Complex analysis

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Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

October 3, 2023

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations)**.** Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a, f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Definition 1.1.2. *Wirtinger derivatives* are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a. The function f is *holomorphic* if it is holomorphic in every point of Ω. We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with $\mathcal{C}^1$-smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,dw \wedge d\overline{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\Delta(z,\varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\Delta(z,\varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w-z}\,dw$ on Ω_ε. As $d\omega = \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w} \wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w-z}\,dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w-z}\,dw = \oint_{\partial\Omega} \frac{f(w)}{w-z}\,dw - \oint_{\partial\Delta(z,\varepsilon)} \frac{f(w)}{w-z}\,dw.$$

In the limit, we have

$$\lim_{\varepsilon\to 0} \oint\limits_{\partial\Delta(z,\varepsilon)} \frac{f(w)}{w-z}\,dw = \lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{f\left(z+\varepsilon e^{it}\right)}{\varepsilon e^{it}}\cdot \varepsilon i e^{it}\,dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon\to 0}\iint\limits_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega\setminus\{z\}} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i}\oint\limits_{\partial\Omega}\frac{f(w)}{w-z}\,dw - f(z) = -\frac{1}{2\pi i}\iint\limits_{\Omega}\frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,dw\wedge d\overline{w}. \qquad \square$$

Theorem 1.1.5 (Power series expansion)**.** Let $\Omega\subseteq\mathbb{C}$ be an open subset, $f\in\mathcal{O}(\Omega)$ and $a\in\Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\Delta(a,r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i}\oint\limits_{\partial\Delta(a,\rho)}\frac{f(w)}{(w-z)^{k+1}}\,dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k\cdot(z-a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k\to\infty}\sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity)**.** Let $\Omega\subseteq\mathbb{C}$ be a domain and $f\in\mathcal{O}(\Omega)$ a holomorphic function. Let $A\subseteq\Omega$ be a subset such that $f(z)=0$ for all $z\in A$. If A has an accumulation point in Ω, then $f(z)=0$ for all $z\in\Omega$.

Proof. Let $a\in\Omega$ be an accumulation point of A. By continuity, we have $f(a)=0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k(z-a)^k,$$

where we assume $c_{k_0}\neq 0$. But now $g(z)=\frac{f(z)}{(z-a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a)=0$, which is a contradiction. It follows that $c_k=0$ for all $k\in\mathbb{N}_0$. It follows that the set $\operatorname{Int}\{z\in\Omega \mid f(z)=0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to Ω. $\square$

Lemma 1.1.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\Delta(a,r)} \subseteq \Omega$. If

$$|f(a)| < \min_{\partial\Delta(a,r)} |f|,$$

then f has a zero in $\Delta(a,r)$.

Proof. Assume otherwise. From the inequality it follows that f has no zeroes on the boundary either. By continuity, f has no zero on an open set V with $\Delta(a,r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint\limits_{\partial\Delta(a,r)} \frac{g(z)}{z-a}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g\left(a + r \cdot e^{it}\right)}{re^{it}} \cdot rie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} g\left(a + re^{it}\right) dt.$$

We can therefore get a bound on $|g(a)|$ as

$$|g(a)| \leq \max_{\partial\Delta(a,r)} |g|,$$

but as the condition on f can be rewritten as

$$|g(a)| > \max_{\partial\Delta(a,r)} |g|,$$

we have reached a contradiction. $\square$

Theorem 1.1.8 (Open mapping)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If f is not constant, it is an open map.

Proof. Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\Delta(z_0, \rho) \subseteq U$ and z_0 is the only pre-image of w_0 in $\Delta(z_0, 2\rho)$.[1]

Since $\partial\Delta(z_0, \rho)$ is a compact set and

$$|f(z) - w_0| > 0$$

for all $z \in \partial\Delta(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a $w \in \Delta(w_0, \varepsilon)$. As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$

on the boundary and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

by the above lemma, $f(z_0) - w$ has a root on $\Delta(z, \rho)$. $\square$

Theorem 1.1.9 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function f is constant.

[1] If such a disk does not exist, f is constant by the identity theorem.

Proof. Suppose that f is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As f is an open map, it also attains the value $(1+\varepsilon) \cdot f(z)$, which is a contradiction as the modulus then equals $(1+\varepsilon) \cdot |f(z)| > |f(z)|$. $\square$

Theorem 1.1.10 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

Proof. As $\overline{\Omega}$ is compact, f attains a global maximum on this set. If the maximum is attained in the interior, f is constant, therefore it is also attained on the boundary. $\square$

Definition 1.1.11. A function $f \colon \Omega \setminus \{a\} \to \mathbb{C}$ is *locally bounded* near a if there exists an open neighbourhood $U \subseteq \Omega$ of a such that $f|_{U \setminus \{a\}}$ is bounded.

Theorem 1.1.12 (Riemann removable singularity theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$. If f is locally bounded near a, then there exists a unique function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus \{a\}} = f$.

Proof. Define the function $F \colon \Omega \to \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \dfrac{1}{2\pi i} \displaystyle\oint_{\partial\Delta(a,\rho)} \dfrac{f(w)}{w-a}\, dw & z = a. \end{cases}$$

It remains to check that F is complex differentiable at a. Indeed, for $z \in \Delta(a,\rho)$ we have

$$\begin{aligned} \lim_{z \to a} \frac{F(z) - F(a)}{z-a} &= \lim_{z \to a} \frac{1}{z-a} \oint_{\partial\Delta(a,\rho)} \left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a} \right) dw \\ &= \lim_{z \to a} \frac{1}{2\pi i} \cdot \frac{1}{z-a} \cdot \oint_{\partial\Delta(a,\rho)} f(w) \cdot \frac{z-a}{(w-z)(w-a)}\, dw \\ &= \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{(w-a)^2}\, dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem. $\square$

Theorem 1.1.13 (Schwarz lemma)**.** Let $f \colon \Delta \to \Delta$ be a holomorphic function with $f(0) = 0$. Then, $|f'(0)| \le 1$ and the inequality $|f(z)| \le |z|$ holds for all $z \in \Delta$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ holds for any $z \neq 0$, then $f(z) = \beta z$ for some $\beta \in \partial\Delta$.

Proof. We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for g on the domain $\Delta(\rho)$. We get

$$\sup_{z \in \Delta(\rho)} |g(z)| \le \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as $\rho \to 1$, it follows that

$$\sup_{z \in \mathbb{\Delta}} |g(z)| \leq 1.$$

It immediately follows that $|f'(0)| = |g(0)| \leq 1$. Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z|.$$

Suppose we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. As then $|g(z_0)| = 1$, it follows that g is constant, therefore $f(z) = \beta z$ for some $\beta \in \partial \mathbb{\Delta}$. If we have $|f'(0)| = 0$, the same argument works for $z_0 = 0$. $\square$

1.2 The $\overline{\partial}$ equation

Lemma 1.2.1. Let $g \in \mathcal{C}^\infty(\mathbb{C})$ be a function with compact support. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Let

$$f(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{g(w)}{w - z} \, dw \wedge d\overline{w}.$$

As

$$dw \wedge d\overline{w} = -2ri \, dr \wedge d\varphi$$

holds for polar coordinates centered at z, we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{rg\left(z + re^{i\varphi}\right)}{re^{i\varphi}} \, dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some R such that $g|_{\mathbb{C} \setminus \Delta(z,R)} = 0$. We get

$$f(z) = -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi,$$

which obviously converges. The function f is therefore well defined. As we are integrating a smooth function on a compact set, the function f is smooth as well.

For $u = re^{i\varphi}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}}(z) &= -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} \frac{\partial}{\partial \overline{z}} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial}{\partial \overline{z}} g(u + z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial g}{\partial \overline{u}}(u + z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w - z} \, dw \wedge d\overline{w}.
\end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint\limits_{\partial \Delta(z,R)} \frac{g(w)}{w - z} \, dw + \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w - z} \, dw \wedge d\overline{w}.$$

by the choice of R, we get

$$\frac{\partial f}{\partial \overline{z}}(z) = g(z).$$

□

Lemma 1.2.2. Given bounded domain $U \subset V \subset \mathbb{R}^n$ such that $\partial U \cap \partial V = \emptyset$, there exists a smooth function $\chi \colon \mathbb{R}^n \to [0, 1]$ such that $\chi|_U = 1$ and $\operatorname{supp} \chi \subseteq V$.

Proof. There is a partition of unity on the sets V and $\mathbb{R}^n \setminus \overline{U}$. □

Lemma 1.2.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $h_j \colon \Omega \to \mathbb{C}$ be holomorphic functions. If the sequence $(h_j)_{j\in\mathbb{N}}$ converges uniformly on compact sets, the limit is also holomorphic on Ω.

Proof. Apply Morera's theorem.[2] $\square$

Theorem 1.2.4 (Dolbeault lemma)**.** Let $g \in \mathcal{C}^\infty(\Delta(R))$ for some $R \in (0, \infty]$. Then there exists a function $f \in \mathcal{C}^\infty(\Delta(R))$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Define disks X_j as follows:

i) If $R = \infty$, set $X_j = \Delta(j)$.

ii) If $R < \infty$, set $X_j = \Delta\left(R - \frac{1}{j}\right)$ (for large enough j).

Applying the above lemma, define functions χ_j with $\chi_j|_{X_j} = 1$ and $\operatorname{supp} \chi_j \subseteq X_{j+1}$ and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \Delta(R), \\ 0 & z \notin \Delta(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function $f_j \in \mathcal{C}^\infty(\mathbb{C})$ with

$$\frac{\partial f_j}{\partial \overline{z}} = g_j.$$

We inductively construct a new sequence $\widetilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$ such that

$$\frac{\partial \widetilde{f}_j}{\partial \overline{z}} = g$$

on X_j and

$$\left\| \widetilde{f}_j - \widetilde{f}_{j-1} \right\|_{X_{j-2}} \leq 2^{-j}.$$

Set $\widetilde{f}_1 = f_1$. Observe the function $F = f_{j+1} - \widetilde{f}_j$ on X_j. By construction, we have $\frac{\partial F}{\partial \overline{z}} = 0$ on X_j. It follows that F can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on X_j. As power series converge uniformly on compact sets, there exists some polynomial $p \in \mathbb{C}[z]$ such that

$$\|F - p\|_{X_{j-1}} \leq 2^{-j}.$$

Now just set $\widetilde{f}_{j+1} = f_{j+1} - p$.

Let $z \in \Delta(R)$ be arbitrary. By construction, it is contained in some X_{j_0}, therefore, $\widetilde{f}_j$ is defined for $j \geq j_0$. As $\left(\widetilde{f}_j(z)\right)_{j \geq j_0}$ is a Cauchy sequence, we can define

$$f(z) = \lim_{j\to\infty} \widetilde{f}_j(z).$$

[2] Analysis 2b, theorem 3.4.6.

But as

$$f - \widetilde{f}_j = \sum_{k=j}^{\infty} \left(\widetilde{f}_{j+1} - \widetilde{f}_j\right)$$

is a sum of holomorphic functions that converges uniformly, the function $f - \widetilde{f}_j$ is a holomorphic function. Therefore, f is smooth and satisfies $\frac{\partial f}{\partial \bar{z}} = g$. $\square$

1.3 Meromorphic functions

Definition 1.3.1. Let $\Omega \subset \mathbb{C}$ be an open subset. We call a function f *meromorphic* of Ω if there exists $A \subset \Omega$ such that $f \in \mathcal{O}(\Omega \setminus A)$, A has no accumulation points in Ω and for all $a \in A$ there exists some $k \in \mathbb{N}$ such that

$$\lim_{z \to a} f(z) \cdot (z-a)^k \neq 0$$

exists. We call A the set of *poles* of the function f. We denote the set of meromorphic functions on Ω with $\mathcal{M}(\Omega)$.

Theorem 1.3.2. Let $0 \leq r < R \leq \infty$. Suppose that $f \in \mathcal{O}(D_{R,r}(a))$ is a holomorphic function, where

$$D_{R,r}(a) = \{z \in \mathbb{C} \mid r < |z-a| < R\}.$$

Then there exists a uniquely determined *Laurent series*

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

that converges to f uniformly and absolutely on compact subsets of $D_{R,r}(a)$. We have

$$c_k = \frac{1}{2\pi i} \oint_{\partial \Delta(a,\rho)} \frac{f(w)}{(w-a)^k} \, dw$$

for $r < \rho < R$.

Definition 1.3.3. Let

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

be a Laurent series. The series

$$\sum_{k=-\infty}^{-1} c_k (z-a)^k$$

is called the *principle part*.

Lemma 1.3.4. Let $f \in \mathcal{O}(\Omega \setminus \{a\})$ be a holomorphic function. Then f is meromorphic on Ω if and only if f has a finite principle part in a.

Proof. Suppose that f is meromorphic on Ω. If a is a removable singularity, f is holomorphic in a, therefore the principle part is trivial. Otherwise, set $m \in \mathbb{N}$ such that

$$\lim_{z \to a} (z-a)^m f(z) \neq 0$$

exists and set $g(z) = (z-a)^m f(z)$. As g is bounded near a, we can extend it to Ω by the Riemann removable singularity theorem. The power series of g corresponds to a finite Laurent series of f.

The converse is obvious. □

Theorem 1.3.5. If $f \in \mathcal{M}(\mathbb{C})$ is a meromorphic function, there exist entire functions g and h such that $f = \frac{g}{h}$.

Definition 1.3.6. Let $\Omega \subseteq \mathbb{C}$ be an open set. An *additive Cousin problem* on Ω is an open cover $\{U_j\}_{j\in J}$ of Ω and functions $f_j \in \mathcal{M}(U_j)$ such that $f_j - f_k|_{U_j \cap U_k}$ is holomorphic for all $j, k \in J$. A function $f \in \mathcal{M}(\Omega)$ is a solution to the additive Cousin problem if $f|_{U_j} - f_j$ is holomorphic for all $j \in J$.

Definition 1.3.7. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A *generalized additive Cousin problem* is an open cover $\{U_j\}_{j\in J}$ of Ω and functions $f_{j,k} \in \mathcal{O}(U_j \cap U_k)$ for each $(j,k) \in J^2$, such that

i) $f_{j,k} = -f_{k,j}$ on $U_j \cap U_k$ for all $(j,k) \in J^2$ and

ii) $f_{j,k} + f_{k,\ell} + f_{\ell,j} = 0$ on $U_j \cap U_k \cap U_\ell$ for all $(j,k,\ell) \in J^3$.

A solution to the generalized additive Cousin problem is given by functions $f_j \in \mathcal{O}(U_j)$ for each $j \in J$ such that $f_{j,k} = f_j - f_k$ for each $(j,k) \in J^2$.

Lemma 1.3.8 (Partition of unity)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $\{U_j\}_{j\in J}$ be an open cover of Ω. Then there exists a partition of unity subordinate to $\{U_j\}_{j\in J}$.

Lemma 1.3.9. Given a generalized additive Cousin problem on $\Omega \subseteq \mathbb{C}$, there exist functions $g_j \in \mathcal{C}^\infty(U_j)$ such that $f_{j,k} = g_j - g_k$ for all $(j,k) \in J^2$.

Proof. Let $\{(V_a, \chi_a)\}_{a\in A}$ be a partition of unity, subordinate to $\{U_j\}_{j\in J}$. For all $a \in A$ choose a $j(a) \in J$ such that $V_a \subseteq U_{j(a)}$. For all $k \in J$, define

$$g_k = -\sum_{a\in A} \chi_a \cdot f_{j(a),k}.$$

This is of course a smooth function on U_k. Now note that

$$g_k - g_\ell = \sum_{a\in A} \chi_a \cdot \left(-f_{j(a),k} + f_{j(a),\ell}\right) = \sum_{a\in A} \chi_a \cdot f_{k,\ell} = f_{k,\ell}. \qquad \square$$

Proposition 1.3.10. The generalized additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

Proof. Let $f_{j,k} = g_j - g_k$ for $g_j \in \mathcal{C}^\infty(U_j)$. Note that

$$\frac{\partial g_j}{\partial \overline{z}} = \frac{\partial g_k}{\partial \overline{z}},$$

therefore,

$$h|_{U_j} = \frac{\partial g_j}{\partial \overline{z}}$$

induces a smooth function $h\colon \Omega \to \mathbb{C}$. By the Dolbeault lemma, there exists a function $g \in \mathcal{C}^\infty(\Omega)$ such that $\frac{\partial g}{\partial \overline{z}} = h$. It is clear that $f_j = g_j - g$ solves the generalized additive Cousin problem. $\square$

Proposition 1.3.11. The additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

Proof. An additive Cousin problem induces a generalized additive Cousin problem for functions $f_{j,k} = f_j - f_k$. Let g_j be a solution to the generalized problem. As $f_j - f_k = f_{j,k} = g_j - g_k$ on $U_j \cap U_k$, we can define a function $f \in \mathcal{M}(\Omega)$ with $f|_{U_j} = f_j - g_j$. This function is of course well defined. As $f|_{U_j} - f_j = g_j \in \mathcal{O}(U_j)$, this function indeed solves the additive Cousin problem. $\square$

Theorem 1.3.12 (Mittag-Leffler)**.** Let $(a_k)_{k\in\mathbb{N}}$ be a sequence without repetition and accumulation points. Let

$$f_k(z) = \sum_{\ell=-m_k}^{-1} c_{k,\ell}(z-a_k)^\ell$$

be finite principal parts. Then there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ with poles in $(a_k)_{k\in\mathbb{N}}$ such that f has principle part f_k in a_k for each $k \in \mathbb{N}$.

Proof. For each a_k choose a disk U_k containing no other a_k. Also set $U_0 = \mathbb{C}\setminus\{a_k \mid k \in \mathbb{N}\}$ and $f_0 = 0$. As $\{U_k \mid k \in \mathbb{N}_0\}$ is an open cover of $\mathbb{C}$, there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ that solves the corresponding additive Cousin problem. It is easy to see that the principle parts of f at a_k are precisely f_k. $\square$

1.4 Sequences of holomorphic functions

Definition 1.4.1. A family of functions $\mathcal{F}$ from Ω to $\mathbb{C}$ is *locally bounded*, if for all $p \in \Omega$ there exist a $\rho > 0$ and $M > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{z \in \mathbb{\Delta}(p,\rho)} |f(z)| < M.$$

Lemma 1.4.2. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ a locally bounded family of functions. Then for all $p \in \Omega$ there exists a $\rho > 0$ such that $\mathcal{F}$ is equi-continuous on $\Omega \cap \mathbb{\Delta}(p, \rho)$.

Proof. Fix $p \in \Omega$ and choose $r > 0$ such that $D = \overline{\mathbb{\Delta}(p, 2r)} \subseteq \Omega$. For any $z, w \in D$ and $f \in \mathcal{F}$ we have

$$f((z) - f(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - w} \, d\xi = \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} \, d\xi.$$

Note that the family $\mathcal{F}$ is bounded on every compact. Therefore, we can write

$$\sup_{f \in \mathcal{F}} \sup_{z \in \partial D} |f(z)| < M.$$

Now, for $z, w \in \mathbb{\Delta}(p, r)$ we have

$$|f((z) - f(w)| = \left| \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} \, d\xi \right| \le |z - w| \cdot \frac{2M}{r}.$$ □

Theorem 1.4.3 (Arzelà-Ascoli)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be an infinite family such that the following conditions hold:

i) $\mathcal{F}$ is point-wise bounded.

ii) $\mathcal{F}$ is locally equi-continuous.

Then there $\mathcal{F}$ contains a sequence that converges uniformly on compacts of Ω.

Proof. Choose a dense countable subset $A \subseteq \Omega$ and enumerate it as a sequence $(a_k)_{k \in \mathbb{N}}$. Pick any sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ with pairwise distinct terms. As $|f_n(a_1)| < M$ for all n, we can choose a subsequence $(f_{1,n})_{n \in \mathbb{N}}$ such that $f_{1,n}(a_1)$ converges by Bolzano-Weierstraß.

Similarly, for every $k \in \mathbb{N}$ there exists a subsequence $(f_{k,n})_n$ of $(f_{k-1,n})_n$ such that $(f_{k,n}(a_k))_n$ converges. Now define $F_n = f_{n,n}$. Observe that (F_n) converges at every point in A.

Fix a $p \in \Omega$. By local equi-continuity, there exists a $\rho > 0$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta < \rho$ and $|F_n(z) - F_n(w)| < \frac{\varepsilon}{3}$ for all $z, w \in \mathbb{\Delta}(p, \rho)$ such that $|z - w| < \delta$. Choose an element $a \in A \cap \mathbb{\Delta}(z, \delta)$.[3] Then,we have

$$|F_n(z) - F_m(z)| \le |F_n(z) - F_n(a)| + |F_n(a) - F_m(a)| + |F_m(a) - F_m(z)| < 3 \cdot \frac{\varepsilon}{3}.$$

It follows that (F_n) is locally uniformly convergent, therefore it converges uniformly on compact sets. □

[3] By compactness of $\overline{\mathbb{\Delta}(p, \rho)}$ we can choose a from a finite set.

Theorem 1.4.4 (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f_n \colon \Omega \to \mathbb{C}$ be a locally bounded sequence of holomorphic functions. Then $(f_n)_n$ contains a subsequence that converges uniformly on compacts.

Proof. As the sequence is locally bounded, it is locally equi-continuous by lemma 1.4.2. By Arzelà-Ascoli, there exists a convergent subsequence. □

Definition 1.4.5. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family of functions $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is *normal* if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on compacts.

Theorem 1.4.6 (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is normal if and only if it is locally bounded.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.4.7 (Vitali)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $(f_n)_n \subseteq \mathcal{O}(\Omega)$ a locally bounded sequence of holomorphic functions. The following statements are equivalent:

i) The sequence $(f_n)_n$ converges uniformly on compact subsets of Ω.

ii) For a point $p \in \Omega$ the sequence $(f_n^{(k)}(p))_n$ converges for all $k \in \mathbb{N}_0$.

iii) The set
$$A = \left\{z \in \Omega \;\middle|\; \lim_{n\to\infty} f_n(z) \text{ converges}\right\}$$
has an accumulation point in Ω.

Proof. Suppose that the sequence converges uniformly on compact subsets. Given a $p \in \Omega$, choose a $\delta > 0$ such that $D = \overline{\Delta(p, \delta)} \subseteq \Omega$. Note that

$$\left|g^{(k)}(p)\right| \leq \frac{k!}{\delta^k} \cdot \|g\|_D$$

holds for all holomorphic functions g. As $\|f - f_n\|$ converges to 0, the derivatives of f_n converge.

Suppose that the sequences of derivatives converge at a point $p \in \Omega$ and choose a $\delta > 0$ such that $D = \overline{\Delta(p, \delta)} \subseteq \Omega$. As the sequence is locally bounded, there exists a constant M such that $\|f_n\|_D \leq M$ holds for all $n \in \mathbb{N}$. We can now develop power series

$$f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z-p)^k.$$

They converge uniformly on compact subsets of $\Delta(p, \delta)$. Note that

$$a_{k,n} = \frac{f_n^{(k)}(p)}{k!}.$$

As derivatives converge, we can define the limit

$$a_k = \lim_{n\to\infty} a_{k,n}.$$

Now define the formal power series

$$f(z)=\sum_{k=0}^{\infty} a_k(z-p)^k.$$

The Cauchy bounds give us the inequality

$$|a_{k,n}|=\frac{\left|f^{(k)}(p)\right|}{k!}\leq\frac{M}{\delta^k},$$

therefore

$$\limsup_{k\to\infty}\sqrt[k]{|a_k|}\leq\limsup_{k\to\infty}\frac{\sqrt[k]{M}}{\delta}=\frac{1}{\delta}.$$

We conclude that the radius of convergence is at least δ. Consider some $\rho\in(0,\delta)$ and $z\in\mathbb{\Delta}(p,\rho)$. We have

$$\begin{aligned}
|f_n(z)-f(z)| &\leq \left|\sum_{k=0}^{m}(a_{k,n}-a_k)\cdot(p-z)^k\right|+\left|\sum_{k=m+1}^{\infty}(a_{k,n}-a_k)\cdot(p-z)^k\right| \\
&\leq \sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+\sum_{k=m+1}^{\infty}2M\cdot\frac{\rho^k}{\delta^k} \\
&= \sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+2M\cdot\left(\frac{\rho}{\delta}\right)^{m+1}\cdot\frac{\delta}{\delta-\rho} \\
&= 2\cdot\frac{\varepsilon}{2}
\end{aligned}$$

for large enough m and n. It follows that p is an accumulation point of A.

Suppose now that A has an accumulation point in Ω. By Montel's theorem there exists a subsequence $(f_{n_m})_m$ that converges uniformly on compact subsets of Ω to a limit function f. Note that all such subsequences have the same limit by the identity principle.

Assume that the sequence $(f_n)_n$ does not converge uniformly on a compact subset $K\subseteq\Omega$. We can therefore construct another subsequence $(g_n)_n$ of $(f_n)_n$ such that

$$\|g_n-f\|_K>\varepsilon$$

for all $n\in\mathbb{N}$. But note that $(g_n)_n$ also has a convergent subsequence by Montel's theorem, which is of course a contradiction, as it cannot converge to f. $\square$

2 Theorems about holomorphic functions

2.1 Riemann mapping theorem

Definition 2.1.1. A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if every closed path in Ω is homotopic to a constant path in Ω.

Lemma 2.1.2. Let $\Omega \subset \mathbb{C}$ be a domain and $a \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Then there exists a holomorphic injection $f \colon \Omega \to \mathbb{A}$ such that $f(a) = 0$.

Proof. Fix a point $p \in \mathbb{C} \setminus \Omega$. By our assumption, there exists a function $v \in \mathcal{O}^*(\Omega)$ such that $v(z)^2 = z - p$. Note that v is injective. Similarly, we have $v(\Omega) \cap -v(\Omega) = \emptyset$. Now choose a point $b \in -v(\Omega)$. As v is not constant, it is an open map. Therefore, there exists some $r > 0$ such that $\mathbb{A}(b, r) \cap v(\Omega) = \emptyset$. The Möbius transformation

$$h(w) = r \cdot \left(\frac{1}{w - b} - \frac{1}{v(a) - b} \right)$$

thus maps $v(\Omega)$ into $\mathbb{A}$. The map f is therefore given as $f = h \circ v$. $\square$

Definition 2.1.3. An *expansion* if a map $\kappa \colon \Omega \to \mathbb{A}$ where $0 \in \Omega \subset \mathbb{A}$ such that $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ holds for all $z \neq 0$.

Lemma 2.1.4. Let $\Omega \subset \mathbb{A}$ be a domain with $0 \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Choose $c \in \mathbb{A}$ such that $c^2 \notin \Omega$. For all $a \in \mathbb{A}$, let

$$g_a = \frac{z - a}{\overline{a} z - 1}$$

and choose $v \in \mathcal{O}(\Omega)$ such that $v(z)^2 = g_{c^2}(z)$ and $v(0) = c$. Then the map $\kappa = g_c \circ v$ is an expansion and

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = \mathrm{id}_\Omega .$$

Proof. Note that v is indeed well-defined. Also note that

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = g_{c^2} \circ (z \mapsto z^2) \circ v = g_{c^2} \circ g_{c^2} = \mathrm{id} .$$

We of course have $\kappa(0) = 0$. Denote $\psi_c = g_{c^2} \circ (z \mapsto z^2) \circ g_c$. It remains to check that $|\kappa(z)| > |z|$, which is equivalent to $|\psi_c(z)| < |z|$ for $z \neq 0$ as $\psi_c \circ \kappa = \mathrm{id}$. Note that $\psi_c \colon \mathbb{A} \to \mathbb{A}$ is holomorphic. As it is not a rotation (it is not injective), the conclusion follows from the Schwarz lemma. $\square$

Lemma 2.1.5 (Hurwitz)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f_n \colon \Omega \to \mathbb{C}$ be holomorphic functions. Suppose that the sequence $(f_n)_n$ converges uniformly on compacts of Ω to a non-constant function $f \colon \Omega \to \mathbb{C}$. Then for all points $p \in \Omega$ there exists a sequence $(p_n)_n \subseteq \Omega$ with limit p such that $f_n(p_n) = f(p)$ for all $n > N$.

Proof. Let $w = f(p)$. There exists a disk $\mathbb{A}(p, \delta)$ such that $f(z) \neq w$ for all points $z \in \overline{\mathbb{A}(p, \delta)} \setminus \{p\}$. Note that we have

$$\min_{z \in \partial \mathbb{A}(p,\delta)} |f(z) - w| > |f(p) - w| = 0.$$

As $(f_n)_n$ converges uniformly on $\overline{\mathbb{\Delta}(p, \delta)}$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\min_{z \in \partial \mathbb{\Delta}(p,\delta)} |f_n(z) - w| > |f_n(p) - w| \, .$$

By lemma 1.1.7, $f_n(z) - w$ has a root $p_n \in \mathbb{\Delta}(p, \delta)$. For any convergent subsequence $(p_{n_k})_k$ with limit q we have

$$f(p) = \lim_{k \to \infty} f_{n_k}(p_{n_k}) = f(q),$$

therefore $p = q$. □

Corollary 2.1.5.1. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n \colon \Omega \to \mathbb{C}$ be holomorphic functions such that $(f_n)_n$ converges uniformly on compacts of Ω to $f \colon \Omega \to \mathbb{C}$. If all the f_n are nowhere vanishing and $f \neq 0$, then f is nowhere vanishing.

Proof. The proof is obvious and need not be mentioned. □

Theorem 2.1.6 (Hurwitz)**.** Let $\Omega, \Omega' \subseteq \Omega$ be domains and $f_n \colon \Omega \to \Omega'$ be holomorphic functions that converge uniformly on compacts of Ω to $f \colon \Omega \to \Omega'$. Assume that f is not constant.

i) If $f_n \colon \Omega \to \Omega'$ is injective, f is also injective.

ii) We have $f(\Omega) \subseteq \Omega'$.

Proof.

i) Let $p \in \Omega$ and observe the functions $g_n(z) = f_n(z) - f_n(p)$. This is a sequence of nowhere vanishing functions. As f is not constant, $f(z) - f(p)$ is nowhere vanishing as well. It follows that f is injective.

ii) Suppose otherwise and apply the Hurwitz lemma for a point p with $f(p) \notin \Omega'$. □

Theorem 2.1.7 (Riemann mapping)**.** For a proper domain $\Omega \subset \mathbb{C}$ the following are equivalent:

i) Ω is simply connected.

ii) Ω admits a logarithm for any $f \in \mathcal{O}^*(\Omega)$.

iii) Ω admits a square root for any $f \in \mathcal{O}^*(\Omega)$.

iv) Ω is biholomorphic to $\mathbb{\Delta}$.

Proof. Note that if Ω is biholomorphic to $\mathbb{\Delta}$, it is of course simply connected. Suppose that Ω is simply connected. Then

$$F(z) = a + \int_{z_0}^{z} \frac{f'(z)}{f(z)} \, dz$$

defines a logarithm for any $f \in \mathcal{O}^*(\Omega)$. Given a logarithm of a function, we can of course construct a square root with $\sqrt{f} = e^{\frac{1}{2} \ln f}$. It remains to check that all domains admitting square roots are biholomorphic to $\mathbb{\Delta}$.

By lemma 2.1.2 we can assume that $\Omega \subseteq \mathbb{A}$ and $0 \in \Omega$. Now define the family of functions

$$\mathcal{F} = \{f : \Omega \to \mathbb{A} \mid f \in \mathcal{O}(\Omega) \land f(0) = 0 \land f \text{ is injective}\}.$$

If $\mathcal{F}$ has no biholomorphic map, it is infinite. Note that $\mathcal{F}$ is bounded, so it is normal by Montel.

Choose a point $p \in \Omega$ with $p \neq 0$. We claim that if $h \in \mathcal{F}$ and

$$|h(p)| = \sup_{f \in \mathcal{F}} |f(p)|,$$

we have $h(\Omega) = \mathbb{A}$. Indeed, if that were not the case, we'd reach a contradiction with the expansion κ of Ω as

$$|\kappa(h(p))| > |h(p)|$$

and $\kappa \circ h \in \mathcal{F}$.

Let

$$M = \sup_{f \in \mathcal{F}} |f(p)|$$

and find a sequence $(f_n)_n \subseteq \mathcal{F}$ with

$$\lim_{n \to \infty} |f_n(p)| = M.$$

As $\mathcal{F}$ is a normal family, there exists a convergent subsequence. The limit is not constant as $f(p) \neq 0$. By Hurwitz, f is injective and $f(\Omega) \subseteq \mathbb{A}$. By the above claim, we have $f(\Omega) = \mathbb{A}$. $\square$

2.2 Bloch's theorem

Lemma 2.2.1. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $f \colon \overline{\Omega} \to \mathbb{C}$ a continuous map such that $f|_\Omega$ is an open map. Let $a \in \Omega$ be a point such that

$$s = \min_{z \in \partial\Omega} |f(z) - f(a)| > 0.$$

Then $f(\Omega)$ contains the disk $\mathbb{\Delta}(f(a), s)$.

Proof. By compactness, there exists a $w_0 \in \partial f(\Omega)$ such that $d(\partial f(\Omega), f(a)) = |w_0 - f(a)|$. Let $(z_k)_k \subseteq \Omega$ be a sequence, convergent to z_0, such that

$$\lim_{k\to\infty} f(z_k) = w_0.$$

Of course $f(z_0) = w_0$. Note that, as $f|_\Omega$ is open, we have $z_0 \in \partial\Omega$. But then

$$d(\partial f(\Omega), f(a)) = |f(z_0) - f(a)| \geq s. \qquad \square$$

Lemma 2.2.2. Let f be a non-constant function, holomorphic in a neighbourhood of $\overline{\mathbb{\Delta}(a, r)}$. Assume that

$$\sup_{z \in \overline{\mathbb{\Delta}(a,r)}} |f'(z)| \leq 2\,|f'(a)|\,.$$

Then $\mathbb{\Delta}(f(a), R) \subseteq f(\mathbb{\Delta}(a, r))$, where

$$R = \left(3 - 2\sqrt{2}\right) \cdot r \cdot |f'(a)|\,.$$

Proof. Without loss of generality assume that $a = f(a) = 0$. Define

$$A(z) = f(z) - f'(0)z = \int_0^1 (f'(tz) - f'(0))\, z\, dt.$$

Note that

$$f'(v) - f'(0) = \frac{1}{2\pi i} \oint_{\partial\mathbb{\Delta}(a,r)} f'(\xi) \cdot \left(\frac{1}{\xi - v} - \frac{1}{\zeta}\right) d\xi,$$

therefore

$$|f'(v) - f'(0)| \leq \frac{1}{2\pi} \cdot |v| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r \cdot (r - |v|)} \cdot 2\pi r = |v| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r - |v|}.$$

It follows that

$$\begin{aligned}
|A(z)| &\leq \int_0^1 |z| \cdot |f'(tz) - f'(0)|\, dt \\
&\leq |z| \cdot \int_0^1 |tz| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r - |tz|}\, dt \\
&\leq |z|^2 \cdot \|f'\|_{\mathbb{\Delta}(a,r)} \cdot \int_0^1 t \cdot \frac{1}{r - |z|} \\
&= |z|^2 \cdot 2\frac{|f'(0)|}{r - |z|}.
\end{aligned}$$

Now, using the triangle inequality, we get

$$|f(z)| \geq |z| \cdot |f'(0)| - |A(z)|\,.$$

Let $|z| = \rho \in (0, r)$. We get

$$|f(z)| \geq \rho \cdot |f'(0)| - |A(z)| \geq \rho \cdot |f'(0)| - \frac{\rho^2}{r-\rho} \cdot |f'(0)| \geq |f'(0)| \cdot \left(\rho - \frac{\rho^2}{r-\rho}\right).$$

Note that there exists a ρ_0 such that

$$\rho_0 - \frac{\rho_0^2}{r-\rho_0} = r \cdot \left(3 - 2\sqrt{2}\right).$$

Therefore, we get

$$|f(z)| \geq |f'(0)| \cdot r \cdot \left(3 - 2\sqrt{2}\right).$$

Now just apply the previous lemma to the disk $\mathbb{\Delta}(0, \rho_0)$. $\square$

Theorem 2.2.3 (Bloch)**.** Let f be a function, holomorphic in a neighbourhood of $\overline{\mathbb{\Delta}}$, with $f'(0) = 1$. Then $f(\mathbb{\Delta})$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.

Proof. Define $h(z) = |f'(z)|\,(1 - |z|) \geq 0$. Not that $h \not\equiv 0$ as f is not constant. Therefore h attains a maximum in a point $p \in \overline{\mathbb{\Delta}}$. In particular, as $h|_{\partial\mathbb{\Delta}} = 0$, we have $p \in \mathbb{\Delta}$. Observe $\Omega = \mathbb{\Delta}\,(p, t)$ for $t = \frac{1}{2} \cdot (1 - |p|)$. For all $z \in \Omega$, we have $1 - |z| \geq t$ and

$$|f'(z)| \cdot (1 - |z|) \leq |f'(p)| \cdot (1 - |p|) = |f'(p)| \cdot 2t \leq |f'(p)| \cdot 2 \cdot (1 - |z|).$$

Now, applying lemma 2.2.2, we have $\mathbb{\Delta}(f(p), R) \subseteq f(\mathbb{\Delta})$ with

$$R = \left(3 - 2\sqrt{2}\right) \cdot \frac{1}{2} \cdot (1 - |p|) \cdot |f'(p)| \geq \frac{3}{2} - \sqrt{2}$$

by choice of p. $\square$

Remark 2.2.3.1. Let

$$\mathcal{F} = \left\{ f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{\Delta}} \;\middle|\; f'(0) = 1 \right\}.$$

For $f \in \mathcal{F}$, denote by L_f the supremum of radii of disks contained in $f(\mathbb{\Delta})$, and by B_f the supremum of radii of disks contained in $f(\mathbb{\Delta})$ that is a biholomorphic image of some subdomain of $\mathbb{\Delta}$. We then define the *Landau's constant*

$$L = \inf_{f \in \mathcal{F}} L_f$$

and the *Bloch's constant*

$$B = \inf_{f \in \mathcal{F}} B_f.$$

The current known bounds for the constants are

$$0.5 < L < 0.544 \quad \text{and} \quad \frac{\sqrt{3}}{4} + 10^{-14} < B \leq \sqrt{\frac{\sqrt{3}-1}{2}} \cdot \frac{\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

Corollary 2.2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{O}(\Omega)$ a function and $p \in \Omega$. Let $r = d(p, \partial\Omega)$. Then $f(\Omega)$ contains a disk of radius

$$\left(\frac{3}{2} - \sqrt{2}\right) \cdot r \cdot |f'(p)|.$$

Proof. The proof is obvious and need not be mentioned. □

Remark 2.2.3.3. Liouville's theorem follows from this corollary.

Lemma 2.2.4. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $1, -1 \notin f(\Omega)$. Then there exists a function $F \in \mathcal{O}(\Omega)$ such that $f = \cos(F)$.

Proof. Note that, as Ω is simply connected, we can define

$$F(z) = \frac{1}{i} \cdot \ln\left(f(z) + \sqrt{f(z)^2 - 1}\right).$$

□

Theorem 2.2.5. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let $f \in \mathcal{O}(\Omega)$. Suppose that $0, 1 \notin f(\Omega)$. Then the following statements are true:

i) There exists a function $g \in \mathcal{O}(\Omega)$ such that

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right).$$

ii) If any $g \in \mathcal{O}(\Omega)$ satisfies the above equality, then $g(\Omega)$ contains no disk of radius 1.

Proof.

i) Apply the previous lemma twice.

ii) Define

$$A = \left\{ m \pm \frac{i}{\pi} \ln\left(n + \sqrt{n^2 - 1}\right) \;\middle|\; m \in \mathbb{Z} \land n \in \mathbb{N} \right\}.$$

We claim that $g(\Omega) \cap A = \emptyset$. Indeed, for $a \in A$ we have

$$f(a) = \frac{1}{2}\left(1 + \cos(\pm\pi \cdot n)\right) \in \{0, 1\}.$$

Now note that

$$\begin{aligned}\ln\left(n + 1 + \sqrt{n^2 + 2n}\right) - \ln\left(n + \sqrt{n^2 - 1}\right) &= \ln\left(\frac{n + 1 + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 - 1}}\right) \\ &\le \ln\left(\frac{2n + 2}{n}\right) \\ &\le \ln(4) \\ &< \pi.\end{aligned}$$

It's straightforward to check that every disk of radius 1 intersects A. □

Theorem 2.2.6 (Picard's little theorem)**.** Every non-constant entire function omits at most one complex value.

Proof. Without loss of generality assume that f omits 0 and 1. Applying the above theorem, we can write

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right).$$

Recall that $g(\mathbb{C})$ contains no disk of radius 1. If g is not constant, $g(\mathbb{C})$ contains arbitrarily large disks by corollary 2.2.3.2, which is a contradiction. □

Corollary 2.2.6.1. Suppose that $f \in \mathcal{M}(\mathbb{C})$ is a non-constant function. Then f omits at most 2 values.

Proof. Suppose that f omits distinct values a, b and c. Then

$$g(z) = \frac{1}{f(z) - a}$$

is an entire function that omits values $\frac{1}{b-a}$ and $\frac{1}{c-a}$, therefore it is constant. $\square$

Theorem 2.2.7. Let $f \in \mathcal{O}(\mathbb{C})$ be an entire function. Then either $f \circ f$ has a fixed point of $f(z) = z + c$.

Proof. If $f \circ f$ has no fixed point, the same holds for f. We can therefore define an entire holomorphic function g with

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}.$$

Note that g omits both 0 and 1, therefore it is constant. But then

$$f(f(z)) - z = \lambda(f(z) - z)$$

for some $\lambda \notin \{0, 1\}$ by Picard's little theorem. Taking the derivative, we get

$$f'(f(z)) \cdot f'(z) - 1 = \lambda(f'(z) - 1),$$

or equivalently

$$f'(z) \cdot (f'(f(z)) - \lambda) = 1 - \lambda \neq 0.$$

Note that $f' \circ f$ omits both λ and 0, therefore it is constant. But then f' is constant as well. The only option is $f'(z) = 1$. $\square$

Lemma 2.2.8. For all $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos(\pi v) = w$ and $|v| \leq 1 + |w|$.

Proof. Let $v = \alpha + i\beta$ and note that

$$|w|^2 = \cos(\pi\alpha)^2 + \sinh(\pi\beta)^2 \geq \pi^2\beta^2.$$

Observe that we can choose some α such that $|\alpha| \leq 1$, therefore

$$1 + |w| \geq 1 + \pi \cdot |\beta| \geq |\alpha| + |\beta| \geq |v|.$$

$\square$

Theorem 2.2.9. Let f be a function, holomorphic on a neighbourhood of $\overline{\mathbb{A}}$, such that $0, 1 \notin f(\Omega)$. There exists a function g, holomorphic on a neighbouhood of $\overline{\mathbb{A}}$, such that

i) the equality

$$f = \frac{1}{2}(1 + \cos(\pi \cdot \cos(\pi \cdot g)))$$

holds with $|g(0)| \leq 3 + 2|f(0)|$, and

ii) the inequality

$$|g(z)| \leq |g(0)| + \frac{\theta}{\gamma(1 - \theta)}$$

holds for all $|z| \leq \theta$.

Proof. Again, apply lemma 2.2.4 and let

$$2f - 1 = \cos(\pi \cdot F).$$

Using the above lemma, we can transform F such that $|F(0)| \leq 1 + |2f(0) - 1|$. Applying lemma 2.2.4 again, we define g such that

$$F = \cos(\pi g).$$

Again, using the above lemma, set $|g(0)| \leq 1 + |F(0)|$. We therefore have

$$|g(0)| \leq 1 + |F(0)| \leq 2 + |2f(0) - 1| \leq 3 + 2\,|f(0)|\,.$$

Recall that $g(\mathbb{\Delta})$ does not contain a disk of radius 1. Let $z \in \overline{\mathbb{\Delta}(\theta)}$. Then, by Bloch's theorem, $g(\mathbb{\Delta})$ contains a disk of radius $R = \gamma \cdot |g'(z)| \cdot (1 - \theta)$. Therefore, we must have

$$|g'(z)| < \frac{1}{\gamma(1 - \theta)}.$$

It follows that

$$|g(z)| = \left|g(0) + \int_0^z g'(\xi)\, d\xi\right| \leq |g(0)| + \int_0^z |g'(\xi)|\; d\xi \leq |g(0)| + |z| \cdot \frac{1}{\gamma(1 - \theta)}. \qquad \square$$

Definition 2.2.10. For $r \geq 0$, let

$$S(r) = \left\{f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{\Delta}} \;\middle|\; 0, 1 \notin f\left(\overline{\mathbb{\Delta}}\right) \wedge |f(0)| \leq r\right\}.$$

For $\theta \in (0, 1)$ and $r > 0$, let

$$L(\theta, r) = \exp\left(\pi \cdot \exp\left(3 + 2r + \frac{\theta}{\gamma(1 - \theta)}\right)\right),$$

where γ is any constant such that Bloch's theorem holds, e.g. $\gamma = \frac{3}{2} - \sqrt{2}$.

Theorem 2.2.11 (Schottky)**.** Let $f \in S(r)$. Then for all $z \in \mathbb{\Delta}$ such that $|z| < \theta$ we have

$$|f(z)| \leq L(\theta, r).$$

Proof. Let g be a holomorphic function as in the previous theorem. Note that $|\cos(w)| \leq e^{|w|}$. We must therefore also have

$$\frac{1}{2} \cdot |1 + \cos(w)| \leq e^{|w|}.$$

Using this inequality, we get

$$|f(z)| \leq \exp\left(\pi \cdot \exp\left(\pi \cdot |g(z)|\right)\right) \leq L(\theta, r). \qquad \square$$

2.3 The great Picard theorem

Lemma 2.3.1. Let $\Omega \subseteq \mathbb{C}$ be a domain, $\omega \in \Omega$ and $r \in (0, \infty)$. Let

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

and $\mathcal{F}_{\omega,r} \subseteq \mathcal{F}$ a subfamily with $|f(\omega)| \leq r$ for all $f \in \mathcal{F}_{\omega,r}$.

i) There exists some $t > 0$ such that $\mathcal{F}_{\omega,r}|_{\Delta(\omega,t)}$ is bounded.

ii) The family $\mathcal{F}_{\omega,1}$ is locally bounded in Ω.

Proof.

i) Choose a $t > 0$ such that $\overline{\Delta(\omega, 2t)} \subseteq \Omega$ and set $\varphi(z) = 2tz + \omega$. By Schottky's theorem, we have

$$|f \circ \varphi(z)| \leq L\left(\frac{1}{2}, r\right)$$

for $|z| < \frac{1}{2}$, or equivalently

$$\sup_{v \in \Delta(w,t)} |f(v)| \leq L\left(\frac{1}{2}, r\right).$$

The family $\mathcal{F}_{\omega,r}$ is therefore bounded.

ii) Let

$$\mathcal{U} = \{u \in \Omega \mid \mathcal{F}_{\omega,1} \text{ is bounded in a neighbourhood of } u\}.$$

Note that $\omega \in \mathcal{U}$, therefore the set is non-empty. Also observe that $\mathcal{U}$ is open. Suppose that $\mathcal{U} \neq \Omega$ and let $v \in \partial\mathcal{U} \cap \Omega$. Then there exists a sequence $(f_n)_n \subseteq \mathcal{F}_{\omega,1}$ such that

$$\lim_{n\to\infty} |f_n(v)| = \infty.$$

Define $g_n = \frac{1}{f_n}$. These functions are holomorphic and omit both 0 and 1 by definition, therefore $g_n \in \mathcal{F}$. Applying the item i) for the sequence $(g_n)_n$ at point v, the sequence is bounded in a neighbourhood of v. By Montel's theorem, there exists a subsequence $(g_{n_k})_k$ that converges to a function g uniformly on compacts of $\Delta(v, s)$. By corollary 2.1.5.1, the function g is constant. But then

$$\lim_{k\to\infty} |f_{n_k}(z)| = \infty$$

for all $z \in \Delta(v, s)$, which is not possible as v is a boundary point. It follows that $\mathcal{U} = \Omega$. □

Definition 2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n \colon \Omega \to \mathbb{C}$ a sequence of functions. We say that f_n *converges to* ∞ if

$$\lim_{n\to\infty} \|f_n\|_K = \infty$$

for every compact $K \subset \Omega$.

Theorem 2.3.3 (Montel – sharp)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

Then $\mathcal{F}$ is normal in Ω where we also allow convergence to ∞.

Proof. Let $\Omega \subseteq \mathbb{C}$ be a domain and $p \in \Omega$. Consider the family $\mathcal{F}_{p,1}$. Let $(f_n)_n \subseteq \mathcal{F}$ be a sequence. If there exists a subsequence $(f_{n_k})_k \subseteq \mathcal{F}_{p,1}$, we can apply the above lemma. By the classical Montel's theorem, this subsequence has a convergent subsequence.

Suppose now that no such subsequence exists, that is $(f_n)_n$ has only finitely many terms in $\mathcal{F}_{p,1}$. But then there exists a subsequence $\left(\frac{1}{f_{n_k}}\right)_k \subseteq \mathcal{F}_{p,1}$. As before, this sequence has a convergent subsequence with limit g. If g is nowhere-vanishing, then $\frac{1}{g}$ is the limit of a subsequence of $(f_n)_n$. Otherwise, by corollary 2.1.5.1, we have $g = 0$ and therefore $(f_n)_n$ converges to ∞. $\square$

Definition 2.3.4. Let $\Omega \subseteq \mathbb{C}$ be an open set and $p \in \Omega$. A function $f \in \mathcal{O}(\Omega \setminus \{p\})$ has an *essential singularity* in p if the limit

$$\lim_{z \to p} f(z)$$

does not exist and

$$\lim_{z \to p} |f(z)| \neq \infty.$$

Theorem 2.3.5 (Picard's great theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set $p \in \Omega$ a point and $f \in \mathcal{O}(\Omega \setminus \{p\})$ a function. If f has an essential singularity at p, then f assumes every complex number as a value infinitely many times with at most one exception.

Proof. Without loss of generality assume that $p = 0$ and consider $\Omega = \mathbb{\Delta}(\varepsilon)$. Suppose that f omits two values on $\mathbb{\Delta}(\varepsilon)$, without loss of generality 0 and 1.

We now claim that f or $\frac{1}{f}$ is bounded in a neighbourhood of 0. Define the sequence of holomorphic functions $(f_n)_n$ with $f_n(z) = f\left(\frac{z}{n}\right)$. This sequence also omits 0 and 1, therefore either $(f_n)_n$ or $\left(\frac{1}{f_n}\right)_n$ has a convergent subsequence that converges uniformly on compacts by the sharp version of Montel's theorem. Denote the subsequence by $(g_{n_k})_k$ and set $g = f$ or $g = \frac{1}{f}$ accordingly.

Observe that there exists a constant M such that

$$\|g_{n_k}\|_{\partial \mathbb{\Delta}\left(\frac{\varepsilon}{2}\right)} \leq M$$

holds for all $k \in \mathbb{N}$. This is equivalent to

$$|g(z)| \leq M$$

for $|z| = \frac{1}{n_k} \cdot \frac{\varepsilon}{2}$. By the maximum principle, we have

$$|g(z)| \leq M$$

for all z such that

$$\frac{\varepsilon}{2} \cdot \frac{1}{n_k} \leq |z| \leq \frac{\varepsilon}{2}.$$

But as $(n_k)_k$ diverges, the inequality $g(z) \leq M$ holds for all z such that $|z| \leq \frac{\varepsilon}{2}$, therefore f or $\frac{1}{f}$ is bounded near 0.

Observe that f is not bounded in a neighbourhood of 0, as otherwise 0 is a removable singularity, which is not possible. Similarly, if $\frac{1}{f}$ is bounded, then f has either a removable singularity or a pole at 0, which is again a contradiction. $\square$

3 Infinite products

3.1 Definition and convergence

Definition 3.1.1. Let $(a_k)_k$ be a sequence of complex numbers. The sequence

$$n \mapsto \prod_{k=1}^{n} a_k$$

is called the *sequence of partial products* with factors a_k. We denote

$$p_{m,n} = \prod_{k=m}^{n} a_k.$$

We say that the infinite product is *convergent* if there exists an index $m \in \mathbb{N}$ such that the limit

$$\widehat{a}_m = \lim_{n \to \infty} p_{m,n}$$

exists and is non-zero. We then define

$$\prod_{k=1}^{\infty} a_k = p_{1,m-1} \cdot \widehat{a}_m.$$

as the limit of the infinite product.

Remark 3.1.1.1. The limit is uniquely defined.

Remark 3.1.1.2. An infinite product is convergent if and only if the product of all its non-zero factors has a non-zero limit and only finitely many factors are non-zero.

Lemma 3.1.2. Let $(a_k)_k \subseteq \mathbb{R}_{\geq 0}$ be a sequence such that

$$\sum_{k=1}^{\infty} (1 - a_k) = \infty.$$

Then

$$\lim_{n \to \infty} \prod_{k=p}^{n} a_k = 0$$

for all $p \in \mathbb{N}$. In particular, the infinite product is divergent.

Proof. Observe that

$$0 \leq \prod_{k=p}^{n} a_k \leq \prod_{k=p}^{n} e^{a_k - 1},$$

which converges to 0. □

Definition 3.1.3. Let $X \subseteq \mathbb{C}$ be a set.

i) A series

$$\sum_{k=1}^{\infty} g_k$$

of continuous functions $g_k \in \mathcal{C}(X)$ is *normally convergent* if for every compact $K \subseteq X$ the series

$$\sum_{k=1}^{\infty} \|g_k\|_K$$

converges.

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ii) A product

$$\prod_{k=1}^{\infty} f_k$$

of continuous functions $f_k = 1 + g_k \in \mathcal{C}(X)$ is *normally convergent* if the series

$$\sum_{k=1}^{\infty} g_k$$

is normally convergent.

Definition 3.1.4. Let $X \subseteq \mathbb{C}$ be a set and $f_k \in \mathcal{C}(X)$ be continuous functions. Denote

$$p_{m,n} = \prod_{k=m}^{n} f_k.$$

We say that the infinite product

$$\prod_{k=1}^{\infty} f_k$$

converges *uniformly* on a set $L \subseteq X$ if there exists an index $m \in \mathbb{N}$ such that $f_k|_L$ has no zeroes for $k \geq m$ and

$$\lim_{n\to\infty} p_{m,n} = \widehat{f}_k$$

exists, is uniform on L and has no zeroes on L. We define

$$\prod_{k=1}^{\infty} f_k = p_{1,m-1} \cdot \widehat{f}_m$$

on L.

Theorem 3.1.5 (Reordering of infinite products)**.** Let

$$\prod_{k=1}^{\infty} f_k$$

be a normally convergent product in $X \subseteq \mathbb{C}$. Then there exists a functions $f \colon X \to \mathbb{C}$ such that for all bijections $\tau \colon \mathbb{N} \to \mathbb{N}$ the product

$$\prod_{k=1}^{\infty} f_{\tau(k)}$$

converges to f uniformly on compacts of X. In particular, the infinite product converges uniformly on compacts.

Proof. Recall that, for $w \in \mathbb{A}$, we can define

$$\log(1+w) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} w^k.$$

Then,

$$|\log(1+w)| \leq |w| \cdot \sum_{k=0}^{\infty} |w|^k = \frac{|w|}{1-|w|}.$$

In particular, if $|w| \leq \frac{1}{2}$, we have

$$|\log(1+w)| \leq 2\,|w|\,.$$

Let $L \subseteq X$ be a compact and write $f_k = 1 + g_k$. For all $k > N$ we have $\|g_k\|_L \leq \frac{1}{2}$, therefore we can write

$$\log f_k = \log(1+g_k) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} g_k^{\ell}.$$

But then

$$\|\log f_k\|_L \leq 2\,\|g_k\|_L\,.$$

It follows that the series

$$\sum_{k=N}^{\infty} \|\log f_k\|_L$$

converges. But then the series

$$h_N = \sum_{k=N}^{\infty} \log f_k$$

converges absolutely, and therefore all reorderings of the series converge as well to the same limit h_N.

Observe that

$$e^{h_N} = \prod_{k=N}^{\infty} e^{\log f_k} = \prod_{k=N}^{\infty} f_k.$$

This product therefore converges uniformly on L, independently of reorderings. We now define

$$f = \prod_{k=1}^{N-1} f_k \cdot e^{h_N}.$$

Note that this holds for all reorderings, as they differ from a suitable one by only finitely many transpositions. $\square$

3.2 Zeroes of infinite products

Definition 3.2.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{O}(\Omega)$. The *zero set* of f is the set

$$Z(f) = \{z \in \Omega \mid f(z) = 0\}.$$

For all $c \in \Omega$, define the *zero order* of f in c as follows: if

$$f(z) = (z - c)^k \cdot g(z)$$

where $g(c) \neq 0$ is a holomorphic function, then $\operatorname{ord}_c(f) = k$.

Remark 3.2.1.1. For non-zero $f \in \mathcal{O}(\Omega)$, the set $Z(f)$ is discrete in Ω.

Remark 3.2.1.2. We have

$$\operatorname{ord}_c\left(\prod_{k=1}^{n} f_k\right) = \sum_{k=1}^{n} \operatorname{ord}_c(f_k).$$

Lemma 3.2.2. Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^{\infty} f_k$$

be a normally convergent product in Ω, where $f_k \in \mathcal{O}(\Omega)$ are non-zero holomorphic functions. Then f is a non-zero function with

$$Z(f) = \bigcup_{k=1}^{\infty} Z(f_k)$$

and

$$\operatorname{ord}_c(f) = \sum_{k=1}^{\infty} \operatorname{ord}_c(f_k).$$

Proof. Recall that normally convergent products converge uniformly on compacts of Ω. In particular, f is a holomorphic function.

Pick a point $c \in \Omega$. By definition of convergence, there exists some $m \in \mathbb{N}$ such that $\widehat{f}_m(c) \neq 0$. As $\widehat{f}_m$ is holomorphic as well, we have

$$f(c) = \left(p_{1,m-1} \cdot \widehat{f}_m\right)(c),$$

but then

$$\operatorname{ord}_c(f) = \sum_{k=1}^{m-1} \operatorname{ord}_c(f_k) = \sum_{k=1}^{\infty} \operatorname{ord}_c(f_k). \qquad \square$$

Lemma 3.2.3. Let $\Omega \subseteq \mathbb{C}$ be a domain. If

$$f = \prod_{k=1}^{\infty} f_k$$

is a normally convergent product in Ω, where $f_k \in \mathcal{O}(\Omega)$ are holomorphic functions, then the sequence $\left(\widehat{f}_n\right)_n$ converges to 1 uniformly on compacts.

Proof. Choose $m \in \mathbb{N}$ such that $\widehat{f}_m \neq 0$. Then the set $Z\left(\widehat{f}_m\right)$ has no accumulation points in Ω. We can therefore write

$$\widehat{f}_n = \frac{\widehat{f}_m}{p_{m,n-1}}$$

on $\Omega \setminus Z\left(\widehat{f}_m\right)$. As $p_{m,n-1}$ converges to $\widehat{f}_m$ on compacts of Ω,

$$\lim_{n\to\infty} \widehat{f}_n = 1$$

uniformly on compacts of $\Omega \setminus Z\left(\widehat{f}_m\right)$. For any compact set $K \subseteq \Omega$, taking m large enough, we have $Z\left(\widehat{f}_m\right) \cap K = \emptyset$. The conclusion follows. $\square$

Definition 3.2.4. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. The meromorphic function $\frac{f'}{f}$ is called the *logarithmic derivative* of f.

Remark 3.2.4.1. For holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(\Omega)$ we have

$$\left(\prod_{k=1}^{n} f_k\right)' \cdot \left(\prod_{k=1}^{n} f_k\right)^{-1} = \sum_{k=1}^{n} \frac{f_k'}{f_k}.$$

Definition 3.2.5. Let $g_k \in \mathcal{M}(\Omega)$ be meromorphic functions. The series

$$\sum_{k=1}^{\infty} g_k$$

is *normally convergent* in Ω if for every compact $L \subseteq \Omega$ there exists some $m \in \mathbb{N}$ such that

$$\sum_{k=m}^{\infty} \|g_k\|_L$$

converges.

Theorem 3.2.6 (Logarithmic differentiation)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^{\infty} f_k$$

be a normally convergent product in Ω, where $f_k \in \mathcal{O}(\Omega)$ are non-zero functions. Then

$$\sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

is normally convergent in Ω and

$$\sum_{k=1}^{\infty} \frac{f_k'}{f_k} = \frac{f'}{f}.$$

Proof. As $\widehat{f}_n$ converges to 1 uniformly on compacts, the sequence $\left(f_n'\right)_n$ converges to 0 uniformly on compacts by Cauchy estimates. Then for any compact L, $\frac{\widehat{f}_n'}{\widehat{f}_n}$ converges to 0 as $\widehat{f}_n$ has no zeroes in L for n large enough. It follows that

$$\lim_{n\to\infty} \frac{f'}{f} - \sum_{k=1}^{n} \frac{f_k'}{f_k} = \lim_{n\to\infty} \frac{\widehat{f}_{n+1}'}{\widehat{f}_{n+1}} = 0.$$

Write $f_k = 1 + g_k$ and fix a compact set $L \subseteq \Omega$. Choose an index m such that we have $Z\left(\widehat{f}_m\right) \cap L = \emptyset$ and

$$\min_{z \in L} |f_k(z)| \geq \frac{1}{2}.$$

Choose $\varepsilon > 0$ such that

$$L_\varepsilon = \{z \in \mathbb{C} \mid d(z, L) \leq \varepsilon\} \subseteq \Omega.$$

By the Cauchy estimates, we have $\|g_k'\|_L \leq \frac{1}{\varepsilon} \|g_k\|_L$. But then

$$\sum_{k=m}^{\infty} \left\| \frac{f_k'}{f_k} \right\|_L = \sum_{k=m}^{\infty} \left\| \frac{g_k'}{f_k} \right\|_L \leq 2 \cdot \sum_{k=m}^{\infty} \|g_k'\|_L \leq \frac{2}{\varepsilon} \cdot \sum_{k=m}^{\infty} \|g_k\|,$$

which is convergent by our assumptions. □

Lemma 3.2.7. Let g be meromorphic on $\mathbb{C}$ with poles in $\mathbb{Z}$ with principal parts $\frac{1}{z-m}$. Moreover, assume that g is an odd function that satisfies

$$2g(2z) = g(z) + g\left(z + \frac{1}{2}\right).$$

Then $g(z) = \pi \cdot \cot(\pi z)$.

Proof. Simple calculations show that $\pi \cdot \cot(\pi z)$ is indeed a solution of the functional equation. Define $h(z) = g(z) - \pi \cdot \cot(\pi z)$. This another solution of the functional equation, and an odd function. In particular, $h(0) = 0$. Observe that the principal parts of h are 0, therefore $h \in \mathcal{O}(\mathbb{C})$ is an entire function.

Suppose that h is not constant. In particular, there exists some $c \in \partial\mathbb{A}(2)$ such that

$$|h(z)| < |h(c)|$$

for all $z \in \mathbb{A}(2)$. As $\frac{c}{2}, \frac{c+1}{2} \in \mathbb{A}(2)$, we can write

$$2\,|h(c)| = \left| h\left(\frac{c}{2}\right) + h\left(\frac{c+1}{2}\right) \right| \leq \left| h\left(\frac{c}{2}\right) \right| + \left| h\left(\frac{c+1}{2}\right) \right| < 2\,|h(c)|,$$

which is a contradiction. It follows that $h = 0$. □

Corollary 3.2.7.1. We have

$$\pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

Proof. Note that

$$\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2} = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k} \right),$$

therefore the series has poles in $\mathbb{Z}$ with principal parts $\frac{1}{z-m}$. It is also an odd function. A calculation shows that, for

$$r_n(z) = \frac{1}{z} + \sum_{k=1}^{n} \frac{2z}{z^2 - k^2},$$

we have

$$r_n(z) + r_n\left(z + \frac{1}{2}\right) = 2r_{2n}(2z) + \frac{2}{2z + 2n + 1}.$$

Taking $n \to \infty$, the conclusion follows. □

Theorem 3.2.8. We have

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Proof. The above product is obviously normally convergent, therefore we can take its logarithmic derivative. A simple calculation shows that it is equal to $\pi \cot(\pi z)$. As logarithmic derivatives are equal only for scalar multiples, we only have to check equality in one point. □

3.3 The Euler gamma function

Lemma 3.3.1. The infinite product

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-\frac{z}{k}}$$

in normally convergent in $\mathbb{C}$.

Proof. Write

$$\begin{aligned}
|1 - (1-\omega)e^{\omega}| &= |1 - e^{\omega} + \omega e^{\omega}| \\
&= \left| -\sum_{k=1}^{\infty} \frac{\omega^k}{k!} + \sum_{k=0}^{\infty} \frac{\omega^{k+1}}{k!} \right| \\
&= \left| \omega^2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \omega^{k-1} \right| \\
&\leq |\omega|^2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \\
&= |\omega|^2
\end{aligned}$$

for $|\omega| \leq 1$. But then the sum

$$\sum_{k=\lceil |z| \rceil}^{\infty} \left| 1 - \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right| \leq \sum_{k=\lceil |z| \rceil}^{\infty} \left| \frac{z^2}{k^2} \right|$$

converges normally. The infinite product must then converge normally in $\mathbb{C}$ as well. $\square$

Lemma 3.3.2. Let

$$H(z) = z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Then $H(1) = e^{-\gamma}$, where γ is the *Euler-Mascheroni constant*, that is

$$\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \log(n).$$

Proof. First note that

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = \prod_{k=1}^{n} \frac{k+1}{k} = n + 1.$$

We therefore have

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) e^{-\frac{1}{k}} = \exp\left(\log(n+1) - \sum_{k=1}^{n} \frac{1}{k} \right),$$

therefore

$$H(1) = \lim_{n \to \infty} \exp\left(\log(n+1) - \sum_{k=1}^{n} \frac{1}{k} \right) = e^{-\gamma}.$$ $\square$

Lemma 3.3.3. Let $\Delta(z) = e^{\gamma z} H(z)$.

i) We have $\Delta(1) = 1$ and $\Delta(z) = z\Delta(z+1)$.

ii) We have $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$.

Proof. Note that $\Delta(1) = 1$ by the previous lemma. Rewrite the partial products as

$$z \cdot \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} = \frac{z}{n!} \cdot \prod_{k=1}^{n} (z+k) \cdot \exp\left(-z \sum_{k=1}^{n} \frac{1}{k}\right).$$

We therefore have

$$\begin{aligned}
\Delta(z) &= \lim_{n\to\infty} \frac{e^{\gamma z}}{n!} \cdot \prod_{k=0}^{n} (z+k) \cdot \exp\left(-z \sum_{k=1}^{n} \frac{1}{k}\right) \\
&= \lim_{n\to\infty} \frac{e^{\gamma z}}{n! \cdot n^z} \cdot \prod_{k=0}^{n} (z+k) \cdot \exp\left(z \log(n) - z \sum_{k=1}^{n} \frac{1}{k}\right) \\
&= \lim_{n\to\infty} \frac{1}{n! \cdot n^z} \cdot \prod_{k=0}^{n} (z+k).
\end{aligned}$$

We can now calculate

$$z \cdot \Delta(z+1) = \lim_{n\to\infty} z \cdot \frac{1}{n! \cdot n^{z+1}} \cdot \prod_{k=1}^{n+1} (z+k) = \Delta(z) \cdot \lim_{n\to\infty} \frac{z+n+1}{n} = \Delta(z).$$

It remains to check the equality $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$. We have

$$\begin{aligned}
\pi \cdot \Delta(z)\Delta(1-z) &= \pi \cdot \Delta(z) \cdot \frac{\Delta(-z)}{-z} \\
&= \pi e^{\gamma z} \cdot z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \cdot e^{-\gamma z} \cdot \frac{-z}{-z} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{\frac{z}{k}} \\
&= \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \\
&= \sin(\pi z).
\end{aligned}$$

□

Definition 3.3.4. The *Euler gamma function* is defined as

$$\Gamma(z) = \frac{1}{\Delta(z)}.$$

Theorem 3.3.5. The Γ function satisfies the following properties:

1. The function Γ is meromorphic with simple poles in $-\mathbb{N}_0$.
2. We have $\Gamma(1) = 1$.
3. The function Γ satisfies $\Gamma(z+1) = z\Gamma(z)$.
4. The function Γ satisfies
$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$
5. We have
$$\Gamma(z) = \lim_{n\to\infty} n! \cdot n^z \cdot \left(\prod_{k=0}^{n} (z+k)\right)^{-1}.$$

Proof. The proof is obvious and need not be mentioned. □

Theorem 3.3.6. Let F be holomorphic in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ and assume $F(z+1) = z \cdot F(z)$. Furthermore, assume that F is bounded on the strip $1 \leq \operatorname{Re}(z) < 2$ and $F(1) = 1$. Then $F = \Gamma$.

3.4 Weierstraß factors

Definition 3.4.1. The *Weierstraß factors* are functions

$$E_n(z) = (1 - z) \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^n}{n}\right).$$

Lemma 3.4.2. The Weierstraß factors satisfy the following:

i) For $n \geq 1$ we have

$$E_n'(z) = -z^n \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^n}{n}\right).$$

ii) For $n \geq 0$ we have

$$E_n(z) = 1 + \sum_{k=n+1}^{\infty} a_k z^k,$$

where

$$\sum_{k=n+1}^{\infty} |a_k| = 1.$$

iii) For $n \geq 0$ and $|z| \leq 1$ we have

$$|E_n(z) - 1| \leq |z|^{n+1}.$$

Proof.

i) Evident.

ii) Observing the derivative, we see that $a_1 = a_2 = \cdots = a_n = 0$, and $a_k \leq 0$ for $k > n$. But then

$$\sum_{k=n+1}^{\infty} |a_k| = -\sum_{k=n+1}^{\infty} a_k = 1 - E_n(1) = 1.$$

iii) We have

$$|E_n(z) - 1| = \left|\sum_{k=n+1}^{\infty} a_k z^k\right| \leq \sum_{k=n+1}^{\infty} |a_k| \cdot |z|^k \leq |z|^{n+1}. \qquad \square$$

Lemma 3.4.3. Let $(a_k)_k \subset \mathbb{C}^*$ be a sequence of complex numbers with no accumulation point and let $(p_k)_k \subseteq \mathbb{N}_0$ be non-negative integers with

$$\sum_{k=1}^{\infty} \left|\frac{r}{a_k}\right|^{p_k+1}$$

converges for every $r > 0$. Then the *Weierstraß product*

$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$$

converges normally on $\mathbb{C}$.

Proof. Note that $|a_k| > |z|$ for all but finitely many k. Now just apply the previous lemma. $\square$

Theorem 3.4.4 (Weierstraß factorization theorem)**.** For any sequence $(a_k)_k \subset \mathbb{C}$ with no accumulation point there exists a Weierstraß product

$$z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$$

that converges normally on $\mathbb{C}$.

Proof. Set $p_k = k - 1$. For any $r > 0$ choose $m \in \mathbb{N}_0$ such that $|a_k| > 2r$ for all $k \geq m$. We then have

$$\sum_{k=m}^{\infty} \left|\frac{r}{a_k}\right|^{p_k+1} \leq \sum_{k=m}^{\infty} \frac{1}{2^k} \leq 2.$$ $\square$

Theorem 3.4.5 (Weierstraß product theorem)**.** Let $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$ be a holomorphic function. Then there exists a function $g \in \mathcal{O}(\mathbb{C})$ such that

$$f = e^g \cdot z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{k-1}\left(\frac{z}{a_k}\right),$$

where a_k are zeroes of f on $\mathbb{C} \setminus \{0\}$, counted with multiplicities, and $q = \operatorname{ord}_0(f)$.

Proof. The proof is obvious and need not be mentioned. $\square$

Lemma 3.4.6. Let $\Omega \subset \mathbb{C}$ be an open subset, $(a_k)_k \subset \Omega$ a sequence with no accumulation point in Ω and $A = \{a_k \mid k \in \mathbb{N}\}$. Let $(b_k)_k \subset \mathbb{C} \setminus \Omega$ and $(p_k)_k \subseteq \mathbb{N}$ be sequences such that the series

$$\sum_{k=1}^{\infty} |r(a_k - b_k)|^{p_k+1}$$

converges for all $r > 0$ and denote $B = \{b_k \mid k \in \mathbb{N}\}$. Then the infinite product

$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{a_k - b_k}{z - b_k}\right)$$

converges normally on $\mathbb{C} \setminus \overline{B}$.

Proof. Let $L \subseteq \mathbb{C} \setminus \overline{B}$ be a compact set and let $\ell = d\left(L, \overline{B}\right) > 0$. We then have $|z - b_k| \geq \ell$ for all $z \in L$ and $k \in \mathbb{N}$.

We can now bound

$$\left\|\frac{a_k - b_k}{z - b_k}\right\|_L \leq \frac{|a_k - b_k|}{\ell}.$$

By the assumption of convergence for $r = \frac{1}{\ell}$, we must have

$$|r \cdot (a_k - b_k)| < 1$$

for all $k \geq n(L)$, but then

$$\sum_{k=n(L)}^{\infty} \left\| E_{p_k}\left(\frac{a_k - b_k}{z - b_k}\right) - 1 \right\|_L \leq \sum_{k=n(L)}^{\infty} \left\| \frac{a_k - b_k}{z - b_k} \right\|_L^{p_k+1} \leq \sum_{k=n(L)}^{\infty} |r \cdot (a_k - b_k)|^{p_k+1},$$

which converges. □

Remark 3.4.6.1. The Weierstraß factor $E_{p_k}\left(\frac{a_k-b_k}{z-b_k}\right)$ is zero if and only if $z = a_k$.

Lemma 3.4.7. Let $A \subset \mathbb{C}$ be a discrete set and define $A' = \overline{A} \setminus A$. Suppose that $A' \neq \emptyset$ and let

$$A_1 = \{z \in A \mid |z| \cdot d(z, A') \geq 1\}$$

and $A_2 = A \setminus A_1$. Now let

$$A_2(\varepsilon) = \{z \in A_2 \mid d(z, A') \geq \varepsilon\}.$$

Then A_1 is a closed set and $A_2(\varepsilon)$ is finite for any $\varepsilon > 0$.

Proof. Assume A_1 has an accumulation point a and let $(a_k)_k \subseteq A$ be a sequence, converging to a. But then

$$\lim_{k \to \infty} |a_k| \cdot d(a_k, A') = 0,$$

which is a contradiction.

Note that, for all $z \in A_2(\varepsilon)$, we have $|z| < \frac{1}{\varepsilon}$. If the set is infinite, it has an accumulation point, which is impossible as $d(z, A') \geq \varepsilon$. □

Remark 3.4.7.1. If $A \subset \mathbb{C}$ is a discrete set, then A' is a closed set in $\mathbb{C}$.

Theorem 3.4.8 (Weierstraß product theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k)_k \subset \Omega$ be a sequence without accumulation points in Ω and denote $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$. Then there exists a Weierstraß product for $(a_k)_k$ that converges normally in $\mathbb{C} \setminus A'$. This product has zeros precisely in $(a_k)_k$, counted with multiplicities.

Proof. Assume that $\Omega \neq \mathbb{C}$ and $A' \neq \emptyset$.[4] Write $A = A_1 \cup A_2$ as in the above lemma. Recall that A_1 has no accumulation points, therefore we can apply theorem 3.4.5 for A_1. It remains to construct a Weierstraß product for A_2.

Observe that $A' = A_2'$. As this is a closed space, for all $a_k \in A_2$ there exists some $b_k \in A_2'$ such that

$$|a_k - b_k| = d(a_k, A_2').$$

Observe that

$$\lim_{\substack{k \to \infty \\ a_k \in A_2}} |a_k - b_k| = 0,$$

as the sets $A_2(\varepsilon)$ are finite. Now set $p_k = k$ and apply lemma 3.4.6. □

[4] Otherwise just apply theorem 3.4.5.

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Corollary 3.4.8.1 (Blaschke products)**.** Let $(a_k)_k \subset \mathbb{\Delta} \setminus \{0\}$ be a sequence without accumulation points in $\mathbb{\Delta}$. If the series

$$\sum_{k=1}^{\infty} (1 - |a_k|)$$

converges, then the product

$$\prod_{k=1}^{\infty} E_0 \left(\frac{a_k - \frac{1}{\overline{a}_k}}{z - \frac{1}{\overline{a}_k}} \right)$$

converges normally in $\mathbb{\Delta}$ and has zeros precisely in $(a_k)_k$, counted with multiplicities.

Proof. Note that

$$|a_k - b_k| = \left| a_k - \frac{1}{\overline{a}_k} \right| = \left| \frac{1}{\overline{a}_k} \right| \cdot \left| |a_k|^2 - 1 \right| = \left| \frac{1}{\overline{a}_k} \right| \cdot (1 - |a_k|)(1 + |a_k|) \leq \frac{2}{m} \cdot (1 - |a_k|),$$

where

$$m = \min \{ |a_k| \mid k \in \mathbb{N} \}.$$

It follows that the series

$$\sum_{k=1}^{\infty} r \cdot |a_k - b_k|$$

converges, therefore we can apply lemma 3.4.6. ◻

Theorem 3.4.9. Let $\Omega \subseteq \Omega$ be a domain and $f \in \mathcal{O}(\Omega) \setminus \{0\}$. Then we can write

$$f = g \cdot \prod_{k=1}^{\infty} f_k,$$

where $g \in \mathcal{O}^*(\Omega)$ and f_k are Weierstraß factors.

Proof. The proof is obvious and need not be mentioned. ◻

Theorem 3.4.10. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{M}(\Omega)$. Then we can write $f = \frac{g}{h}$, where $g, h \in \mathcal{O}(\Omega)$.

Proof. Define h as the Weierstraß product of the poles of f. ◻

Remark 3.4.10.1. Let $\Omega \subseteq \mathbb{C}$ be a domain. Then $\mathcal{O}(\Omega)$ is not a factorial ring,[5] but $\gcd(f, g) \in \mathcal{O}(\Omega)$ exists.

Definition 3.4.11. Let Ω be an open subset and $\{a_k\}_k$ be a sequence without accumulation pints and without repetition. Let

$$q_k(z) = \sum_{n=1}^{\infty} c_{k,m} (z - a)^{-m}$$

be a principal part in a_k for each k.

[5] "Kolobar z enolično faktorizacijo."

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If there exist functions $g_k \in \mathcal{O}(\Omega)$ for each k such that

$$\sum_{k=1}^{\infty} q_k - g_k$$

converges normally in Ω, we call it the *Mittag-Leffler series* for the distribution of principal part (a_k, q_k).

Remark 3.4.11.1. We adopt the following conventions: If $0 \in \{a_k\}_k$, then $a_1 = 0$.

Theorem 3.4.12 (Mittag-Leffler for $\mathbb{C}$)**.** For every distribution of principal parts in $\mathbb{C}$ there exists a corresponding Mittag-Leffler series.

Proof. Let g_k be the Taylor series of q_k about 0 in the disk $\mathbb{\Delta}(|a_k|)$ such that the inequality $\|q_k - g_k\| < 2^{-k}$ holds for each $k \geq 2$. Note that

$$\lim_{k\to\infty} |a_k| = \infty$$

as the points don't accumulate. For each $r > 0$ we can therefore find an integer n such that $r < \frac{1}{2}|a_k|$ for all $k \geq n$. Then

$$\sum_{k=n}^{\infty} \|q_k - g_k\|_{\overline{\mathbb{\Delta}(r)}} \leq 1. \qquad \square$$

Remark 3.4.12.1. The above series $f \in \mathcal{O}(\mathbb{C} \setminus \{a_1, a_2, \dots\})$ with principal parts q_k in a_k for each $k \in \mathbb{N}$. If the principal part are finite, then $f \in \mathcal{M}$.

Lemma 3.4.13. Let $a \in \mathbb{C}$, $q \in \mathcal{O}(\Omega \setminus \{a\})$ be a principal part and $b \in \mathbb{C} \setminus \{a\}$. Then q has a Laurent series expansion about b in the annulus $\{z \in \mathbb{C} \mid |z-b| > |a-b|\}$ of the form

$$q(z) = \sum_{m=1}^{\infty} c_m (z-b)^{-m}$$

that converges uniformly for $|z-b| > r > |a-b|$.

Proof. Choose a path γ_r that goes around the circle centered at b of radius r. We claim that

$$c_m = \frac{1}{2\pi} \int_{\gamma_r} \frac{q(z)}{(z-b)^{-m+1}}\, dz$$

for $m \in \mathbb{Z}$ suffice. We can estimate

$$|c_m| \geq \frac{1}{2\pi} \cdot 2\pi \frac{\|q\|_{\gamma_r}}{r^{-m}} = \frac{\|q\|_{\gamma_r}}{r^{-m}}.$$

We know that $q(z)$ is of the form

$$q(z) = \sum_{m=1}^{\infty} d_m (z-a)^{-m}$$

for some $d_m \in \mathbb{C}$ when developed into a Laurent series around a. It is trivial to show that

$$\lim_{|z|\to\infty} q(z) = 0.$$

Thus, $\|q\|_{\gamma_r}$ approaches zero as r goes to infinity. If $m \leq 0$, then

$$\lim_{r\to\infty} \|q\|_{\gamma_r} r^m = 0.$$

Therefore, $c_m = 0$ for $m \leq 0$ and

$$q(z) = \sum_{m=1}^{\infty} c_m (z-b)^{-m}$$

is indeed a power series in $z-b$ which converges uniformly for $|z-b|^{-1} \leq r$. $\square$

Definition 3.4.14. The partial sums of

$$q_\ell(z) = \sum_{m=1}^{\ell} c_m (z-b)^{-m}$$

are called the *ℓ-th Laurent terms* of q about b.

Lemma 3.4.15. Let $(a_k, q_k)_k$ be a distribution of principal parts in an open set $\Omega \subseteq \mathbb{C}$, $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$.[6] Assume there exists a sequence $(b_k)_k \subseteq A'$ with

$$\lim_{k\to\infty} |a_k - b_k| = 0.$$

Let $q_{k,\ell}$ be the ℓ-th Laurent term of q_k about b_k. Then there exists a sequence $(\ell_k)_k \subseteq \mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty} (q_k - q_{k,\ell_k})$$

is a Mittag-Leffler series for $(a_k, q_k)_k$.

Proof. For a principal part q_k the Laurent series converges uniformly on $|z - b_k| > r$ for any $r > |a_k - b_k|$ by the previous lemma. Thus, we can choose ℓ_k large enough such that

$$|q_k(z) - q_{k,\ell}(z)| < 2^{-k}$$

for all z such that $|z - b_k| \geq 2\,|a_k - b_k|$.

For any compact set $L \subseteq \mathbb{C} \setminus A'$, the distance to A' is strictly positive. Since

$$\lim_{k\to\infty} |a_k - b_k| = 0,$$

the point b_k must lie outside L for large enough k. Thus, there exists some $n(L) \in \mathbb{N}$ such that

$$L \subseteq \bigcap_{k \geq n(L)} \{z \in \mathbb{C} \mid |z - b_k| \geq 2\,|a_k - b_k|\}.$$

Therefore, we can use the previous estimate on L, to get

$$\sum_{k \geq n(L)} \|q_k - q_{k,\ell_k}\|_L \leq \sum_{k \geq n(L)} 2^{-k} \leq 2.$$ $\square$

[6] The closure is taken in $\mathbb{C}$.

Theorem 3.4.16 (Mittag-Leffler for open subsets)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k, q_k)_k$ be a distribution of principal parts in $\Omega \subseteq$ and $A = \{a_k \mid k \in \mathbb{N}\}$. Then there exists a Mittag-Leffler series for (a_k, q_k) that converges normally in $\mathbb{C} \subsetneq A' = \overline{A} \setminus A$.

Proof. By lemma 3.4.7, $(A_1)'$ is empty and $(A_2)' = A'$. If A' is empty, then

$$\lim_{k\to\infty} |a_k| = \infty$$

and we can apply theorem 3.4.12. Similarly, we can assume $\Omega \neq \mathbb{C}$. Again by the lemma, $A_2(\epsilon)$ is finite. Hence, there exist $(b_k)_k \in A'$ such that

$$\lim_{k\to\infty} |a_k - b_k| = 0.$$

We can apply lemma 3.4.15 to obtain a Mittag-Leffler series. Now we apply theorem 3.4.12 for $\mathbb{C}$ to A_1. Sum up this two series to get the series from the statement. □

Theorem 3.4.17 (Mittag-Leffler osculation theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $(a_k)_k \subseteq \Omega$ be a sequence without accumulation points and without repetition. Furthermore, let

$$f_k(z) = \sum_{\ell=-\infty}^{n(k)} c_{k,\ell}(z - a_k)^\ell,$$

where $n(k) \in \mathbb{N}_0$, be normally convergent on $\mathbb{C} \setminus A$, where A is the set of a_k. Then there exists a function $f \in \mathcal{O}(\Omega \setminus A)$ such that $\operatorname{ord}_{a_k}(f - f_k) > n(k)$ for all $k \in \mathbb{N}$.

Proof. By the Weierstraß product theorem, there exists a function $h \in \mathcal{O}(\Omega)$ such that $\operatorname{ord}_{a_k}(h) > n(k)$ and h has no zeroes on $\mathbb{C} \setminus A$. Then $\left(a_k, \frac{f_k}{h}\right)_k$ is a distribution of principal parts. By theorem 3.4.16, there exists a $g \in \mathcal{O}(\Omega \setminus A)$ with these principal parts.

Now define $f = g \cdot h$. Then

$$f - f_k = g \cdot h - f_k = h \cdot \left(g - \frac{f_k}{h}\right),$$

which vanishes to order larger than $n(k)$ in a_k. □

Corollary 3.4.17.1. For every sequence $(a_k)_k \subseteq \Omega$ without accumulation points and without repetition and every sequence $(c_k)_k \subseteq \mathbb{C}$ there exists a function $f \in \mathcal{O}(\Omega)$ such that $f(a_k) = c_k$ for each $k \in \mathbb{N}$.

3.5 Ring structure of holomorphic functions

Definition 3.5.1. Let $\Omega \subseteq \mathbb{C}$ be an open set. A *divisor* of a meromorphic function $f \in \mathcal{M}^*(\Omega)$ is the function $(f) \colon \Omega \to \mathbb{Z}$, given by

$$(f)(z) = \begin{cases} n & f \text{ has a zero of order } n \text{ in } z \\ -n & f \text{ has a pole of order } n \text{ in } z \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.5.1.1. The divisor of a product is the sum of divisors, i.e. $(f \cdot g) = (f) + (g)$.

Definition 3.5.2. Let $S \subseteq \mathcal{O}(\Omega)$ be a subset that contains a non-zero holomorphic function.

Define

$$d(z) = \min_{f \in S \setminus \{0\}} (f)(z) \in \mathbb{N}_0.$$

By Weierstraß product theorem there exists a function $g \in \mathcal{O}(\Omega)$ such that $(g) = d$. We define $\gcd(S) = g$.[7]

Lemma 3.5.3 (Wedderburn)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f, g \in \mathcal{O}(\Omega)$ be functions with $\gcd(f, g) = 1$. Then there exist functions $a, b \in \mathcal{O}(\Omega)$ so that $af + bg = 1$. Moreover, we can choose a to be nonwhere vanishing.

Proof. If $g = 0$ and $f \neq 0$ then f cannot vanish by assumption on gcd, therefore $a = \frac{1}{f}$ and $b = 1$ suffice. Therefore, we can assume both f, g and are nonzero. Note that $Z(f) \cap Z(g)$ is empty, since $(z - p)$ divides $\gcd(f, g)$ for any $p \in Z(f) \cap Z(g)$. The set $Z(f) \cup Z(g)$ is thus discrete. Further, for each zero p of g, there exists a disk of radius ε and a holomorphic function $f_p \in \mathcal{O}(\Delta(p, \varepsilon))$ such that

$$f = e^{f_p}.$$

By the Mittag-Leffler osculation theorem there exists a function $h \in \mathcal{O}(\Omega)$ such that $\operatorname{ord}_p(h - f_p) > \operatorname{ord}_p(g)$.

Here we stop for a short observation. Developing into the power series, we get that $e^{w^n} - 1 = w^n + O(w^{2n})$. Then,

$$\operatorname{ord}_p \left(f - e^h\right) = \operatorname{ord}_p \left(e^h \cdot \left(e^{f_p - h} - 1\right)\right) = \operatorname{ord}_p \left(\left(e^{f_p - h} - 1\right)\right) = \operatorname{ord}_p \left(f_p - h\right) > \operatorname{ord}_p(g).$$

Define $k = \frac{f - e^h}{g} \in \mathcal{O}(\Omega)$. We claim that $a = e^{-h}$ and $b = -ke^{-h}$ satisfy the conditions. Clearly, a doesn't vanish, and

$$af + bg = e^{-h} f - ke^{-h} g = e^{-h}(f - kg) = e^{-h} \left(f - \frac{f - e^h}{g} g\right) = e^{-h} e^h = 1. \qquad \square$$

Corollary 3.5.3.1. For holomorphic functions $f_j \in \mathcal{O}(\Omega)$, where $j \leq n$, we can write $f = \gcd(f_1, f_2, \dots, f_n)$ as

$$f = \sum_{j=1}^{n} a_j f_j$$

[7] There are of course multiple possible functions that satisfy this condition, but their quotients are invertible.

Proof. We proceed by induction. The base case is just Wedderburn's lemma. Now let $\hat{f} = \gcd(f_2, f_3, \dots f_n)$, which can be written as

$$\hat{f} = \sum_{j=2}^{n} \hat{a}_j f_j$$

by the induction hypothesis. Then $\frac{f_1}{f}, \frac{\hat{f}}{f} \in \mathcal{O}(\Omega)$ are holomorphic functions with gcd equal to 1. We can therefore apply Wedderburn' lemma to get functions a and b such that

$$a\frac{f_1}{f} + b\frac{\hat{f}}{f} = 1.$$

The conclusion follows $\square$

Theorem 3.5.4. Let $I \triangleleft \mathcal{O}(\Omega)$ be the ideal generated by holomorphic functions $f_1, f_2, \dots f_n$ on Ω. Then there exists a holomorphic function f such that $I = (f)$.

Proof. Take $f = \gcd(f_j)$. This function is an element of I by the previous corollary. Since $f \mid f_j$, this implies that $I = (f)$. $\square$

Definition 3.5.5. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal.

i) We call I *closed* if for every sequence $(f_n)_n \subseteq I$ that converges uniformly on compacts of Ω to some function f, we also have $f \in I$.

ii) We call $p \in \Omega$ a *zero* of I if $f(p) = 0$ for every $f \in I$.

Lemma 3.5.6. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ and ideal. Let $p \in \Omega$ be a point that is not a zero of I. Let $f, g \in \mathcal{O}(\Omega)$ be functions such that $f(z) \neq 0$ for all $z \neq p$. If $fg \in I$, then $g \in I$.

Proof. Since p is not a zero of I, then there exists a function $h \in I$ such that $h(p) \neq 0$. Let $n = \operatorname{ord}_p(f)$. If $n = 0$, then f is a unit, so $g \in I$. Otherwise, we have

$$\frac{f(z)}{z-p} g = -\frac{1}{h(p)} \cdot \left(\frac{h - h(p)}{z-p} fg - \frac{fg}{z-p} h \right) \in I$$

since $\frac{f}{z-p}$ is holomorphic.

We can iterate this process to find $\frac{f}{(z-p)^n} g \in I$. Since $\frac{f}{(z-p)^n}$ is a unit, g must be an element of I. $\square$

Theorem 3.5.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal. If I has no zeroes and is closed, then $I = \mathcal{O}(\Omega)$.

Proof. Let f be an arbitrary nonzero element of I. By the Weierstraß product theorem, we can write

$$f = \prod_{k=1}^{\infty} f_k,$$

where each f_k has exactly one zero in Ω, and the tails

$$\widehat{f_n} = \prod_{k=n}^{\infty} f_k$$

converge to 1 uniformly on compacts of Ω. As $f = \widehat{f}_1 = f_1 \widehat{f}_2$, we can apply the preivous lemma to find $\widehat{f}_2 \in I$. Inductively, $\widehat{f}_n \in I$ and since the ideal I is assumed to be closed, we have

$$1 = \lim_{k \to \infty} \widehat{f}_k \in I. \qquad \square$$

4 Approximation of holomorphic functions

4.1 Runge's little theorem

Lemma 4.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $K \subseteq \Omega$ a non-empty compact. Then there exist finitely many horizontal or vertical line segments $\sigma_1, \dots, \sigma_n$ of equal length in $\Omega \setminus K$ such that for all $f \in \mathcal{O}(\Omega)$ we have

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\sigma_k} \frac{f(\xi)}{\xi - z} \, d\xi$$

for all $z \in K$.

Proof. Define

$$\delta = \begin{cases} 1, & \Omega = \mathbb{C}, \\ d(K, \partial\Omega), & \Omega \neq \mathbb{C}. \end{cases}$$

Let Q be a grid of squares, parallel to the coordinate axes, with side length $d < \frac{\delta}{\sqrt{2}}$. As K is compact, it only intersects finitely many of them. Now just choose the boundary of the union of those squares. It is clear that all those segments are subsets of $\Omega \setminus K$.

Let Q_k denote the above squares. Then,

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \oint_{\partial Q_k} \frac{f(\xi)}{\xi - z} \, d\xi = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\sigma_k} \frac{f(\xi)}{\xi - z} \, d\xi.$$

If $z \in \operatorname{Int} Q_\ell$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q_\ell} \frac{f(\xi)}{\xi - z} \, d\xi$$

and

$$\frac{1}{2\pi i} \int_{\partial Q_k} \frac{f(\xi)}{\xi - z} \, d\xi = 0$$

for all $k \neq \ell$, therefore the lemma holds for all such z. As both sides of the equations are continuous functions, they must agree on the whole set K. $\square$

Lemma 4.1.2. Let σ be a compact line segment and $K \subseteq \mathbb{C}$ be a compact set such that $K \cap \sigma = \emptyset$. Let $h \in \mathcal{C}(\sigma)$ be a function. Then for all $\varepsilon > 0$ there exist points $c_1, \dots, c_m \in \mathbb{C}$ and $w_1, \dots, w_m \in \sigma$ such that

$$\left\| \int_{\sigma} \frac{h(\xi)}{\xi - z} \, d\xi - \sum_{k=1}^{m} \frac{c_k}{z - w_k} \right\|_K < \varepsilon.$$

Proof. Let ℓ be the length of σ and define the function $v \colon \sigma \times K \to \mathbb{C}$ with

$$v(\xi, z) = \frac{h(\xi)}{\xi - z}.$$

It is clearly continuous, therefore it is uniformly continuous. In particular, there exists a $\delta > 0$ such that

$$|v(\xi, z) - v(\xi', z)| < \frac{\varepsilon}{\ell}$$

for all $z \in \mathbb{C}$ and $|\xi - \xi'| < \delta$.

Let $\tau_1, \dots, \tau_m$ be the partition of σ into line segments of length $d < \sigma$. Choose points $w_k \in \tau_k$ and set $c_k = -h(w_k)d$. We therefore have

$$\left| \int_{\tau_k} v(\xi, z)\, d\xi - \frac{c_k}{z - w_k} \right| = \left| \int_{\tau_k} v(\xi, z)\, d\xi - d \cdot v(w_k, z) \right| < d \cdot \frac{\varepsilon}{\ell}.$$

Summing up, we get the desired inequality. □

Lemma 4.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open set and $K \subseteq \Omega$ be a non-empty compact. Then there exist finitely many line segments $\sigma_1, \dots, \sigma_n$ in $\Omega \setminus K$ such that for any holomorphic function $f \in \mathcal{O}(\Omega)$ and $\varepsilon > 0$ there exists a rational function q of the form

$$q(z) = \sum_{k=1}^{m} \frac{c_k}{z - w_k},$$

where $c_k \in \mathbb{C}$ and $w_k \in \sigma$, such that

$$\|f - q\|_K < \varepsilon.$$

Proof. By lemma 4.1.1 there exist line segments σ_k such that

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\sigma_k} \frac{f(\xi)}{\xi - z}\, d\xi$$

for all $z \in K$. Now just apply lemma 4.1.2 to each line segment separately. □

Lemma 4.1.4 (Shifting poles)**.** Let $K \subseteq \mathbb{C}$ be a compact set, Z a connected component of $\mathbb{C} \setminus K$ and $a, b \in Z$. Then, for all $\varepsilon > 0$ there exists a polynomial q such that

$$\left\| \frac{1}{z - a} - q\left(\frac{1}{z - b}\right) \right\|_K < \varepsilon.$$

If Z is the unbounded component, we can approximate $\frac{1}{z-a}$ by $q(z)$ instead.

Proof. Let L_ω be the family of all functions that are holomorphic in a neighbourhood of K that can be approximated uniformly on K by polynomials in $\frac{1}{z-\omega}$. Note that, if $\frac{1}{z-p} \in L_q$, then $L_p \subseteq L_q$.

Consider the set

$$S = \left\{ s \in Z \;\middle|\; \frac{1}{z - s} \in L_b \right\}.$$

We claim that $S = Z$. First note that $b \in S$, therefore $S \neq \emptyset$. Take any point $p \in S$ and $\delta > 0$ such that $\mathbb{A}(p, \delta) \subseteq Z$. For any $s \in \mathbb{A}(p, \delta)$, we can write

$$\frac{1}{z - s} = \frac{1}{z - p} \cdot \frac{1}{1 - \frac{s-p}{z-p}} = \frac{1}{z - p} \cdot \sum_{k=0}^{\infty} \left(\frac{s - p}{z - p} \right)^k,$$

which converges as $|s - p| < |z - p|$ by choice of δ. We conclude that $\mathbb{A}(p, \delta) \subseteq S$. In particular, S is an open set. Take any point $p \in \partial S \cap Z$. Suppose that $\mathbb{A}(p, 3\delta) \subseteq Z$ and

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choose a point $p' \in \mathbb{\Delta}(p, \delta) \cap S$. Then, $\mathbb{\Delta}(p', 2\delta) \subseteq S$ and therefore $\mathbb{\Delta}(p, \delta) \subseteq S$, which is a contradiction and therefore proves our claim.

Now suppose that Z is the unbounded component and take a point $d \in Z$ such that $K \subseteq \mathbb{\Delta}(0, |d|)$. Then all functions $\left(\frac{1}{z-d}\right)^n$ can be approximated uniformly on K by Taylor's polynomials about 0. $\square$

Definition 4.1.5. For any set $P \subseteq \mathbb{C}$ denote by $\mathbb{C}_P[z]$ the family of rational functions with poles in P.

Theorem 4.1.6 (Runge approximation)**.** Let $K \subseteq \mathbb{C}$ be a compact. If P intersects every bounded connected component of $\mathbb{C} \setminus K$, then for every function f, holomorphic in a neighbourhood of K, and every $\varepsilon > 0$ there exists a function $q \in \mathbb{C}_P[z]$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Be lemma 4.1.3, we can find a compact union σ of line segments such that every such f can be approximated by

$$\widetilde{q} = \sum_{k=1}^{m} \frac{c_k}{z - w_k},$$

where $c_k \in \mathbb{C}$ and $w_k \in \sigma$. Suppose then

$$\|f - \widetilde{q}\|_K < \frac{\varepsilon}{2}.$$

Let Z_k be the connected component of $\mathbb{C} \setminus K$ that contains w_k.

If Z_k is bounded, then choose $t_k \in P \cap Z_k$. By the pole shifting lemma, we can find a polynomial g_k such that

$$\left\| \frac{c_k}{z - w_k} - g_k\left(1\right) z - t_k \right\|_K < \frac{\varepsilon}{2m}.$$

If Z_k is unbounded, we can instead approximate $\frac{c_k}{z-w_k}$ by a polynomial instead. Choosing

$$q = \sum_{k=1}^{m} g_k,$$

we find

$$\|f - q\|_K \leq \|f - \widetilde{q}\|_K + \|\widetilde{q} - q\|_K < \frac{\varepsilon}{2} + m \cdot \frac{\varepsilon}{2m} = \varepsilon.$$ $\square$

Theorem 4.1.7 (Runge approximation)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $K \subseteq \Omega$ be a compact. If every bounded component of $\mathbb{C} \setminus K$ intersects $\mathbb{C} \setminus \Omega$, then for every function f that is holomorphic on a neighbourhood of K and every $\varepsilon > 0$ there exists a function $q \in \mathcal{O}(\Omega)$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Choose $P = \mathbb{C} \setminus \Omega$ in theorem 4.1.6. $\square$

Corollary 4.1.7.1 (Runge's little theorem)**.** Let $K \subseteq \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. Then for every holomorphic function on a neighbourhood of K and $\varepsilon > 0$ there exists a polynomial $q \in \mathbb{C}[z]$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Choose $P = \emptyset$ in theorem 4.1.6. $\square$

Definition 4.1.8. Let V be a vector space over $\mathbb{C}$, equipped with a topology. Let $T \colon V \to V$ be a linear map.

i) We call T *cyclic* if there exists some $f \in V$, called a *cyclic vector*, such that

$$\operatorname{span}_{\mathbb{C}} \{T^n(f) \mid n \in \mathbb{N}_0\} = V.$$

ii) We call T *hypercyclic* if there exists some $f \in V$, such that

$$\overline{\{T^n(f) \mid n \in \mathbb{N}_0\}} = V.$$

Theorem 4.1.9 (Birkhoff)**.** Let $\tau \colon \mathbb{C} \to \mathbb{C}$ be given by $\tau(z) = z + a$ for some $a \neq 0$. Then the map $T \colon \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$, given by $T(f) = f \circ \tau$, is hypercyclic.

Proof. Set $K_n = \overline{\mathbb{\Delta}(\ell_n \cdot a, n)}$ for a sequence $(\ell_n)_n \subseteq \mathbb{N}$, such that all K_n are pairwise disjoint and $K_n \cap \mathbb{\Delta}(0, n) = \emptyset$ for all $n \in \mathbb{N}$. Choose a sequence $(\varepsilon_n)_n \subseteq \mathbb{R}^+$, converging to 0. Furthermore, let $(p_n)_n$ be a sequence of all polynomials $(\mathbb{Q} \oplus i\mathbb{Q})[z]$.

We first construct a sequence of holomorphic functions $(f_n)_n$, such that the following conditions hold:

i) For all $n \in \mathbb{N}$, we have

$$\left\| \sum_{k=1}^{n} f_k - \tau^{-\ell_n} \circ p_n \right\|_{K_n} < \frac{\varepsilon_n}{2^n}.$$

ii) For all $m < n$, we have

$$\|f_n\|_{K_m} < \frac{\varepsilon_m}{2^n}.$$

iii) For all $n \in \mathbb{N}$, we have

$$\|f_n\|_{\mathbb{\Delta}(0,n)} < \frac{1}{2^n}.$$

Choose ℓ_n such that $\mathbb{\Delta}(0, n) \cap K_m = \emptyset$ for all $m < n$. Then the set

$$\mathbb{C} \setminus \left(\bigcup_{m=1}^{n} K_m \cup \mathbb{\Delta}(0, n) \right)$$

is obviously connected. We can now just apply Runge's little theorem by choosing the function 0 on $\mathbb{\Delta}(0, n)$ and K_m for $m < n$, and

$$\tau^{-\ell_n} \circ p_n - \sum_{k=1}^{n-1}$$

on K_n.

Note that the series

$$f = \sum_{n=1}^{\infty} f_n$$

is uniformly convergent by the third condition. We now compute

$$\begin{aligned}\left\|f \circ \tau^{\ell_n} - p_n\right\|_{\tau^{-\ell_n}(K_n)} &= \left\|f - \tau^{-\ell_n} \circ p_n\right\|_{K_n} \\ &\le \sum_{k>n} \|f_k\|_{K_n} + \left\|\sum_{k=1}^{n} f_k - \tau^{-\ell_n} \circ p_n\right\|_{K_n} \\ &< \varepsilon_n.\end{aligned}$$

It follows that we can approximate every rational polynomial with iterations $T^n(f)$. □

this is the remainder of the proof that we did not finish last time, TODO merge

Proof. Suppose $\Omega \setminus K$ has no relatively compact components. Let $p \in \Omega \setminus K$. Consider $\Omega \setminus (K \cup \{p\})$. This has the same connnected components not containing p which are still not relatively compact. Removing the point p does not disconnect and open connected component, so the connected component that contained p is still relatively compact. We can use the fourth item from the statement on the function $g|_K = 0$ and $g(p) = 1$, which is holomorphic in a neighbourhood of $K \cup p$. Thus there exist h, a holomorphic function on Ω such that $\|g - h\|_{K \cup p} < \frac{1}{2}$. Then $\|h\|_K < \frac{1}{2} < |h(p)|$ which is exactly the fifth item. □

Lemma 4.1.10. Let $\Omega \subseteq \mathbb{C}$ is an open subset and $K \subseteq \Omega$ a compact. Then there exists a compact K' such that $K \subseteq K' \subseteq \Omega$ and every bounded component of $\mathbb{C} \setminus K'$ intersects $\mathbb{C} \setminus \Omega$ and contains a compact component of $\mathbb{C} \setminus \Omega$.

Proof. For the case $\Omega = \mathbb{C}$, we can take $K' = \overline{D}_r(0)$, where r is large enough so that $K \subseteq K'$. So suppose $\Omega \neq \mathbb{C}$. Choose $0 < \rho < d(K, \partial\Omega) = d(K, \mathbb{C} \setminus \Omega)$. Let $M = \{z \in \Omega \mid d(z\mathbb{C} \setminus \Omega) \ge \rho\}$. Choose $r > 0$ such that $K \subseteq \overline{D}_r(0)$. Let $K' = M \cap \overline{D}_r(0)$, which is clearly compact, contained inside Ω, and contains K. We are left to prove the statement about the bounded components.

Let Z be a bounded component of $\mathbb{C} \setminus K'$. Note that $\mathbb{C} \setminus \overline{D}_r(0)$ is connected and unbounded and contains Z. If Z were also not in $C \setminus M$, then it would be in the unbounded component of $\mathbb{C} \setminus K'$. Thus $Z \subseteq \mathbb{C} \setminus M$.

We can rewrite $\mathbb{C} \setminus M = \bigcup_{w \in \mathbb{C} \setminus \Omega} D_\rho(w)$ by the definition of M. We have that $D_\rho(w) \subseteq Z$ or $D_\rho(w) \cap Z = \emptyset$. Thus $Z = \bigcup_{w \in Z \setminus \Omega} D_\rho(w)$ so Z intersects $\mathbb{C} \setminus \Omega$. Let S be the component of $\mathbb{C} \setminus \Omega$ that Z intersects $\mathbb{C} \setminus \Omega \subseteq \mathbb{C} \setminus K'$. Then $S \subseteq Z$ since Z is a max connected set in $\mathbb{C} \setminus K'$. Since Z is bounded, S is bounded. □

Definition 4.1.11. Let $\Omega \subseteq \mathbb{C}$ be an open subset. We call every bounded component of $\mathbb{C} \setminus \Omega$ a *hole* of Ω.

Remark 4.1.11.1. If K is the Cantor set, then $\Omega = \mathbb{C} \setminus K$ has uncountably many holes.

Theorem 4.1.12 (Runge's Theorem for rational apporximations)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset $P \subseteq \mathbb{C} \setminus \Omega$ such that $\overline{P}$ intersects every hole of Ω. Then $\mathbb{C}_P[z]$ is dense in $\mathcal{O}(\Omega)$.

Proof. Let $K \subseteq \Omega$ be compact. Let K' be from the previous lemma. Then $\overline{P}$ intersects every bounded componont of $\mathbb{C} \setminus K'$. We can thus apply the lemma **??** on K'. □

Corollary 4.1.12.1 (Runge's theorem for polynomial approximation)**.** If Ω has no holes then $\mathbb{C}[z]$ is dense in $\mathcal{O}(\Omega)$.

Proof. Take $P = \emptyset$ □

Remark 4.1.12.2. This gives us another proof of the Mittag-Leffler theorem 3.4.16.

Proof. Let $A = \bigcup_{k\in\mathbb{N}} \{a_k\} \subseteq \partial\Omega$. Let $K_j = \overline{\left\{z \in \Omega \setminus A \;\middle|\; |z| \leq j, d(z, \partial\Omega \setminus A) \geq \frac{1}{j}\right\}}$, which form an exhaustion by compacts. Let K_j' be the compact furnished by the previous lemma for K_j. Choose a subsequence $(L_j)_{j\in\mathbb{N}}$ of $(K_j')_{j\in\mathbb{N}}$ that is an exhaustion by compacts, which can be done by the properties of exhaustion, i. e. for every K_j' there exists $k \in \mathbb{N}$ so that $K_j' \subseteq K_k \subseteq K_k'$ and apply induction.

We may assume $K_1 \cap \overline{A} = \emptyset$. Let $A_n = A \cap (L_{n+1} \setminus L_n)$. For each $a \in A_n$ the principal part of q is holomorphic in L_n. By the approximation with holomorphic functions on Ω, there exists $g \in \mathcal{O}(\Omega)$ such that $\|q - g\|_{L_n} < \max(1, |A_n|)2^{-n}$. Then

$$\sum_{n=1}^{\infty} \|q_n - g_n\|_{L_n} \leq \sum_{n=1}^{\infty} 2^{-n} - 1.$$

Thus the series converges uniformly on compacts. □

Definition 4.1.13. (Ω, Ω') is a *Runge pair* if for every $f \in \mathcal{O}(\Omega)$ and each $\epsilon > 0$ and compact $K \subseteq K$ there exists $F\mathcal{O}(\Omega')$ such that $\|f - F\|_K < \epsilon$.

Theorem 4.1.14. Let $\Omega \subseteq \Omega' \subseteq \mathbb{C}$ be open sets. The following statements are equivalent:

1. $\Omega' \setminus \Omega$ has no compact components,
2. $\mathbb{C}_P[z]$ with $P \subseteq \mathbb{C} \setminus \Omega'$ is dense in $\mathcal{O}(\Omega)$,
3. (Ω, Ω') is a Runge pair,
4. $\Omega' \setminus \Omega$ contains no open compact.

Proof. □

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