb{C}$. $\square$

Definition 2.2.8. If

$$f(x) = \sum_{j=0}^{n} a_j x^j$$

is a polynomial and $a \in A$ an element of an algebra, we define

$$f(a) = \sum_{j=0}^{n} a_j a^j \in A.$$

Theorem 2.2.9 (Spectral mapping theorem for polynomials)**.** Let A be an algebra and $f \in \mathbb{C}[x]$. Then

$$f(\sigma_A(a)) = \sigma_A(f(a))$$

holds for all $a \in A$.

Proof. Let $\lambda \in \sigma_A(a)$ and

$$f(x) = \sum_{j=0}^{n} a_j x^j.$$

We can write

$$f - f(\lambda) = (x - \lambda) \cdot \sum_{j=1}^{n} a_j \sum_{k=0}^{j-1} x^k \lambda^{j-i-k}.$$

It follows that

$$f(a) - f(\lambda) = (a - \lambda) \cdot \sum_{j=1}^{n} a_j \sum_{k=0}^{j-1} a^k \lambda^{j-1-k}.$$

As $a - \lambda$ is not invertible and commutes with the second factor, it follows that $f(a) - f(\lambda)$ is also not invertible.

Conversely, if $\mu \notin f(\sigma_A(a))$, we can write

$$f - \mu = a_n \cdot \prod_{j=1}^{n} (x - \lambda_j).$$

As $f(\lambda) - \mu \neq 0$ for all $\lambda \in \sigma_A(a)$, we have $\lambda_i \notin \sigma_A(a)$ for all i. Therefore, it follows that $f(a) - \mu \in \mathrm{GL}(A)$. $\square$

Definition 2.2.10. Let A be a Banach algebra and $x \in A$. The *spectral radius* of x is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

Theorem 2.2.11 (Spectral radius formula)**.** Let A be a Banach algebra and $x \in A$. Then, the limit

$$\lim_{n\to\infty} \sqrt[n]{\|x^n\|}$$

exists and

$$r(x) = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

Proof. Introduction to functional analysis, theorem 6.1.20. □

Definition 2.2.12. Let A be a Banach *-algebra and $x \in A$.

i) The element x is *normal* if $xx^* = x^*x$.

ii) The element x is *selfadjoint* if $x = x^*$.

iii) The element x is *skew selfadjoint* if $x = -x^*$.

Corollary 2.2.12.1. Let A be a Banach *-algebra and $x \in A$ a normal element. Then

$$r(x^*x) \le r(x)^2.$$

If A is a C^*-algebra, then $r(x^*x) = r(x)^2$.

Proof. Note that

$$r(x^*x) = \lim_{n\to\infty} \sqrt[n]{\|(x^*x)^n\|} \le \lim_{n\to\infty} \sqrt[n]{\|x^n\|}^2 = r(x)^2.$$

If A is a C^*-algebra, we have equality. □

Proposition 2.2.13. Let A be a C^*-algebra and $x \in A$ a normal element. Then

$$r(x) = \|x\|.$$

Proof. The statement holds for selfadjoint elements by Introduction to functional analysis, corollary 6.1.20.1. We can therefore write

$$\|x\|^2 = \|x^*x\| = r(x^*x) = r(x)^2.$$ □

Corollary 2.2.13.1. Let A and B be C^*-algebras and $\Phi\colon A \to B$ a *-homomorphism.[7] Then Φ is a contraction. Furthermore, if Φ is a *-isomorphism, it is isometric.

Proof. Note that $\Phi(\mathrm{GL}(A)) \subseteq \mathrm{GL}(B)$, therefore $\sigma_B(\Phi(x)) \subseteq \sigma_A(x)$. It follows that $r(\Phi(x)) \le r(x)$. Now observe that

$$\|\Phi(x)\|^2 = \|\Phi(x) \cdot \Phi(x)^*\| = \|\Phi(xx^*)\| = r(\Phi(xx^*)) \le r(xx^*) = \|x\|^2.$$ □

Corollary 2.2.13.2. If A is a *-algebra, there exists at most one norm on A such that A is a C^*-algebra.

Proof. The proof is obvious and need not be mentioned. □

[7] $\Phi(x^*) = \Phi(x)^*$ for all $x \in A$.

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Lemma 2.2.14. Let A be a C^*-algebra and $x \in A$ a selfadjoint element. Then $\sigma_A(x) \subseteq \mathbb{R}$.

Proof. Let $\lambda = \alpha + i\beta \in \sigma_A(x)$ for some $\alpha, \beta \in \mathbb{R}$. Define $y = x - \alpha + it$ for some $t \in \mathbb{R}$. Note that $i \cdot (\beta + t) \in \sigma_A(y)$ and that y is normal. It follows that

$$|i \cdot (\beta + t)|^2 \leq r(y)^2 = \|y\|^2 = \|yy^*\| = \left\|(x - \alpha)^2 + t^2\right\| \leq \|x - \alpha\|^2 + t^2.$$

Rearranging the above inequality, we get

$$\beta^2 + 2\beta t \leq \|x - \alpha\|^2 .$$

As $t \in \mathbb{R}$ was arbitrary, it follows that $\beta = 0$. □

Lemma 2.2.15. Let A be a Banach algebra and $x \in A \setminus \mathrm{GL}(A)$. If the sequence $(x_n)_n$ of elements of $\mathrm{GL}(A)$ converges to x, then

$$\lim_{n \to \infty} \left\|x_n^{-1}\right\| = \infty.$$

Proof. If the sequence were bounded, we'd have

$$\lim_{n \to \infty} \left\|1 - xx_n^{-1}\right\| \leq \lim_{n \to \infty} \|x_n - x\| \cdot \left\|x_n^{-1}\right\| = 0.$$

In particular, $\|1 - xx_n^{-1}\| < 1$ for some $n \in \mathbb{N}$. It follows that xx_n^{-1} is invertible, therefore x is also invertible. □

Proposition 2.2.16. Let B be a C^*-algebra and $A \subseteq B$ a unital C^*-subalgebra. Then for all $x \in A$ we have $\sigma_A(x) = \sigma_B(x)$.

Proof. Note that $\mathrm{GL}(A) \subseteq \mathrm{GL}(B)$. Let $x \in A \setminus \mathrm{GL}(A)$ be a selfadjoint element. Note that $it \notin \sigma_A(x)$ for $t \in \mathbb{R}^*$. As x is not invertible in A, we have

$$\lim_{n \to \infty} \left\|(x - it)^{-1}\right\| = \infty$$

by the previous lemma. Since the inversion map is continuous, it follows that $x \notin \mathrm{GL}(B)$.

Let now $x \in A \cap \mathrm{GL}(B)$ be an arbitrary element. We know that x^* is also invertible, therefore $x^*x \in \mathrm{GL}(B) \cap A$. From the first part of the proof we now know $x^*x \in \mathrm{GL}(A)$, so we can write

$$x^{-1} = (x^*x)^{-1} \cdot x^*.$$ □

2.3 Gelfald transform

Definition 2.3.1. Let A be an abelian Banach algebra. The *spectrum* of A is the set

$$\sigma(A) = \{\varphi\colon A \to \mathbb{C} \mid \varphi \neq 0 \text{ is a continuous homomorphism}\}$$

with the weak-* topology. We call elements of $\sigma(A)$ *characters*.

Remark 2.3.1.1. For all characters $\varphi \in \sigma(A)$ we have $\ker\varphi \cap \mathrm{GL}(A) = \emptyset$.

Proposition 2.3.2. For all $\varphi \in \sigma(A)$ and $x \in A$ we have $\varphi(x) \in \sigma_A(x)$.

Proof. We have $\varphi(x - \varphi(x)) = 0$. □

Corollary 2.3.2.1. We have $\sigma(A) \subseteq (A^*)_1$. In particular, $\sigma(A)$ is a compact Hausdorff space.

Proof. For all characters $\varphi \in \sigma(A)$ we have $|\varphi(x)| \leq r(x) \leq \|x\|$, therefore $\|\varphi\| \leq 1$.[8] □

Proposition 2.3.3. Let A be an abelian Banach algebra. The map $\varphi \mapsto \ker\varphi$ is a bijection between $\sigma(A)$ and maximal ideals $I \triangleleft A$.

Proof. Let $\varphi \in \sigma(A)$ be an arbitrary character. Note that $\ker\varphi \triangleleft A$. Suppose that $\ker\varphi \subset I \triangleleft A$ and let $x \in I \setminus \ker\varphi$. Let $y = 1 - \frac{x}{\varphi(x)} \in \ker\varphi$. It follows that

$$1 = y + \frac{1}{\varphi(x)} \cdot x \in I,$$

therefore $\ker\varphi$ is maximal.

Let $I \triangleleft A$ be a maximal ideal. Note that $I \cap \mathrm{GL}(A) = \emptyset$ as $I \neq A$. It follows that $\|1 - y\| \geq 1$ for all $y \in I$. The set $\overline{I}$ is again an ideal, but as $1 \notin \overline{I}$, it follows that $I = \overline{I}$. Note that A/I is an abelian Banach algebra. Since I was maximal, it is also a division ring. By Gelfald-Mazur theorem, we have $A/I \cong \mathbb{C}$. The projection $\pi\colon A \to A/I$ is therefore a character with $\ker\pi = I$. □

Corollary 2.3.3.1. Let A be an abelian Banach algebra and $x \in A \setminus \mathrm{GL}(A)$. Then there exists a $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$.

Proof. By Zorn's lemma, there exists a maximal ideal containing $\langle x \rangle$. □

Theorem 2.3.4 (Stone-Čech)**.** Let X be a topological space. For each $x \in X$ define $\beta_x\colon \mathcal{C}_b(X) \to \mathbb{C}$ as the evaluation homomorphism. Then the map $\beta\colon X \to \sigma(\mathcal{C}_b(X))$ given by $\beta(x) = \beta_x$ is continuous with dense image and the following universal property: for every continuous map $\pi\colon X \to K$, where K is a compact Hausdorff topological space, there exists a unique continuous map $\beta_\pi\colon \sigma(\mathcal{C}_b(X)) \to K$ such that $\pi = \beta_\pi \circ \beta$. In particular, if X is a compact Hausdorff space, then β is a homeomorphism.

$$\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[dr, "\pi"'] & \sigma(\mathcal{C}_b(X)) \arrow[d, dashed, "\beta_\pi"] \\
& K
\end{tikzcd}$$

[8] In fact, as $\varphi(1) = 1$, we have $\|\varphi\| = 1$.

Proof. Let $(x_i)_I$ be a net with limit x. For all $f \in \mathcal{C}_b(X)$

$$\lim_{i\in I} \beta_{x_i}(f) = \lim_{i\in I} f(x_i) = \beta_x(f),$$

therefore β is continuous. Suppose that $\varphi \in \sigma(\mathcal{C}_b(X)) \setminus \overline{\beta(X)}$ and denote $I = \ker \varphi$. For all $\psi \in \overline{\beta(X)}$ there exists a $f_\psi \in I$ such that $f_\psi \notin \ker \psi$. Therefore there exist $c_\psi > 0$ and an open neighbourhood U_ψ of ψ such that $\left|\widetilde{\psi}\,(f_\psi)\right| > c_\psi$ for all $\widetilde{\psi} \in U_\psi$. This is of course an open cover of $\overline{\beta(X)}$, therefore there exists a finite subcover. Equivalently, there exist elements $f_1, f_2, \dots f_n \in I$ and $c > 0$ such that

$$\sum_{i=1}^n \psi\left(|f_i|^2\right) > c$$

for all $\psi \in \overline{\beta(X)}$. In particular, we have

$$\sum_{i=1}^n |f_i(x)|^2 = \sum_{i=1}^n \beta_x\left(|f_i|^2\right) > c$$

for all $x \in X$. It follows that the function

$$\sum_{i=1}^n |f_i|^2 \in I$$

is invertible, therefore we have $I = \mathcal{C}_b(X)$. This is of course impossible.

If X is compact and Hausdorff, then β is surjective, as its image is a closed dense set. As X is compact and Hausdorff, it is normal and therefore $\mathcal{C}_b(X)$ separates points by Tietze. It follows that β is injective as well. As β is a map between compact Hausdorff spaces, it is closed, therefore a homeomorphism.

Let $\pi\colon X \to K$ be a continuous map, where K is a compact Hausdorff space. Note that there exists a continuous map $\pi^*\colon \mathcal{C}(K) \to \mathcal{C}_b(X)$ with $\pi^*(f) = f \circ \pi$. Similarly, there exists a continuous map $\widetilde{\pi}\colon \sigma(\mathcal{C}_b(X)) \to \sigma(\mathcal{C}(K))$ with $\widetilde{\pi}(\varphi) = \varphi \circ \pi^*$.

As K is a compact Hausdorff space, the map $\beta^K\colon K \to \sigma(\mathcal{C}(K))$ is a homeomorphism by the previous part of the proof. Now define $\beta_\pi\colon \sigma(\mathcal{C}_b(X)) \to K$ by $\beta_\pi = \left(\beta^K\right)^{-1} \circ \widetilde{\pi}$. Indeed,

$$\widetilde{\pi}(\beta_x)(g) = \beta_x(\pi^*(g)) = g(\pi(x)) = \beta^K_{\pi(x)}(g),$$

therefore $\pi(x) = \beta_\pi(\beta(x))$. □

Definition 2.3.5. Let A be an abelian Banach algebra. The *Gelfald transform* of A is the map $\Gamma\colon A \to \mathcal{C}(\sigma(A))$ with $\Gamma(x) = (\varphi \mapsto \varphi(x))$.

Theorem 2.3.6. The Gelfald transform of an abelian Banach algebra A is a homomorphism and a contraction. For $x \in A$, we have

$$\Gamma(x) \in \mathrm{GL}(\mathcal{C}(\sigma(A))) \iff x \in \mathrm{GL}(A).$$

Proof. The Gelfald transform is obviously a homomorphism. Note that

$$\|\Gamma(x)\| = \sup_{\varphi\in\sigma(A)} \|\Gamma(x)(\varphi)\| = \sup_{\varphi\in\sigma(A)} \|\varphi(x)\| \le \|x\|\,.$$

If x is invertible, then so is $\Gamma(x)$. Now suppose that $x \in A$ is not invertible. By corollary 2.3.3.1 there exists a character $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. As $\Gamma(x)(\varphi) = 0$, the map $\Gamma(x)$ is not invertible either. □

Corollary 2.3.6.1. Let A be an abelian Banach algebra. Then $\sigma(\Gamma(x)) = \sigma(x)$ and $\|\Gamma(x)\| = r(x)$.

Theorem 2.3.7 (Gelfald)**.** Let A be an abelian C^*-algebra. Then Γ is a $*$-isomorphism.

Proof. Let $x \in A$ be a selfadjoint element. We then have $\sigma(\Gamma(x)) = \sigma(x) \subseteq \mathbb{R}$, therefore $\overline{\Gamma(x)} = \Gamma(x)$. For an arbitrary $x \in A$, write $x = a + bi$ for selfadjoint $a, b \in A$. It follows that

$$\Gamma(x^*) = \Gamma(a - ib) = \overline{\Gamma(x)},$$

therefore Γ is a $*$-homomorphism.

Since A is abelian, every element is normal. We therefore have

$$\|x\| = r(x) = r(\Gamma(x)) = \|\Gamma(x)\|,$$

so Γ is injective and therefore injective. Note that $\Gamma(A)$ is closed under $*$, closed as a subalgebra in $\mathcal{C}(\sigma(A))$ and separates points. By Stone-Weierstrass, $\Gamma(A) = \mathcal{C}(\sigma(A))$. □

Remark 2.3.7.1. If A is a C^*-algebra and $x \in A$ a normal element, then the C^*-subalgebra generated by[9] x is abelian.

Corollary 2.3.7.2. Let A be an abelian C^*-algebra, generated by x. Then $\sigma(A) \approx \sigma(x)$.

Proof. Define a map $\Phi\colon \sigma(A) \to \sigma(x)$ with $\Phi(\varphi) = \varphi(x) = \Gamma(x)(\varphi)$. Observe that Φ is well defined as $\varphi(x) \in \sigma(x)$. □

[9] Also denoted by $C^*(x)$.

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