Algebraic topology 1

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Introduction

These are my lecture notes on the course Algebraic topology 1 in the year 2023/24. The lecturer that year was prof. dr. Petar Pavešić.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Basic homotopy theory

1.1 Definition

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Definition 1.1.1. Continuous maps $f, g\colon X \to Y$ of topological spaces are *homotopic*, if there is a continuous map $H\colon X \times I \to Y$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such H is called a *homotopy*. We write $H\colon f \simeq g$.

Remark 1.1.1.1. If X is a locally compact and Hausdorff space, homotopies coincide with paths in the space $\mathcal{C}(X, Y)$.

Proposition 1.1.2. Homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.

Proof. The proof is obvious and need not be mentioned. □

Definition 1.1.3. We denote the set of equivalence classes of the homotopy relation on $\mathcal{C}(X, Y)$ by $[X, Y]$.

Remark 1.1.3.1. If X is a locally compact and Hausdorff space, $[X, Y]$ is the set of path components of $\mathcal{C}(X, Y)$.

Definition 1.1.4. With $f\colon (X, A) \to (Y, B)$ we denote maps $f\colon X \to Y$ such that $f(A) \subseteq f(B)$. Similarly, we define $\mathcal{C}((X, A), (Y, B))$ and $[(X, A), (Y, B)]$.

Definition 1.1.5. Let $A \subseteq X$ and $f, g\colon X \to Y$ be maps satisfying $f|_A = g|_A$. The map $G\colon X \times I \to Y$ is a *homotopy relative to* A if $H\colon f \simeq g$ and $H_t|_A = f|_A$ for all $t \in I$. We write $H\colon f \simeq g$ (rel A).

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Definition 1.1.6. A map $f\colon X \to Y$ is *null-homotopic* if it is homotopic to a constant.

Definition 1.1.7. Let X be a topological space. The *cone* on X is the space

$$CX = {X \times I}\big/{X \times \{1\}} .$$

Proposition 1.1.8. A map $f\colon X \to Y$ is null-homotopic if and only if it extends to the cone CX.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\
CX \arrow[ur, dashed, "F"'] &
\end{tikzcd}$$

Proof. In the following diagram, H exists if and only if F exists.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_0"'] & Y \\
X \times I \arrow[ur, dashed, "H"] \arrow[r, "q"'] & CX \arrow[u, dashed, "F"']
\end{tikzcd}$$

□

1.2 Homotopy equivalence

Proposition 1.2.1. Let $f, f' \colon X \to Y$ and $g, g' \colon Y \to Z$ be maps. If $f \simeq g$ and $f' \simeq g'$, we also have $g \circ f \simeq g' \circ f'$.

Proof. Let $H \colon X \times I \to Y$ and $K \colon Y \times I \to Z$ be the two homotopies. It is trivial to check that

$$L(x,t) = K(H(x,t),t)$$

is a homotopy of the compositions. □

Definition 1.2.2. The *homotopy category* $\underline{\text{HoTop}}$ is the category with topological spaces as objects and homotopies as morphisms. Operations are induced by the compositions of maps.

Definition 1.2.3. The category $\underline{\text{Top}}^2$ has pairs of spaces (X, A) with $A \subseteq X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}^2$.

Definition 1.2.4. The category $\underline{\text{Top}}_\bullet$ has pairs (X, x_0) with $x_0 \in X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}_\bullet$.

Definition 1.2.5. *Homotopy equivalence* is an isomorphism in the category $\underline{\text{HoTop}}$. If spaces X and Y are homotopy equivalent, we write $X \simeq Y$.

Remark 1.2.5.1. A map $f \colon X \to Y$ is a homotopy equivalence if there exists a map $g \colon Y \to X$ such that both $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ hold. The map g is a *homotopy inverse* of f.

Definition 1.2.6. A space X is *contractible* if it is homotopy equivalent to a point.

Remark 1.2.6.1. Every cone is contractible.

Proposition 1.2.7. Let X be a topological space. The following statements are equivalent:

i) The space X is contractible.

ii) The map id_X is homotopy equivalent to a constant map.

iii) The space X is a retract of CX.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.2.8. Let X and Y be closed surfaces. If $X \simeq Y$, then $X \approx Y$.

Definition 1.2.9. A subspace $A \subseteq X$ is a *deformation retract* of X if it is a retract and the retraction is a homotopy inverse of the inclusion.

Definition 1.2.10. A subspace $A \subseteq X$ is a *strong deformation retract* if it is a retract and the retraction is a homotopy inverse of the inclusion relative to A.

1.3 Extensions of homotopies

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Definition 1.3.1. A closed subspace $A \subseteq X$ has the *homotopy extension property* if for every space Y, map $f\colon X \to Y$ and homotopy $H\colon A \times I \to Y$ with $H_0 = f|_A$ there exists a homotopy $\overline{H}\colon X \times I \to Y$ such that $\overline{H}_0 = f$ and $\overline{H}\Big|_A = H$.

Proposition 1.3.2. A closed subspace $A \subseteq X$ has the homotopy extension property if an only if the space

$$L = A \times I \cup X \times \{0\}$$

is a retract of $X \times I$.

Proof. Suppose that L is a retract of X. It is easy to see that $\overline{H} = (H \cup f) \circ r$ is the required homotopy extension, where $r\colon X \times I \to L$ is a retraction.

Now suppose that A has the homotopy extension property. Define $i_0\colon X \hookrightarrow L$ and $H\colon A \times I \hookrightarrow L$ as the inclusions. By the homotopy extension property, there exists a homotopy $\overline{H}\colon X \times I \to L$, which is of course a retraction. $\square$

Proposition 1.3.3. Let $A \subseteq X$ be a contractible subspace. If A has the homotopy extension property, then $q\colon X \to X/A$ is a homotopy equivalence.

Proof. Let $K\colon A \times I \to A$ be a homotopy equivalence between id_A and the constant map. Then $K \cup \mathrm{id}_X\colon A \times I \cup X \to X$ is a well defined map. Let $H\colon X \times I \to X$ be its extension.

$$\begin{tikzcd}
X \times I \arrow[r, "H"] \arrow[d, "q \times \mathrm{id}_I"'] & X \arrow[d, "q"] \\
X/A \times I \arrow[r, dashed, "\overline{H}"'] & X/A
\end{tikzcd}$$

The map $q \times \mathrm{id}_I$ is a quotient map because I is locally compact. The induced map $\overline{H}$ is therefore well defined. But now H_1 induces a map $h\colon X/A \to X$. Note that, by definition, $h \circ q = H_1 \simeq H_0 = \mathrm{id}_X$, and $q \circ h = \overline{H}_1 \simeq \overline{H}_0 = \mathrm{id}_{X/A}$. $\square$

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Proposition 1.3.4. Suppose the map $A \hookrightarrow X$ has the homotopy extension property and $f, g\colon A \to Y$ are homotopic. Then we have $X \cup_f Y \simeq X \cup_g Y$.

Proposition 1.3.5. Suppose the map $A \hookrightarrow X$ has the homotopy extension property. If $A \hookrightarrow X$ is a homotopy equivalence, then A is a strong deformation retract of X.

Definition 1.3.6. Let $f\colon X \to Y$ be a map. The *mapping cylinder* of f is the space

$$M_f = X \times I \sqcup Y \Big/ (x, 0) \sim f(x) \, .$$

Remark 1.3.6.1. The subspace $X \approx X \times \{1\} \subseteq M_f$ has the homotopy extension property.

Proposition 1.3.7. Suppose that $f\colon X \to Y$ is a homotopy equivalence. Then both X and Y are strong deformation retracts of M_f.

Proof. Note that the map $H((x,s),t) = [(x, s(1-t)]$ is a homotopy between id and a retraction $r\colon M_f \to Y$, therefore Y is a strong deformation retract of M_f. In particular, r is a homotopy equivalence. Denote by $i\colon X \hookrightarrow M_f$ the inclusion of $X \simeq X \times \{1\}$ in M_f. Note that $f = r \circ i$, therefore i is also a homotopy equivalence. It follows that X is also a strong deformation retract of M_f. □

2 Fundamental group

2.1 Definition

Definition 2.1.1. Let $\alpha, \beta \colon I \to X$ be two paths with $\alpha(1) = \beta(0)$. The *concatenation product* is the path

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & t \le \frac{1}{2}, \\ \beta(2t-1), & t \ge \frac{1}{2}. \end{cases}$$

Definition 2.1.2. Let X be a topological space and $x_0 \in X$. The set of *loops* with starting point x_0 is the set

$$\Omega(X, x_0) = \mathcal{C}((I, \partial I), (X, x_0)).$$

Theorem 2.1.3. Concatenation on $\Omega(X, x_0)$ induces a group structure on

$$\Omega(X, x_0) \big/ \simeq \ \operatorname{rel} \partial I \ .$$

Definition 2.1.4. The above group is called the *fundamental group* and is denoted by $\pi_1(X, x_0)$.

Remark 2.1.4.1. Suppose that C_{x_0} is the path-component of X containing x_0. Then we have $\pi_1(X, x_0) = \pi_1(C_{x_0}, x_0)$.

Definition 2.1.5. Suppose $x_0, x_1 \in X$ are connected by a path $\gamma \colon (I, 0, 1) \to (X, x_0, x_1)$. Define the map $\operatorname{tr}_\gamma \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ as $\operatorname{tr}_\gamma([\alpha]) = [\overline{\gamma} \cdot \alpha \cdot \gamma]$.

Proposition 2.1.6. Suppose x_0, x_1 are connected by a path $\gamma \colon (I, 0, 1) \to (X, x_0, x_1)$. Then the map tr_γ is an isomorphism with inverse $(\operatorname{tr}_\gamma)^{-1} = \operatorname{tr}_{\overline{\gamma}}$.

Proof. The proof is obvious and need not be mentioned. □

Proposition 2.1.7. For paths $\gamma, \delta \colon (I, 0, 1) \to (X, x_0, x_1)$ we have

$$\operatorname{tr}_\delta = \operatorname{tr}_{\overline{\gamma}\delta} \circ \operatorname{tr}_\gamma \ .$$

Proof. The proof is obvious and need not be mentioned. □

Remark 2.1.7.1. If $\pi_1(X, x_1)$ is commutative, the tr-isomorphisms are independent from the chosen paths. We write $\pi_1(X) = \pi_1(X, x_0)$.

Proposition 2.1.8. Let $f \colon (X, x_0) \to (Y, y_0)$ be a map. Then f induces a homomorphism $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Proof. The proof is obvious and need not be mentioned. □

Proposition 2.1.9. The fundamental group is a functor $\pi_1 \colon \underline{\mathrm{HoTop}}_\bullet \to \underline{\mathrm{Grp}}$.

Corollary 2.1.9.1. The following statements are true:

i) If $f \colon (X, x_0) \to (Y, y_0)$ is a homotopy equivalence relative to x_0, the induced map $f_* \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism.

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ii) If $f\colon X \to Y$ is a homotopy equivalence, then the map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

iii) If X is contractible, then $\pi(X, x_0) \cong \{1\}$.

iv) For $x_0 \in X$ and $y_0 \in Y$ we have that

$$((p_X)_*, (p_Y)_*) : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is an isomorphism.

v) If A is a retract of X, then $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a monomorphism for all $x_0 \in A$. If $\pi_1(X, x_0)$ is commutative, the group $\pi_1(A, x_0)$ is a direct summand.

Proof. Assume that A is a retract of X. As $\pi_1\colon \underline{\text{HoTop}_\bullet} \to \underline{\text{Grp}}$ is a functor, we can apply it to the diagram

$$(A, x_0) \xhookrightarrow{i} (X, x_0) \xrightarrow{r} (A, x_0).$$

It follows that $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a monomorphism. □

Theorem 2.1.10 (Fundamental group of S^1)**.** The fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

Proof. Let $\alpha\colon (I, 0, 1) \to (S^1, 1, 1)$ be a loop. We can lift the map α to a unique map $\widetilde{\alpha}\colon (I, 0) \to (\mathbb{R}, 0)$, as $\mathbb{R}$ is a covering space of S^1. Observe the map $\Phi_0\colon \Omega(S^1, 1) \to \mathbb{Z}$, given by $\Phi_0(\alpha) = \alpha(1)$. As any homotopy $h\colon \alpha \simeq \beta \ (\mathrm{rel}\, \partial I)$ can be lifted to a unique homotopy $H\colon \widetilde{\alpha} \simeq \widetilde{\beta} \ (\mathrm{rel}\, \partial I)$, the map Φ_0 is constant on equivalence classes and therefore induces a map $\Phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$. Note that it is an isomorphism, therefore, $\pi_1(S^1, 1) \cong \mathbb{Z}$. □

Corollary 2.1.10.1 (Brouwer)**.** Every map $f\colon B^2 \to B^2$ has a fixed point.

Proof. If f had no fixed point, we could construct a retraction $r\colon B^2 \to S^1$. This is of course not possible, as $\mathbb{Z}$ is not a subgroup of the trivial group. □

Proposition 2.1.11. Let $q\colon I \to S^1$ be the quotient map. Then, the induced map $q^*\colon [(S^1, 1), (X, x_0)] \to \pi_1(X, x_0)$ is an isomorphism of groups.

Definition 2.1.12. The *degree* of α is the number $\deg \alpha$, where $\deg\colon [(S^1, 1), (S^1, 1)] \to \mathbb{Z}$ is the above isomorphism.

Theorem 2.1.13 (Borsuk-Ulam)**.** For any continuous map $f\colon S^1 \to \mathbb{R}$ there exists some $x \in S^1$ such that $f(x) = f(-x)$.

Proof. Let $g(x) = f(x) - f(-x)$. Note that g is an odd function and assume that $g(x) \neq 0$ for all x. Observe that

$$\frac{g}{|g|}\colon S^1 \to \{-1, 1\}$$

is a well defined, continuous odd function. This is of course impossible, as S^1 is connected. □

Lemma 2.1.14. For any odd function $f\colon S^1 \to S^1$ we have $2 \nmid \deg(f)$.

Proof. Without loss of generality assume $f(1) = 1$. Let $q\colon I \to S^1$ be the quotient map. Note that we can lift the map $f \circ q$ to a map $F\colon I \to \mathbb{R}$ with $F(0) = 0$. Then, $\deg(f) = F(1)$. Note that

$$f \circ q\left(\frac{1}{2}\right) = f(-1) = -1,$$

so $F\left(\frac{1}{2}\right) = k + \frac{1}{2}$. As f is odd, the map

$$G(x) = \begin{cases} F(x), & x \le \frac{1}{2}, \\ F\left(\frac{1}{2}\right) + F\left(x - \frac{1}{2}\right), & x \ge \frac{1}{2} \end{cases}$$

is also a lift of $f \circ q$. By uniqueness, we have $F = G$ and therefore $F(1) = 2k + 1$. □

Corollary 2.1.14.1. There are no odd maps $f\colon S^n \to S^1$ for $n > 1$.

Proof. Assume that such a map f exists. Note that the inclusion $\iota\colon S^1 \hookrightarrow S^n$ is null-homotopic. But then $f \circ \iota$ should also be null-homotopic, hence have degree 0. □

Theorem 2.1.15 (Borsuk-Ulam)**.** For any map $f\colon S^2 \to \mathbb{R}^2$ there exists some $x \in S^2$ such that $f(x) = -f(-x)$.

Proof. Suppose otherwise and note that $g\colon S^2 \to S^1$, given by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|},$$

is an odd map. □

Remark 2.1.15.1. The theorem holds for all maps $f\colon S^n \to \mathbb{R}^n$.

Theorem 2.1.16 (Stone-Tukey)**.** Let A, B and C be bounded measurable subsets of $\mathbb{R}^3$. Then there exists a plane which bisects all of A, B and C.[1]

Theorem 2.1.17 (Lusternik-Schnirelmann)**.** Let A, B and C form a closed cover of S^2. Then, at least one of them contains a pair of antipodal points.

Theorem 2.1.18 (Fundamental theorem of algebra)**.** Every non-constant polynomial $p \in \mathbb{C}[x]$ has a complex root.

Proof. Let $n = \deg p$ and assume $0 \notin p(\mathbb{C})$. Define

$$H(z,t) = \sum_{i=0}^{n} a_i z^i t^{n-i}.$$

Note that $H\colon S^1 \times I \to \mathbb{C} \setminus \{0\}$ is a homotopy between z^n and $p(z)$. Also, the map

$$K(z,t) = p(zt)$$

is a homotopy between a_0 and $p(z)$. It follows that p is null-homotopic, so $\deg p = 0$. □

[1] Also called the *Ham Sandwich theorem*.

2.2 Computation of the fundamental group

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Definition 2.2.1. The *coproduct* of groups G and H is the group

$$G * H = \left\{ \prod_{i=1}^{n} g_i h_i \;\middle|\; n \in \mathbb{N} \wedge g_i \in G \wedge h_i \in H \right\}.$$

Remark 2.2.1.1. The definition coincides with coproducts in the category Grp.

Theorem 2.2.2 (Seifert-van Kampen)**.** Let X and Y be open subspaces in $X \cup Y$ and $x_0 \in X \cap Y$. Suppose that X, Y and $X \cap Y$ are path-connected. Denote by $i_Z \colon X \cap Y \hookrightarrow Z$ and $j_Z \colon Z \hookrightarrow X \cup Y$ the inclusions for $Z \in \{X, Y\}$. Then

$$\varphi = ((j_X)_*, (j_Y)_*) \colon \pi_1(X, x_0) * \pi_1(Y, x_0) \to \pi_1(X \cup Y, x_0)$$

is an epimorphism with

$$\ker \varphi = \left\langle \left\{ (i_X)_*(\alpha) \cdot (i_Y)_* \left(\alpha^{-1}\right) \;\middle|\; \alpha \in \pi_1(X \cap Y, x_0) \right\} \right\rangle.$$

Proof. Let $\alpha \in \pi_1(X \cup Y, x_0)$ be an arbitrary loop. Note that, using the Lebesgue number, we can split α into finitely many paths in X and Y. Using the path-connectedness of $X \cap Y$, we can join the path segments into loops starting at x_0, thus constructing an element of $\pi_1(X, x_0) * \pi_1(Y, x_0)$ that maps to α.

Let

$$N = \left\langle \left\{ (i_X)_*(\alpha) \cdot (i_Y)_* \left(\alpha^{-1}\right) \;\middle|\; \alpha \in \pi_1(X \cap Y, x_0) \right\} \right\rangle.$$

Note that

$$\begin{aligned} \varphi\left((i_X)_*(\alpha) \cdot (i_Y)_* \left(\alpha^{-1}\right)\right) &= (j_X)_*((i_X)_*(\alpha)) \cdot (j_Y)_* \left((i_Y)_* \left(\alpha^{-1}\right)\right) \\ &= (j_Y)_*((i_Y)_*(\alpha)) \cdot (j_Y)_* \left((i_Y)_* (\alpha)\right)^{-1} \\ &= 1, \end{aligned}$$

as $j_X \circ i_X = j_Y \circ i_Y$. It follows that $N \leq \ker \varphi$. It remains to check that $\ker \varphi \leq N$. □

Definition 2.2.3. A path-connected topological space X is *simply connected* if any two paths $\alpha, \beta \colon I \to X$, such that $\alpha|_{\partial I} = \beta|_{\partial I}$, are homotopic relative to ∂I.

Remark 2.2.3.1. Equivalently, X is path-connected with $\pi_1(X) \cong \{1\}$.

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Proposition 2.2.4. Let X, Y and $X \cap Y$ be path-connected and $x_0 \in X \cap Y$. Assume that $X \cap Y$ is a strong deformation retract of its open neighbourhood U. The conclusion of Seifert-van Kampen then holds for $\pi_1(X \cup Y, x_0)$.

Proof. Set $X' = X \cup U$ and $Y' = Y \cup U$. Note that $X' \cap Y' = U$, therefore, X' and Y' satisfy the assumptions of Seifert-van Kampen theorem. Observe that X is a strong deformation retract of X', therefore $\pi_1(X, x_0) \cong \pi_1(X', x_0)$. □

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