# Complex analysis

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# Contents

# Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

# 1 Holomorphic functions

## 1.1 Properties of holomorphic functions

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**Definition 1.1.1.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists.

**Remark 1.1.1.1** (Cauchy-Riemann equations)**.** Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where $f$ is real differentiable in $a$, $f$ is complex differentiable in $a$ if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

**Definition 1.1.2.** *Wirtinger derivatives* are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$

**Remark 1.1.2.1.** A function $f$ is complex differentiable in $a$ if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

**Definition 1.1.3.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *holomorphic in $a$* if it is complex differentiable in an open neighbourhood of $a$. The function $f$ is *holomorphic* if it is holomorphic in every point of $\Omega$. We denote the set of holomorphic functions in $\Omega$ as $\mathcal{O}(\Omega)$.

**Theorem 1.1.4** (Inhomogeneous Cauchy integral formula)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with $\mathcal{C}^1$-smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f(w)}{w - z}\, dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z}\, dw \wedge d\overline{w}$$

*Proof.* As $\Omega$ is an open set, there exists an $\varepsilon > 0$ such that $\overline{\Delta(z, \varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\Delta(z, \varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w-z}\, dw$ on $\Omega_\varepsilon$. As $d\omega = \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w} \wedge dw$, we have

$$\oint_{\partial \Omega_\varepsilon} \frac{f(w)}{w - z}\, dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z}\, d\overline{w} \wedge dw.$$

Note that

$$\oint_{\partial \Omega_\varepsilon} \frac{f(w)}{w - z}\, dw = \oint_{\partial \Omega} \frac{f(w)}{w - z}\, dw - \oint_{\partial \Delta(z, \varepsilon)} \frac{f(w)}{w - z}\, dw.$$

In the limit, we have

$$\lim_{\varepsilon\to 0} \oint\limits_{\partial\mathbb{A}(z,\varepsilon)} \frac{f(w)}{w-z}\,dw = \lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{f\left(z+\varepsilon e^{it}\right)}{\varepsilon e^{it}}\cdot \varepsilon i e^{it}\,dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon\to 0}\iint\limits_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega\setminus\{z\}} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i}\oint\limits_{\partial\Omega}\frac{f(w)}{w-z}\,dw - f(z) = -\frac{1}{2\pi i}\iint\limits_{\Omega}\frac{\partial f}{\partial\overline{w}}\cdot\frac{1}{w-z}\,dw\wedge d\overline{w}.$$ □

**Theorem 1.1.5** (Power series expansion)**.** Let $\Omega\subseteq\mathbb{C}$ be an open subset, $f\in\mathcal{O}(\Omega)$ and $a\in\Omega$. The function $f$ can be developed into a power series about $a$ that converges absolutely and uniformly to $f$ in compacts inside $\mathbb{A}(a,r)$, where $r$ is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i}\oint\limits_{\partial\mathbb{A}(a,\rho)}\frac{f(w)}{(w-z)^{k+1}}\,dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k\cdot(z-a)^k.$$

**Remark 1.1.5.1.** The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

**Remark 1.1.5.2.** The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k\to\infty}\sqrt[k]{|c_k|}.$$

**Theorem 1.1.6** (Identity)**.** Let $\Omega\subseteq\mathbb{C}$ be a domain and $f\in\mathcal{O}(\Omega)$ a holomorphic function. Let $A\subseteq\Omega$ be a subset such that $f(z)=0$ for all $z\in A$. If $A$ has an accumulation point in $\Omega$, then $f(z)=0$ for all $z\in\Omega$.

*Proof.* Let $a\in\Omega$ be an accumulation point of $A$. By continuity, we have $f(a)=0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k(z-a)^k,$$

where we assume $c_{k_0}\neq 0$. But now $g(z)=\frac{f(z)}{(z-a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a)=0$, which is a contradiction. It follows that $c_k=0$ for all $k\in\mathbb{N}_0$. It follows that the set $\operatorname{Int}\{z\in\Omega\mid f(z)=0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to $\Omega$. □

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**Lemma 1.1.7.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\mathbb{A}(a,r)} \subseteq \Omega$. If
$$|f(a)| < \min_{\partial \mathbb{A}(a,r)} |f|,$$
then $f$ has a zero in $\mathbb{A}(a,r)$.

*Proof.* Assume otherwise. From the inequality it follows that $f$ has no zeroes on the boundary either. By continuity, $f$ has no zero on an open set $V$ with $\mathbb{A}(a,r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint_{\partial \mathbb{A}(a,r)} \frac{g(z)}{z-a}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g\left(a + r \cdot e^{it}\right)}{re^{it}} \cdot rie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} g\left(a + re^{it}\right) dt.$$

We can therefore get a bound on $|g(a)|$ as
$$|g(a)| \leq \max_{\partial \mathbb{A}(a,r)} |g|,$$
but as the condition on $f$ can be rewritten as
$$|g(a)| > \max_{\partial \mathbb{A}(a,r)} |g|,$$
we have reached a contradiction. $\square$

**Theorem 1.1.8** (Open mapping)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If $f$ is not constant, it is an open map.

*Proof.* Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\mathbb{A}(z_0, \rho) \subseteq U$ and $z_0$ is the only pre-image of $w_0$ in $\mathbb{A}(z_0, 2\rho)$.[1]

Since $\partial \mathbb{A}(z_0, \rho)$ is a compact set and
$$|f(z) - w_0| > 0$$
for all $z \in \partial \mathbb{A}(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that
$$|f(z) - w_0| > 2\varepsilon$$
holds on the boundary of the disk. Choose a $w \in \mathbb{A}(w_0, \varepsilon)$. As we have
$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$
on the boundary and
$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$
by the above lemma, $f(z_0) - w$ has a root on $\mathbb{A}(z, \rho)$. $\square$

**Theorem 1.1.9** (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function $f$ is constant.

[1] If such a disk does not exist, $f$ is constant by the identity theorem.

*Proof.* Suppose that $f$ is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As $f$ is an open map, it also attains the value $(1+\varepsilon) \cdot f(z)$, which is a contradiction as the modulus then equals $(1+\varepsilon) \cdot |f(z)| > |f(z)|$. □

**Theorem 1.1.10** (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

*Proof.* As $\overline{\Omega}$ is compact, $f$ attains a global maximum on this set. If the maximum is attained in the interior, $f$ is constant, therefore it is also attained on the boundary. □

**Definition 1.1.11.** A function $f \colon \Omega \setminus \{a\} \to \mathbb{C}$ is *locally bounded* near $a$ if there exists an open neighbourhood $U \subseteq \Omega$ of $a$ such that $f|_{U\setminus\{a\}}$ is bounded.

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**Theorem 1.1.12** (Riemann removable singularity theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$. If $f$ is locally bounded near $a$, then there exists a unique function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega\setminus\{a\}} = f$.

*Proof.* Define the function $F \colon \Omega \to \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \dfrac{1}{2\pi i} \displaystyle\oint_{\partial\mathbb{\Delta}(a,\rho)} \dfrac{f(w)}{w-a}\, dw & z = a. \end{cases}$$

It remains to check that $F$ is complex differentiable at $a$. Indeed, for $z \in \mathbb{\Delta}(a,\rho)$ we have

$$\begin{aligned} \lim_{z\to a} \frac{F(z)-F(a)}{z-a} &= \lim_{z\to a} \frac{1}{z-a} \oint_{\partial\mathbb{\Delta}(a,\rho)} \left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a}\right) dw \\ &= \lim_{z\to a} \frac{1}{2\pi i} \cdot \frac{1}{z-a} \cdot \oint_{\partial\mathbb{\Delta}(a,\rho)} f(w) \cdot \frac{z-a}{(w-z)(w-a)}\, dw \\ &= \frac{1}{2\pi i} \oint_{\partial\mathbb{\Delta}(a,\rho)} \frac{f(w)}{(w-a)^2}\, dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem. □

**Theorem 1.1.13** (Schwarz lemma)**.** Let $f \colon \mathbb{\Delta} \to \mathbb{\Delta}$ be a holomorphic function with $f(0) = 0$. Then, $|f'(0)| \le 1$ and the inequality $|f(z)| \le |z|$ holds for all $z \in \mathbb{\Delta}$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ holds for any $z \neq 0$, then $f(z) = \beta z$ for some $\beta \in \partial\mathbb{\Delta}$.

*Proof.* We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for $g$ on the domain $\mathbb{\Delta}(\rho)$. We get

$$\sup_{z\in\mathbb{\Delta}(\rho)} |g(z)| \le \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as $\rho \to 1$, it follows that

$$\sup_{z \in \mathbb{\Delta}} |g(z)| \leq 1.$$

It immediately follows that $|f'(0)| = |g(0)| \leq 1$. Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z| \, .$$

Suppose we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. As then $|g(z_0)| = 1$, it follows that $g$ is constant, therefore $f(z) = \beta z$ for some $\beta \in \partial \mathbb{\Delta}$. If we have $|f'(0)| = 0$, the same argument works for $z_0 = 0$. $\square$

## 1.2 The $\overline{\partial}$ equation

**Lemma 1.2.1.** Let $g \in \mathcal{C}^\infty(\mathbb{C})$ be a function with compact support. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

*Proof.* Let

$$f(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{g(w)}{w - z} \, dw \wedge d\overline{w}.$$

As

$$dw \wedge d\overline{w} = -2ri \, dr \wedge d\varphi$$

holds for polar coordinates centered at $z$, we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{rg\left(z + re^{i\varphi}\right)}{re^{i\varphi}} \, dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some $R$ such that $g|_{\mathbb{C} \setminus \Delta(z,R)} = 0$. We get

$$f(z) = -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi,$$

which obviously converges. The function $f$ is therefore well defined. As we are integrating a smooth function on a compact set, the function $f$ is smooth as well.

For $u = re^{i\varphi}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}}(z) &= -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} \frac{\partial}{\partial \overline{z}} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial}{\partial \overline{z}} g(u + z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial g}{\partial \overline{u}}(u + z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w - z} \, dw \wedge d\overline{w}.
\end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint\limits_{\partial\Delta(z,R)} \frac{g(w)}{w - z} \, dw + \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w - z} \, dw \wedge d\overline{w}.$$

by the choice of $R$, we get

$$\frac{\partial f}{\partial \overline{z}}(z) = g(z).$$

□

**Lemma 1.2.2.** Given bounded domain $U \subset V \subset \mathbb{R}^n$ such that $\partial U \cap \partial V = \emptyset$, there exists a smooth function $\chi \colon \mathbb{R}^n \to [0, 1]$ such that $\chi|_U = 1$ and $\operatorname{supp} \chi \subseteq V$.

*Proof.* There is a partition of unity on the sets $V$ and $\mathbb{R}^n \setminus \overline{U}$. □

**Lemma 1.2.3.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $h_j \colon \Omega \to \mathbb{C}$ be holomorphic functions. If the sequence $(h_j)_{j\in\mathbb{N}}$ converges uniformly on compact sets, the limit is also holomorphic on $\Omega$.

*Proof.* Apply Morera's theorem.[2] □

**Theorem 1.2.4** (Dolbeault lemma)**.** Let $g \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ for some $R \in (0, \infty]$. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

*Proof.* Define disks $X_j$ as follows:

i) If $R = \infty$, set $X_j = \mathbb{\Delta}(j)$.

ii) If $R < \infty$, set $X_j = \mathbb{\Delta}\left(R - \frac{1}{j}\right)$ (for large enough $j$).

Applying the above lemma, define functions $\chi_j$ with $\chi_j|_{X_j} = 1$ and $\operatorname{supp} \chi_j \subseteq X_{j+1}$ and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \mathbb{\Delta}(R), \\ 0 & z \notin \mathbb{\Delta}(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function $f_j \in \mathcal{C}^\infty(\mathbb{C})$ with

$$\frac{\partial f_j}{\partial \overline{z}} = g_j.$$

We inductively construct a new sequence $\widetilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$ such that

$$\frac{\partial \widetilde{f}_j}{\partial \overline{z}} = g$$

on $X_j$ and

$$\left\| \widetilde{f}_j - \widetilde{f}_{j-1} \right\|_{X_{j-2}} \le 2^{-j}.$$

Set $\widetilde{f}_1 = f_1$. Observe the function $F = f_{j+1} - \widetilde{f}_j$ on $X_j$. By construction, we have $\frac{\partial F}{\partial \overline{z}} = 0$ on $X_j$. It follows that $F$ can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on $X_j$. As power series converge uniformly on compact sets, there exists some polynomial $p \in \mathbb{C}[z]$ such that

$$\|F - p\|_{X_{j-1}} \le 2^{-j}.$$

Now just set $\widetilde{f}_{j+1} = f_{j+1} - p$.

Let $z \in \mathbb{\Delta}(R)$ be arbitrary. By construction, it is contained in some $X_{j_0}$, therefore, $\widetilde{f}_j$ is defined for $j \ge j_0$. As $\left(\widetilde{f}_j(z)\right)_{j \ge j_0}$ is a Cauchy sequence, we can define

$$f(z) = \lim_{j\to\infty} \widetilde{f}_j(z).$$

[2] Analysis 2b, theorem 3.4.6.

But as

$$f - \widetilde{f}_j = \sum_{k=j}^{\infty} \left( \widetilde{f}_{j+1} - \widetilde{f}_j \right)$$

is a sum of holomorphic functions that converges uniformly, the function $f - \widetilde{f}_j$ is a holomorphic function. Therefore, $f$ is smooth and satisfies $\frac{\partial f}{\partial \bar{z}} = g$. $\square$

## 1.3 Meromorphic functions

**Definition 1.3.1.** Let $\Omega \subset \mathbb{C}$ be an open subset. We call a function $f$ *meromorphic* of $\Omega$ if there exists $A \subset \Omega$ such that $f \in \mathcal{O}(\Omega \setminus A)$, $A$ has no accumulation points in $\Omega$ and for all $a \in A$ there exists some $k \in \mathbb{N}$ such that

$$\lim_{z \to a} f(z) \cdot (z-a)^k \neq 0$$

exists. We call $A$ the set of *poles* of the function $f$. We denote the set of meromorphic functions on $\Omega$ with $\mathcal{M}(\Omega)$.

**Theorem 1.3.2.** Let $0 \le r < R \le \infty$. Suppose that $f \in \mathcal{O}(D_{R,r}(a))$ is a holomorphic function, where

$$D_{R,r}(a) = \{z \in \mathbb{C} \mid r < |z-a| < R\}.$$

Then there exists a uniquely determined *Laurent series*

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

that converges to $f$ uniformly and absolutely on compact subsets of $D_{R,r}(a)$. We have

$$c_k = \frac{1}{2\pi i} \oint_{\partial \Delta(a,\rho)} \frac{f(w)}{(w-a)^k} \, dw$$

for $r < \rho < R$.

**Definition 1.3.3.** Let

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

be a Laurent series. The series

$$\sum_{k=-\infty}^{-1} c_k (z-a)^k$$

is called the *principle part*.

**Lemma 1.3.4.** Let $f \in \mathcal{O}(\Omega \setminus \{a\})$ be a holomorphic function. Then $f$ is meromorphic on $\Omega$ if and only if $f$ has a finite principle part in $a$.

*Proof.* Suppose that $f$ is meromorphic on $\Omega$. If $a$ is a removable singularity, $f$ is holomorphic in $a$, therefore the principle part is trivial. Otherwise, set $m \in \mathbb{N}$ such that

$$\lim_{z \to a} (z-a)^m f(z) \neq 0$$

exists and set $g(z) = (z-a)^m f(z)$. As $g$ is bounded near $a$, we can extend it to $\Omega$ by the Riemann removable singularity theorem. The power series of $g$ corresponds to a finite Laurent series of $f$.

The converse is obvious. $\square$

**Theorem 1.3.5.** If $f \in \mathcal{M}(\mathbb{C})$ is a meromorphic function, there exist entire functions $g$ and $h$ such that $f = \frac{g}{h}$.

**Definition 1.3.6.** Let $\Omega \subseteq \mathbb{C}$ be an open set. An *additive Cousin problem* on $\Omega$ is an open cover $\{U_j\}_{j\in J}$ of $\Omega$ and functions $f_j \in \mathcal{M}(U_j)$ such that $f_j - f_k|_{U_j\cap U_k}$ is holomorphic for all $j, k \in J$. A function $f \in \mathcal{M}(\Omega)$ is a solution to the additive Cousin problem if $f|_{U_j} - f_j$ is holomorphic for all $j \in J$.

**Definition 1.3.7.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A *generalized additive Cousin problem* is an open cover $\{U_j\}_{j\in J}$ of $\Omega$ and functions $f_{j,k} \in \mathcal{O}(U_j \cap U_k)$ for each $(j, k) \in J^2$, such that

i) $f_{j,k} = -f_{k,j}$ on $U_j \cap U_k$ for all $(j, k) \in J^2$ and

ii) $f_{j,k} + f_{k,\ell} + f_{\ell,j} = 0$ on $U_j \cap U_k \cap U_\ell$ for all $(j, k, \ell) \in J^3$.

A solution to the generalized additive Cousin problem is given by functions $f_j \in \mathcal{O}(U_j)$ for each $j \in J$ such that $f_{j,k} = f_j - f_k$ for each $(j, k) \in J^2$.

**Lemma 1.3.8** (Partition of unity)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $\{U_j\}_{j\in J}$ be an open cover of $\Omega$. Then there exists a partition of unity subordinate to $\{U_j\}_{j\in J}$.

**Lemma 1.3.9.** Given a generalized additive Cousin problem on $\Omega \subseteq \mathbb{C}$, there exist functions $g_j \in \mathcal{C}^\infty(U_j)$ such that $f_{j,k} = g_j - g_k$ for all $(j, k) \in J^2$.

*Proof.* Let $\{(V_a, \chi_a)\}_{a\in A}$ be a partition of unity, subordinate to $\{U_j\}_{j\in J}$. For all $a \in A$ choose a $j(a) \in J$ such that $V_a \subseteq U_{j(a)}$. For all $k \in J$, define

$$g_k = -\sum_{a\in A} \chi_a \cdot f_{j(a),k}.$$

This is of course a smooth function on $U_k$. Now note that

$$g_k - g_\ell = \sum_{a\in A} \chi_a \cdot \left(-f_{j(a),k} + f_{j(a),\ell}\right) = \sum_{a\in A} \chi_a \cdot f_{k,\ell} = f_{k,\ell}. \qquad \square$$

**Proposition 1.3.10.** The generalized additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

*Proof.* Let $f_{j,k} = g_j - g_k$ for $g_j \in \mathcal{C}^\infty(U_j)$. Note that

$$\frac{\partial g_j}{\partial \overline{z}} = \frac{\partial g_k}{\partial \overline{z}},$$

therefore,

$$h|_{U_j} = \frac{\partial g_j}{\partial \overline{z}}$$

induces a smooth function $h\colon \Omega \to \mathbb{C}$. By the Dolbeault lemma, there exists a function $g \in \mathcal{C}^\infty(\Omega)$ such that $\frac{\partial g}{\partial \overline{z}} = h$. It is clear that $f_j = g_j - g$ solves the generalized additive Cousin problem. $\square$

**Proposition 1.3.11.** The additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

*Proof.* An additive Cousin problem induces a generalized additive Cousin problem for functions $f_{j,k} = f_j - f_k$. Let $g_j$ be a solution to the generalized problem. As $f_j - f_k = f_{j,k} = g_j - g_k$ on $U_j \cap U_k$, we can define a function $f \in \mathcal{M}(\Omega)$ with $f|_{U_j} = f_j - g_j$. This function is of course well defined. As $f|_{U_j} - f_j = g_j \in \mathcal{O}(U_j)$, this function indeed solves the additive Cousin problem. $\square$

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**Theorem 1.3.12** (Mittag-Leffler)**.** Let $(a_k)_{k\in\mathbb{N}}$ be a sequence without repetition and accumulation points. Let

$$f_k(z) = \sum_{\ell=-m_k}^{-1} c_{k,\ell}(z-a_k)^\ell$$

be finite principal parts. Then there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ with poles in $(a_k)_{k\in\mathbb{N}}$ such that $f$ has principle part $f_k$ in $a_k$ for each $k \in \mathbb{N}$.

*Proof.* For each $a_k$ choose a disk $U_k$ containing no other $a_k$. Also set $U_0 = \mathbb{C}\setminus\{a_k \mid k \in \mathbb{N}\}$ and $f_0 = 0$. As $\{U_k \mid k \in \mathbb{N}_0\}$ is an open cover of $\mathbb{C}$, there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ that solves the corresponding additive Cousin problem. It is easy to see that the principle parts of $f$ at $a_k$ are precisely $f_k$. □

## 1.4 Sequences of holomorphic functions

**Definition 1.4.1.** A family of functions $\mathcal{F}$ from $\Omega$ to $\mathbb{C}$ is *locally bounded*, if for all $p \in \Omega$ there exist a $\rho > 0$ and $M > 0$ such that

$$\sup_{f\in\mathcal{F}} \sup_{z\in\Delta(p,\rho)} |f(z)| < M.$$

**Lemma 1.4.2.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ a locally bounded family of functions. Then for all $p \in \Omega$ there exists a $\rho > 0$ such that $\mathcal{F}$ is equi-continuous on $\Omega \cap \Delta(p, \rho)$.

*Proof.* Fix $p \in \Omega$ and choose $r > 0$ such that $D = \overline{\Delta(p, 2r)} \subseteq \Omega$. For any $z, w \in D$ and $f \in \mathcal{F}$ we have

$$f((z) - f(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - w}\, d\xi = \frac{z-w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)}\, d\xi.$$

Note that the family $\mathcal{F}$ is bounded on every compact. Therefore, we can write

$$\sup_{f\in\mathcal{F}} \sup_{z\in\partial D} |f(z)| < M.$$

Now, for $z, w \in \Delta(p, r)$ we have

$$|f((z) - f(w)| = \left| \frac{z-w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)}\, d\xi \right| \le |z - w| \cdot \frac{2M}{r}. \qquad \square$$

**Theorem 1.4.3** (Arzelà-Ascoli)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be an infinite family such that the following conditions hold:

i) $\mathcal{F}$ is point-wise bounded.

ii) $\mathcal{F}$ is locally equi-continuous.

Then there $\mathcal{F}$ contains a sequence that converges uniformly on compacts of $\Omega$.

*Proof.* Choose a dense countable subset $A \subseteq \Omega$ and enumerate it as a sequence $(a_k)_{k\in\mathbb{N}}$. Pick any sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ with pairwise distinct terms. As $|f_n(a_1)| < M$ for all $n$, we can choose a subsequence $(f_{1,n})_{n\in\mathbb{N}}$ such that $f_{1,n}(a_1)$ converges by Bolzano-Weierstraß.

Similarly, for every $k \in \mathbb{N}$ there exists a subsequence $(f_{k,n})_n$ of $(f_{k-1,n})_n$ such that $(f_{k,n}(a_k))_n$ converges. Now define $F_n = f_{n,n}$. Observe that $(F_n)$ converges at every point in $A$.

Fix a $p \in \Omega$. By local equi-continuity, there exists a $\rho > 0$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta < \rho$ and $|F_n(z) - F_n(w)| < \frac{\varepsilon}{3}$ for all $z, w \in \Delta(p, \rho)$ such that $|z - w| < \delta$. Choose an element $a \in A \cap \Delta(z, \delta)$.[3] Then,we have

$$|F_n(z) - F_m(z)| \le |F_n(z) - F_n(a)| + |F_n(a) - F_m(a)| + |F_m(a) - F_m(z)| < 3 \cdot \frac{\varepsilon}{3}.$$

It follows that $(F_n)$ is locally uniformly convergent, therefore it converges uniformly on compact sets. $\square$

[3] By compactness of $\overline{\Delta(p, \rho)}$ we can choose $a$ from a finite set.

**Theorem 1.4.4** (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f_n\colon \Omega \to \mathbb{C}$ be a locally bounded sequence of holomorphic functions. Then $(f_n)_n$ contains a subsequence that converges uniformly on compacts.

*Proof.* As the sequence is locally bounded, it is locally equi-continuous by lemma 1.4.2. By Arzelà-Ascoli, there exists a convergent subsequence. □

**Definition 1.4.5.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family of functions $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is *normal* if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on compacts.

**Theorem 1.4.6** (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is normal if and only if it is locally bounded.

*Proof.* The proof is obvious and need not be mentioned. □

**Theorem 1.4.7** (Vitali)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $(f_n)_n \subseteq \mathcal{O}(\Omega)$ a locally bounded sequence of holomorphic functions. The following statements are equivalent:

i) The sequence $(f_n)_n$ converges uniformly on compact subsets of $\Omega$.

ii) For a point $p \in \Omega$ the sequence $(f_n^{(k)}(p))_n$ converges for all $k \in \mathbb{N}_0$.

iii) The set

$$A = \left\{ z \in \Omega \;\middle|\; \lim_{n\to\infty} f_n(z) \text{ converges} \right\}$$

has an accumulation point in $\Omega$.

*Proof.* Suppose that the sequence converges uniformly on compact subsets. Given a $p \in \Omega$, choose a $\delta > 0$ such that $D = \overline{\mathbb{\Delta}(p, \delta)} \subseteq \Omega$. Note that

$$\left| g^{(k)}(p) \right| \leq \frac{k!}{\delta^k} \cdot \|g\|_D$$

holds for all holomorphic functions $g$. As $\|f - f_n\|$ converges to 0, the derivatives of $f_n$ converge.

Suppose that the sequences of derivatives converge at a point $p \in \Omega$ and choose a $\delta > 0$ such that $D = \overline{\mathbb{\Delta}(p, \delta)} \subseteq \Omega$. As the sequence is locally bounded, there exists a constant $M$ such that $\|f_n\|_D \leq M$ holds for all $n \in \mathbb{N}$. We can now develop power series

$$f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z-p)^k.$$

They converge uniformly on compact subsets of $\mathbb{\Delta}(p, \delta)$. Note that

$$a_{k,n} = \frac{f_n^{(k)}(p)}{k!}.$$

As derivatives converge, we can define the limit

$$a_k = \lim_{n\to\infty} a_{k,n}.$$

Now define the formal power series

$$f(z)=\sum_{k=0}^{\infty}a_k(z-p)^k.$$

The Cauchy bounds give us the inequality

$$|a_{k,n}|=\frac{\left|f^{(k)}(p)\right|}{k!}\leq\frac{M}{\delta^k},$$

therefore

$$\limsup_{k\to\infty}\sqrt[k]{|a_k|}\leq\limsup_{k\to\infty}\frac{\sqrt[k]{M}}{\delta}=\frac{1}{\delta}.$$

We conclude that the radius of convergence is at least $\delta$. Consider some $\rho\in(0,\delta)$ and $z\in\mathbb{\Delta}(p,\rho)$. We have

$$\begin{aligned}
|f_n(z)-f(z)| &\leq \left|\sum_{k=0}^{m}(a_{k,n}-a_k)\cdot(p-z)^k\right|+\left|\sum_{k=m+1}^{\infty}(a_{k,n}-a_k)\cdot(p-z)^k\right| \\
&\leq \sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+\sum_{k=m+1}^{\infty}2M\cdot\frac{\rho^k}{\delta^k} \\
&= \sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+2M\cdot\left(\frac{\rho}{\delta}\right)^{m+1}\cdot\frac{\delta}{\delta-\rho} \\
&= 2\cdot\frac{\varepsilon}{2}
\end{aligned}$$

for large enough $m$ and $n$. It follows that $p$ is an accumulation point of $A$.

Suppose now that $A$ has an accumulation point in $\Omega$. By Montel's theorem there exists a subsequence $(f_{n_m})_m$ that converges uniformly on compact subsets of $\Omega$ to a limit function $f$. Note that all such subsequences have the same limit by the identity principle.

Assume that the sequence $(f_n)_n$ does not converge uniformly on a compact subset $K\subseteq\Omega$. We can therefore construct another subsequence $(g_n)_n$ of $(f_n)_n$ such that

$$\|g_n-f\|_K>\varepsilon$$

for all $n\in\mathbb{N}$. But note that $(g_n)_n$ also has a convergent subsequence by Montel's theorem, which is of course a contradiction, as it cannot converge to $f$. □

# 2 Theorems about holomorphic functions

## 2.1 Riemann mapping theorem



**Definition 2.1.1.** A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if every closed path in $\Omega$ is homotopic to a constant path in $\Omega$.

**Lemma 2.1.2.** Let $\Omega \subset \mathbb{C}$ be a domain and $a \in \Omega$. Assume that $\Omega$ admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Then there exists a holomorphic injection $f\colon \Omega \to \mathbb{\Delta}$ such that $f(a) = 0$.

*Proof.* Fix a point $p \in \mathbb{C} \setminus \Omega$. By our assumption, there exists a function $v \in \mathcal{O}^*(\Omega)$ such that $v(z)^2 = z - p$. Note that $v$ is injective. Similarly, we have $v(\Omega) \cap -v(\Omega) = \emptyset$. Now choose a point $b \in -v(\Omega)$. As $v$ is not constant, it is an open map. Therefore, there exists some $r > 0$ such that $\mathbb{\Delta}(b, r) \cap v(\Omega) = \emptyset$. The Möbius transformation

$$h(w) = r \cdot \left(\frac{1}{w-b} - \frac{1}{v(a)-b}\right)$$

thus maps $v(\Omega)$ into $\mathbb{\Delta}$. The map $f$ is therefore given as $f = h \circ v$. □

**Definition 2.1.3.** An *expansion* if a map $\kappa\colon \Omega \to \mathbb{\Delta}$ where $0 \in \Omega \subset \mathbb{\Delta}$ such that $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ holds for all $z \neq 0$.

**Lemma 2.1.4.** Let $\Omega \subset \mathbb{\Delta}$ be a domain with $0 \in \Omega$. Assume that $\Omega$ admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Choose $c \in \mathbb{\Delta}$ such that $c^2 \notin \Omega$. For all $a \in \mathbb{\Delta}$, let

$$g_a = \frac{z-a}{\overline{a}z - 1}$$

and choose $v \in \mathcal{O}(\Omega)$ such that $v(z)^2 = g_{c^2}(z)$ and $v(0) = c$. Then the map $\kappa = g_c \circ v$ is an expansion and

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = \mathrm{id}_\Omega .$$

*Proof.* Note that $v$ is indeed well-defined. Also note that

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = g_{c^2} \circ (z \mapsto z^2) \circ v = g_{c^2} \circ g_{c^2} = \mathrm{id} .$$

We of course have $\kappa(0) = 0$. Denote $\psi_c = g_{c^2} \circ (z \mapsto z^2) \circ g_c$. It remains to check that $|\kappa(z)| > |z|$, which is equivalent to $|\psi_c(z)| < |z|$ for $z \neq 0$ as $\psi_c \circ \kappa = \mathrm{id}$. Note that $\psi_c\colon \mathbb{\Delta} \to \mathbb{\Delta}$ is holomorphic. As it is not a rotation (it is not injective), the conclusion follows from the Schwarz lemma. □

**Lemma 2.1.5** (Hurwitz)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f_n\colon \Omega \to \mathbb{C}$ be holomorphic functions. Suppose that the sequence $(f_n)_n$ converges uniformly on compacts of $\Omega$ to a non-constant function $f\colon \Omega \to \mathbb{C}$. Then for all points $p \in \Omega$ there exists a sequence $(p_n)_n \subseteq \Omega$ with limit $p$ such that $f_n(p_n) = f(p)$ for all $n > N$.

*Proof.* Let $w = f(p)$. There exists a disk $\mathbb{\Delta}(p, \delta)$ such that $f(z) \neq w$ for all points $z \in \overline{\mathbb{\Delta}(p, \delta)} \setminus \{p\}$. Note that we have

$$\min_{z \in \partial \mathbb{\Delta}(p,\delta)} |f(z) - w| > |f(p) - w| = 0.$$

As $(f_n)_n$ converges uniformly on $\overline{\mathbb{A}(p, \delta)}$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\min_{z \in \partial \mathbb{A}(p,\delta)} |f_n(z) - w| > |f_n(p) - w| .$$

By lemma 1.1.7, $f_n(z) - w$ has a root $p_n \in \mathbb{A}(p, \delta)$. For any convergent subsequence $(p_{n_k})_k$ with limit $q$ we have

$$f(p) = \lim_{k \to \infty} f_{n_k}(p_{n_k}) = f(q),$$

therefore $p = q$. □

**Corollary 2.1.5.1.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n \colon \Omega \to \mathbb{C}$ be holomorphic functions such that $(f_n)_n$ converges uniformly on compacts of $\Omega$ to $f \colon \Omega \to \mathbb{C}$. If all the $f_n$ are nowhere vanishing and $f \neq 0$, then $f$ is nowhere vanishing.

*Proof.* The proof is obvious and need not be mentioned. □

**Theorem 2.1.6** (Hurwitz)**.** Let $\Omega, \Omega' \subseteq \Omega$ be domains and $f_n \colon \Omega \to \Omega'$ be holomorphic functions that converge uniformly on compacts of $\Omega$ to $f \colon \Omega \to \Omega'$. Assume that $f$ is not constant.

i) If $f_n \colon \Omega \to \Omega'$ is injective, $f$ is also injective.

ii) We have $f(\Omega) \subseteq \Omega'$.

*Proof.*

i) Let $p \in \Omega$ and observe the functions $g_n(z) = f_n(z) - f_n(p)$. This is a sequence of nowhere vanishing functions. As $f$ is not constant, $f(z) - f(p)$ is nowhere vanishing as well. It follows that $f$ is injective.

ii) Suppose otherwise and apply the Hurwitz lemma for a point $p$ with $f(p) \notin \Omega'$. □

**Theorem 2.1.7** (Riemann mapping)**.** For a proper domain $\Omega \subset \mathbb{C}$ the following are equivalent:

i) $\Omega$ is simply connected.

ii) $\Omega$ admits a logarithm for any $f \in \mathcal{O}^*(\Omega)$.

iii) $\Omega$ admits a square root for any $f \in \mathcal{O}^*(\Omega)$.

iv) $\Omega$ is biholomorphic to $\mathbb{A}$.

*Proof.* Note that if $\Omega$ is biholomorphic to $\mathbb{A}$, it is of course simply connected. Suppose that $\Omega$ is simply connected. Then

$$F(z) = a + \int_{z_0}^{z} \frac{f'(z)}{f(z)} \, dz$$

defines a logarithm for any $f \in \mathcal{O}^*(\Omega)$. Given a logarithm of a function, we can of course construct a square root with $\sqrt{f} = e^{\frac{1}{2} \ln f}$. It remains to check that all domains admitting square roots are biholomorphic to $\mathbb{A}$.

By lemma 2.1.2 we can assume that $\Omega \subseteq \mathbb{A}$ and $0 \in \Omega$. Now define the family of functions

$$\mathcal{F} = \{f \colon \Omega \to \mathbb{A} \mid f \in \mathcal{O}(\Omega) \wedge f(0) = 0 \wedge f \text{ is injective}\}.$$

If $\mathcal{F}$ has no biholomorphic map, it is infinite. Note that $\mathcal{F}$ is bounded, so it is normal by Montel.

Choose a point $p \in \Omega$ with $p \neq 0$. We claim that if $h \in \mathcal{F}$ and

$$|h(p)| = \sup_{f \in \mathcal{F}} |f(p)|,$$

we have $h(\Omega) = \mathbb{A}$. Indeed, if that were not the case, we'd reach a contradiction with the expansion $\kappa$ of $\Omega$ as

$$|\kappa(h(p))| > |h(p)|$$

and $\kappa \circ h \in \mathcal{F}$.

Let

$$M = \sup_{f \in \mathcal{F}} |f(p)|$$

and find a sequence $(f_n)_n \subseteq \mathcal{F}$ with

$$\lim_{n \to \infty} |f_n(p)| = M.$$

As $\mathcal{F}$ is a normal family, there exists a convergent subsequence. The limit is not constant as $f(p) \neq 0$. By Hurwitz, $f$ is injective and $f(\Omega) \subseteq \mathbb{A}$. By the above claim, we have $f(\Omega) = \mathbb{A}$. $\square$

## 2.2 Bloch's theorem

**Lemma 2.2.1.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $f : \overline{\Omega} \to \mathbb{C}$ a continuous map such that $f|_\Omega$ is an open map. Let $a \in \Omega$ be a point such that

$$s = \min_{z \in \partial\Omega} |f(z) - f(a)| > 0.$$

Then $f(\Omega)$ contains the disk $\mathbb{\Delta}(f(a), s)$.

*Proof.* By compactness, there exists a $w_0 \in \partial f(\Omega)$ such that $d(\partial f(\Omega), f(a)) = |w_0 - f(a)|$. Let $(z_k)_k \subseteq \Omega$ be a sequence, convergent to $z_0$, such that

$$\lim_{k\to\infty} f(z_k) = w_0.$$

Of course $f(z_0) = w_0$. Note that, as $f|_\Omega$ is open, we have $z_0 \in \partial\Omega$. But then

$$d(\partial f(\Omega), f(a)) = |f(z_0) - f(a)| \geq s.$$ □

**Lemma 2.2.2.** Let $f$ be a non-constant function, holomorphic in a neighbourhood of $\overline{\mathbb{\Delta}(a, r)}$. Assume that

$$\sup_{z \in \overline{\mathbb{\Delta}(a,r)}} |f'(z)| \leq 2\,|f'(a)|\,.$$

Then $\mathbb{\Delta}(f(a), R) \subseteq f(\mathbb{\Delta}(a, r))$, where

$$R = \left(3 - 2\sqrt{2}\right) \cdot r \cdot |f'(a)|\,.$$

*Proof.* Without loss of generality assume that $a = f(a) = 0$. Define

$$A(z) = f(z) - f'(0)z = \int_0^1 (f'(tz) - f'(0))\, z\, dt.$$

Note that

$$f'(v) - f'(0) = \frac{1}{2\pi i} \oint_{\partial \mathbb{\Delta}(a,r)} f'(\xi) \cdot \left(\frac{1}{\xi - v} - \frac{1}{\zeta}\right) d\xi,$$

therefore

$$|f'(v) - f'(0)| \leq \frac{1}{2\pi} \cdot |v| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r \cdot (r - |v|)} \cdot 2\pi r = |v| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r - |v|}.$$

It follows that

$$\begin{aligned}
|A(z)| &\leq \int_0^1 |z| \cdot |f'(tz) - f'(0)|\; dt \\
&\leq |z| \cdot \int_0^1 |tz| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r - |tz|}\, dt \\
&\leq |z|^2 \cdot \|f'\|_{\mathbb{\Delta}(a,r)} \cdot \int_0^1 t \cdot \frac{1}{r - |z|} \\
&= |z|^2 \cdot \frac{|f'(0)|}{r - |z|}.
\end{aligned}$$

Now, using the triangle inequality, we get

$$|f(z)| \geq |z| \cdot |f'(0)| - |A(z)|\,.$$

Let $|z| = \rho \in (0, r)$. We get

$$|f(z)| \geq \rho \cdot |f'(0)| - |A(z)| \geq \rho \cdot |f'(0)| - \frac{\rho^2}{r-\rho} \cdot |f'(0)| \geq |f'(0)| \cdot \left(\rho - \frac{\rho^2}{r-\rho}\right).$$

Note that there exists a $\rho_0$ such that

$$\rho_0 - \frac{\rho_0^2}{r-\rho_0} = r \cdot \left(3 - 2\sqrt{2}\right).$$

Therefore, we get

$$|f(z)| \geq |f'(0)| \cdot r \cdot \left(3 - 2\sqrt{2}\right).$$

Now just apply the previous lemma to the disk $\mathbb{\Delta}(0, \rho_0)$. $\square$

**Theorem 2.2.3** (Bloch)**.** Let $f$ be a function, holomorphic in a neighbourhood of $\overline{\mathbb{\Delta}}$, with $f'(0) = 1$. Then $f(\mathbb{\Delta})$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.

*Proof.* Define $h(z) = |f'(z)| \, (1 - |z|) \geq 0$. Not that $h \not\equiv 0$ as $f$ is not constant. Therefore $h$ attains a maximum in a point $p \in \overline{\mathbb{\Delta}}$. In particular, as $h|_{\partial \mathbb{\Delta}} = 0$, we have $p \in \mathbb{\Delta}$. Observe $\Omega = \mathbb{\Delta}\,(p, t)$ for $t = \frac{1}{2} \cdot (1 - |p|)$. For all $z \in \Omega$, we have $1 - |z| \geq t$ and

$$|f'(z)| \cdot (1 - |z|) \leq |f'(p)| \cdot (1 - |p|) = |f'(p)| \cdot 2t \leq |f'(p)| \cdot 2 \cdot (1 - |z|).$$

Now, applying lemma 2.2.2, we have $\mathbb{\Delta}(f(p), R) \subseteq f(\mathbb{\Delta})$ with

$$R = \left(3 - 2\sqrt{2}\right) \cdot \frac{1}{2} \cdot (1 - |p|) \cdot |f'(p)| \geq \frac{3}{2} - \sqrt{2}$$

by choice of $p$. $\square$

**Remark 2.2.3.1.** Let

$$\mathcal{F} = \left\{ f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{\Delta}} \;\middle|\; f'(0) = 1 \right\}.$$

For $f \in \mathcal{F}$, denote by $L_f$ the supremum of radii of disks contained in $f(\mathbb{\Delta})$, and by $B_f$ the supremum of radii of disks contained in $f(\mathbb{\Delta})$ that is a biholomorphic image of some subdomain of $\mathbb{\Delta}$. We then define the *Landau's constant*

$$L = \inf_{f \in \mathcal{F}} L_f$$

and the *Bloch's constant*

$$B = \inf_{f \in \mathcal{F}} B_f.$$

The current known bounds for the constants are

$$0.5 < L < 0.544 \quad \text{and} \quad \frac{\sqrt{3}}{4} + 10^{-14} < B \leq \sqrt{\frac{\sqrt{3}-1}{2}} \cdot \frac{\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

**Corollary 2.2.3.2.** Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{O}(\Omega)$ a function and $p \in \Omega$. Let $r = d(p, \partial\Omega)$. Then $f(\Omega)$ contains a disk of radius

$$\left(\frac{3}{2} - \sqrt{2}\right) \cdot r \cdot |f'(p)|.$$

*Proof.* The proof is obvious and need not be mentioned. □

**Remark 2.2.3.3.** Liouville's theorem follows from this corollary.

**Lemma 2.2.4.** Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $1, -1 \notin f(\Omega)$. Then there exists a function $F \in \mathcal{O}(\Omega)$ such that $f = \cos(F)$.

*Proof.* Note that, as $\Omega$ is simply connected, we can define

$$F(z) = \frac{1}{i} \cdot \ln\left(f(z) + \sqrt{f(z)^2 - 1}\right).$$

□

**Theorem 2.2.5.** Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let $f \in \mathcal{O}(\Omega)$. Suppose that $0, 1 \notin f(\Omega)$. Then the following statements are true:

i) There exists a function $g \in \mathcal{O}(\Omega)$ such that

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right).$$

ii) If any $g \in \mathcal{O}(\Omega)$ satisfies the above equality, then $g(\Omega)$ contains no disk of radius 1.

*Proof.*

i) Apply the previous lemma twice.

ii) Define

$$A = \left\{ m \pm \frac{i}{\pi} \ln\left(n + \sqrt{n^2 - 1}\right) \,\middle|\, m \in \mathbb{Z} \land n \in \mathbb{N} \right\}.$$

We claim that $g(\Omega) \cap A = \emptyset$. Indeed, for $a \in A$ we have

$$f(a) = \frac{1}{2}\left(1 + \cos(\pm\pi \cdot n)\right) \in \{0, 1\}.$$

Now note that

$$\begin{aligned} \ln\left(n + 1 + \sqrt{n^2 + 2n}\right) - \ln\left(n + \sqrt{n^2 - 1}\right) &= \ln\left(\frac{n + 1 + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 - 1}}\right) \\ &\le \ln\left(\frac{2n + 2}{n}\right) \\ &\le \ln(4) \\ &< \pi. \end{aligned}$$

It's straightforward to check that every disk of radius 1 intersects $A$. □

**Theorem 2.2.6** (Picard's little theorem)**.** Every non-constant entire function omits at most one complex value.

*Proof.* Without loss of generality assume that $f$ omits 0 and 1. Applying the above theorem, we can write

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right).$$

Recall that $g(\mathbb{C})$ contains no disk of radius 1. If $g$ is not constant, $g(\mathbb{C})$ contains arbitrarily large disks by corollary 2.2.3.2, which is a contradiction. □

**Corollary 2.2.6.1.** Suppose that $f \in \mathcal{M}(\mathbb{C})$ is a non-constant function. Then $f$ omits at most 2 values.

*Proof.* Suppose that $f$ omits distinct values $a$, $b$ and $c$. Then

$$g(z) = \frac{1}{f(z) - a}$$

is an entire function that omits values $\frac{1}{b-a}$ and $\frac{1}{c-a}$, therefore it is constant. $\square$

**Theorem 2.2.7.** Let $f \in \mathcal{O}(\mathbb{C})$ be an entire function. Then either $f \circ f$ has a fixed point of $f(z) = z + c$.

*Proof.* If $f \circ f$ has no fixed point, the same holds for $f$. We can therefore define an entire holomorphic function $g$ with

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}.$$

Note that $g$ omits both 0 and 1, therefore it is constant. But then

$$f(f(z)) - z = \lambda(f(z) - z)$$

for some $\lambda \notin \{0, 1\}$ by Picard's little theorem. Taking the derivative, we get

$$f'(f(z)) \cdot f'(z) - 1 = \lambda(f'(z) - 1),$$

or equivalently

$$f'(z) \cdot (f'(f(z)) - \lambda) = 1 - \lambda \neq 0.$$

Note that $f' \circ f$ omits both $\lambda$ and 0, therefore it is constant. But then $f'$ is constant as well. The only option is $f'(z) = 1$. $\square$

**Lemma 2.2.8.** For all $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos(\pi v) = w$ and $|v| \leq 1 + |w|$.

*Proof.* Let $v = \alpha + i\beta$ and note that

$$|w|^2 = \cos(\pi\alpha)^2 + \sinh(\pi\beta)^2 \geq \pi^2\beta^2.$$

Observe that we can choose some $\alpha$ such that $|\alpha| \leq 1$, therefore

$$1 + |w| \geq 1 + \pi \cdot |\beta| \geq |\alpha| + |\beta| \geq |v|. \quad \square$$

**Theorem 2.2.9.** Let $f$ be a function, holomorphic on a neighbourhood of $\overline{\mathbb{\Delta}}$, such that $0, 1 \notin f(\Omega)$. There exists a function $g$, holomorphic on a neighbouhood of $\overline{\mathbb{\Delta}}$, such that

i) the equality

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right)$$

holds with $|g(0)| \leq 3 + 2\,|f(0)|$, and

ii) the inequality

$$|g(z)| \leq |g(0)| + \frac{\theta}{\gamma(1 - \theta)}$$

holds for all $|z| \leq \theta$.

*Proof.* Again, apply lemma 2.2.4 and let

$$2f - 1 = \cos(\pi \cdot F).$$

Using the above lemma, we can transform $F$ such that $|F(0)| \leq 1 + |2f(0) - 1|$. Applying lemma 2.2.4 again, we define $g$ such that

$$F = \cos(\pi g).$$

Again, using the above lemma, set $|g(0)| \leq 1 + |F(0)|$. We therefore have

$$|g(0)| \leq 1 + |F(0)| \leq 2 + |2f(0) - 1| \leq 3 + 2\,|f(0)|\,.$$

Recall that $g(\mathbb{\Delta})$ does not contain a disk of radius 1. Let $z \in \overline{\mathbb{\Delta}(\theta)}$. Then, by Bloch's theorem, $g(\mathbb{\Delta})$ contains a disk of radius $R = \gamma \cdot |g'(z)| \cdot (1 - \theta)$. Therefore, we must have

$$|g'(z)| < \frac{1}{\gamma(1 - \theta)}.$$

It follows that

$$|g(z)| = \left|g(0) + \int_0^z g'(\xi)\,d\xi\right| \leq |g(0)| + \int_0^z |g'(\xi)|\,d\xi \leq |g(0)| + |z| \cdot \frac{1}{\gamma(1 - \theta)}. \qquad \square$$

**Definition 2.2.10.** For $r \geq 0$, let

$$S(r) = \left\{f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{\Delta}} \;\middle|\; 0, 1 \notin f\left(\overline{\mathbb{\Delta}}\right) \land |f(0)| \leq r\right\}.$$

For $\theta \in (0, 1)$ and $r > 0$, let

$$L(\theta, r) = \exp\left(\pi \cdot \exp\left(3 + 2r + \frac{\theta}{\gamma(1 - \theta)}\right)\right),$$

where $\gamma$ is any constant such that Bloch's theorem holds, e.g. $\gamma = \frac{3}{2} - \sqrt{2}$.

**Theorem 2.2.11** (Schottky)**.** Let $f \in S(r)$. Then for all $z \in \mathbb{\Delta}$ such that $|z| < \theta$ we have

$$|f(z)| \leq L(\theta, r).$$

*Proof.* Let $g$ be a holomorphic function as in the previous theorem. Note that $|\cos(w)| \leq e^{|w|}$. We must therefore also have

$$\frac{1}{2} \cdot |1 + \cos(w)| \leq e^{|w|}.$$

Using this inequality, we get

$$|f(z)| \leq \exp\left(\pi \cdot \exp\left(\pi \cdot |g(z)|\right)\right) \leq L(\theta, r). \qquad \square$$

## 2.3 The great Picard theorem

**Lemma 2.3.1.** Let $\Omega \subseteq \mathbb{C}$ be a domain, $\omega \in \Omega$ and $r \in (0, \infty)$. Let

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

and $\mathcal{F}_{\omega,r} \subseteq \mathcal{F}$ a subfamily with $|f(\omega)| \leq r$ for all $f \in \mathcal{F}_{\omega,r}$.

i) There exists some $t > 0$ such that $\mathcal{F}_{\omega,r}|_{\mathbb{\Delta}(\omega,t)}$ is bounded.

ii) The family $\mathcal{F}_{\omega,1}$ is locally bounded in $\Omega$.

*Proof.*

i) Choose a $t > 0$ such that $\overline{\mathbb{\Delta}(\omega, 2t)} \subseteq \Omega$ and set $\varphi(z) = 2tz + \omega$. By Schottky's theorem, we have

$$|f \circ \varphi(z)| \leq L\left(\frac{1}{2}, r\right)$$

for $|z| < \frac{1}{2}$, or equivalently

$$\sup_{v \in \mathbb{\Delta}(w,t)} |f(v)| \leq L\left(\frac{1}{2}, r\right).$$

The family $\mathcal{F}_{\omega,r}$ is therefore bounded.

ii) Let

$$\mathcal{U} = \{u \in \Omega \mid \mathcal{F}_{\omega,1} \text{ is bounded in a neighbourhood of } u\}.$$

Note that $\omega \in \mathcal{U}$, therefore the set is non-empty. Also observe that $\mathcal{U}$ is open. Suppose that $\mathcal{U} \neq \Omega$ and let $v \in \partial\mathcal{U} \cap \Omega$. Then there exists a sequence $(f_n)_n \subseteq \mathcal{F}_{\omega,1}$ such that

$$\lim_{n\to\infty} |f_n(v)| = \infty.$$

Define $g_n = \frac{1}{f_n}$. These functions are holomorphic and omit both 0 and 1 by definition, therefore $g_n \in \mathcal{F}$. Applying the item i) for the sequence $(g_n)_n$ at point $v$, the sequence is bounded in a neighbourhood of $v$. By Montel's theorem, there exists a subsequence $(g_{n_k})_k$ that converges to a function $g$ uniformly on compacts of $\mathbb{\Delta}(v, s)$. By corollary 2.1.5.1, the function $g$ is constant. But then

$$\lim_{k\to\infty} |f_{n_k}(z)| = \infty$$

for all $z \in \mathbb{\Delta}(v, s)$, which is not possible as $v$ is a boundary point. It follows that $\mathcal{U} = \Omega$. $\square$

**Definition 2.3.2.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n \colon \Omega \to \mathbb{C}$ a sequence of functions. We say that $f_n$ *converges to* $\infty$ if

$$\lim_{n\to\infty} \|f_n\|_K = \infty$$

for every compact $K \subset \Omega$.

**Theorem 2.3.3** (Montel – sharp)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

Then $\mathcal{F}$ is normal in $\Omega$ where we also allow convergence to $\infty$.

*Proof.* Let $\Omega \subseteq \mathbb{C}$ be a domain and $p \in \Omega$. Consider the family $\mathcal{F}_{p,1}$. Let $(f_n)_n \subseteq \mathcal{F}$ be a sequence. If there exists a subsequence $(f_{n_k})_k \subseteq \mathcal{F}_{p,1}$, we can apply the above lemma. By the classical Montel's theorem, this subsequence has a convergent subsequence.

Suppose now that no such subsequence exists, that is $(f_n)_n$ has only finitely many terms in $\mathcal{F}_{p,1}$. But then there exists a subsequence $\left(\frac{1}{f_{n_k}}\right)_k \subseteq \mathcal{F}_{p,1}$. As before, this sequence has a convergent subsequence with limit $g$. If $g$ is nowhere-vanishing, then $\frac{1}{g}$ is the limit of a subsequence of $(f_n)_n$. Otherwise, by corollary 2.1.5.1, we have $g = 0$ and therefore $(f_n)_n$ converges to $\infty$. $\square$

**Definition 2.3.4.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $p \in \Omega$. A function $f \in \mathcal{O}(\Omega \setminus \{p\})$ has an *essential singularity* in $p$ if the limit

$$\lim_{z \to p} f(z)$$

does not exist and

$$\lim_{z \to p} |f(z)| \neq \infty.$$

**Theorem 2.3.5** (Picard's great theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set $p \in \Omega$ a point and $f \in \mathcal{O}(\Omega \setminus \{p\})$ a function. If $f$ has an essential singularity at $p$, then $f$ assumes every complex number as a value infinitely many times with at most one exception.

*Proof.* Without loss of generality assume that $p = 0$ and consider $\Omega = \mathbb{\Delta}(\varepsilon)$. Suppose that $f$ omits two values on $\mathbb{\Delta}(\varepsilon)$, without loss of generality 0 and 1.

We now claim that $f$ or $\frac{1}{f}$ is bounded in a neighbourhood of 0. Define the sequence of holomorphic functions $(f_n)_n$ with $f_n(z) = f\left(\frac{z}{n}\right)$. This sequence also omits 0 and 1, therefore either $(f_n)_n$ or $\left(\frac{1}{f_n}\right)_n$ has a convergent subsequence that converges uniformly on compacts by the sharp version of Montel's theorem. Denote the subsequence by $(g_{n_k})_k$ and set $g = f$ or $g = \frac{1}{f}$ accordingly.

Observe that there exists a constant $M$ such that

$$\|g_{n_k}\|_{\partial \mathbb{\Delta}\left(\frac{\varepsilon}{2}\right)} \leq M$$

holds for all $k \in \mathbb{N}$. This is equivalent to

$$|g(z)| \leq M$$

for $|z| = \frac{1}{n_k} \cdot \frac{\varepsilon}{2}$. By the maximum principle, we have

$$|g(z)| \leq M$$

for all $z$ such that

$$\frac{\varepsilon}{2} \cdot \frac{1}{n_k} \leq |z| \leq \frac{\varepsilon}{2}.$$

But as $(n_k)_k$ diverges, the inequality $g(z) \leq M$ holds for all $z$ such that $|z| \leq \frac{\varepsilon}{2}$, therefore $f$ or $\frac{1}{f}$ is bounded near 0.

Observe that $f$ is not bounded in a neighbourhood of 0, as otherwise 0 is a removable singularity, which is not possible. Similarly, if $\frac{1}{f}$ is bounded, then $f$ has either a removable singularity or a pole at 0, which is again a contradiction. $\square$

# 3 Infinite products

## 3.1 Definition and convergence

**Definition 3.1.1.** Let $(a_k)_k$ be a sequence of complex numbers. The sequence

$$n \mapsto \prod_{k=1}^{n} a_k$$

is called the *sequence of partial products* with factors $a_k$. We denote

$$p_{m,n} = \prod_{k=m}^{n} a_k.$$

We say that the infinite product is *convergent* if there exists an index $m \in \mathbb{N}$ such that the limit

$$\widehat{a}_m = \lim_{n\to\infty} p_{m,n}$$

exists and is non-zero. We then define

$$\prod_{k=1}^{\infty} a_k = p_{1,m-1} \cdot \widehat{a}_m.$$

as the limit of the infinite product.

**Remark 3.1.1.1.** The limit is uniquely defined.

**Remark 3.1.1.2.** An infinite product is convergent if and only if the product of all its non-zero factors has a non-zero limit and only finitely many factors are non-zero.

**Lemma 3.1.2.** Let $(a_k)_k \subseteq \mathbb{R}_{\geq 0}$ be a sequence such that

$$\sum_{k=1}^{\infty} (1 - a_k) = \infty.$$

Then

$$\lim_{n\to\infty} \prod_{k=p}^{n} a_k = 0$$

for all $p \in \mathbb{N}$. In particular, the infinite product is divergent.

*Proof.* Observe that

$$0 \leq \prod_{k=p}^{n} a_k \leq \prod_{k=p}^{n} e^{a_k - 1},$$

which converges to 0. □

**Definition 3.1.3.** Let $X \subseteq \mathbb{C}$ be a set.

i) A series

$$\sum_{k=1}^{\infty} g_k$$

of continuous functions $g_k \in \mathcal{C}(X)$ is *normally convergent* if for every compact $K \subseteq X$ the series

$$\sum_{k=1}^{\infty} \|g_k\|_K$$

converges.

ii) A product

$$\prod_{k=1}^{\infty} f_k$$

of continuous functions $f_k = 1 + g_k \in \mathcal{C}(X)$ is *normally convergent* if the series

$$\sum_{k=1}^{\infty} g_k$$

is normally convergent.

**Definition 3.1.4.** Let $X \subseteq \mathbb{C}$ be a set and $f_k \in \mathcal{C}(X)$ be continuous functions. Denote

$$p_{m,n} = \prod_{k=m}^{n} f_k.$$

We say that the infinite product

$$\prod_{k=1}^{\infty} f_k$$

converges *uniformly* on a set $L \subseteq X$ if there exists an index $m \in \mathbb{N}$ such that $f_k|_L$ has no zeroes for $k \geq m$ and

$$\lim_{n \to \infty} p_{m,n} = \widehat{f}_k$$

exists, is uniform on $L$ and has no zeroes on $L$. We define

$$\prod_{k=1}^{\infty} f_k = p_{1,m-1} \cdot \widehat{f}_m$$

on $L$.

**Theorem 3.1.5** (Reordering of infinite products)**.** Let

$$\prod_{k=1}^{\infty} f_k$$

be a normally convergent product in $X \subseteq \mathbb{C}$. Then there exists a functions $f \colon X \to \mathbb{C}$ such that for all bijections $\tau \colon \mathbb{N} \to \mathbb{N}$ the product

$$\prod_{k=1}^{\infty} f_{\tau(k)}$$

converges to $f$ uniformly on compacts of $X$. In particular, the infinite product converges uniformly on compacts.

*Proof.* Recall that, for $w \in \mathbb{\Delta}$, we can define

$$\log(1+w) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} w^k.$$

Then,

$$|\log(1+w)| \leq |w| \cdot \sum_{k=0}^{\infty} |w|^k = \frac{|w|}{1-|w|}.$$

In particular, if $|w| \leq \frac{1}{2}$, we have

$$|\log(1+w)| \leq 2\,|w|\,.$$

Let $L \subseteq X$ be a compact and write $f_k = 1 + g_k$. For all $k > N$ we have $\|g_k\|_L \leq \frac{1}{2}$, therefore we can write

$$\log f_k = \log(1+g_k) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} g_k^{\ell}.$$

But then

$$\|\log f_k\|_L \leq 2\,\|g_k\|_L\,.$$

It follows that the series

$$\sum_{k=N}^{\infty} \|\log f_k\|_L$$

converges. But then the series

$$h_N = \sum_{k=N}^{\infty} \log f_k$$

converges absolutely, and therefore all reorderings of the series converge as well to the same limit $h_N$.

Observe that

$$e^{h_N} = \prod_{k=N}^{\infty} e^{\log f_k} = \prod_{k=N}^{\infty} f_k.$$

This product therefore converges uniformly on $L$, independently of reorderings. We now define

$$f = \prod_{k=1}^{N-1} f_k \cdot e^{h_N}.$$

Note that this holds for all reorderings, as they differ from a suitable one by only finitely many transpositions. $\square$

## 3.2 Zeroes of infinite products

**Definition 3.2.1.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{O}(\Omega)$. The *zero set* of $f$ is the set

$$Z(f) = \{z \in \Omega \mid f(z) = 0\}.$$

For all $c \in \Omega$, define the *zero order* of $f$ in $c$ as follows: if

$$f(z) = (z-c)^k \cdot g(z)$$

where $g(c) \neq 0$ is a holomorphic function, then $\operatorname{ord}_c(f) = k$.

**Remark 3.2.1.1.** For non-zero $f \in \mathcal{O}(\Omega)$, the set $Z(f)$ is discrete in $\Omega$.

**Remark 3.2.1.2.** We have

$$\operatorname{ord}_c\left(\prod_{k=1}^n f_k\right) = \sum_{k=1}^n \operatorname{ord}_c(f_k).$$

**Lemma 3.2.2.** Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^\infty f_k$$

be a normally convergent product in $\Omega$, where $f_k \in \mathcal{O}(\Omega)$ are non-zero holomorphic functions. Then $f$ is a non-zero function with

$$Z(f) = \bigcup_{k=1}^\infty Z(f_k)$$

and

$$\operatorname{ord}_c(f) = \sum_{k=1}^\infty \operatorname{ord}_c(f_k).$$

*Proof.* Recall that normally convergent products converge uniformly on compacts of $\Omega$. In particular, $f$ is a holomorphic function.

Pick a point $c \in \Omega$. By definition of convergence, there exists some $m \in \mathbb{N}$ such that $\widehat{f}_m(c) \neq 0$. As $\widehat{f}_m$ is holomorphic as well, we have

$$f(c) = \left(p_{1,m-1} \cdot \widehat{f}_m\right)(c),$$

but then

$$\operatorname{ord}_c(f) = \sum_{k=1}^{m-1} \operatorname{ord}_c(f_k) = \sum_{k=1}^\infty \operatorname{ord}_c(f_k). \qquad \square$$

**Lemma 3.2.3.** Let $\Omega \subseteq \mathbb{C}$ be a domain. If

$$f = \prod_{k=1}^\infty f_k$$

is a normally convergent product in $\Omega$, where $f_k \in \mathcal{O}(\Omega)$ are holomorphic functions, then the sequence $\left(\widehat{f}_n\right)_n$ converges to 1 uniformly on compacts.

*Proof.* Choose $m \in \mathbb{N}$ such that $\widehat{f}_m \neq 0$. Then the set $Z\left(\widehat{f}_m\right)$ has no accumulation points in $\Omega$. We can therefore write

$$\widehat{f}_n = \frac{\widehat{f}_m}{p_{m,n-1}}$$

on $\Omega \setminus Z\left(\widehat{f}_m\right)$. As $p_{m,n-1}$ converges to $\widehat{f}_m$ on compacts of $\Omega$,

$$\lim_{n\to\infty} \widehat{f}_n = 1$$

uniformly on compacts of $\Omega \setminus Z\left(\widehat{f}_m\right)$. For any compact set $K \subseteq \Omega$, taking $m$ large enough, we have $Z\left(\widehat{f}_m\right) \cap K = \emptyset$. The conclusion follows. $\square$

**Definition 3.2.4.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. The meromorphic function $\frac{f'}{f}$ is called the *logarithmic derivative* of $f$.

**Remark 3.2.4.1.** For holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(\Omega)$ we have

$$\left(\prod_{k=1}^n f_k\right)' \cdot \left(\prod_{k=1}^n f_k\right)^{-1} = \sum_{k=1}^n \frac{f_k'}{f_k}.$$

**Definition 3.2.5.** Let $g_k \in \mathcal{M}(\Omega)$ be meromorphic functions. The series

$$\sum_{k=1}^\infty g_k$$

is *normally convergent* in $\Omega$ if for every compact $L \subseteq \Omega$ there exists some $m \in \mathbb{N}$ such that

$$\sum_{k=m}^\infty \|g_k\|_L$$

converges.

**Theorem 3.2.6** (Logarithmic differentiation)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^\infty f_k$$

be a normally convergent product in $\Omega$, where $f_k \in \mathcal{O}(\Omega)$ are non-zero functions. Then

$$\sum_{k=1}^\infty \frac{f_k'}{f_k}$$

is normally convergent in $\Omega$ and

$$\sum_{k=1}^\infty \frac{f_k'}{f_k} = \frac{f'}{f}.$$

*Proof.* As $\widehat{f}_n$ converges to 1 uniformly on compacts, the sequence $(f_n')_n$ converges to 0 uniformly on compacts by Cauchy estimates. Then for any compact $L$, $\frac{\widehat{f}_n'}{\widehat{f}_n}$ converges to 0 as $\widehat{f}_n$ has no zeroes in $L$ for $n$ large enough. It follows that

$$\lim_{n\to\infty} \frac{f'}{f} - \sum_{k=1}^n \frac{f_k'}{f_k} = \lim_{n\to\infty} \frac{\widehat{f}_{n+1}'}{\widehat{f}_{n+1}} = 0.$$

Write $f_k = 1 + g_k$ and fix a compact set $L \subseteq \Omega$. Choose an index $m$ such that we have $Z\left(\widehat{f}_m\right) \cap L = \emptyset$ and

$$\min_{z \in L} |f_k(z)| \geq \frac{1}{2}.$$

Choose $\varepsilon > 0$ such that

$$L_\varepsilon = \{z \in \mathbb{C} \mid d(z, L) \leq \varepsilon\} \subseteq \Omega.$$

By the Cauchy estimates, we have $\|g_k'\|_L \leq \frac{1}{\varepsilon} \|g_k\|_L$. But then

$$\sum_{k=m}^{\infty} \left\| \frac{f_k'}{f_k} \right\|_L = \sum_{k=m}^{\infty} \left\| \frac{g_k'}{f_k} \right\|_L \leq 2 \cdot \sum_{k=m}^{\infty} \|g_k'\|_L \leq \frac{2}{\varepsilon} \cdot \sum_{k=m}^{\infty} \|g_k\|,$$

which is convergent by our assumptions. □

**Lemma 3.2.7.** Let $g$ be meromorphic on $\mathbb{C}$ with poles in $\mathbb{Z}$ with principal parts $\frac{1}{z-m}$. Moreover, assume that $g$ is an odd function that satisfies

$$2g(2z) = g(z) + g\left(z + \frac{1}{2}\right).$$

Then $g(z) = \pi \cdot \cot(\pi z)$.

*Proof.* Simple calculations show that $\pi \cdot \cot(\pi z)$ is indeed a solution of the functional equation. Define $h(z) = g(z) - \pi \cdot \cot(\pi z)$. This another solution of the functional equation, and an odd function. In particular, $h(0) = 0$. Observe that the principal parts of $h$ are 0, therefore $h \in \mathcal{O}(\mathbb{C})$ is an entire function.

Suppose that $h$ is not constant. In particular, there exists some $c \in \partial\mathbb{\Delta}(2)$ such that

$$|h(z)| < |h(c)|$$

for all $z \in \mathbb{\Delta}(2)$. As $\frac{c}{2}, \frac{c+1}{2} \in \mathbb{\Delta}(2)$ , we can write

$$2|h(c)| = \left|h\left(\frac{c}{2}\right) + h\left(\frac{c+1}{2}\right)\right| \leq \left|h\left(\frac{c}{2}\right)\right| + \left|h\left(\frac{c+1}{2}\right)\right| < 2|h(c)|,$$

which is a contradiction. It follows that $h = 0$. □

**Corollary 3.2.7.1.** We have

$$\pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

*Proof.* Note that

$$\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2} = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k}\right),$$

therefore the series has poles in $\mathbb{Z}$ with principal parts $\frac{1}{z-m}$. It is also an odd function. A calculation shows that, for

$$r_n(z) = \frac{1}{z} + \sum_{k=1}^{n} \frac{2z}{z^2 - k^2},$$

we have

$$r_n(z) + r_n\left(z + \frac{1}{2}\right) = 2r_{2n}(2z) + \frac{2}{2z + 2n + 1}.$$

Taking $n \to \infty$, the conclusion follows. □

**Theorem 3.2.8.** We have

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

*Proof.* The above product is obviously normally convergent, therefore we can take its logarithmic derivative. A simple calculation shows that it is equal to $\pi \cot(\pi z)$. As logarithmic derivatives are equal only for scalar multiples, we only have to check equality in one point. □

## 3.3 The Euler gamma function

**Lemma 3.3.1.** The infinite product

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)e^{-\frac{z}{k}}$$

in normally convergent in $\mathbb{C}$.

*Proof.* Write

$$\begin{aligned}
|1-(1-\omega)e^{\omega}| &= |1-e^{\omega}+\omega e^{\omega}| \\
&= \left|-\sum_{k=1}^{\infty}\frac{\omega^k}{k!}+\sum_{k=0}^{\infty}\frac{\omega^{k+1}}{k!}\right| \\
&= \left|\omega^2\cdot\sum_{k=1}^{\infty}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right)\omega^{k-1}\right| \\
&\leq |\omega|^2\cdot\sum_{k=1}^{\infty}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right) \\
&= |\omega|^2
\end{aligned}$$

for $|\omega|\leq 1$. But then the sum

$$\sum_{k=\lceil|z|\rceil}^{\infty}\left|1-\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\right|\leq\sum_{k=\lceil|z|\rceil}^{\infty}\left|\frac{z^2}{k^2}\right|$$

converges normally. The infinite product must then converge normally in $\mathbb{C}$ as well. ☐

**Lemma 3.3.2.** Let

$$H(z)=z\cdot\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}.$$

Then $H(1)=e^{-\gamma}$, where $\gamma$ is the *Euler-Mascheroni constant*, that is

$$\gamma=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\log(n).$$

*Proof.* First note that

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)=\prod_{k=1}^{n}\frac{k+1}{k}=n+1.$$

We therefore have

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-\frac{1}{k}}=\exp\left(\log(n+1)-\sum_{k=1}^{n}\frac{1}{k}\right),$$

therefore

$$H(1)=\lim_{n\to\infty}\exp\left(\log(n+1)-\sum_{k=1}^{n}\frac{1}{k}\right)=e^{-\gamma}.$$ ☐

**Lemma 3.3.3.** Let $\Delta(z)=e^{\gamma z}H(z)$.

i) We have $\Delta(1) = 1$ and $\Delta(z) = z\Delta(z+1)$.

ii) We have $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$.

*Proof.* Note that $\Delta(1) = 1$ by the previous lemma. Rewrite the partial products as

$$z \cdot \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} = \frac{z}{n!} \cdot \prod_{k=1}^{n} (z+k) \cdot \exp\left(-z \sum_{k=1}^{n} \frac{1}{k}\right).$$

We therefore have

$$\begin{aligned}
\Delta(z) &= \lim_{n\to\infty} \frac{e^{\gamma z}}{n!} \cdot \prod_{k=0}^{n} (z+k) \cdot \exp\left(-z \sum_{k=1}^{n} \frac{1}{k}\right) \\
&= \lim_{n\to\infty} \frac{e^{\gamma z}}{n! \cdot n^z} \cdot \prod_{k=0}^{n} (z+k) \cdot \exp\left(z \log(n) - z \sum_{k=1}^{n} \frac{1}{k}\right) \\
&= \lim_{n\to\infty} \frac{1}{n! \cdot n^z} \cdot \prod_{k=0}^{n} (z+k).
\end{aligned}$$

We can now calculate

$$z \cdot \Delta(z+1) = \lim_{n\to\infty} z \cdot \frac{1}{n! \cdot n^{z+1}} \cdot \prod_{k=1}^{n+1} (z+k) = \Delta(z) \cdot \lim_{n\to\infty} \frac{z+n+1}{n} = \Delta(z).$$

It remains to check the equality $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$. We have

$$\begin{aligned}
\pi \cdot \Delta(z)\Delta(1-z) &= \pi \cdot \Delta(z) \cdot \frac{\Delta(-z)}{-z} \\
&= \pi e^{\gamma z} \cdot z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \cdot e^{-\gamma z} \cdot \frac{-z}{-z} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{\frac{z}{k}} \\
&= \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \\
&= \sin(\pi z).
\end{aligned}$$

□

**Definition 3.3.4.** The *Euler gamma function* is defined as

$$\Gamma(z) = \frac{1}{\Delta(z)}.$$

**Theorem 3.3.5.** The $\Gamma$ function satisfies the following properties:

1. The function $\Gamma$ is meromorphic with simple poles in $-\mathbb{N}_0$.
2. We have $\Gamma(1) = 1$.
3. The function $\Gamma$ satisfies $\Gamma(z+1) = z\Gamma(z)$.
4. The function $\Gamma$ satisfies
$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$
5. We have
$$\Gamma(z) = \lim_{n\to\infty} n! \cdot n^z \cdot \left(\prod_{k=0}^{n} (z+k)\right)^{-1}.$$

*Proof.* The proof is obvious and need not be mentioned. $\square$

**Theorem 3.3.6.** Let $F$ be holomorphic in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ and assume $F(z+1) = z \cdot F(z)$. Furthermore, assume that $F$ is bounded on the strip $1 \leq \operatorname{Re}(z) < 2$ and $F(1) = 1$. Then $F = \Gamma$.

## 3.4 Weierstraß factors

**Definition 3.4.1.** The *Weierstraß factors* are functions

$$E_n(z) = (1-z) \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^n}{n}\right).$$

**Lemma 3.4.2.** The Weierstraß factors satisfy the following:

i) For $n \geq 1$ we have
$$E_n'(z) = -z^n \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^n}{n}\right).$$

ii) For $n \geq 0$ we have
$$E_n(z) = 1 + \sum_{k=n+1}^{\infty} a_k z^k,$$
where
$$\sum_{k=n+1}^{\infty} |a_k| = 1.$$

iii) For $n \geq 0$ and $|z| \leq 1$ we have
$$|E_n(z) - 1| \leq |z|^{n+1}.$$

*Proof.*

i) Evident.

ii) Observing the derivative, we see that $a_1 = a_2 = \cdots = a_n = 0$, and $a_k \leq 0$ for $k > n$. But then
$$\sum_{k=n+1}^{\infty} |a_k| = -\sum_{k=n+1}^{\infty} a_k = 1 - E_n(1) = 1.$$

iii) We have
$$|E_n(z) - 1| = \left|\sum_{k=n+1}^{\infty} a_k z^k\right| \leq \sum_{k=n+1}^{\infty} |a_k| \cdot |z|^k \leq |z|^{n+1}. \qquad \square$$

**Lemma 3.4.3.** Let $(a_k)_k \subset \mathbb{C}^*$ be a sequence of complex numbers with no accumulation point and let $(p_k)_k \subseteq \mathbb{N}_0$ be non-negative integers with

$$\sum_{k=1}^{\infty} \left|\frac{r}{a_k}\right|^{p_k+1}$$

converges for every $r > 0$. Then the *Weierstraß product*

$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$$

converges normally on $\mathbb{C}$.

*Proof.* Note that $|a_k| > |z|$ for all but finitely many $k$. Now just apply the previous lemma. $\square$

**Theorem 3.4.4** (Weierstraß factorization theorem)**.** For any sequence $(a_k)_k \subset \mathbb{C}$ with no accumulation point there exists a Weierstraß product

$$z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$$

that converges normally on $\mathbb{C}$.

*Proof.* Set $p_k = k - 1$. For any $r > 0$ choose $m \in \mathbb{N}_0$ such that $|a_k| > 2r$ for all $k \geq m$. We then have

$$\sum_{k=m}^{\infty} \left|\frac{r}{a_k}\right|^{p_k+1} \leq \sum_{k=m}^{\infty} \frac{1}{2^k} \leq 2.$$

$\square$

**Theorem 3.4.5** (Weierstraß product theorem)**.** Let $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$ be a holomorphic function. Then there exists a function $g \in \mathcal{O}(\mathbb{C})$ such that

$$f = e^g \cdot z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{k-1}\left(\frac{z}{a_k}\right),$$

where $a_k$ are zeroes of $f$ on $\mathbb{C} \setminus \{0\}$, counted with multiplicities, and $q = \operatorname{ord}_0(f)$.

*Proof.* The proof is obvious and need not be mentioned. $\square$

**Lemma 3.4.6.** Let $\Omega \subset \mathbb{C}$ be an open subset, $(a_k)_k \subset \Omega$ a sequence with no accumulation point in $\Omega$ and $A = \{a_k \mid k \in \mathbb{N}\}$. Let $(b_k)_k \subset \mathbb{C} \setminus \Omega$ and $(p_k)_k \subseteq \mathbb{N}$ be sequences such that the series

$$\sum_{k=1}^{\infty} |r(a_k - b_k)|^{p_k+1}$$

converges for all $r > 0$ and denote $B = \{b_k \mid k \in \mathbb{N}\}$. Then the infinite product

$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{a_k - b_k}{z - b_k}\right)$$

converges normally on $\mathbb{C} \setminus \overline{B}$.

*Proof.* Let $L \subseteq \mathbb{C} \setminus \overline{B}$ be a compact set and let $\ell = d\left(L, \overline{B}\right) > 0$. We then have $|z - b_k| \geq \ell$ for all $z \in L$ and $k \in \mathbb{N}$.

We can now bound

$$\left\|\frac{a_k - b_k}{z - b_k}\right\|_L \leq \frac{|a_k - b_k|}{\ell}.$$

By the assumption of convergence for $r = \frac{1}{\ell}$, we must have

$$|r \cdot (a_k - b_k)| < 1$$

for all $k \geq n(L)$, but then

$$\sum_{k=n(L)}^{\infty} \left\| E_{p_k}\left(\frac{a_k - b_k}{z - b_k}\right) - 1 \right\|_L \leq \sum_{k=n(L)}^{\infty} \left\| \frac{a_k - b_k}{z - b_k} \right\|_L^{p_k+1} \leq \sum_{k=n(L)}^{\infty} |r \cdot (a_k - b_k)|^{p_k+1},$$

which converges. □

**Remark 3.4.6.1.** The Weierstraß factor $E_{p_k}\left(\frac{a_k - b_k}{z - b_k}\right)$ is zero if and only if $z = a_k$.

**Lemma 3.4.7.** Let $A \subset \mathbb{C}$ be a discrete set and define $A' = \overline{A} \setminus A$. Suppose that $A' \neq \emptyset$ and let

$$A_1 = \{z \in A \mid |z| \cdot d(z, A') \geq 1\}$$

and $A_2 = A \setminus A_1$. Now let

$$A_2(\varepsilon) = \{z \in A_2 \mid d(z, A') \geq \varepsilon\}.$$

Then $A = A_1 \cup A_2$, $A_1$ is a closed set and $A_2(\varepsilon)$ is finite for any $\varepsilon > 0$.

*Proof.* Assume $A_1$ has an accumulation point $a$ and let $(a_k)_k \subseteq A$ be a sequence, converging to $a$. But then

$$\lim_{k \to \infty} |a_k| \cdot d(a_k, A') = 0,$$

which is a contradiction.

Note that, for all $z \in A_2(\varepsilon)$, we have $|z| < \frac{1}{\varepsilon}$. If the set is infinite, it has an accumulation point, which is impossible as $d(z, A') \geq \varepsilon$. □

**Remark 3.4.7.1.** If $A \subset \mathbb{C}$ is a discrete set, then $A'$ is a closed set in $\mathbb{C}$.

**Theorem 3.4.8** (Weierstraß product theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k)_k \subset \Omega$ be a sequence without accumulation points in $\Omega$ and denote $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$. Then there exists a Weierstraß product for $(a_k)_k$ that converges normally in $\mathbb{C} \setminus A'$. This product has zeros precisely in $(a_k)_k$, counted with multiplicities.

*Proof.* Assume that $\Omega \neq \mathbb{C}$ and $A' \neq \emptyset$.[4] Write $A = A_1 \cup A_2$ as in the above lemma. Recall that $A_1$ has no accumulation points, therefore we can apply theorem 3.4.5 for $A_1$. It remains to construct a Weierstraß product for $A_2$.

Observe that $A' = A_2'$. As this is a closed space, for all $a_k \in A_2$ there exists some $b_k \in A_2'$ such that

$$|a_k - b_k| = d(a_k, A_2').$$

Observe that

$$\lim_{\substack{k \to \infty \\ a_k \in A_2}} |a_k - b_k| = 0,$$

as the sets $A_2(\varepsilon)$ are finite. Now set $p_k = k$ and apply lemma 3.4.6. □

[4] Otherwise just apply theorem 3.4.5.

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**Corollary 3.4.8.1** (Blaschke products)**.** Let $(a_k)_k \subset \mathbb{A} \setminus \{0\}$ be a sequence without accumulation points in $\mathbb{A}$. If the series

$$\sum_{k=1}^{\infty} (1 - |a_k|)$$

converges, then the product

$$\prod_{k=1}^{\infty} E_0 \left( \frac{a_k - \frac{1}{\bar{a}_k}}{z - \frac{1}{\bar{a}_k}} \right)$$

converges normally in $\mathbb{A}$ and has zeros precisely in $(a_k)_k$, counted with multiplicities.

*Proof.* Note that

$$|a_k - b_k| = \left| a_k - \frac{1}{\bar{a}_k} \right| = \left| \frac{1}{\bar{a}_k} \right| \cdot \left| |a_k|^2 - 1 \right| = \left| \frac{1}{\bar{a}_k} \right| \cdot (1 - |a_k|)(1 + |a_k|) \le \frac{2}{m} \cdot (1 - |a_k|)\,,$$

where

$$m = \min \{ |a_k| \mid k \in \mathbb{N} \}\,.$$

It follows that the series

$$\sum_{k=1}^{\infty} r \cdot |a_k - b_k|$$

converges, therefore we can apply lemma 3.4.6. □

**Theorem 3.4.9.** Let $\Omega \subseteq \Omega$ be a domain and $f \in \mathcal{O}(\Omega) \setminus \{0\}$. Then we can write

$$f = g \cdot \prod_{k=1}^{\infty} f_k,$$

where $g \in \mathcal{O}^*(\Omega)$ and $f_k$ are Weierstraß factors.

*Proof.* The proof is obvious and need not be mentioned. □

**Theorem 3.4.10.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{M}(\Omega)$. Then we can write $f = \frac{g}{h}$, where $g, h \in \mathcal{O}(\Omega)$.

*Proof.* Define $h$ as the Weierstraß product of the poles of $f$. □

**Remark 3.4.10.1.** Let $\Omega \subseteq \mathbb{C}$ be a domain. Then $\mathcal{O}(\Omega)$ is not a factorial ring,[5] but $\gcd(f, g) \in \mathcal{O}(\Omega)$ exists.

**Definition 3.4.11.** Let $\Omega$ be an open subset and $\{a_k\}_k$ be a sequence without accumulation pints and without repetition. Let

$$q_k(z) = \sum_{n=1}^{\infty} c_{k,m} (z - a)^{-m}$$

be a principal part in $a_k$ for each $k$.

[5] "Kolobar z enolično faktorizacijo."

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If there exist functions $g_k \in \mathcal{O}(\Omega)$ for each $k$ such that

$$\sum_{k=1}^{\infty} q_k - g_k$$

converges normally in $\Omega$, we call it the *Mittag-Leffler series* for the distribution of principal part $(a_k, q_k)$.

**Remark 3.4.11.1.** We adopt the following conventions: If $0 \in \{a_k\}_k$, then $a_1 = 0$.

**Theorem 3.4.12** (Mittag-Leffler for $\mathbb{C}$)**.** For every distribution of principal parts in $\mathbb{C}$ there exists a corresponding Mittag-Leffler series.

*Proof.* Let $g_k$ be the Taylor series of $q_k$ about 0 in the disk $\mathbb{\Delta}(|a_k|)$ such that the inequality $\|q_k - g_k\| < 2^{-k}$ holds for each $k \geq 2$. Note that

$$\lim_{k \to \infty} |a_k| = \infty$$

as the points don't accumulate. For each $r > 0$ we can therefore find an integer $n$ such that $r < \frac{1}{2}|a_k|$ for all $k \geq n$. Then

$$\sum_{k=n}^{\infty} \|q_k - g_k\|_{\overline{\mathbb{\Delta}(r)}} \leq 1. \qquad \square$$

**Remark 3.4.12.1.** The above series $f \in \mathcal{O}(\mathbb{C} \setminus \{a_1, a_2, \dots\})$ with principal parts $q_k$ in $a_k$ for each $k \in \mathbb{N}$. If the principal part are finite, then $f \in \mathcal{M}$.

**Lemma 3.4.13.** Let $a \in \mathbb{C}$, $q \in \mathcal{O}(\Omega \setminus \{a\})$ be a principal part and $b \in \mathbb{C} \setminus \{a\}$. Then $q$ has a Laurent series expansion about $b$ in the annulus $\{z \in \mathbb{C} \mid |z - b| > |a - b|\}$ of the form

$$q(z) = \sum_{m=1}^{\infty} c_m (z - b)^{-m}$$

that converges uniformly for $|z - b| > r > |a - b|$.

*Proof.* Choose a path $\gamma_r$ that goes around the circle centered at $b$ of radius $r$. We claim that

$$c_m = \frac{1}{2\pi} \int_{\gamma_r} \frac{q(z)}{(z - b)^{-m+1}}\, dz$$

for $m \in \mathbb{Z}$ suffice. We can estimate

$$|c_m| \geq \frac{1}{2\pi} \cdot 2\pi \frac{\|q\|_{\gamma_r}}{r^{-m}} = \frac{\|q\|_{\gamma_r}}{r^{-m}}.$$

We know that $q(z)$ is of the form

$$q(z) = \sum_{m=1}^{\infty} d_m (z - a)^{-m}$$

for some $d_m \in \mathbb{C}$ when developed into a Laurent series around $a$. It is trivial to show that

$$\lim_{|z| \to \infty} q(z) = 0.$$

Thus, $\|q\|_{\gamma_r}$ approaches zero as $r$ goes to infinity. If $m \leq 0$, then

$$\lim_{r \to \infty} \|q\|_{\gamma_r} r^m = 0.$$

Therefore, $c_m = 0$ for $m \leq 0$ and

$$q(z) = \sum_{m=1}^{\infty} c_m (z-b)^{-m}$$

is indeed a power series in $z - b$ which converges uniformly for $|z-b|^{-1} \leq r$. $\square$

**Definition 3.4.14.** The partial sums of

$$q_\ell(z) = \sum_{m=1}^{\ell} c_m (z-b)^{-m}$$

are called the *$\ell$-th Laurent terms* of $q$ about $b$.

**Lemma 3.4.15.** Let $(a_k, q_k)_k$ be a distribution of principal parts in an open set $\Omega \subseteq \mathbb{C}$, $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$.[6] Assume there exists a sequence $(b_k)_k \subseteq A'$ with

$$\lim_{k \to \infty} |a_k - b_k| = 0.$$

Let $q_{k,\ell}$ be the $\ell$-th Laurent term of $q_k$ about $b_k$. Then there exists a sequence $(\ell_k)_k \subseteq \mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty} (q_k - q_{k,\ell_k})$$

is a Mittag-Leffler series for $(a_k, q_k)_k$.

*Proof.* For a principal part $q_k$ the Laurent series converges uniformly on $|z - b_k| > r$ for any $r > |a_k - b_k|$ by the previous lemma. Thus, we can choose $\ell_k$ large enough such that

$$|q_k(z) - q_{k,\ell}(z)| < 2^{-k}$$

for all $z$ such that $|z - b_k| \geq 2\,|a_k - b_k|$.

For any compact set $L \subseteq \mathbb{C} \setminus A'$, the distance to $A'$ is strictly positive. Since

$$\lim_{k \to \infty} |a_k - b_k| = 0,$$

the point $b_k$ must lie outside $L$ for large enough $k$. Thus, there exists some $n(L) \in \mathbb{N}$ such that

$$L \subseteq \bigcap_{k \geq n(L)} \{z \in \mathbb{C} \mid |z - b_k| \geq 2\,|a_k - b_k|\}.$$

Therefore, we can use the previous estimate on $L$, to get

$$\sum_{k \geq n(L)} \|q_k - q_{k,\ell_k}\|_L \leq \sum_{k \geq n(L)} 2^{-k} \leq 2.$$ $\square$

[6] The closure is taken in $\mathbb{C}$.

**Theorem 3.4.16** (Mittag-Leffler for open subsets)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k, q_k)_k$ be a distribution of principal parts in $\Omega \subseteq$ and $A = \{a_k \mid k \in \mathbb{N}\}$. Then there exists a Mittag-Leffler series for $(a_k, q_k)$ that converges normally in $\mathbb{C} \subsetneq A' = \overline{A} \setminus A$.

*Proof.* By lemma 3.4.7, $(A_1)'$ is empty and $(A_2)' = A'$. If $A'$ is empty, then

$$\lim_{k\to\infty} |a_k| = \infty$$

and we can apply theorem 3.4.12. Similarly, we can assume $\Omega \neq \mathbb{C}$. Again by the lemma, $A_2(\epsilon)$ is finite. Hence, there exist $(b_k)_k \in A'$ such that

$$\lim_{k\to\infty} |a_k - b_k| = 0.$$

We can apply lemma 3.4.15 to obtain a Mittag-Leffler series. Now we apply theorem 3.4.12 for $\mathbb{C}$ to $A_1$. Sum up this two series to get the series from the statement. $\square$

**Theorem 3.4.17** (Mittag-Leffler osculation theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $(a_k)_k \subseteq \Omega$ be a sequence without accumulation points and without repetition. Furthermore, let

$$f_k(z) = \sum_{\ell=-\infty}^{n(k)} c_{k,\ell}(z - a_k)^\ell,$$

where $n(k) \in \mathbb{N}_0$, be normally convergent on $\mathbb{C} \setminus A$, where $A$ is the set of $a_k$. Then there exists a function $f \in \mathcal{O}(\Omega \setminus A)$ such that $\operatorname{ord}_{a_k}(f - f_k) > n(k)$ for all $k \in \mathbb{N}$.

*Proof.* By the Weierstraß product theorem, there exists a function $h \in \mathcal{O}(\Omega)$ such that $\operatorname{ord}_{a_k}(h) > n(k)$ and $h$ has no zeroes on $\mathbb{C} \setminus A$. Then $\left(a_k, \frac{f_k}{h}\right)_k$ is a distribution of principal parts. By theorem 3.4.16, there exists a $g \in \mathcal{O}(\Omega \setminus A)$ with these principal parts.

Now define $f = g \cdot h$. Then

$$f - f_k = g \cdot h - f_k = h \cdot \left(g - \frac{f_k}{h}\right),$$

which vanishes to order larger than $n(k)$ in $a_k$. $\square$

**Corollary 3.4.17.1.** For every sequence $(a_k)_k \subseteq \Omega$ without accumulation points and without repetition and every sequence $(c_k)_k \subseteq \mathbb{C}$ there exists a function $f \in \mathcal{O}(\Omega)$ such that $f(a_k) = c_k$ for each $k \in \mathbb{N}$.

# 4 Ring structure of holomorphic functions

## 4.1 Ideals of holomorphic functions

**Definition 4.1.1.** Let $\Omega \subseteq \mathbb{C}$ be an open set. A *divisor* of a meromorphic function $f \in \mathcal{M}^*(\Omega)$ is the function $(f)\colon \Omega \to \mathbb{Z}$, given by

$$(f)(z) = \begin{cases} n & f \text{ has a zero of order } n \text{ in } z \\ -n & f \text{ has a pole of order } n \text{ in } z \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 4.1.1.1.** The divisor of a product is the sum of divisors, i.e. $(f \cdot g) = (f) + (g)$.

**Definition 4.1.2.** Let $S \subseteq \mathcal{O}(\Omega)$ be a subset that contains a non-zero holomorphic function.

Define

$$d(z) = \min_{f \in S \setminus \{0\}} (f)(z) \in \mathbb{N}_0.$$

By Weierstraß product theorem there exists a function $g \in \mathcal{O}(\Omega)$ such that $(g) = d$. We define $\gcd(S) = g$.[7]

**Lemma 4.1.3** (Wedderburn)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f, g \in \mathcal{O}(\Omega)$ be functions with $\gcd(f, g) = 1$. Then there exist functions $a, b \in \mathcal{O}(\Omega)$ so that $af + bg = 1$. Moreover, we can choose $a$ to be nonwhere vanishing.

*Proof.* If $g = 0$ and $f \neq 0$ then $f$ cannot vanish by assumption on gcd, therefore $a = \frac{1}{f}$ and $b = 1$ suffice. Therefore, we can assume both $f, g$ and are nonzero. Note that $Z(f) \cap Z(g)$ is empty, since $(z - p)$ divides $\gcd(f, g)$ for any $p \in Z(f) \cap Z(g)$. The set $Z(f) \cup Z(g)$ is thus discrete. Further, for each zero $p$ of $g$, there exists a disk of radius $\varepsilon$ and a holomorphic function $f_p \in \mathcal{O}(\Delta(p, \varepsilon))$ such that

$$f = e^{f_p}.$$

By the Mittag-Leffler osculation theorem there exists a function $h \in \mathcal{O}(\Omega)$ such that $\operatorname{ord}_p(h - f_p) > \operatorname{ord}_p(g)$.

Here we stop for a short observation. Developing into the power series, we get that $e^{w^n} - 1 = w^n + O(w^{2n})$. Then,

$$\operatorname{ord}_p\left(f - e^h\right) = \operatorname{ord}_p\left(e^h \cdot \left(e^{f_p - h} - 1\right)\right) = \operatorname{ord}_p\left(\left(e^{f_p - h} - 1\right)\right) = \operatorname{ord}_p\left(f_p - h\right) > \operatorname{ord}_p(g).$$

Define $k = \frac{f - e^h}{g} \in \mathcal{O}(\Omega)$. We claim that $a = e^{-h}$ and $b = -ke^{-h}$ satisfy the conditions. Clearly, $a$ doesn't vanish, and

$$af + bg = e^{-h}f - ke^{-h}g = e^{-h}(f - kg) = e^{-h}\left(f - \frac{f - e^h}{g}g\right) = e^{-h}e^h = 1. \qquad \square$$

[7] There are of course multiple possible functions that satisfy this condition, but their quotients are invertible.

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**Corollary 4.1.3.1.** For holomorphic functions $f_j \in \mathcal{O}(\Omega)$, where $j \leq n$, we can write $f = \gcd(f_1, f_2, \ldots, f_n)$ as

$$f = \sum_{j=1}^{n} a_j f_j$$

*Proof.* We proceed by induction. The base case is just Wedderburn's lemma. Now let $\hat{f} = \gcd(f_2, f_3, \ldots f_n)$, which can be written as

$$\hat{f} = \sum_{j=2}^{n} \hat{a}_j f_j$$

by the induction hypothesis. Then $\frac{f_1}{f}, \frac{\hat{f}}{f} \in \mathcal{O}(\Omega)$ are holomorphic functions with gcd equal to 1. We can therefore apply Wedderburn' lemma to get functions $a$ and $b$ such that

$$a\frac{f_1}{f} + b\frac{\hat{f}}{f} = 1.$$

The conclusion follows $\square$

**Theorem 4.1.4.** Let $I \triangleleft \mathcal{O}(\Omega)$ be the ideal generated by holomorphic functions $f_1, f_2, \ldots f_n$ on $\Omega$. Then there exists a holomorphic function $f$ such that $I = (f)$.

*Proof.* Take $f = \gcd(f_j)$. This function is an element of $I$ by the previous corollary. Since $f \mid f_j$, this implies that $I = (f)$. $\square$

**Definition 4.1.5.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal.

i) We call $I$ *closed* if for every sequence $(f_n)_n \subseteq I$ that converges uniformly on compacts of $\Omega$ to some function $f$, we also have $f \in I$.

ii) We call $p \in \Omega$ a *zero* of $I$ if $f(p) = 0$ for every $f \in I$.

**Lemma 4.1.6.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ and ideal. Let $p \in \Omega$ be a point that is not a zero of $I$. Let $f, g \in \mathcal{O}(\Omega)$ be functions such that $f(z) \neq 0$ for all $z \neq p$. If $fg \in I$, then $g \in I$.

*Proof.* Since $p$ is not a zero of $I$, then there exists a function $h \in I$ such that $h(p) \neq 0$. Let $n = \operatorname{ord}_p(f)$. If $n = 0$, then $f$ is a unit, so $g \in I$. Otherwise, we have

$$\frac{f(z)}{z-p} g = -\frac{1}{h(p)} \cdot \left( \frac{h - h(p)}{z-p} fg - \frac{fg}{z-p} h \right) \in I$$

since $\frac{f}{z-p}$ is holomorphic.

We can iterate this process to find $\frac{f}{(z-p)^n} g \in I$. Since $\frac{f}{(z-p)^n}$ is a unit, $g$ must be an element of $I$. $\square$

**Theorem 4.1.7.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal. If $I$ has no zeroes and is closed, then $I = \mathcal{O}(\Omega)$.

*Proof.* Let $f$ be an arbitrary nonzero element of $I$. By the Weierstraß product theorem, we can write

$$f = \prod_{k=1}^{\infty} f_k,$$

where each $f_k$ has exactly one zero in $\Omega$, and the tails

$$\widehat{f}_n = \prod_{k=n}^{\infty} f_k$$

converge to 1 uniformly on compacts of $\Omega$. As $f = \widehat{f}_1 = f_1 \widehat{f}_2$, we can apply the preivous lemma to find $\widehat{f}_2 \in I$. Inductively, $\widehat{f}_n \in I$ and since the ideal $I$ is assumed to be closed, we have

$$1 = \lim_{k \to \infty} \widehat{f}_k \in I. \qquad \square$$

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