Noncommutative algebra

Luka Horjak (luka1.horjak@gmail.com)

October 6, 2023

Contents

Introduction

These are my lecture notes on the course Noncommutative algebra in the year 2023/24. The lecturer that year was prof. dr. Igor Klep.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Finite-dimensional algebras, Wedderburn's structure theory

1.1 Free algebras

Definition 1.1.1. Let $R = K\langle x, y\rangle$ be a free algebra and $F = \{xy - yx - 1\}$. The quotient

$$\mathcal{A}_1(K) = R/(F)$$

is called the *first Weyl algebra*.

Remark 1.1.1.1. The first Weyl algebra is generated by elements $\overline{x}$ and $\overline{y}$ that satisfy $\overline{x} \cdot \overline{y} - \overline{y} \cdot \overline{x} - 1$.

Remark 1.1.1.2. The first Weyl algebra is the algebra of differential operators – for $D, L \colon K[y] \to K[y]$, defined as $D(p) = \frac{\partial p}{\partial y}$ and $L(p) = yp$, we have $DL - LD = I$.

Definition 1.1.2. Let R be a ring and $\sigma \in \operatorname{End}(R)$. The *skew polynomial ring* is the set

$$R[x, \sigma] = \left\{ \sum_{i=0}^{n} b_i x^i \;\middle|\; n \in \mathbb{N} \wedge b_i \in R \right\}$$

in which for all $b \in R$ the equality in $xb = \sigma(b)x$ holds.

Definition 1.1.3. Let R be a ring and σ a derivation[1] on R. The *skew polynomial ring* is the set

$$R[x, \sigma] = \left\{ \sum_{i=0}^{n} b_i x^i \;\middle|\; n \in \mathbb{N} \wedge b_i \in R \right\}$$

in which for all $b \in R$ the equality in $xb = bx + \sigma(b)$ holds.

[1] $\sigma(a + b) = \sigma(a) + \sigma(b)$, $\sigma(ab) = a\sigma(b) + \sigma(a)b$.

1.2 Chain conditions

Definition 1.2.1. Let C be a set and $\{C_i \mid i \in I\}$ a set of subsets of C. The set $\{C_i \mid i \in I\}$ satisfies the *ascending chain condition* if there does not exist an infinite strictly increasing chain

$$C_{i_1} \subset C_{i_2} \subset C_{i_3} \subset \dots$$

The *descending chain condition* is defined analogously.

Definition 1.2.2. Let R be a ring and M an R-module.

i) M is *noetherian* if the set of submodules of M satisfies the ascending chain condition.

ii) M is *artinian* if the set of submodules of M satisfies the descending chain condition.

Proposition 1.2.3. The following statements are true:

i) A module M is noetherian if and only if each submodule of M is finitely generated.

ii) Let $N \leq M$ be a submodule. Then M is noetherian if and only if both N and ${}^{M}/_{N}$ are noetherian.

iii) Let $N \leq M$ be a submodule. Then M is artinian if and only if both N and ${}^{M}/_{N}$ are artinian.

Proof.

i) Suppose that each submodule of M is finitely generated and $M_1 \leq M_2 \leq \cdots \leq M$. Define the submodule
$$N = \bigcup_{j \in \mathbb{N}} M_j.$$
By assumption, N is finitely generated. But then there exists some $j \in \mathbb{N}$ such that M_j contains all generators of N, so $M_j = N$. Therefore, the chain cannot be strictly increasing.

Now assume that M is noetherian and let $N \leq M$ be a submodule. Define
$$\mathcal{C} = \{S \leq N \mid S \text{ is finitely generated}\}.$$
This set must have some maximal element $N_0 \leq N$. Suppose $N_0 < N$ and consider some element $b \in N \setminus N_0$. The module $N + Rb$ is also finitely generated and contained in N, which is a contradiction as N_0 was maximal. Therefore we must have $N = N_0$ and N is finitely generated.

ii) Suppose that M is noetherian. Consider the following short exact sequence:
$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} {}^{M}/_{N} \longrightarrow 0.$$
It is easy to see that N is also noetherian, as the inclusion of a chain in N is also a chain in M. As preimages of submodules are also submodules, the same conclusion follows for ${}^{M}/_{N}$.

Now suppose that both N and ${}^{M}/_{N}$ are noetherian and consider a chain $M_1 \leq M_2 \leq \cdots \leq M$ of submodules. As $f^{-1}(M_i)$ and $g(M_i)$ form increasing chains in their respective modules, it follows that there exists some $n \in \mathbb{N}$ such that both $f^{-1}(M_i)$ and $g(M_i)$ are constant for all $i \geq n$. Now consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & f^{-1}(M_n) \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & M_n \arrow[r, "g"] \arrow[d, hook, "i"] & g(M_n) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & f^{-1}(M_i) \arrow[r, "f"] & M_i \arrow[r, "g"] & g(M_i) \arrow[r] & 0.
\end{tikzcd}$$

By the short five lemma, i is an isomorphism, so $M_n = M_i$.

iii) Same as ii). □

Index