Algebraic topology 1

Luka Horjak (luka1.horjak@gmail.com)

October 9, 2023

Contents

Introduction

These are my lecture notes on the course Algebraic topology 1 in the year 2023/24. The lecturer that year was prof. dr. Petar Pavešić.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Basic homotopy theory

1.1 Definition

October 2, 2023

Definition 1.1.1. Continuous maps $f, g\colon X \to Y$ of topological spaces are *homotopic*, if there is a continuous map $H\colon X \times I \to Y$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such H is called a *homotopy*. We write $H\colon f \simeq g$.

Remark 1.1.1.1. If X is a locally compact and Hausdorff space, homotopies coincide with paths in the space $\mathcal{C}(X, Y)$.

Proposition 1.1.2. Homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.

Proof. The proof is obvious and need not be mentioned. □

Definition 1.1.3. We denote the set of equivalence classes of the homotopy relation on $\mathcal{C}(X, Y)$ by $[X, Y]$.

Remark 1.1.3.1. If X is a locally compact and Hausdorff space, $[X, Y]$ is the set of path components of $\mathcal{C}(X, Y)$.

Definition 1.1.4. With $f\colon (X, A) \to (Y, B)$ we denote maps $f\colon X \to Y$ such that $f(A) \subseteq f(B)$. Similarly, we define $\mathcal{C}((X, A), (Y, B))$ and $[(X, A), (Y, B)]$.

Definition 1.1.5. Let $A \subseteq X$ and $f, g\colon X \to Y$ be maps, such that $f|_A = g|_A$. $G\colon X \times I \to Y$ is a *homotopy relative to A* if $H\colon f \simeq g$ and $H_t|_A = f|_A$ for all $t \in I$.

October 6, 2023

Definition 1.1.6. A map $f\colon X \to Y$ is *null-homotopic* if it is homotopic to a constant.

Definition 1.1.7. Let X be a topological space. The *cone* on X is the space

$$CX = {X \times I}\big/{X \times \{1\}}\ .$$

Proposition 1.1.8. A map $f\colon X \to Y$ is null-homotopic if and only if it extends to the cone CX.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\
CX \arrow[ur, dashed, "F"'] &
\end{tikzcd}$$

Proof. In the following diagram, H exists if and only if F exists.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_0"'] & Y \\
X \times I \arrow[r, "q"'] \arrow[ur, dashed, "H"] & CX \arrow[u, dashed, "F"']
\end{tikzcd}$$

□

1.2 Homotopy equivalence

Proposition 1.2.1. Let $f, f' \colon X \to Y$ and $g, g' \colon Y \to Z$ be maps. If $f \simeq g$ and $f' \simeq g'$, we also have $g \circ f \simeq g' \circ f'$.

Proof. Let $H \colon X \times I \to Y$ and $K \colon Y \times I \to Z$ be the two homotopies. It is trivial to check that

$$L(x, t) = K(H(x, t), t)$$

is a homotopy of the compositions. □

Definition 1.2.2. The *homotopy category* $\underline{\text{HoTop}}$ is the category with topological spaces as objects and homotopies as morphisms. Operations are induced by the compositions of maps.

Definition 1.2.3. The category $\underline{\text{Top}}^2$ has pairs of spaces (X, A) with $A \subseteq X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}^2$.

Definition 1.2.4. The category $\underline{\text{Top}}_\bullet$ has pairs (X, x_0) with $x_0 \in X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category $\underline{\text{HoTop}}_\bullet$.

Definition 1.2.5. *Homotopy equivalence* is an isomorphism in the category $\underline{\text{HoTop}}$. If spaces X and Y are homotopy equivalent, we write $X \simeq Y$.

Remark 1.2.5.1. A map $f \colon X \to Y$ is a homotopy equivalence if there exists a map $g \colon Y \to X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. The map g is a *homotopy inverse* of f.

Definition 1.2.6. A space X is *contractible* if it is homotopy equivalent to a point.

Remark 1.2.6.1. Every cone is contractible.

Proposition 1.2.7. Let X be a topological space. The following statements are equivalent:

i) The space X is contractible.

ii) The map id_X is homotopy equivalent to a constant map.

iii) The space X is a retract of CX.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.2.8. Let X and Y be closed surfaces. If $X \simeq Y$, then $X \approx Y$.

Definition 1.2.9. A subspace $A \subseteq X$ is a *deformation retract* of X if it is a retract and the retraction is a homotopy inverse of the inclusion.

Definition 1.2.10. A subspace $A \subseteq X$ is a *strong deformation retract* is a retract and the retraction is a homotopy inverse of the inclusion relative to A.

1.3 Extensions of homotopies

Definition 1.3.1. A closed subspace $A \subseteq X$ has the *homotopy extension property* if for every space Y, map $f\colon X \to Y$ and homotopy $H\colon A \times I \to Y$ with $H_0 = f|_A$ there exists a homotopy $\overline{H}\colon X \times I \to Y$ such that $\overline{H}_0 = f$ and $\overline{H}\Big|_A = H$.

Proposition 1.3.2. A closed subspace $A \subseteq X$ has the homotopy extension property if an only if the space

$$L = A \times I \cup X \times \{0\}$$

is a retract of $X \times I$.

Proof. Suppose that L is a retract of X. It is easy to see that $\overline{H} = (H \cup f) \circ r$ is the required homotopy extension, where $r\colon X \times I \to L$ is a retraction.

Now suppose that A has the homotopy extension property. Let $i_0\colon X \hookrightarrow L$ and $H\colon A \times I \hookrightarrow L$ be inclusions. By the homotopy extension property, there exists a homotopy $\overline{H}\colon X \times I \to L$, which is of course a retraction. $\square$

Proposition 1.3.3. Let $A \subseteq X$ be a contractible subspace. If A has the homotopy extension property, then $q\colon X \to X/A$ is a homotopy equivalence.

Proof. Let $K\colon A \times I \to A$ be a homotopy equivalence between id_A and the constant map. Then $K \cup \mathrm{id}_X\colon A \times I \cup X \to X$ is a well defined map. Let $H\colon X \times I \to X$ be its extension.

$$\begin{tikzcd}
X \times I \arrow[r, "H"] \arrow[d, "q \times \mathrm{id}_I"'] & X \arrow[d, "q"] \\
X/A \times I \arrow[r, dashed, "\overline{H}"'] & X/A
\end{tikzcd}$$

The map $q \times \mathrm{id}_I$ is a quotient map because I is locally compact. The induced map $\overline{H}$ is therefore well defined. But now H_1 induces a map $h\colon X/A \to X$. Note that, by definition, $h \circ q = H_1 \simeq H_0 = \mathrm{id}_X$, and $q \circ h = \overline{H}_1 \simeq \overline{H}_0 = \mathrm{id}_{X/A}$. $\square$

October 9, 2023

Index