

# Complex analysis

Luka Horjak ([luka1.horjak@gmail.com](mailto:luka1.horjak@gmail.com))

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## Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

# 1 Holomorphic functions

## 1.1 Properties of holomorphic functions

**Definition 1.1.1.** Let  $\Omega \subseteq \mathbb{C}$  be an open subset. A function  $f: \Omega \rightarrow \mathbb{C}$  is *complex differentiable* in a point  $a \in \Omega$  if the limit

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists.

**Remark 1.1.1.1** (Cauchy-Riemann equations). Denoting  $u = \operatorname{Re} f$  and  $v = \operatorname{Im} f$  where  $f$  is real differentiable in  $a$ ,  $f$  is complex differentiable in  $a$  if and only if  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ .

**Definition 1.1.2.** Wirtinger derivatives are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

**Remark 1.1.2.1.** A function  $f$  is complex differentiable in  $a$  if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

**Definition 1.1.3.** Let  $\Omega \subseteq \mathbb{C}$  be an open subset. A function  $f: \Omega \rightarrow \mathbb{C}$  is *holomorphic in  $a$*  if it is complex differentiable in an open neighbourhood of  $a$ . The function  $f$  is *holomorphic* if it is holomorphic in every point of  $\Omega$ . We denote the set of holomorphic functions in  $\Omega$  as  $\mathcal{O}(\Omega)$ .

**Theorem 1.1.4** (Inhomogeneous Cauchy integral formula). Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain with  $\mathcal{C}^1$ -smooth boundary and  $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ . Then, for all  $z \in \Omega$ , we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}$$

*Proof.* As  $\Omega$  is an open set, there exists an  $\varepsilon > 0$  such that  $\overline{\Delta(z, \varepsilon)} \subseteq \Omega$ . Define a new domain  $\Omega_\varepsilon = \Omega \setminus \overline{\Delta(z, \varepsilon)}$ .

We now apply Stokes' theorem to  $\omega = \frac{f(w)}{w - z} dw$  on  $\Omega_\varepsilon$ . As  $d\omega = \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw$ , we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \oint_{\partial\Omega} \frac{f(w)}{w - z} dw - \oint_{\partial\Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw.$$

In the limit, we have

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial \Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{f(z + \varepsilon e^{it})}{\varepsilon e^{it}} \cdot \varepsilon i e^{it} dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega \setminus \{z\}} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f(w)}{w - z} dw - f(z) = -\frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}. \quad \square$$

**Theorem 1.1.5** (Power series expansion). Let  $\Omega \subseteq \mathbb{C}$  be an open subset,  $f \in \mathcal{O}(\Omega)$  and  $a \in \Omega$ . The function  $f$  can be developed into a power series about  $a$  that converges absolutely and uniformly to  $f$  in compacts inside  $\Delta(a, r)$ , where  $r$  is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \oint_{\partial \Delta(a, \rho)} \frac{f(w)}{(w - a)^{k+1}} dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z - a)^k.$$

**Remark 1.1.5.1.** The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

**Remark 1.1.5.2.** The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}.$$

**Theorem 1.1.6** (Identity). Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f \in \mathcal{O}(\Omega)$  a holomorphic function. Let  $A \subseteq \Omega$  be a subset such that  $f(z) = 0$  for all  $z \in A$ . If  $A$  has an accumulation point in  $\Omega$ , then  $f(z) = 0$  for all  $z \in \Omega$ .

*Proof.* Let  $a \in \Omega$  be an accumulation point of  $A$ . By continuity, we have  $f(a) = 0$ . We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k (z - a)^k,$$

where we assume  $c_{k_0} \neq 0$ . But now  $g(z) = \frac{f(z)}{(z - a)^{k_0}}$  is also holomorphic. Again, by continuity, we must have  $g(a) = 0$ , which is a contradiction. It follows that  $c_k = 0$  for all  $k \in \mathbb{N}_0$ . It follows that the set  $\text{Int} \{z \in \Omega \mid f(z) = 0\}$  is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to  $\Omega$ .  $\square$

**Lemma 1.1.7.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f \in \mathcal{O}(\Omega)$ . Suppose that for  $a \in \Omega$  and  $r > 0$  we have  $\overline{\Delta(a, r)} \subseteq \Omega$ . If

$$|f(a)| < \min_{\partial\Delta(a, r)} |f|,$$

then  $f$  has a zero in  $\Delta(a, r)$ .

*Proof.* Assume otherwise. From the inequality it follows that  $f$  has no zeroes on the boundary either. By continuity,  $f$  has no zero on an open set  $V$  with  $\Delta(a, r) \subseteq V$ . We can therefore define  $g \in \mathcal{O}(V)$  with  $g(z) = \frac{1}{f(z)}$ . We now have

$$g(a) = \frac{1}{2\pi i} \oint_{\partial\Delta(a, r)} \frac{g(z)}{z - a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(a + r \cdot e^{it})}{r e^{it}} \cdot r i e^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} g(a + r e^{it}) dt.$$

We can therefore get a bound on  $|g(a)|$  as

$$|g(a)| \leq \max_{\partial\Delta(a, r)} |g|,$$

but as the condition on  $f$  can be rewritten as

$$|g(a)| > \max_{\partial\Delta(a, r)} |g|,$$

we have reached a contradiction. □

**Theorem 1.1.8** (Open mapping). Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f \in \mathcal{O}(\Omega)$  a function. If  $f$  is not constant, it is an open map.

*Proof.* Let  $U \subseteq \Omega$  be an open set and  $w_0 \in f(U)$ . Choose a  $z_0 \in U$  such that  $f(z_0) = w_0$ . Choose a  $\rho > 0$  such that  $\Delta(z_0, \rho) \subseteq U$  and  $z_0$  is the only pre-image of  $w_0$  in  $\Delta(z_0, 2\rho)$ .<sup>1</sup>

Since  $\partial\Delta(z_0, \rho)$  is a compact set and

$$|f(z) - w_0| > 0$$

for all  $z \in \partial\Delta(z_0, \rho)$ , we can choose some  $\varepsilon > 0$  such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a  $w \in \Delta(w_0, \varepsilon)$ . As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$

on the boundary and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

by the above lemma,  $f(z_0) - w$  has a root on  $\Delta(z, \rho)$ . □

**Theorem 1.1.9** (Maximum principle). Let  $\Omega \subseteq \mathbb{C}$  be a domain. If the modulus  $|f|$  of a function  $f \in \mathcal{O}(\Omega)$  attains a local maximum, the function  $f$  is constant.

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<sup>1</sup> If such a disk does not exist,  $f$  is constant by the identity theorem.

*Proof.* Suppose that  $f$  is non-constant and that its modulus attains a local maximum at  $z \in \Omega$ . As  $f$  is an open map, it also attains the value  $(1 + \varepsilon) \cdot f(z)$ , which is a contradiction as the modulus then equals  $(1 + \varepsilon) \cdot |f(z)| > |f(z)|$ .  $\square$

**Theorem 1.1.10** (Maximum principle). Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain and assume that  $f \in \mathcal{O}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ . Then, the maximum of  $|f|$  is attained in the boundary  $\partial\Omega$ .

*Proof.* As  $\overline{\Omega}$  is compact,  $f$  attains a global maximum on this set. If the maximum is attained in the interior,  $f$  is constant, therefore it is also attained on the boundary.  $\square$

**Definition 1.1.11.** A function  $f: \Omega \setminus \{a\} \rightarrow \mathbb{C}$  is *locally bounded* near  $a$  if there exists an open neighbourhood  $U \subseteq \Omega$  of  $a$  such that  $f|_{U \setminus \{a\}}$  is bounded.

**Theorem 1.1.12** (Riemann removable singularity theorem). Let  $\Omega \subseteq \mathbb{C}$  be an open subset,  $a \in \Omega$  and  $f \in \mathcal{O}(\Omega \setminus \{a\})$ . If  $f$  is locally bounded near  $a$ , then there exists a unique function  $F \in \mathcal{O}(\Omega)$  such that  $F|_{\Omega \setminus \{a\}} = f$ .

*Proof.* Define the function  $F: \Omega \rightarrow \mathbb{C}$  as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{w-a} dw & z = a. \end{cases}$$

It remains to check that  $F$  is complex differentiable at  $a$ . Indeed, for  $z \in \Delta(a, \rho)$  we have

$$\begin{aligned} \lim_{z \rightarrow a} \frac{F(z) - F(a)}{z - a} &= \lim_{z \rightarrow a} \frac{1}{z - a} \oint_{\partial\Delta(a,\rho)} \left( \frac{f(w)}{w-z} - \frac{f(w)}{w-a} \right) dw \\ &= \lim_{z \rightarrow a} \frac{1}{2\pi i} \cdot \frac{1}{z - a} \cdot \oint_{\partial\Delta(a,\rho)} f(w) \cdot \frac{z - a}{(w - z)(w - a)} dw \\ &= \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{(w - a)^2} dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem.  $\square$

**Theorem 1.1.13** (Schwarz lemma). Let  $f: \Delta \rightarrow \Delta$  be a holomorphic function with  $f(0) = 0$ . Then,  $|f'(0)| \leq 1$  and the inequality  $|f(z)| \leq |z|$  holds for all  $z \in \Delta$ . If  $|f'(0)| = 1$  or  $|f(z)| = |z|$  holds for any  $z \neq 0$ , then  $f(z) = \beta z$  for some  $\beta \in \partial\Delta$ .

*Proof.* We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for  $g$  on the domain  $\Delta(\rho)$ . We get

$$\sup_{z \in \Delta(\rho)} |g(z)| \leq \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as  $\rho \rightarrow 1$ , it follows that

$$\sup_{z \in \mathbb{A}} |g(z)| \leq 1.$$

It immediately follows that  $|f'(0)| = |g(0)| \leq 1$ . Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z|.$$

Suppose we have  $|f(z_0)| = |z_0|$  for some  $z_0 \neq 0$ . As then  $|g(z_0)| = 1$ , it follows that  $g$  is constant, therefore  $f(z) = \beta z$  for some  $\beta \in \partial \mathbb{A}$ . If we have  $|f'(0)| = 0$ , the same argument works for  $z_0 = 0$ .  $\square$



## 1.2 The $\bar{\partial}$ equation

**Lemma 1.2.1.** Let  $g \in \mathcal{C}^\infty(\mathbb{C})$  be a function with compact support. Then there exists a function  $f \in \mathcal{C}^\infty(\mathbb{C})$  such that  $\frac{\partial f}{\partial \bar{z}} = g$ .

*Proof.* Let

$$f(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(w)}{w - z} dw \wedge d\bar{w}.$$

As

$$dw \wedge d\bar{w} = -2ri dr \wedge d\varphi$$

holds for polar coordinates centered at  $z$ , we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{rg(z + re^{i\varphi})}{re^{i\varphi}} dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some  $R$  such that  $g|_{\mathbb{C} \setminus \Delta(z, R)} = 0$ . We get

$$f(z) = -\frac{1}{\pi} \iint_{\Delta(z, R)} g(z + re^{i\varphi}) e^{-i\varphi} dr \wedge d\varphi,$$

which obviously converges. The function  $f$  is therefore well defined. As we are integrating a smooth function on a compact set, the function  $f$  is smooth as well.

For  $u = re^{i\varphi}$ , we have

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}}(z) &= -\frac{1}{\pi} \iint_{\Delta(z, R)} \frac{\partial}{\partial \bar{z}} g(z + re^{i\varphi}) e^{-i\varphi} dr \wedge d\varphi \\ &= \frac{1}{2\pi i} \iint_{\Delta(0, R)} \frac{\partial}{\partial \bar{z}} g(u + z) \frac{1}{u} du \wedge d\bar{u} \\ &= \frac{1}{2\pi i} \iint_{\Delta(0, R)} \frac{\partial g}{\partial \bar{u}}(u + z) \frac{1}{u} du \wedge d\bar{u} \\ &= \frac{1}{2\pi i} \iint_{\Delta(z, R)} \frac{\partial g}{\partial \bar{w}}(w) \frac{1}{w - z} dw \wedge d\bar{w}. \end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint_{\partial \Delta(z, R)} \frac{g(w)}{w - z} dw + \frac{1}{2\pi i} \iint_{\Delta(z, R)} \frac{\partial g}{\partial \bar{w}}(w) \frac{1}{w - z} dw \wedge d\bar{w}.$$

by the choice of  $R$ , we get

$$\frac{\partial f}{\partial \bar{z}}(z) = g(z). \quad \square$$

**Lemma 1.2.2.** Given bounded domain  $U \subset V \subset \mathbb{R}^n$  such that  $\partial U \cap \partial V = \emptyset$ , there exists a smooth function  $\chi: \mathbb{R}^n \rightarrow [0, 1]$  such that  $\chi|_U = 1$  and  $\text{supp } \chi \subseteq V$ .

*Proof.* There is a partition of unity on the sets  $V$  and  $\mathbb{R}^n \setminus \bar{U}$ .  $\square$

**Lemma 1.2.3.** Let  $\Omega \subseteq \mathbb{C}$  be an open subset. Let  $h_j: \Omega \rightarrow \mathbb{C}$  be holomorphic functions. If the sequence  $(h_j)_{j \in \mathbb{N}}$  converges uniformly on compact sets, the limit is also holomorphic on  $\Omega$ .

*Proof.* Apply Morera's theorem.<sup>2</sup> □

**Theorem 1.2.4** (Dolbeault lemma). Let  $g \in \mathcal{C}^\infty(\Delta(R))$  for some  $R \in (0, \infty]$ . Then there exists a function  $f \in \mathcal{C}^\infty(\Delta(R))$  such that  $\frac{\partial f}{\partial \bar{z}} = g$ .

*Proof.* Define disks  $X_j$  as follows:

- i) If  $R = \infty$ , set  $X_j = \Delta(j)$ .
- ii) If  $R < \infty$ , set  $X_j = \Delta\left(R - \frac{1}{j}\right)$  (for large enough  $j$ ).

Applying the above lemma, define functions  $\chi_j$  with  $\chi_j|_{X_j} = 1$  and  $\text{supp } \chi_j \subseteq X_{j+1}$  and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \Delta(R), \\ 0 & z \notin \Delta(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function  $f_j \in \mathcal{C}^\infty(\mathbb{C})$  with

$$\frac{\partial f_j}{\partial \bar{z}} = g_j.$$

We inductively construct a new sequence  $\tilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$  such that

$$\frac{\partial \tilde{f}_j}{\partial \bar{z}} = g$$

on  $X_j$  and

$$\|\tilde{f}_j - \tilde{f}_{j-1}\|_{X_{j-2}} \leq 2^{-j}.$$

Set  $\tilde{f}_1 = f_1$ . Observe the function  $F = f_{j+1} - \tilde{f}_j$  on  $X_j$ . By construction, we have  $\frac{\partial F}{\partial \bar{z}} = 0$  on  $X_j$ . It follows that  $F$  can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on  $X_j$ . As power series converge uniformly on compact sets, there exists some polynomial  $p \in \mathbb{C}[z]$  such that

$$\|F - p\|_{X_{j-1}} \leq 2^{-j}.$$

Now just set  $\tilde{f}_{j+1} = f_{j+1} - p$ .

Let  $z \in \Delta(R)$  be arbitrary. By construction, it is contained in some  $X_{j_0}$ , therefore,  $\tilde{f}_j$  is defined for  $j \geq j_0$ . As  $(\tilde{f}_j(z))_{j \geq j_0}$  is a Cauchy sequence, we can define

$$f(z) = \lim_{j \rightarrow \infty} \tilde{f}_j(z).$$

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<sup>2</sup> Analysis 2b, theorem 3.4.6.

But as

$$f - \tilde{f}_j = \sum_{k=j}^{\infty} (\tilde{f}_{k+1} - \tilde{f}_k)$$

is a sum of holomorphic functions that converges uniformly, the function  $f - \tilde{f}_j$  is a holomorphic function. Therefore,  $f$  is smooth and satisfies  $\frac{\partial f}{\partial \bar{z}} = g$ .  $\square$

### 1.3 Meromorphic functions

**Definition 1.3.1.** Let  $\Omega \subset \mathbb{C}$  be an open subset. We call a function  $f$  *meromorphic* of  $\Omega$  if there exists  $A \subset \Omega$  such that  $f \in \mathcal{O}(\Omega \setminus A)$ ,  $A$  has no accumulation points in  $\Omega$  and for all  $a \in A$  there exists some  $k \in \mathbb{N}$  such that

$$\lim_{z \rightarrow a} f(z) \cdot (z - a)^k \neq 0$$

exists. We call  $A$  the set of *poles* of the function  $f$ . We denote the set of meromorphic functions on  $\Omega$  with  $\mathcal{M}(\Omega)$ .

**Theorem 1.3.2.** Let  $0 \leq r < R \leq \infty$ . Suppose that  $f \in \mathcal{O}(D_{R,r}(a))$  is a holomorphic function, where

$$D_{R,r}(a) = \{z \in \mathbb{C} \mid r < |z - a| < R\}.$$

Then there exists a uniquely determined *Laurent series*

$$\sum_{k \in \mathbb{Z}} c_k (z - a)^k$$

that converges to  $f$  uniformly and absolutely on compact subsets of  $D_{R,r}(a)$ . We have

$$c_k = \frac{1}{2\pi i} \oint_{\partial \Delta(a, \rho)} \frac{f(w)}{(w - a)^k} dw$$

for  $r < \rho < R$ .

**Definition 1.3.3.** Let

$$\sum_{k \in \mathbb{Z}} c_k (z - a)^k$$

be a Laurent series. The series

$$\sum_{k=-\infty}^{-1} c_k (z - a)^k$$

is called the *principle part*.

**Lemma 1.3.4.** Let  $f \in \mathcal{O}(\Omega \setminus \{a\})$  be a holomorphic function. Then  $f$  is meromorphic on  $\Omega$  if and only if  $f$  has a finite principle part in  $a$ .

*Proof.* Suppose that  $f$  is meromorphic on  $\Omega$ . If  $a$  is a removable singularity,  $f$  is holomorphic in  $a$ , therefore the principle part is trivial. Otherwise, set  $m \in \mathbb{N}$  such that

$$\lim_{z \rightarrow a} (z - a)^m f(z) \neq 0$$

exists and set  $g(z) = (z - a)^m f(z)$ . As  $g$  is bounded near  $a$ , we can extend it to  $\Omega$  by the Riemann removable singularity theorem. The power series of  $g$  corresponds to a finite Laurent series of  $f$ .

The converse is obvious. □

**Theorem 1.3.5.** If  $f \in \mathcal{M}(\mathbb{C})$  is a meromorphic function, there exist entire functions  $g$  and  $h$  such that  $f = \frac{g}{h}$ .

**Definition 1.3.6.** Let  $\Omega \subseteq \mathbb{C}$  be an open set. An *additive Cousin problem* on  $\Omega$  is an open cover  $\{U_j\}_{j \in J}$  of  $\Omega$  and functions  $f_j \in \mathcal{M}(U_j)$  such that  $f_j - f_k|_{U_j \cap U_k}$  is holomorphic for all  $j, k \in J$ . A function  $f \in \mathcal{M}(\Omega)$  is a solution to the additive Cousin problem if  $f|_{U_j} - f_j$  is holomorphic for all  $j \in J$ .

**Definition 1.3.7.** Let  $\Omega \subseteq \mathbb{C}$  be an open subset. A *generalized additive Cousin problem* is an open cover  $\{U_j\}_{j \in J}$  of  $\Omega$  and functions  $f_{j,k} \in \mathcal{O}(U_j \cap U_k)$  for each  $(j, k) \in J^2$ , such that

- i)  $f_{j,k} = -f_{k,j}$  on  $U_j \cap U_k$  for all  $(j, k) \in J^2$  and
- ii)  $f_{j,k} + f_{k,\ell} + f_{\ell,j} = 0$  on  $U_j \cap U_k \cap U_\ell$  for all  $(j, k, \ell) \in J^3$ .

A solution to the generalized additive Cousin problem is given by functions  $f_j \in \mathcal{O}(U_j)$  for each  $j \in J$  such that  $f_{j,k} = f_j - f_k$  for each  $(j, k) \in J^2$ .

**Lemma 1.3.8** (Partition of unity). Let  $\Omega \subseteq \mathbb{C}$  be an open set and  $\{U_j\}_{j \in J}$  be an open cover of  $\Omega$ . Then there exists a partition of unity subordinate to  $\{U_j\}_{j \in J}$ .

**Lemma 1.3.9.** Given a generalized additive Cousin problem on  $\Omega \subseteq \mathbb{C}$ , there exist functions  $g_j \in \mathcal{C}^\infty(U_j)$  such that  $f_{j,k} = g_j - g_k$  for all  $(j, k) \in J^2$ .

*Proof.* Let  $\{(V_a, \chi_a)\}_{a \in A}$  be a partition of unity, subordinate to  $\{U_j\}_{j \in J}$ . For all  $a \in A$  choose a  $j(a) \in J$  such that  $V_a \subseteq U_{j(a)}$ . For all  $k \in J$ , define

$$g_k = - \sum_{a \in A} \chi_a \cdot f_{j(a),k}.$$

This is of course a smooth function on  $U_k$ . Now note that

$$g_k - g_\ell = \sum_{a \in A} \chi_a \cdot (-f_{j(a),k} + f_{j(a),\ell}) = \sum_{a \in A} \chi_a \cdot f_{k,\ell} = f_{k,\ell}. \quad \square$$

**Proposition 1.3.10.** The generalized additive Cousin problem is solvable for  $\Omega = \Delta(r)$  and  $\Omega = \mathbb{C}$ .

*Proof.* Let  $f_{j,k} = g_j - g_k$  for  $g_j \in \mathcal{C}^\infty(U_j)$ . Note that

$$\frac{\partial g_j}{\partial \bar{z}} = \frac{\partial g_k}{\partial \bar{z}},$$

therefore,

$$h|_{U_j} = \frac{\partial g_j}{\partial \bar{z}}$$

induces a smooth function  $h: \Omega \rightarrow \mathbb{C}$ . By the Dolbeault lemma, there exists a function  $g \in \mathcal{C}^\infty(\Omega)$  such that  $\frac{\partial g}{\partial \bar{z}} = h$ . It is clear that  $f_j = g_j - g$  solves the generalized additive Cousin problem.  $\square$

**Proposition 1.3.11.** The additive Cousin problem is solvable for  $\Omega = \Delta(r)$  and  $\Omega = \mathbb{C}$ .

*Proof.* An additive Cousin problem induces a generalized additive Cousin problem for functions  $f_{j,k} = f_j - f_k$ . Let  $g_j$  be a solution to the generalized problem. As  $f_j - f_k = f_{j,k} = g_j - g_k$  on  $U_j \cap U_k$ , we can define a function  $f \in \mathcal{M}(\Omega)$  with  $f|_{U_j} = f_j - g_j$ . This function is of course well defined. As  $f|_{U_j} - f_j = g_j \in \mathcal{O}(U_j)$ , this function indeed solves the additive Cousin problem.  $\square$

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**Theorem 1.3.12** (Mittag-Leffler). Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence without repetition and accumulation points. Let

$$f_k(z) = \sum_{\ell=-m_k}^{-1} c_{k,\ell} (z - a_k)^\ell$$

be finite principal parts. Then there exists a meromorphic function  $f \in \mathcal{M}(\mathbb{C})$  with poles in  $(a_k)_{k \in \mathbb{N}}$  such that  $f$  has principle part  $f_k$  in  $a_k$  for each  $k \in \mathbb{N}$ .

*Proof.* For each  $a_k$  choose a disk  $U_k$  containing no other  $a_k$ . Also set  $U_0 = \mathbb{C} \setminus \{a_k \mid k \in \mathbb{N}\}$  and  $f_0 = 0$ . As  $\{U_k \mid k \in \mathbb{N}_0\}$  is an open cover of  $\mathbb{C}$ , there exists a meromorphic function  $f \in \mathcal{M}(\mathbb{C})$  that solves the corresponding additive Cousin problem. It is easy to see that the principle parts of  $f$  at  $a_k$  are precisely  $f_k$ .  $\square$

## 1.4 Sequences of holomorphic functions

**Definition 1.4.1.** A family of functions  $\mathcal{F}$  from  $\Omega$  to  $\mathbb{C}$  is *locally bounded*, if for all  $p \in \Omega$  there exist a  $\rho > 0$  and  $M > 0$  such that

$$\sup_{f \in \mathcal{F}} \sup_{z \in \Delta(p, \rho)} |f(z)| < M.$$

**Lemma 1.4.2.** Let  $\Omega \subseteq \mathbb{C}$  be an open subset and  $\mathcal{F} \subseteq \mathcal{O}(\Omega)$  a locally bounded family of functions. Then for all  $p \in \Omega$  there exists a  $\rho > 0$  such that  $\mathcal{F}$  is equi-continuous on  $\Omega \cap \Delta(p, \rho)$ .

*Proof.* Fix  $p \in \Omega$  and choose  $r > 0$  such that  $D = \overline{\Delta(p, 2r)} \subseteq \Omega$ . For any  $z, w \in D$  and  $f \in \mathcal{F}$  we have

$$f(z) - f(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - w} d\xi = \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi.$$

Note that the family  $\mathcal{F}$  is bounded on every compact. Therefore, we can write

$$\sup_{f \in \mathcal{F}} \sup_{z \in \partial D} |f(z)| < M.$$

Now, for  $z, w \in \Delta(p, r)$  we have

$$|f(z) - f(w)| = \left| \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi \right| \leq |z - w| \cdot \frac{2M}{r}. \quad \square$$

**Theorem 1.4.3** (Arzelà-Ascoli). Let  $\Omega \subseteq \mathbb{C}$  be an open subset and let  $\mathcal{F} \subseteq \mathcal{O}(\Omega)$  be an infinite family such that the following conditions hold:

- i)  $\mathcal{F}$  is point-wise bounded.
- ii)  $\mathcal{F}$  is locally equi-continuous.

Then there  $\mathcal{F}$  contains a sequence that converges uniformly on compacts of  $\Omega$ .

*Proof.* Choose a dense countable subset  $A \subseteq \Omega$  and enumerate it as a sequence  $(a_k)_{k \in \mathbb{N}}$ . Pick any sequence  $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$  with pairwise distinct terms. As  $|f_n(a_1)| < M$  for all  $n$ , we can choose a subsequence  $(f_{1,n})_{n \in \mathbb{N}}$  such that  $f_{1,n}(a_1)$  converges by Bolzano-Weierstraß.

Similarly, for every  $k \in \mathbb{N}$  there exists a subsequence  $(f_{k,n})_n$  of  $(f_{k-1,n})_n$  such that  $(f_{k,n}(a_k))_n$  converges. Now define  $F_n = f_{n,n}$ . Observe that  $(F_n)$  converges at every point in  $A$ .

Fix a  $p \in \Omega$ . By local equi-continuity, there exists a  $\rho > 0$  such that for all  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $\delta < \rho$  and  $|F_n(z) - F_n(w)| < \frac{\varepsilon}{3}$  for all  $z, w \in \Delta(p, \rho)$  such that  $|z - w| < \delta$ . Choose an element  $a \in A \cap \Delta(p, \delta)$ .<sup>3</sup> Then, we have

$$|F_n(z) - F_m(z)| \leq |F_n(z) - F_n(a)| + |F_n(a) - F_m(a)| + |F_m(a) - F_m(z)| < 3 \cdot \frac{\varepsilon}{3}.$$

It follows that  $(F_n)$  is locally uniformly convergent, therefore it converges uniformly on compact sets.  $\square$

<sup>3</sup> By compactness of  $\overline{\Delta(p, \rho)}$  we can choose  $a$  from a finite set.

**Theorem 1.4.4** (Montel). Let  $\Omega \subseteq \mathbb{C}$  be an open subset and  $f_n: \Omega \rightarrow \mathbb{C}$  be a locally bounded sequence of holomorphic functions. Then  $(f_n)_n$  contains a subsequence that converges uniformly on compacts.

*Proof.* As the sequence is locally bounded, it is locally equi-continuous by lemma 1.4.2. By Arzelà-Ascoli, there exists a convergent subsequence.  $\square$

**Definition 1.4.5.** Let  $\Omega \subseteq \mathbb{C}$  be an open subset. A family of functions  $\mathcal{F} \subseteq \mathcal{O}(\Omega)$  is *normal* if every sequence in  $\mathcal{F}$  contains a subsequence that converges uniformly on compacts.

**Theorem 1.4.6** (Montel). Let  $\Omega \subseteq \mathbb{C}$  be an open subset. A family  $\mathcal{F} \subseteq \mathcal{O}(\Omega)$  is normal if and only if it is locally bounded.

*Proof.* The proof is obvious and need not be mentioned.  $\square$

**Theorem 1.4.7** (Vitali). Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $(f_n)_n \subseteq \mathcal{O}(\Omega)$  a locally bounded sequence of holomorphic functions. The following statements are equivalent:

- i) The sequence  $(f_n)_n$  converges uniformly on compact subsets of  $\Omega$ .
- ii) For a point  $p \in \Omega$  the sequence  $(f_n^{(k)}(p))_n$  converges for all  $k \in \mathbb{N}_0$ .
- iii) The set

$$A = \left\{ z \in \Omega \mid \lim_{n \rightarrow \infty} f_n(z) \text{ converges} \right\}$$

has an accumulation point in  $\Omega$ .

*Proof.* Suppose that the sequence converges uniformly on compact subsets. Given a  $p \in \Omega$ , choose a  $\delta > 0$  such that  $D = \overline{\Delta}(p, \delta) \subseteq \Omega$ . Note that

$$|g^{(k)}(p)| \leq \frac{k!}{\delta^k} \cdot \|g\|_D$$

holds for all holomorphic functions  $g$ . As  $\|f - f_n\|$  converges to 0, the derivatives of  $f_n$  converge.

Suppose that the sequences of derivatives converge at a point  $p \in \Omega$  and choose a  $\delta > 0$  such that  $D = \overline{\Delta}(p, \delta) \subseteq \Omega$ . As the sequence is locally bounded, there exists a constant  $M$  such that  $\|f_n\|_D \leq M$  holds for all  $n \in \mathbb{N}$ . We can now develop power series

$$f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z-p)^k.$$

They converge uniformly on compact subsets of  $\Delta(p, \delta)$ . Note that

$$a_{k,n} = \frac{f_n^{(k)}(p)}{k!}.$$

As derivatives converge, we can define the limit

$$a_k = \lim_{n \rightarrow \infty} a_{k,n}.$$



Now define the formal power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - p)^k.$$

The Cauchy bounds give us the inequality

$$|a_{k,n}| = \frac{|f^{(k)}(p)|}{k!} \leq \frac{M}{\delta^k},$$

therefore

$$\limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} \leq \limsup_{k \rightarrow \infty} \frac{\sqrt[k]{M}}{\delta} = \frac{1}{\delta}.$$

We conclude that the radius of convergence is at least  $\delta$ . Consider some  $\rho \in (0, \delta)$  and  $z \in \mathbb{A}(p, \rho)$ . We have

$$\begin{aligned} |f_n(z) - f(z)| &\leq \left| \sum_{k=0}^m (a_{k,n} - a_k) \cdot (p - z)^k \right| + \left| \sum_{k=m+1}^{\infty} (a_{k,n} - a_k) \cdot (p - z)^k \right| \\ &\leq \sum_{k=0}^m |a_{k,n} - a_k| \rho^k + \sum_{k=m+1}^{\infty} 2M \cdot \frac{\rho^k}{\delta^k} \\ &= \sum_{k=0}^m |a_{k,n} - a_k| \rho^k + 2M \cdot \left(\frac{\rho}{\delta}\right)^{m+1} \cdot \frac{\delta}{\delta - \rho} \\ &= 2 \cdot \frac{\varepsilon}{2} \end{aligned}$$

for large enough  $m$  and  $n$ . It follows that  $p$  is an accumulation point of  $A$ .

Suppose now that  $A$  has an accumulation point in  $\Omega$ . By Montel's theorem there exists a subsequence  $(f_{n_m})_m$  that converges uniformly on compact subsets of  $\Omega$  to a limit function  $f$ . Note that all such subsequences have the same limit by the identity principle.

Assume that the sequence  $(f_n)_n$  does not converge uniformly on a compact subset  $K \subseteq \Omega$ . We can therefore construct another subsequence  $(g_n)_n$  of  $(f_n)_n$  such that

$$\|g_n - f\|_K > \varepsilon$$

for all  $n \in \mathbb{N}$ . But note that  $(g_n)_n$  also has a convergent subsequence by Montel's theorem, which is of course a contradiction, as it cannot converge to  $f$ .  $\square$

## 2 Theorems about holomorphic functions

### 2.1 Riemann mapping theorem

**Definition 2.1.1.** A domain  $\Omega \subseteq \mathbb{C}$  is *simply connected* if every closed path in  $\Omega$  is homotopic to a constant path in  $\Omega$ .

**Lemma 2.1.2.** Let  $\Omega \subset \mathbb{C}$  be a domain and  $a \in \Omega$ . Assume that  $\Omega$  admits a square root for all function  $g \in \mathcal{O}^*(\Omega)$ . Then there exists a holomorphic injection  $f: \Omega \rightarrow \mathbb{A}$  such that  $f(a) = 0$ .

*Proof.* Fix a point  $p \in \mathbb{C} \setminus \Omega$ . By our assumption, there exists a function  $v \in \mathcal{O}^*(\Omega)$  such that  $v(z)^2 = z - p$ . Note that  $v$  is injective. Similarly, we have  $v(\Omega) \cap -v(\Omega) = \emptyset$ . Now choose a point  $b \in -v(\Omega)$ . As  $v$  is not constant, it is an open map. Therefore, there exists some  $r > 0$  such that  $\mathbb{A}(b, r) \cap v(\Omega) = \emptyset$ . The Möbius transformation

$$h(w) = r \cdot \left( \frac{1}{w - b} - \frac{1}{v(a) - b} \right)$$

thus maps  $v(\Omega)$  into  $\mathbb{A}$ . The map  $f$  is therefore given as  $f = h \circ v$ .  $\square$

**Definition 2.1.3.** An *expansion* is a map  $\kappa: \Omega \rightarrow \mathbb{A}$  where  $0 \in \Omega \subset \mathbb{A}$  such that  $\kappa(0) = 0$  and  $|\kappa(z)| > |z|$  holds for all  $z \neq 0$ .

**Lemma 2.1.4.** Let  $\Omega \subset \mathbb{A}$  be a domain with  $0 \in \Omega$ . Assume that  $\Omega$  admits a square root for all function  $g \in \mathcal{O}^*(\Omega)$ . Choose  $c \in \mathbb{A}$  such that  $c^2 \notin \Omega$ . For all  $a \in \mathbb{A}$ , let

$$g_a = \frac{z - a}{\bar{a}z - 1}$$

and choose  $v \in \mathcal{O}(\Omega)$  such that  $v(z)^2 = g_{c^2}(z)$  and  $v(0) = c$ . Then the map  $\kappa = g_c \circ v$  is an expansion and

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = \text{id}_\Omega.$$

*Proof.* Note that  $v$  is indeed well-defined. Also note that

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = g_{c^2} \circ (z \mapsto z^2) \circ v = g_{c^2} \circ g_{c^2} = \text{id}.$$

We of course have  $\kappa(0) = 0$ . Denote  $\psi_c = g_{c^2} \circ (z \mapsto z^2) \circ g_c$ . It remains to check that  $|\kappa(z)| > |z|$ , which is equivalent to  $|\psi_c(z)| < |z|$  for  $z \neq 0$  as  $\psi_c \circ \kappa = \text{id}$ . Note that  $\psi_c: \mathbb{A} \rightarrow \mathbb{A}$  is holomorphic. As it is not a rotation (it is not injective), the conclusion follows from the Schwarz lemma.  $\square$

**Lemma 2.1.5** (Hurwitz). Let  $\Omega \subseteq \mathbb{C}$  be a domain and let  $f_n: \Omega \rightarrow \mathbb{C}$  be holomorphic functions. Suppose that the sequence  $(f_n)_n$  converges uniformly on compacts of  $\Omega$  to a non-constant function  $f: \Omega \rightarrow \mathbb{C}$ . Then for all points  $p \in \Omega$  there exists a sequence  $(p_n)_n \subseteq \Omega$  with limit  $p$  such that  $f_n(p_n) = f(p)$  for all  $n > N$ .

*Proof.* Let  $w = f(p)$ . There exists a disk  $\mathbb{A}(p, \delta)$  such that  $f(z) \neq w$  for all points  $z \in \mathbb{A}(p, \delta) \setminus \{p\}$ . Note that we have

$$\min_{z \in \partial \mathbb{A}(p, \delta)} |f(z) - w| > |f(p) - w| = 0.$$

As  $(f_n)_n$  converges uniformly on  $\overline{\Delta(p, \delta)}$ , there exists some  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have

$$\min_{z \in \partial \Delta(p, \delta)} |f_n(z) - w| > |f_n(p) - w|.$$

By lemma 1.1.7,  $f_n(z) - w$  has a root  $p_n \in \Delta(p, \delta)$ . For any convergent subsequence  $(p_{n_k})_k$  with limit  $q$  we have

$$f(p) = \lim_{k \rightarrow \infty} f_{n_k}(p_{n_k}) = f(q),$$

therefore  $p = q$ . □

**Corollary 2.1.5.1.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f_n: \Omega \rightarrow \mathbb{C}$  be holomorphic functions such that  $(f_n)_n$  converges uniformly on compacts of  $\Omega$  to  $f: \Omega \rightarrow \mathbb{C}$ . If all the  $f_n$  are nowhere vanishing and  $f \neq 0$ , then  $f$  is nowhere vanishing.

*Proof.* The proof is obvious and need not be mentioned. □

**Theorem 2.1.6** (Hurwitz). Let  $\Omega, \Omega' \subseteq \mathbb{C}$  be domains and  $f_n: \Omega \rightarrow \Omega'$  be holomorphic functions that converge uniformly on compacts of  $\Omega$  to  $f: \Omega \rightarrow \Omega'$ . Assume that  $f$  is not constant.

- i) If  $f_n: \Omega \rightarrow \Omega'$  is injective,  $f$  is also injective.
- ii) We have  $f(\Omega) \subseteq \Omega'$ .

*Proof.*

- i) Let  $p \in \Omega$  and observe the functions  $g_n(z) = f_n(z) - f_n(p)$ . This is a sequence of nowhere vanishing functions. As  $f$  is not constant,  $f(z) - f(p)$  is nowhere vanishing as well. It follows that  $f$  is injective.
- ii) Suppose otherwise and apply the Hurwitz lemma for a point  $p$  with  $f(p) \notin \Omega'$ . □

**Theorem 2.1.7** (Riemann mapping). For a proper domain  $\Omega \subset \mathbb{C}$  the following are equivalent:

- i)  $\Omega$  is simply connected.
- ii)  $\Omega$  admits a logarithm for any  $f \in \mathcal{O}^*(\Omega)$ .
- iii)  $\Omega$  admits a square root for any  $f \in \mathcal{O}^*(\Omega)$ .
- iv)  $\Omega$  is biholomorphic to  $\Delta$ .

*Proof.* Note that if  $\Omega$  is biholomorphic to  $\Delta$ , it is of course simply connected. Suppose that  $\Omega$  is simply connected. Then

$$F(z) = a + \int_{z_0}^z \frac{f'(z)}{f(z)} dz$$

defines a logarithm for any  $f \in \mathcal{O}^*(\Omega)$ . Given a logarithm of a function, we can of course construct a square root with  $\sqrt{f} = e^{\frac{1}{2} \ln f}$ . It remains to check that all domains admitting square roots are biholomorphic to  $\Delta$ .

By lemma 2.1.2 we can assume that  $\Omega \subseteq \mathbb{A}$  and  $0 \in \Omega$ . Now define the family of functions

$$\mathcal{F} = \{f: \Omega \rightarrow \mathbb{A} \mid f \in \mathcal{O}(\Omega) \wedge f(0) = 0 \wedge f \text{ is injective}\}.$$

If  $\mathcal{F}$  has no biholomorphic map, it is infinite. Note that  $\mathcal{F}$  is bounded, so it is normal by Montel.

Choose a point  $p \in \Omega$  with  $p \neq 0$ . We claim that if  $h \in \mathcal{F}$  and

$$|h(p)| = \sup_{f \in \mathcal{F}} |f(p)|,$$

we have  $h(\Omega) = \mathbb{A}$ . Indeed, if that were not the case, we'd reach a contradiction with the expansion  $\kappa$  of  $\Omega$  as

$$|\kappa(h(p))| > |h(p)|$$

and  $\kappa \circ h \in \mathcal{F}$ .

Let

$$M = \sup_{f \in \mathcal{F}} |f(p)|$$

and find a sequence  $(f_n)_n \subseteq \mathcal{F}$  with

$$\lim_{n \rightarrow \infty} |f_n(p)| = M.$$

As  $\mathcal{F}$  is a normal family, there exists a convergent subsequence. The limit is not constant as  $f(p) \neq 0$ . By Hurwitz,  $f$  is injective and  $f(\Omega) \subseteq \mathbb{A}$ . By the above claim, we have  $f(\Omega) = \mathbb{A}$ .  $\square$

## 2.2 Bloch's theorem

**Lemma 2.2.1.** Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain and  $f: \overline{\Omega} \rightarrow \mathbb{C}$  a continuous map such that  $f|_{\Omega}$  is an open map. Let  $a \in \Omega$  be a point such that

$$s = \min_{z \in \partial\Omega} |f(z) - f(a)| > 0.$$

Then  $f(\Omega)$  contains the disk  $\Delta(f(a), s)$ .

*Proof.* By compactness, there exists a  $w_0 \in \partial f(\Omega)$  such that  $d(\partial f(\Omega), f(a)) = |w_0 - f(a)|$ . Let  $(z_k)_k \subseteq \Omega$  be a sequence, convergent to  $z_0$ , such that

$$\lim_{k \rightarrow \infty} f(z_k) = w_0.$$

Of course  $f(z_0) = w_0$ . Note that, as  $f|_{\Omega}$  is open, we have  $z_0 \in \partial\Omega$ . But then

$$d(\partial f(\Omega), f(a)) = |f(z_0) - f(a)| \geq s. \quad \square$$

**Lemma 2.2.2.** Let  $f$  be a non-constant function, holomorphic in a neighbourhood of  $\overline{\Delta(a, r)}$ . Assume that

$$\sup_{z \in \overline{\Delta(a, r)}} |f'(z)| \leq 2 |f'(a)|.$$

Then  $\Delta(f(a), R) \subseteq f(\Delta(a, r))$ , where

$$R = (3 - 2\sqrt{2}) \cdot r \cdot |f'(a)|.$$

*Proof.* Without loss of generality assume that  $a = f(a) = 0$ . Define

$$A(z) = f(z) - f'(0)z = \int_0^1 (f'(tz) - f'(0)) z \, dt.$$

Note that

$$f'(v) - f'(0) = \frac{1}{2\pi i} \oint_{\partial\Delta(a, r)} f'(\xi) \cdot \left( \frac{1}{\xi - v} - \frac{1}{\xi} \right) d\xi,$$

therefore

$$|f'(v) - f'(0)| \leq \frac{1}{2\pi} \cdot |v| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r \cdot (r - |v|)} \cdot 2\pi r = |v| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r - |v|}.$$

It follows that

$$\begin{aligned} |A(z)| &\leq \int_0^1 |z| \cdot |f'(tz) - f'(0)| \, dt \\ &\leq |z| \cdot \int_0^1 |tz| \cdot \frac{\|f'\|_{\Delta(a, r)}}{r - |tz|} \, dt \\ &\leq |z|^2 \cdot \|f'\|_{\Delta(a, r)} \cdot \int_0^1 t \cdot \frac{1}{r - |z|} \\ &= |z|^2 \cdot \frac{|f'(0)|}{r - |z|}. \end{aligned}$$

Now, using the triangle inequality, we get

$$|f(z)| \geq |z| \cdot |f'(0)| - |A(z)|.$$

Let  $|z| = \rho \in (0, r)$ . We get

$$|f(z)| \geq \rho \cdot |f'(0)| - |A(z)| \geq \rho \cdot |f'(0)| - \frac{\rho^2}{r - \rho} \cdot |f'(0)| \geq |f'(0)| \cdot \left( \rho - \frac{\rho^2}{r - \rho} \right).$$

Note that there exists a  $\rho_0$  such that

$$\rho_0 - \frac{\rho_0^2}{r - \rho_0} = r \cdot (3 - 2\sqrt{2}).$$

Therefore, we get

$$|f(z)| \geq |f'(0)| \cdot r \cdot (3 - 2\sqrt{2}).$$

Now just apply the previous lemma to the disk  $\Delta(0, \rho_0)$ . □

**Theorem 2.2.3** (Bloch). Let  $f$  be a function, holomorphic in a neighbourhood of  $\bar{\Delta}$ , with  $f'(0) = 1$ . Then  $f(\Delta)$  contains a disk of radius  $\frac{3}{2} - \sqrt{2}$ .

*Proof.* Define  $h(z) = |f'(z)| (1 - |z|) \geq 0$ . Note that  $h \not\equiv 0$  as  $f$  is not constant. Therefore  $h$  attains a maximum in a point  $p \in \bar{\Delta}$ . In particular, as  $h|_{\partial\Delta} = 0$ , we have  $p \in \Delta$ . Observe  $\Omega = \Delta(p, t)$  for  $t = \frac{1}{2} \cdot (1 - |p|)$ . For all  $z \in \Omega$ , we have  $1 - |z| \geq t$  and

$$|f'(z)| \cdot (1 - |z|) \leq |f'(p)| \cdot (1 - |p|) = |f'(p)| \cdot 2t \leq |f'(p)| \cdot 2 \cdot (1 - |z|).$$

Now, applying lemma 2.2.2, we have  $\Delta(f(p), R) \subseteq f(\Delta)$  with

$$R = (3 - 2\sqrt{2}) \cdot \frac{1}{2} \cdot (1 - |p|) \cdot |f'(p)| \geq \frac{3}{2} - \sqrt{2}$$

by choice of  $p$ . □

**Remark 2.2.3.1.** Let

$$\mathcal{F} = \left\{ f \text{ holomorphic on a neighbourhood of } \bar{\Delta} \mid f'(0) = 1 \right\}.$$

For  $f \in \mathcal{F}$ , denote by  $L_f$  the supremum of radii of disks contained in  $f(\Delta)$ , and by  $B_f$  the supremum of radii of disks contained in  $f(\Delta)$  that is a biholomorphic image of some subdomain of  $\Delta$ . We then define the *Landau's constant*

$$L = \inf_{f \in \mathcal{F}} L_f$$

and the *Bloch's constant*

$$B = \inf_{f \in \mathcal{F}} B_f.$$

The current known bounds for the constants are

$$0.5 < L < 0.544 \quad \text{and} \quad \frac{\sqrt{3}}{4} + 10^{-14} < B \leq \sqrt{\frac{\sqrt{3}-1}{2}} \cdot \frac{\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

**Corollary 2.2.3.2.** Let  $\Omega \subseteq \mathbb{C}$  be a domain,  $f \in \mathcal{O}(\Omega)$  a function and  $p \in \Omega$ . Let  $r = d(p, \partial\Omega)$ . Then  $f(\Omega)$  contains a disk of radius

$$\left( \frac{3}{2} - \sqrt{2} \right) \cdot r \cdot |f'(p)|.$$

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*Proof.* The proof is obvious and need not be mentioned.  $\square$

**Remark 2.2.3.3.** Liouville's theorem follows from this corollary.

**Lemma 2.2.4.** Let  $\Omega \subseteq \mathbb{C}$  be a simply connected domain and  $1, -1 \notin f(\Omega)$ . Then there exists a function  $F \in \mathcal{O}(\Omega)$  such that  $f = \cos(F)$ .

*Proof.* Note that, as  $\Omega$  is simply connected, we can define

$$F(z) = \frac{1}{i} \cdot \ln \left( f(z) + \sqrt{f(z)^2 - 1} \right). \quad \square$$

**Theorem 2.2.5.** Let  $\Omega \subseteq \mathbb{C}$  be a simply connected domain and let  $f \in \mathcal{O}(\Omega)$ . Suppose that  $0, 1 \notin f(\Omega)$ . Then the following statements are true:

i) There exists a function  $g \in \mathcal{O}(\Omega)$  such that

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g))).$$

ii) If any  $g \in \mathcal{O}(\Omega)$  satisfies the above equality, then  $g(\Omega)$  contains no disk of radius 1.

*Proof.*

i) Apply the previous lemma twice.

ii) Define

$$A = \left\{ m \pm \frac{i}{\pi} \ln \left( n + \sqrt{n^2 - 1} \right) \mid m \in \mathbb{Z} \wedge n \in \mathbb{N} \right\}.$$

We claim that  $g(\Omega) \cap A = \emptyset$ . Indeed, for  $a \in A$  we have

$$f(a) = \frac{1}{2} (1 + \cos(\pm \pi \cdot n)) \in \{0, 1\}.$$

Now note that

$$\begin{aligned} \ln \left( n + 1 + \sqrt{n^2 + 2n} \right) - \ln \left( n + \sqrt{n^2 - 1} \right) &= \ln \left( \frac{n + 1 + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 - 1}} \right) \\ &\leq \ln \left( \frac{2n + 2}{n} \right) \\ &\leq \ln(4) \\ &< \pi. \end{aligned}$$

It's straightforward to check that every disk of radius 1 intersects  $A$ .  $\square$

**Theorem 2.2.6** (Picard's little theorem). Every non-constant entire function omits at most one complex value.

*Proof.* Without loss of generality assume that  $f$  omits 0 and 1. Applying the above theorem, we can write

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g))).$$

Recall that  $g(\mathbb{C})$  contains no disk of radius 1. If  $g$  is not constant,  $g(\mathbb{C})$  contains arbitrarily large disks by corollary 2.2.3.2, which is a contradiction.  $\square$

**Corollary 2.2.6.1.** Suppose that  $f \in \mathcal{M}(\mathbb{C})$  is a non-constant function. Then  $f$  omits at most 2 values.

*Proof.* Suppose that  $f$  omits distinct values  $a, b$  and  $c$ . Then

$$g(z) = \frac{1}{f(z) - a}$$

is an entire function that omits values  $\frac{1}{b-a}$  and  $\frac{1}{c-a}$ , therefore it is constant.  $\square$

**Theorem 2.2.7.** Let  $f \in \mathcal{O}(\mathbb{C})$  be an entire function. Then either  $f \circ f$  has a fixed point of  $f(z) = z + c$ .

*Proof.* If  $f \circ f$  has no fixed point, the same holds for  $f$ . We can therefore define an entire holomorphic function  $g$  with

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}.$$

Note that  $g$  omits both 0 and 1, therefore it is constant. But then

$$f(f(z)) - z = \lambda(f(z) - z)$$

for some  $\lambda \notin \{0, 1\}$  by Picard's little theorem. Taking the derivative, we get

$$f'(f(z)) \cdot f'(z) - 1 = \lambda(f'(z) - 1),$$

or equivalently

$$f'(z) \cdot (f'(f(z)) - \lambda) = 1 - \lambda \neq 0.$$

Note that  $f' \circ f$  omits both  $\lambda$  and 0, therefore it is constant. But then  $f'$  is constant as well. The only option is  $f'(z) = 1$ .  $\square$

**Lemma 2.2.8.** For all  $w \in \mathbb{C}$  there exists a  $v \in \mathbb{C}$  such that  $\cos(\pi v) = w$  and  $|v| \leq 1 + |w|$ .

*Proof.* Let  $v = \alpha + i\beta$  and note that

$$|w|^2 = \cos(\pi\alpha)^2 + \sinh(\pi\beta)^2 \geq \pi^2\beta^2.$$

Observe that we can choose some  $\alpha$  such that  $|\alpha| \leq 1$ , therefore

$$1 + |w| \geq 1 + \pi \cdot |\beta| \geq |\alpha| + |\beta| \geq |v|. \quad \square$$

**Theorem 2.2.9.** Let  $f$  be a function, holomorphic on a neighbourhood of  $\overline{\Delta}$ , such that  $0, 1 \notin f(\Omega)$ . There exists a function  $g$ , holomorphic on a neighbourhood of  $\overline{\Delta}$ , such that

i) the equality

$$f = \frac{1}{2} (1 + \cos(\pi \cdot \cos(\pi \cdot g)))$$

holds with  $|g(0)| \leq 3 + 2|f(0)|$ , and

ii) the inequality

$$|g(z)| \leq |g(0)| + \frac{\theta}{\gamma(1 - \theta)}$$

holds for all  $|z| \leq \theta$ .



*Proof.* Again, apply lemma 2.2.4 and let

$$2f - 1 = \cos(\pi \cdot F).$$

Using the above lemma, we can transform  $F$  such that  $|F(0)| \leq 1 + |2f(0) - 1|$ . Applying lemma 2.2.4 again, we define  $g$  such that

$$F = \cos(\pi g).$$

Again, using the above lemma, set  $|g(0)| \leq 1 + |F(0)|$ . We therefore have

$$|g(0)| \leq 1 + |F(0)| \leq 2 + |2f(0) - 1| \leq 3 + 2|f(0)|.$$

Recall that  $g(\mathbb{A})$  does not contain a disk of radius 1. Let  $z \in \overline{\mathbb{A}(\theta)}$ . Then, by Bloch's theorem,  $g(\mathbb{A})$  contains a disk of radius  $R = \gamma \cdot |g'(z)| \cdot (1 - \theta)$ . Therefore, we must have

$$|g'(z)| < \frac{1}{\gamma(1 - \theta)}.$$

It follows that

$$|g(z)| = \left| g(0) + \int_0^z g'(\xi) d\xi \right| \leq |g(0)| + \int_0^z |g'(\xi)| d\xi \leq |g(0)| + |z| \cdot \frac{1}{\gamma(1 - \theta)}. \quad \square$$

**Definition 2.2.10.** For  $r \geq 0$ , let

$$S(r) = \left\{ f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{A}} \mid 0, 1 \notin f(\overline{\mathbb{A}}) \wedge |f(0)| \leq r \right\}.$$

For  $\theta \in (0, 1)$  and  $r > 0$ , let

$$L(\theta, r) = \exp \left( \pi \cdot \exp \left( 3 + 2r + \frac{\theta}{\gamma(1 - \theta)} \right) \right),$$

where  $\gamma$  is any constant such that Bloch's theorem holds, e.g.  $\gamma = \frac{3}{2} - \sqrt{2}$ .

**Theorem 2.2.11** (Schottky). Let  $f \in S(r)$ . Then for all  $z \in \mathbb{A}$  such that  $|z| < \theta$  we have

$$|f(z)| \leq L(\theta, r).$$

*Proof.* Let  $g$  be a holomorphic function as in the previous theorem. Note that  $|\cos(w)| \leq e^{|w|}$ . We must therefore also have

$$\frac{1}{2} \cdot |1 + \cos(w)| \leq e^{|w|}.$$

Using this inequality, we get

$$|f(z)| \leq \exp(\pi \cdot \exp(\pi \cdot |g(z)|)) \leq L(\theta, r). \quad \square$$

## 2.3 The great Picard theorem

**Lemma 2.3.1.** Let  $\Omega \subseteq \mathbb{C}$  be a domain,  $\omega \in \Omega$  and  $r \in (0, \infty)$ . Let

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

and  $\mathcal{F}_{\omega,r} \subseteq \mathcal{F}$  a subfamily with  $|f(\omega)| \leq r$  for all  $f \in \mathcal{F}_{\omega,r}$ .

- i) There exists some  $t > 0$  such that  $\mathcal{F}_{\omega,r}|_{\Delta(\omega,t)}$  is bounded.
- ii) The family  $\mathcal{F}_{\omega,1}$  is locally bounded in  $\Omega$ .

*Proof.*

- i) Choose a  $t > 0$  such that  $\overline{\Delta(\omega, 2t)} \subseteq \Omega$  and set  $\varphi(z) = 2tz + \omega$ . By Schottky's theorem, we have

$$|f \circ \varphi(z)| \leq L\left(\frac{1}{2}, r\right)$$

for  $|z| < \frac{1}{2}$ , or equivalently

$$\sup_{v \in \Delta(\omega, t)} |f(v)| \leq L\left(\frac{1}{2}, r\right).$$

The family  $\mathcal{F}_{\omega,r}$  is therefore bounded.

- ii) Let

$$\mathcal{U} = \{u \in \Omega \mid \mathcal{F}_{\omega,1} \text{ is bounded in a neighbourhood of } u\}.$$

Note that  $\omega \in \mathcal{U}$ , therefore the set is non-empty. Also observe that  $\mathcal{U}$  is open. Suppose that  $\mathcal{U} \neq \Omega$  and let  $v \in \partial\mathcal{U} \cap \Omega$ . Then there exists a sequence  $(f_n)_n \subseteq \mathcal{F}_{\omega,1}$  such that

$$\lim_{n \rightarrow \infty} |f_n(v)| = \infty.$$

Define  $g_n = \frac{1}{f_n}$ . These functions are holomorphic and omit both 0 and 1 by definition, therefore  $g_n \in \mathcal{F}$ . Applying the item i) for the sequence  $(g_n)_n$  at point  $v$ , the sequence is bounded in a neighbourhood of  $v$ . By Montel's theorem, there exists a subsequence  $(g_{n_k})_k$  that converges to a function  $g$  uniformly on compacts of  $\Delta(v, s)$ . By corollary 2.1.5.1, the function  $g$  is constant. But then

$$\lim_{k \rightarrow \infty} |f_{n_k}(z)| = \infty$$

for all  $z \in \Delta(v, s)$ , which is not possible as  $v$  is a boundary point. It follows that  $\mathcal{U} = \Omega$ .  $\square$

**Definition 2.3.2.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f_n: \Omega \rightarrow \mathbb{C}$  a sequence of functions. We say that  $f_n$  converges to  $\infty$  if

$$\lim_{n \rightarrow \infty} \|f_n\|_K = \infty$$

for every compact  $K \subset \Omega$ .

**Theorem 2.3.3** (Montel – sharp). Let  $\Omega \subseteq \mathbb{C}$  be a domain and

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

Then  $\mathcal{F}$  is normal in  $\Omega$  where we also allow convergence to  $\infty$ .

*Proof.* Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $p \in \Omega$ . Consider the family  $\mathcal{F}_{p,1}$ . Let  $(f_n)_n \subseteq \mathcal{F}$  be a sequence. If there exists a subsequence  $(f_{n_k})_k \subseteq \mathcal{F}_{p,1}$ , we can apply the above lemma. By the classical Montel's theorem, this subsequence has a convergent subsequence.

Suppose now that no such subsequence exists, that is  $(f_n)_n$  has only finitely many terms in  $\mathcal{F}_{p,1}$ . But then there exists a subsequence  $\left(\frac{1}{f_{n_k}}\right)_k \subseteq \mathcal{F}_{p,1}$ . As before, this sequence has a convergent subsequence with limit  $g$ . If  $g$  is nowhere-vanishing, then  $\frac{1}{g}$  is the limit of a subsequence of  $(f_n)_n$ . Otherwise, by corollary 2.1.5.1, we have  $g = 0$  and therefore  $(f_n)_n$  converges to  $\infty$ .  $\square$

**Definition 2.3.4.** Let  $\Omega \subseteq \mathbb{C}$  be an open set and  $p \in \Omega$ . A function  $f \in \mathcal{O}(\Omega \setminus \{p\})$  has an *essential singularity* in  $p$  if the limit

$$\lim_{z \rightarrow p} f(z)$$

does not exist and

$$\lim_{z \rightarrow p} |f(z)| \neq \infty.$$

**Theorem 2.3.5** (Picard's great theorem). Let  $\Omega \subseteq \mathbb{C}$  be an open set  $p \in \Omega$  a point and  $f \in \mathcal{O}(\Omega \setminus \{p\})$  a function. If  $f$  has an essential singularity at  $p$ , then  $f$  assumes every complex number as a value infinitely many times with at most one exception.

*Proof.* Without loss of generality assume that  $p = 0$  and consider  $\Omega = \Delta(\varepsilon)$ . Suppose that  $f$  omits two values on  $\Delta(\varepsilon)$ , without loss of generality 0 and 1.

We now claim that  $f$  or  $\frac{1}{f}$  is bounded in a neighbourhood of 0. Define the sequence of holomorphic functions  $(f_n)_n$  with  $f_n(z) = f\left(\frac{z}{n}\right)$ . This sequence also omits 0 and 1, therefore either  $(f_n)_n$  or  $\left(\frac{1}{f_n}\right)_n$  has a convergent subsequence that converges uniformly on compacts by the sharp version of Montel's theorem. Denote the subsequence by  $(g_{n_k})_k$  and set  $g = f$  or  $g = \frac{1}{f}$  accordingly.

Observe that there exists a constant  $M$  such that

$$\|g_{n_k}\|_{\partial\Delta(\frac{\varepsilon}{2})} \leq M$$

holds for all  $k \in \mathbb{N}$ . This is equivalent to

$$|g(z)| \leq M$$

for  $|z| = \frac{1}{n_k} \cdot \frac{\varepsilon}{2}$ . By the maximum principle, we have

$$|g(z)| \leq M$$

for all  $z$  such that

$$\frac{\varepsilon}{2} \cdot \frac{1}{n_k} \leq |z| \leq \frac{\varepsilon}{2}.$$

But as  $(n_k)_k$  diverges, the inequality  $|g(z)| \leq M$  holds for all  $z$  such that  $|z| \leq \frac{\varepsilon}{2}$ , therefore  $f$  or  $\frac{1}{f}$  is bounded near 0.

Observe that  $f$  is not bounded in a neighbourhood of 0, as otherwise 0 is a removable singularity, which is not possible. Similarly, if  $\frac{1}{f}$  is bounded, then  $f$  has either a removable singularity or a pole at 0, which is again a contradiction.  $\square$

### 3 Infinite products

#### 3.1 Definition and convergence

**Definition 3.1.1.** Let  $(a_k)_k$  be a sequence of complex numbers. The sequence

$$n \mapsto \prod_{k=1}^n a_k$$

is called the *sequence of partial products* with factors  $a_k$ . We denote

$$p_{m,n} = \prod_{k=m}^n a_k.$$

We say that the infinite product is *convergent* if there exists an index  $m \in \mathbb{N}$  such that the limit

$$\hat{a}_m = \lim_{n \rightarrow \infty} p_{m,n}$$

exists and is non-zero. We then define

$$\prod_{k=1}^{\infty} a_k = p_{1,m-1} \cdot \hat{a}_m.$$

as the limit of the infinite product.

**Remark 3.1.1.1.** The limit is uniquely defined.

**Remark 3.1.1.2.** An infinite product is convergent if and only if the product of all its non-zero factors has a non-zero limit and only finitely many factors are non-zero.

**Lemma 3.1.2.** Let  $(a_k)_k \subseteq \mathbb{R}_{\geq 0}$  be a sequence such that

$$\sum_{k=1}^{\infty} (1 - a_k) = \infty.$$

Then

$$\lim_{n \rightarrow \infty} \prod_{k=p}^n a_k = 0$$

for all  $p \in \mathbb{N}$ . In particular, the infinite product is divergent.

*Proof.* Observe that

$$0 \leq \prod_{k=p}^n a_k \leq \prod_{k=p}^n e^{a_k - 1},$$

which converges to 0. □

**Definition 3.1.3.** Let  $X \subseteq \mathbb{C}$  be a set.

i) A series

$$\sum_{k=1}^{\infty} g_k$$

of continuous functions  $g_k \in \mathcal{C}(X)$  is *normally convergent* if for every compact  $K \subseteq X$  the series

$$\sum_{k=1}^{\infty} \|g_k\|_K$$

converges.

ii) A product

$$\prod_{k=1}^{\infty} f_k$$

of continuous functions  $f_k = 1 + g_k \in \mathcal{C}(X)$  is *normally convergent* if the series

$$\sum_{k=1}^{\infty} g_k$$

is normally convergent.

**Definition 3.1.4.** Let  $X \subseteq \mathbb{C}$  be a set and  $f_k \in \mathcal{C}(X)$  be continuous functions. Denote

$$p_{m,n} = \prod_{k=m}^n f_k.$$

We say that the infinite product

$$\prod_{k=1}^{\infty} f_k$$

converges *uniformly* on a set  $L \subseteq X$  if there exists an index  $m \in \mathbb{N}$  such that  $f_k|_L$  has no zeroes for  $k \geq m$  and

$$\lim_{n \rightarrow \infty} p_{m,n} = \hat{f}_m$$

exists, is uniform on  $L$  and has no zeroes on  $L$ . We define

$$\prod_{k=1}^{\infty} f_k = p_{1,m-1} \cdot \hat{f}_m$$

on  $L$ .

**Theorem 3.1.5** (Reordering of infinite products). Let

$$\prod_{k=1}^{\infty} f_k$$

be a normally convergent product in  $X \subseteq \mathbb{C}$ . Then there exists a functions  $f: X \rightarrow \mathbb{C}$  such that for all bijections  $\tau: \mathbb{N} \rightarrow \mathbb{N}$  the product

$$\prod_{k=1}^{\infty} f_{\tau(k)}$$

converges to  $f$  uniformly on compacts of  $X$ . In particular, the infinite product converges uniformly on compacts.

*Proof.* Recall that, for  $w \in \mathbb{D}$ , we can define

$$\log(1 + w) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} w^k.$$

Then,

$$|\log(1 + w)| \leq |w| \cdot \sum_{k=0}^{\infty} |w|^k = \frac{|w|}{1 - |w|}.$$

In particular, if  $|w| \leq \frac{1}{2}$ , we have

$$|\log(1 + w)| \leq 2|w|.$$

Let  $L \subseteq X$  be a compact and write  $f_k = 1 + g_k$ . For all  $k > N$  we have  $\|g_k\|_L \leq \frac{1}{2}$ , therefore we can write

$$\log f_k = \log(1 + g_k) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} g_k^\ell.$$

But then

$$\|\log f_k\|_L \leq 2\|g_k\|_L.$$

It follows that the series

$$\sum_{k=N}^{\infty} \|\log f_k\|_L$$

converges. But then the series

$$h_N = \sum_{k=N}^{\infty} \log f_k$$

converges absolutely, and therefore all reorderings of the series converge as well to the same limit  $h_N$ .

Observe that

$$e^{h_N} = \prod_{k=N}^{\infty} e^{\log f_k} = \prod_{k=N}^{\infty} f_k.$$

This product therefore converges uniformly on  $L$ , independently of reorderings. We now define

$$f = \prod_{k=1}^{N-1} f_k \cdot e^{h_N}.$$

Note that this holds for all reorderings, as they differ from a suitable one by only finitely many transpositions.  $\square$

### 3.2 Zeroes of infinite products

**Definition 3.2.1.** Let  $\Omega \subseteq \mathbb{C}$  be an open set and  $f \in \mathcal{O}(\Omega)$ . The *zero set* of  $f$  is the set

$$Z(f) = \{z \in \Omega \mid f(z) = 0\}.$$

For all  $c \in \Omega$ , define the *zero order* of  $f$  in  $c$  as follows: if

$$f(z) = (z - c)^k \cdot g(z)$$

where  $g(c) \neq 0$  is a holomorphic function, then  $\text{ord}_c(f) = k$ .

**Remark 3.2.1.1.** For non-zero  $f \in \mathcal{O}(\Omega)$ , the set  $Z(f)$  is discrete in  $\Omega$ .

**Remark 3.2.1.2.** We have

$$\text{ord}_c \left( \prod_{k=1}^n f_k \right) = \sum_{k=1}^n \text{ord}_c(f_k).$$

**Lemma 3.2.2.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and

$$f = \prod_{k=1}^{\infty} f_k$$

be a normally convergent product in  $\Omega$ , where  $f_k \in \mathcal{O}(\Omega)$  are non-zero holomorphic functions. Then  $f$  is a non-zero function with

$$Z(f) = \bigcup_{k=1}^{\infty} Z(f_k)$$

and

$$\text{ord}_c(f) = \sum_{k=1}^{\infty} \text{ord}_c(f_k).$$

*Proof.* Recall that normally convergent products converge uniformly on compacts of  $\Omega$ . In particular,  $f$  is a holomorphic function.

Pick a point  $c \in \Omega$ . By definition of convergence, there exists some  $m \in \mathbb{N}$  such that  $\hat{f}_m(c) \neq 0$ . As  $\hat{f}_m$  is holomorphic as well, we have

$$f(c) = (p_{1,m-1} \cdot \hat{f}_m)(c),$$

but then

$$\text{ord}_c(f) = \sum_{k=1}^{m-1} \text{ord}_c(f_k) = \sum_{k=1}^{\infty} \text{ord}_c(f_k). \quad \square$$

**Lemma 3.2.3.** Let  $\Omega \subseteq \mathbb{C}$  be a domain. If

$$f = \prod_{k=1}^{\infty} f_k$$

is a normally convergent product in  $\Omega$ , where  $f_k \in \mathcal{O}(\Omega)$  are holomorphic functions, then the sequence  $(\hat{f}_n)_n$  converges to 1 uniformly on compacts.

*Proof.* Choose  $m \in \mathbb{N}$  such that  $\hat{f}_m \neq 0$ . Then the set  $Z(\hat{f}_m)$  has no accumulation points in  $\Omega$ . We can therefore write

$$\hat{f}_n = \frac{\hat{f}_m}{p_{m,n-1}}$$

on  $\Omega \setminus Z(\hat{f}_m)$ . As  $p_{m,n-1}$  converges to  $\hat{f}_m$  on compacts of  $\Omega$ ,

$$\lim_{n \rightarrow \infty} \hat{f}_n = 1$$

uniformly on compacts of  $\Omega \setminus Z(\hat{f}_m)$ . For any compact set  $K \subseteq \Omega$ , taking  $m$  large enough, we have  $Z(\hat{f}_m) \cap K = \emptyset$ . The conclusion follows.  $\square$

**Definition 3.2.4.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f \in \mathcal{O}(\Omega)$ . The meromorphic function  $\frac{f'}{f}$  is called the *logarithmic derivative* of  $f$ .

**Remark 3.2.4.1.** For holomorphic functions  $f_1, \dots, f_n \in \mathcal{O}(\Omega)$  we have

$$\left( \prod_{k=1}^n f_k \right)' \cdot \left( \prod_{k=1}^n f_k \right)^{-1} = \sum_{k=1}^n \frac{f'_k}{f_k}.$$

**Definition 3.2.5.** Let  $g_k \in \mathcal{M}(\Omega)$  be meromorphic functions. The series

$$\sum_{k=1}^{\infty} g_k$$

is *normally convergent* in  $\Omega$  if for every compact  $L \subseteq \Omega$  there exists some  $m \in \mathbb{N}$  such that

$$\sum_{k=m}^{\infty} \|g_k\|_L$$

converges.

**Theorem 3.2.6** (Logarithmic differentiation). Let  $\Omega \subseteq \mathbb{C}$  be a domain and

$$f = \prod_{k=1}^{\infty} f_k$$

be a normally convergent product in  $\Omega$ , where  $f_k \in \mathcal{O}(\Omega)$  are non-zero functions. Then

$$\sum_{k=1}^{\infty} \frac{f'_k}{f_k}$$

is normally convergent in  $\Omega$  and

$$\sum_{k=1}^{\infty} \frac{f'_k}{f_k} = \frac{f'}{f}.$$

*Proof.* As  $\hat{f}_n$  converges to 1 uniformly on compacts, the sequence  $(f'_n)_n$  converges to 0 uniformly on compacts by Cauchy estimates. Then for any compact  $L$ ,  $\frac{\hat{f}'_n}{\hat{f}_n}$  converges to 0 as  $\hat{f}_n$  has no zeroes in  $L$  for  $n$  large enough. It follows that

$$\lim_{n \rightarrow \infty} \frac{f'}{f} - \sum_{k=1}^n \frac{f'_k}{f_k} = \lim_{n \rightarrow \infty} \frac{\hat{f}'_{n+1}}{\hat{f}_{n+1}} = 0.$$



Write  $f_k = 1 + g_k$  and fix a compact set  $L \subseteq \Omega$ . Choose an index  $m$  such that we have  $Z(\hat{f}_m) \cap L = \emptyset$  and

$$\min_{z \in L} |f_k(z)| \geq \frac{1}{2}.$$

Choose  $\varepsilon > 0$  such that

$$L_\varepsilon = \{z \in \mathbb{C} \mid d(z, L) \leq \varepsilon\} \subseteq \Omega.$$

By the Cauchy estimates, we have  $\|g'_k\|_L \leq \frac{1}{\varepsilon} \|g_k\|_L$ . But then

$$\sum_{k=m}^{\infty} \left\| \frac{f'_k}{f_k} \right\|_L = \sum_{k=m}^{\infty} \left\| \frac{g'_k}{f_k} \right\|_L \leq 2 \cdot \sum_{k=m}^{\infty} \|g'_k\|_L \leq \frac{2}{\varepsilon} \cdot \sum_{k=m}^{\infty} \|g_k\|_L,$$

which is convergent by our assumptions.  $\square$

**Lemma 3.2.7.** Let  $g$  be meromorphic on  $\mathbb{C}$  with poles in  $\mathbb{Z}$  with principal parts  $\frac{1}{z-m}$ . Moreover, assume that  $g$  is an odd function that satisfies

$$2g(2z) = g(z) + g\left(z + \frac{1}{2}\right).$$

Then  $g(z) = \pi \cdot \cot(\pi z)$ .

*Proof.* Simple calculations show that  $\pi \cdot \cot(\pi z)$  is indeed a solution of the functional equation. Define  $h(z) = g(z) - \pi \cdot \cot(\pi z)$ . This another solution of the functional equation, and an odd function. In particular,  $h(0) = 0$ . Observe that the principal parts of  $h$  are 0, therefore  $h \in \mathcal{O}(\mathbb{C})$  is an entire function.

Suppose that  $h$  is not constant. In particular, there exists some  $c \in \partial\mathbb{A}(2)$  such that

$$|h(z)| < |h(c)|$$

for all  $z \in \mathbb{A}(2)$ . As  $\frac{c}{2}, \frac{c+1}{2} \in \mathbb{A}(2)$ , we can write

$$2|h(c)| = \left| h\left(\frac{c}{2}\right) + h\left(\frac{c+1}{2}\right) \right| \leq \left| h\left(\frac{c}{2}\right) \right| + \left| h\left(\frac{c+1}{2}\right) \right| < 2|h(c)|,$$

which is a contradiction. It follows that  $h = 0$ .  $\square$

**Corollary 3.2.7.1.** We have

$$\pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

*Proof.* Note that

$$\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2} = \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{1}{z-k} + \frac{1}{z+k} \right),$$

therefore the series has poles in  $\mathbb{Z}$  with principal parts  $\frac{1}{z-m}$ . It is also an odd function. A calculation shows that, for

$$r_n(z) = \frac{1}{z} + \sum_{k=1}^n \frac{2z}{z^2 - k^2},$$

we have

$$r_n(z) + r_n\left(z + \frac{1}{2}\right) = 2r_{2n}(2z) + \frac{2}{2z + 2n + 1}.$$

Taking  $n \rightarrow \infty$ , the conclusion follows.  $\square$

**Theorem 3.2.8.** We have

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

*Proof.* The above product is obviously normally convergent, therefore we can take its logarithmic derivative. A simple calculation shows that it is equal to  $\pi \cot(\pi z)$ . As logarithmic derivatives are equal only for scalar multiples, we only have to check equality in one point.  $\square$

### 3.3 The Euler gamma function

**Lemma 3.3.1.** The infinite product

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-\frac{z}{k}}$$

is normally convergent in  $\mathbb{C}$ .

*Proof.* Write

$$\begin{aligned} |1 - (1 - \omega)e^{\omega}| &= |1 - e^{\omega} + \omega e^{\omega}| \\ &= \left| -\sum_{k=1}^{\infty} \frac{\omega^k}{k!} + \sum_{k=0}^{\infty} \frac{\omega^{k+1}}{k!} \right| \\ &= \left| \omega^2 \cdot \sum_{k=1}^{\infty} \left( \frac{1}{k!} - \frac{1}{(k+1)!} \right) \omega^{k-1} \right| \\ &\leq |\omega|^2 \cdot \sum_{k=1}^{\infty} \left( \frac{1}{k!} - \frac{1}{(k+1)!} \right) \\ &= |\omega|^2 \end{aligned}$$

for  $|\omega| \leq 1$ . But then the sum

$$\sum_{k=\lceil |z| \rceil}^{\infty} \left| 1 - \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right| \leq \sum_{k=\lceil |z| \rceil}^{\infty} \left| \frac{z^2}{k^2} \right|$$

converges normally. The infinite product must then converge normally in  $\mathbb{C}$  as well.  $\square$

**Lemma 3.3.2.** Let

$$H(z) = z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Then  $H(1) = e^{-\gamma}$ , where  $\gamma$  is the *Euler-Mascheroni constant*, that is

$$\gamma = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \log(n).$$

*Proof.* First note that

$$\prod_{k=1}^n \left(1 + \frac{1}{k}\right) = \prod_{k=1}^n \frac{k+1}{k} = n+1.$$

We therefore have

$$\prod_{k=1}^n \left(1 + \frac{1}{k}\right) e^{-\frac{1}{k}} = \exp \left( \log(n+1) - \sum_{k=1}^n \frac{1}{k} \right),$$

therefore

$$H(1) = \lim_{n \rightarrow \infty} \exp \left( \log(n+1) - \sum_{k=1}^n \frac{1}{k} \right) = e^{-\gamma}. \quad \square$$

**Lemma 3.3.3.** Let  $\Delta(z) = e^{\gamma z} H(z)$ .

i) We have  $\Delta(1) = 1$  and  $\Delta(z) = z\Delta(z+1)$ .

ii) We have  $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$ .

*Proof.* Note that  $\Delta(1) = 1$  by the previous lemma. Rewrite the partial products as

$$z \cdot \prod_{k=1}^n \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} = \frac{z}{n!} \cdot \prod_{k=1}^n (z+k) \cdot \exp\left(-z \sum_{k=1}^n \frac{1}{k}\right).$$

We therefore have

$$\begin{aligned} \Delta(z) &= \lim_{n \rightarrow \infty} \frac{e^{\gamma z}}{n!} \cdot \prod_{k=0}^n (z+k) \cdot \exp\left(-z \sum_{k=1}^n \frac{1}{k}\right) \\ &= \lim_{n \rightarrow \infty} \frac{e^{\gamma z}}{n! \cdot n^z} \cdot \prod_{k=0}^n (z+k) \cdot \exp\left(z \log(n) - z \sum_{k=1}^n \frac{1}{k}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n! \cdot n^z} \cdot \prod_{k=0}^n (z+k). \end{aligned}$$

We can now calculate

$$z \cdot \Delta(z+1) = \lim_{n \rightarrow \infty} z \cdot \frac{1}{n! \cdot n^{z+1}} \cdot \prod_{k=1}^{n+1} (z+k) = \Delta(z) \cdot \lim_{n \rightarrow \infty} \frac{z+n+1}{n} = \Delta(z).$$

It remains to check the equality  $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$ . We have

$$\begin{aligned} \pi \cdot \Delta(z)\Delta(1-z) &= \pi \cdot \Delta(z) \cdot \frac{\Delta(-z)}{-z} \\ &= \pi e^{\gamma z} \cdot z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \cdot e^{-\gamma z} \cdot \frac{-z}{-z} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{\frac{z}{k}} \\ &= \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \\ &= \sin(\pi z). \end{aligned}$$

□

**Definition 3.3.4.** The *Euler gamma function* is defined as

$$\Gamma(z) = \frac{1}{\Delta(z)}.$$

**Theorem 3.3.5.** The  $\Gamma$  function satisfies the following properties:

1. The function  $\Gamma$  is meromorphic with simple poles in  $-\mathbb{N}_0$ .
2. We have  $\Gamma(1) = 1$ .
3. The function  $\Gamma$  satisfies  $\Gamma(z+1) = z\Gamma(z)$ .
4. The function  $\Gamma$  satisfies

$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

5. We have

$$\Gamma(z) = \lim_{n \rightarrow \infty} n! \cdot n^z \cdot \left( \prod_{k=0}^n (z+k) \right)^{-1}.$$

*Proof.* The proof is obvious and need not be mentioned.  $\square$

**Theorem 3.3.6.** Let  $F$  be holomorphic in  $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$  and assume  $F(z+1) = z \cdot F(z)$ . Furthermore, assume that  $F$  is bounded on the strip  $1 \leq \operatorname{Re}(z) < 2$  and  $F(1) = 1$ . Then  $F = \Gamma$ .

### 3.4 Weierstraß factors

**Definition 3.4.1.** The *Weierstraß factors* are functions

$$E_n(z) = (1 - z) \cdot \exp \left( \sum_{\ell=1}^n \frac{z^\ell}{\ell} \right).$$

**Lemma 3.4.2.** The Weierstraß factors satisfy the following:

i) For  $n \geq 1$  we have

$$E'_n(z) = -z^n \cdot \exp \left( \sum_{\ell=1}^n \frac{z^\ell}{\ell} \right).$$

ii) For  $n \geq 0$  we have

$$E_n(z) = 1 + \sum_{k=n+1}^{\infty} a_k z^k,$$

where

$$\sum_{k=n+1}^{\infty} |a_k| = 1.$$

iii) For  $n \geq 0$  and  $|z| \leq 1$  we have

$$|E_n(z) - 1| \leq |z|^{n+1}.$$

*Proof.*

i) Evident.

ii) Observing the derivative, we see that  $a_1 = a_2 = \dots = a_n = 0$ , and  $a_k \leq 0$  for  $k > n$ .  
But then

$$\sum_{k=n+1}^{\infty} |a_k| = - \sum_{k=n+1}^{\infty} a_k = 1 - E_n(1) = 1.$$

iii) We have

$$|E_n(z) - 1| = \left| \sum_{k=n+1}^{\infty} a_k z^k \right| \leq \sum_{k=n+1}^{\infty} |a_k| \cdot |z|^k \leq |z|^{n+1}. \quad \square$$

**Lemma 3.4.3.** Let  $(a_k)_k \subset \mathbb{C}^*$  be a sequence of complex numbers with no accumulation point and let  $(p_k)_k \subseteq \mathbb{N}_0$  be non-negative integers with

$$\sum_{k=1}^{\infty} \left| \frac{r}{a_k} \right|^{p_k+1}$$

converges for every  $r > 0$ . Then the *Weierstraß product*

$$\prod_{k=1}^{\infty} E_{p_k} \left( \frac{z}{a_k} \right)$$

converges normally on  $\mathbb{C}$ .

*Proof.* Note that  $|a_k| > |z|$  for all but finitely many  $k$ . Now just apply the previous lemma.  $\square$

**Theorem 3.4.4** (Weierstraß factorization theorem). For any sequence  $(a_k)_k \subset \mathbb{C}$  with no accumulation point there exists a Weierstraß product

$$z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{p_k} \left( \frac{z}{a_k} \right)$$

that converges normally on  $\mathbb{C}$ .

*Proof.* Set  $p_k = k - 1$ . For any  $r > 0$  choose  $m \in \mathbb{N}_0$  such that  $|a_k| > 2r$  for all  $k \geq m$ . We then have

$$\sum_{k=m}^{\infty} \left| \frac{r}{a_k} \right|^{p_k+1} \leq \sum_{k=m}^{\infty} \frac{1}{2^k} \leq 2. \quad \square$$

**Theorem 3.4.5** (Weierstraß product theorem). Let  $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$  be a holomorphic function. Then there exists a function  $g \in \mathcal{O}(\mathbb{C})$  such that

$$f = e^g \cdot z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{k-1} \left( \frac{z}{a_k} \right),$$

where  $a_k$  are zeroes of  $f$  on  $\mathbb{C} \setminus \{0\}$ , counted with multiplicities, and  $q = \text{ord}_0(f)$ .

*Proof.* The proof is obvious and need not be mentioned.  $\square$

**Lemma 3.4.6.** Let  $\Omega \subset \mathbb{C}$  be an open subset,  $(a_k)_k \subset \Omega$  a sequence with no accumulation point in  $\Omega$  and  $A = \{a_k \mid k \in \mathbb{N}\}$ . Let  $(b_k)_k \subset \mathbb{C} \setminus \Omega$  and  $(p_k)_k \subseteq \mathbb{N}$  be sequences such that the series

$$\sum_{k=1}^{\infty} |r(a_k - b_k)|^{p_k+1}$$

converges for all  $r > 0$  and denote  $B = \{b_k \mid k \in \mathbb{N}\}$ . Then the infinite product

$$\prod_{k=1}^{\infty} E_{p_k} \left( \frac{a_k - b_k}{z - b_k} \right)$$

converges normally on  $\mathbb{C} \setminus \overline{B}$ .

*Proof.* Let  $L \subseteq \mathbb{C} \setminus \overline{B}$  be a compact set and let  $\ell = d(L, \overline{B}) > 0$ . We then have  $|z - b_k| \geq \ell$  for all  $z \in L$  and  $k \in \mathbb{N}$ .

We can now bound

$$\left\| \frac{a_k - b_k}{z - b_k} \right\|_L \leq \frac{|a_k - b_k|}{\ell}.$$

By the assumption of convergence for  $r = \frac{1}{\ell}$ , we must have

$$|r \cdot (a_k - b_k)| < 1$$

for all  $k \geq n(L)$ , but then

$$\sum_{k=n(L)}^{\infty} \left\| E_{p_k} \left( \frac{a_k - b_k}{z - b_k} \right) - 1 \right\|_L \leq \sum_{k=n(L)}^{\infty} \left\| \frac{a_k - b_k}{z - b_k} \right\|_L^{p_k+1} \leq \sum_{k=n(L)}^{\infty} |r \cdot (a_k - b_k)|^{p_k+1},$$

which converges.  $\square$

**Remark 3.4.6.1.** The Weierstraß factor  $E_{p_k} \left( \frac{a_k - b_k}{z - b_k} \right)$  is zero if and only if  $z = a_k$ .

**Lemma 3.4.7.** Let  $A \subset \mathbb{C}$  be a discrete set and define  $A' = \overline{A} \setminus A$ . Suppose that  $A' \neq \emptyset$  and let

$$A_1 = \{z \in A \mid |z| \cdot d(z, A') \geq 1\}$$

and  $A_2 = A \setminus A_1$ . Now let

$$A_2(\varepsilon) = \{z \in A_2 \mid d(z, A') \geq \varepsilon\}.$$

Then  $A_1$  is a closed set and  $A_2(\varepsilon)$  is finite for any  $\varepsilon > 0$ .

*Proof.* Assume  $A_1$  has an accumulation point  $a$  and let  $(a_k)_k \subseteq A$  be a sequence, converging to  $a$ . But then

$$\lim_{k \rightarrow \infty} |a_k| \cdot d(a_k, A') = 0,$$

which is a contradiction.

Note that, for all  $z \in A_2(\varepsilon)$ , we have  $|z| < \frac{1}{\varepsilon}$ . If the set is infinite, it has an accumulation point, which is impossible as  $d(z, A') \geq \varepsilon$ .  $\square$

**Remark 3.4.7.1.** If  $A \subset \mathbb{C}$  is a discrete set, then  $A'$  is a closed set in  $\mathbb{C}$ .

**Theorem 3.4.8** (Weierstraß product theorem). Let  $\Omega \subseteq \mathbb{C}$  be an open subset. Let  $(a_k)_k \subset \Omega$  be a sequence without accumulation points in  $\Omega$  and denote  $A = \{a_k \mid k \in \mathbb{N}\}$  and  $A' = \overline{A} \setminus A$ . Then there exists a Weierstraß product for  $(a_k)_k$  that converges normally in  $\mathbb{C} \setminus A'$ . This product has zeros precisely in  $(a_k)_k$ , counted with multiplicities.

*Proof.* Assume that  $\Omega \neq \mathbb{C}$  and  $A' \neq \emptyset$ .<sup>4</sup> Write  $A = A_1 \cup A_2$  as in the above lemma. Recall that  $A_1$  has no accumulation points, therefore we can apply theorem 3.4.5 for  $A_1$ . It remains to construct a Weierstraß product for  $A_2$ .

Observe that  $A' = A'_2$ . As this is a closed space, for all  $a_k \in A_2$  there exists some  $b_k \in A'_2$  such that

$$|a_k - b_k| = d(a_k, A'_2).$$

Observe that

$$\lim_{\substack{k \rightarrow \infty \\ a_k \in A_2}} |a_k - b_k| = 0,$$

as the sets  $A_2(\varepsilon)$  are finite. Now set  $p_k = k$  and apply lemma 3.4.6.  $\square$

<sup>4</sup> Otherwise just apply theorem 3.4.5.



**Corollary 3.4.8.1** (Blaschke products). Let  $(a_k)_k \subset \mathbb{A} \setminus \{0\}$  be a sequence without accumulation points in  $\mathbb{A}$ . If the series

$$\sum_{k=1}^{\infty} (1 - |a_k|)$$

converges, then the product

$$\prod_{k=1}^{\infty} E_0 \left( \frac{a_k - \frac{1}{\bar{a}_k}}{z - \frac{1}{\bar{a}_k}} \right)$$

converges normally in  $\mathbb{A}$  and has zeros precisely in  $(a_k)_k$ , counted with multiplicities.

*Proof.* Note that

$$|a_k - b_k| = \left| a_k - \frac{1}{\bar{a}_k} \right| = \left| \frac{1}{\bar{a}_k} \right| \cdot \left| |a_k|^2 - 1 \right| = \left| \frac{1}{\bar{a}_k} \right| \cdot (1 - |a_k|)(1 + |a_k|) \leq \frac{2}{m} \cdot (1 - |a_k|),$$

where

$$m = \min \{ |a_k| \mid k \in \mathbb{N} \}.$$

It follows that the series

$$\sum_{k=1}^{\infty} r \cdot |a_k - b_k|$$

converges, therefore we can apply lemma 3.4.6. □

**Theorem 3.4.9.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f \in \mathcal{O}(\Omega) \setminus \{0\}$ . Then we can write

$$f = g \cdot \prod_{k=1}^{\infty} f_k,$$

where  $g \in \mathcal{O}^*(\Omega)$  and  $f_k$  are Weierstraß factors.

*Proof.* The proof is obvious and need not be mentioned. □

**Theorem 3.4.10.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f \in \mathcal{M}(\Omega)$ . Then we can write  $f = \frac{g}{h}$ , where  $g, h \in \mathcal{O}(\Omega)$ .

*Proof.* Define  $h$  as the Weierstraß product of the poles of  $f$ . □

**Remark 3.4.10.1.** Let  $\Omega \subseteq \mathbb{C}$  be a domain. Then  $\mathcal{O}(\Omega)$  is not a factorial ring,<sup>5</sup> but  $\gcd(f, g) \in \mathcal{O}(\Omega)$  exists.

**Definition 3.4.11.** Let  $\Omega$  be an open subset and  $\{a_k\}_k$  be a sequence without accumulation points and without repetition. Let

$$q_k(z) = \sum_{n=1}^{\infty} c_{k,n} (z - a_k)^{-n}$$

be a principal part in  $a_k$  for each  $k$ .

---

<sup>5</sup> “Kolobar z enolično faktorizacijo.”

If there exist functions  $g_k \in \mathcal{O}(\Omega)$  for each  $k$  such that

$$\sum_{k=1}^{\infty} q_k - g_k$$

converges normally in  $\Omega$ , we call it the *Mittag-Leffler series* for the distribution of principal part  $(a_k, q_k)$ .

**Remark 3.4.11.1.** We adopt the following conventions: If  $0 \in \{a_k\}_k$ , then  $a_1 = 0$ .

**Theorem 3.4.12** (Mittag-Leffler for  $\mathbb{C}$ ). For every distribution of principal parts in  $\mathbb{C}$  there exists a corresponding Mittag-Leffler series.

*Proof.* Let  $g_k$  be the Taylor series of  $q_k$  about 0 in the disk  $\Delta(|a_k|)$  such that the inequality  $\|q_k - g_k\| < 2^{-k}$  holds for each  $k \geq 2$ . Note that

$$\lim_{k \rightarrow \infty} |a_k| = \infty$$

as the points don't accumulate. For each  $r > 0$  we can therefore find an integer  $n$  such that  $r < \frac{1}{2}|a_k|$  for all  $k \geq n$ . Then

$$\sum_{k=n}^{\infty} \|q_k - g_k\|_{\overline{\Delta(r)}} \leq 1. \quad \square$$

**Remark 3.4.12.1.** The above series  $f \in \mathcal{O}(\mathbb{C} \setminus \{a_1, a_2, \dots\})$  with principal parts  $q_k$  in  $a_k$  for each  $k \in \mathbb{N}$ . If the principal part are finite, then  $f \in \mathcal{M}$ .

**Lemma 3.4.13.** Let  $a \in \mathbb{C}$ ,  $q \in \mathcal{O}(\Omega \setminus \{a\})$  be a principal part and  $b \in \mathbb{C} \setminus \{a\}$ . Then  $q$  has a Laurent series expansion about  $b$  in the annulus  $\{z \in \mathbb{C} \mid |z - b| > |a - b|\}$  of the form

$$q(z) = \sum_{m=1}^{\infty} c_m (z - b)^{-m}$$

that converges uniformly for  $|z - b| > r > |a - b|$ .

*Proof.* Choose a path  $\gamma_r$  that goes around the circle centered at  $b$  of radius  $r$ . We claim that

$$c_m = \frac{1}{2\pi} \int_{\gamma_r} \frac{q(z)}{(z - b)^{-m+1}} dz$$

for  $m \in \mathbb{Z}$  suffice. We can estimate

$$|c_m| \geq \frac{1}{2\pi} \cdot 2\pi \frac{\|q\|_{\gamma_r}}{r^{-m}} = \frac{\|q\|_{\gamma_r}}{r^{-m}}.$$

We know that  $q(z)$  is of the form

$$q(z) = \sum_{m=1}^{\infty} d_m (z - a)^{-m}$$

for some  $d_m \in \mathbb{C}$  when developed into a Laurent series around  $a$ . It is trivial to show that

$$\lim_{|z| \rightarrow \infty} q(z) = 0.$$

Thus,  $\|q\|_{\gamma_r}$  approaches zero as  $r$  goes to infinity. If  $m \leq 0$ , then

$$\lim_{r \rightarrow \infty} \|q\|_{\gamma_r} r^m = 0.$$

Therefore,  $c_m = 0$  for  $m \leq 0$  and

$$q(z) = \sum_{m=1}^{\infty} c_m (z - b)^{-m}$$

is indeed a power series in  $z - b$  which converges uniformly for  $|z - b|^{-1} \leq r$ .  $\square$

**Definition 3.4.14.** The partial sums of

$$q_\ell(z) = \sum_{m=1}^{\ell} c_m (z - b)^{-m}$$

are called the  $\ell$ -th *Laurent terms* of  $q$  about  $b$ .

**Lemma 3.4.15.** Let  $(a_k, q_k)_k$  be a distribution of principal parts in an open set  $\Omega \subseteq \mathbb{C}$ ,  $A = \{a_k \mid k \in \mathbb{N}\}$  and  $A' = \overline{A} \setminus A$ .<sup>6</sup> Assume there exists a sequence  $(b_k)_k \subseteq A'$  with

$$\lim_{k \rightarrow \infty} |a_k - b_k| = 0.$$

Let  $q_{k,\ell}$  be the  $\ell$ -th Laurent term of  $q_k$  about  $b_k$ . Then there exists a sequence  $(\ell_k)_k \subseteq \mathbb{N}_0$  such that

$$\sum_{k=1}^{\infty} (q_k - q_{k,\ell_k})$$

is a Mittag-Leffler series for  $(a_k, q_k)_k$ .

*Proof.* For a principal part  $q_k$  the Laurent series converges uniformly on  $|z - b_k| > r$  for any  $r > |a_k - b_k|$  by the previous lemma. Thus, we can choose  $\ell_k$  large enough such that

$$|q_k(z) - q_{k,\ell_k}(z)| < 2^{-k}$$

for all  $z$  such that  $|z - b_k| \geq 2|a_k - b_k|$ .

For any compact set  $L \subseteq \mathbb{C} \setminus A'$ , the distance to  $A'$  is strictly positive. Since

$$\lim_{k \rightarrow \infty} |a_k - b_k| = 0,$$

the point  $b_k$  must lie outside  $L$  for large enough  $k$ . Thus, there exists some  $n(L) \in \mathbb{N}$  such that

$$L \subseteq \bigcap_{k \geq n(L)} \{z \in \mathbb{C} \mid |z - b_k| \geq 2|a_k - b_k|\}.$$

Therefore, we can use the previous estimate on  $L$ , to get

$$\sum_{k \geq n(L)} \|q_k - q_{k,\ell_k}\|_L \leq \sum_{k \geq n(L)} 2^{-k} \leq 2. \quad \square$$

---

<sup>6</sup> The closure is taken in  $\mathbb{C}$ .

**Theorem 3.4.16** (Mittag-Leffler for open subsets). Let  $\Omega \subseteq \mathbb{C}$  be an open subset. Let  $(a_k, q_k)_k$  be a distribution of principal parts in  $\Omega \subseteq \mathbb{C}$  and  $A = \{a_k \mid k \in \mathbb{N}\}$ . Then there exists a Mittag-Leffler series for  $(a_k, q_k)$  that converges normally in  $\mathbb{C} \subseteq A' = \overline{A} \setminus A$ .

*Proof.* By lemma 3.4.7,  $(A_1)'$  is empty and  $(A_2)' = A'$ . If  $A'$  is empty, then

$$\lim_{k \rightarrow \infty} |a_k| = \infty$$

and we can apply theorem 3.4.12. Similarly, we can assume  $\Omega \neq \mathbb{C}$ . Again by the lemma,  $A_2(\epsilon)$  is finite. Hence, there exist  $(b_k)_k \in A'$  such that

$$\lim_{k \rightarrow \infty} |a_k - b_k| = 0.$$

We can apply lemma 3.4.15 to obtain a Mittag-Leffler series. Now we apply theorem 3.4.12 for  $\mathbb{C}$  to  $A_1$ . Sum up this two series to get the series from the statement.  $\square$

**Theorem 3.4.17** (Mittag-Leffler osculation theorem). Let  $\Omega \subseteq \mathbb{C}$  be an open subset and  $(a_k)_k \subseteq \Omega$  be a sequence without accumulation points and without repetition. Furthermore, let

$$f_k(z) = \sum_{\ell=-\infty}^{n(k)} c_{k,\ell} (z - a_k)^\ell,$$

where  $n(k) \in \mathbb{N}_0$ , be normally convergent on  $\mathbb{C} \setminus A$ , where  $A$  is the set of  $a_k$ . Then there exists a function  $f \in \mathcal{O}(\Omega \setminus A)$  such that  $\text{ord}_{a_k}(f - f_k) > n(k)$  for all  $k \in \mathbb{N}$ .

*Proof.* By the Weierstraß product theorem, there exists a function  $h \in \mathcal{O}(\Omega)$  such that  $\text{ord}_{a_k}(h) > n(k)$  and  $h$  has no zeroes on  $\mathbb{C} \setminus A$ . Then  $(a_k, \frac{f_k}{h})_k$  is a distribution of principal parts. By theorem 3.4.16, there exists a  $g \in \mathcal{O}(\Omega \setminus A)$  with these principal parts.

Now define  $f = g \cdot h$ . Then

$$f - f_k = g \cdot h - f_k = h \cdot \left( g - \frac{f_k}{h} \right),$$

which vanishes to order larger than  $n(k)$  in  $a_k$ .  $\square$

**Corollary 3.4.17.1.** For every sequence  $(a_k)_k \subseteq \Omega$  without accumulation points and without repetition and every sequence  $(c_k)_k \subseteq \mathbb{C}$  there exists a function  $f \in \mathcal{O}(\Omega)$  such that  $f(a_k) = c_k$  for each  $k \in \mathbb{N}$ .

### 3.5 Ring structure of holomorphic functions

**Definition 3.5.1.** Let  $\Omega \subseteq \mathbb{C}$  be an open set. A *divisor* of a meromorphic function  $f \in \mathcal{M}^*(\Omega)$  is the function  $(f): \Omega \rightarrow \mathbb{Z}$ , given by

$$(f)(z) = \begin{cases} n & f \text{ has a zero of order } n \text{ in } z \\ -n & f \text{ has a pole of order } n \text{ in } z \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 3.5.1.1.** The divisor of a product is the sum of divisors, i.e.  $(f \cdot g) = (f) + (g)$ .

**Definition 3.5.2.** Let  $S \subseteq \mathcal{O}(\Omega)$  be a subset that contains a non-zero holomorphic function.

Define

$$d(z) = \min_{f \in S \setminus \{0\}} (f)(z) \in \mathbb{N}_0.$$

By Weierstraß product theorem there exists a function  $g \in \mathcal{O}(\Omega)$  such that  $(g) = d$ . We define  $\gcd(S) = g$ .<sup>7</sup>

**Lemma 3.5.3** (Wedderburn). Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $f, g \in \mathcal{O}(\Omega)$  be functions with  $\gcd(f, g) = 1$ . Then there exist functions  $a, b \in \mathcal{O}(\Omega)$  so that  $af + bg = 1$ . Moreover, we can choose  $a$  to be nowhere vanishing.

*Proof.* If  $g = 0$  and  $f \neq 0$  then  $f$  cannot vanish by assumption on  $\gcd$ , therefore  $a = \frac{1}{f}$  and  $b = 1$  suffice. Therefore, we can assume both  $f, g$  are nonzero. Note that  $Z(f) \cap Z(g)$  is empty, since  $(z - p)$  divides  $\gcd(f, g)$  for any  $p \in Z(f) \cap Z(g)$ . The set  $Z(f) \cup Z(g)$  is thus discrete. Further, for each zero  $p$  of  $g$ , there exists a disk of radius  $\varepsilon$  and a holomorphic function  $f_p \in \mathcal{O}(\Delta(p, \varepsilon))$  such that

$$f = e^{f_p}.$$

By the Mittag-Leffler osculation theorem there exists a function  $h \in \mathcal{O}(\Omega)$  such that  $\text{ord}_p(h - f_p) > \text{ord}_p(g)$ .

Here we stop for a short observation. Developing into the power series, we get that  $e^{w^n} - 1 = w^n + \mathcal{O}(w^{2n})$ . Then,

$$\text{ord}_p(f - e^h) = \text{ord}_p(e^h \cdot (e^{f_p - h} - 1)) = \text{ord}_p((e^{f_p - h} - 1)) = \text{ord}_p(f_p - h) > \text{ord}_p(g).$$

Define  $k = \frac{f - e^h}{g} \in \mathcal{O}(\Omega)$ . We claim that  $a = e^{-h}$  and  $b = -ke^{-h}$  satisfy the conditions. Clearly,  $a$  doesn't vanish, and

$$af + bg = e^{-h}f - ke^{-h}g = e^{-h}(f - kg) = e^{-h} \left( f - \frac{f - e^h}{g}g \right) = e^{-h}e^h = 1. \quad \square$$

**Corollary 3.5.3.1.** For holomorphic functions  $f_j \in \mathcal{O}(\Omega)$ , where  $j \leq n$ , we can write  $f = \gcd(f_1, f_2, \dots, f_n)$  as

$$f = \sum_{j=1}^n a_j f_j$$

<sup>7</sup> There are of course multiple possible functions that satisfy this condition, but their quotients are invertible.

*Proof.* We proceed by induction. The base case is just Wedderburn's lemma. Now let  $\hat{f} = \gcd(f_2, f_3, \dots, f_n)$ , which can be written as

$$\hat{f} = \sum_{j=2}^n \hat{a}_j f_j$$

by the induction hypothesis. Then  $\frac{f_1}{\hat{f}}, \frac{\hat{f}}{\hat{f}} \in \mathcal{O}(\Omega)$  are holomorphic functions with gcd equal to 1. We can therefore apply Wedderburn's lemma to get functions  $a$  and  $b$  such that

$$a \frac{f_1}{\hat{f}} + b \frac{\hat{f}}{\hat{f}} = 1.$$

The conclusion follows □

**Theorem 3.5.4.** Let  $I \triangleleft \mathcal{O}(\Omega)$  be the ideal generated by holomorphic functions  $f_1, f_2, \dots, f_n$  on  $\Omega$ . Then there exists a holomorphic function  $f$  such that  $I = (f)$ .

*Proof.* Take  $f = \gcd(f_j)$ . This function is an element of  $I$  by the previous corollary. Since  $f \mid f_j$ , this implies that  $I = (f)$ . □

**Definition 3.5.5.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $I \triangleleft \mathcal{O}(\Omega)$  an ideal.

- i) We call  $I$  *closed* if for every sequence  $(f_n)_n \subseteq I$  that converges uniformly on compacts of  $\Omega$  to some function  $f$ , we also have  $f \in I$ .
- ii) We call  $p \in \Omega$  a *zero* of  $I$  if  $f(p) = 0$  for every  $f \in I$ .

**Lemma 3.5.6.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $I \triangleleft \mathcal{O}(\Omega)$  an ideal. Let  $p \in \Omega$  be a point that is not a zero of  $I$ . Let  $f, g \in \mathcal{O}(\Omega)$  be functions such that  $f(z) \neq 0$  for all  $z \neq p$ . If  $fg \in I$ , then  $g \in I$ .

*Proof.* Since  $p$  is not a zero of  $I$ , then there exists a function  $h \in I$  such that  $h(p) \neq 0$ . Let  $n = \text{ord}_p(f)$ . If  $n = 0$ , then  $f$  is a unit, so  $g \in I$ . Otherwise, we have

$$\frac{f(z)}{z-p} g = -\frac{1}{h(p)} \cdot \left( \frac{h-h(p)}{z-p} f g - \frac{f g}{z-p} h \right) \in I$$

since  $\frac{f}{z-p}$  is holomorphic.

We can iterate this process to find  $\frac{f}{(z-p)^n} g \in I$ . Since  $\frac{f}{(z-p)^n}$  is a unit,  $g$  must be an element of  $I$ . □

**Theorem 3.5.7.** Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $I \triangleleft \mathcal{O}(\Omega)$  an ideal. If  $I$  has no zeroes and is closed, then  $I = \mathcal{O}(\Omega)$ .

*Proof.* Let  $f$  be an arbitrary nonzero element of  $I$ . By the Weierstraß product theorem, we can write

$$f = \prod_{k=1}^{\infty} f_k,$$

where each  $f_k$  has exactly one zero in  $\Omega$ , and the tails

$$\hat{f}_n = \prod_{k=n}^{\infty} f_k$$

converge to 1 uniformly on compacts of  $\Omega$ . As  $f = \widehat{f}_1 = f_1 \widehat{f}_2$ , we can apply the previous lemma to find  $\widehat{f}_2 \in I$ . Inductively,  $\widehat{f}_n \in I$  and since the ideal  $I$  is assumed to be closed, we have

$$1 = \lim_{k \rightarrow \infty} \widehat{f}_k \in I. \quad \square$$

## 4 Approximation of holomorphic functions

### 4.1 Runge's little theorem

**Lemma 4.1.1.** Let  $\Omega \subseteq \mathbb{C}$  be an open subset and  $K \subseteq \Omega$  a non-empty compact. Then there exist finitely many horizontal or vertical line segments  $\sigma_1, \dots, \sigma_n$  of equal length in  $\Omega \setminus K$  such that for all  $f \in \mathcal{O}(\Omega)$  we have

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\sigma_k} \frac{f(\xi)}{\xi - z} d\xi$$

for all  $z \in K$ .

*Proof.* Define

$$\delta = \begin{cases} 1, & \Omega = \mathbb{C}, \\ d(K, \partial\Omega), & \Omega \neq \mathbb{C}. \end{cases}$$

Let  $Q$  be a grid of squares, parallel to the coordinate axes, with side length  $d < \frac{\delta}{\sqrt{2}}$ . As  $K$  is compact, it only intersects finitely many of them. Now just choose the boundary of the union of those squares. It is clear that all those segments are subsets of  $\Omega \setminus K$ .

Let  $Q_k$  denote the above squares. Then,

$$\frac{1}{2\pi i} \sum_{k=1}^m \oint_{\partial Q_k} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\sigma_k} \frac{f(\xi)}{\xi - z} d\xi.$$

If  $z \in \text{Int } Q_\ell$ , then

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q_\ell} \frac{f(\xi)}{\xi - z} d\xi$$

and

$$\frac{1}{2\pi i} \int_{\partial Q_k} \frac{f(\xi)}{\xi - z} d\xi = 0$$

for all  $k \neq \ell$ , therefore the lemma holds for all such  $z$ . As both sides of the equations are continuous functions, they must agree on the whole set  $K$ .  $\square$

**Lemma 4.1.2.** Let  $\sigma$  be a compact line segment and  $K \subseteq \mathbb{C}$  be a compact set such that  $K \cap \sigma = \emptyset$ . Let  $h \in \mathcal{C}(\sigma)$  be a function. Then for all  $\varepsilon > 0$  there exist points  $c_1, \dots, c_m \in \mathbb{C}$  and  $w_1, \dots, w_m \in \sigma$  such that

$$\left\| \int_{\sigma} \frac{h(\xi)}{\xi - z} d\xi - \sum_{k=1}^m \frac{c_k}{z - w_k} \right\|_K < \varepsilon.$$

*Proof.* Let  $\ell$  be the length of  $\sigma$  and define the function  $v: \sigma \times K \rightarrow \mathbb{C}$  with

$$v(\xi, z) = \frac{h(\xi)}{\xi - z}.$$

It is clearly continuous, therefore it is uniformly continuous. In particular, there exists a  $\delta > 0$  such that

$$|v(\xi, z) - v(\xi', z)| < \frac{\varepsilon}{\ell}$$



for all  $z \in \mathbb{C}$  and  $|\xi - \xi'| < \delta$ .

Let  $\tau_1, \dots, \tau_m$  be the partition of  $\sigma$  into line segments of length  $d < \sigma$ . Choose points  $w_k \in \tau_k$  and set  $c_k = -h(w_k)d$ . We therefore have

$$\left| \int_{\tau_k} v(\xi, z) d\xi - \frac{c_k}{z - w_k} \right| = \left| \int_{\tau_k} v(\xi, z) d\xi - d \cdot v(w_k, z) \right| < d \cdot \frac{\varepsilon}{\ell}.$$

Summing up, we get the desired inequality.  $\square$

**Lemma 4.1.3.** Let  $\Omega \subseteq \mathbb{C}$  be an open set and  $K \subseteq \Omega$  be a non-empty compact. Then there exist finitely many line segments  $\sigma_1, \dots, \sigma_n$  in  $\Omega \setminus K$  such that for any holomorphic function  $f \in \mathcal{O}(\Omega)$  and  $\varepsilon > 0$  there exists a rational function  $q$  of the form

$$q(z) = \sum_{k=1}^m \frac{c_k}{z - w_k},$$

where  $c_k \in \mathbb{C}$  and  $w_k \in \sigma$ , such that

$$\|f - q\|_K < \varepsilon.$$

*Proof.* By lemma 4.1.1 there exist line segments  $\sigma_k$  such that

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\sigma_k} \frac{f(\xi)}{\xi - z} d\xi$$

for all  $z \in K$ . Now just apply lemma 4.1.2 to each line segment separately.  $\square$

**Lemma 4.1.4** (Shifting poles). Let  $K \subseteq \mathbb{C}$  be a compact set,  $Z$  a connected component of  $\mathbb{C} \setminus K$  and  $a, b \in Z$ . Then, for all  $\varepsilon > 0$  there exists a polynomial  $q$  such that

$$\left\| \frac{1}{z - a} - q\left(\frac{1}{z - b}\right) \right\|_K < \varepsilon.$$

If  $Z$  is the unbounded component, we can approximate  $\frac{1}{z-a}$  by  $q(z)$  instead.

*Proof.* Let  $L_\omega$  be the family of all functions that are holomorphic in a neighbourhood of  $K$  that can be approximated uniformly on  $K$  by polynomials in  $\frac{1}{z-\omega}$ . Note that, if  $\frac{1}{z-p} \in L_q$ , then  $L_p \subseteq L_q$ .

Consider the set

$$S = \left\{ s \in Z \mid \frac{1}{z-s} \in L_b \right\}.$$

We claim that  $S = Z$ . First note that  $b \in S$ , therefore  $S \neq \emptyset$ . Take any point  $p \in S$  and  $\delta > 0$  such that  $\Delta(p, \delta) \subseteq Z$ . For any  $s \in \Delta(p, \delta)$ , we can write

$$\frac{1}{z-s} = \frac{1}{z-p} \cdot \frac{1}{1 - \frac{s-p}{z-p}} = \frac{1}{z-p} \cdot \sum_{k=0}^{\infty} \left( \frac{s-p}{z-p} \right)^k,$$

which converges as  $|s-p| < |z-p|$  by choice of  $\delta$ . We conclude that  $\Delta(p, \delta) \subseteq S$ . In particular,  $S$  is an open set. Take any point  $p \in \partial S \cap Z$ . Suppose that  $\Delta(p, 3\delta) \subseteq Z$  and

choose a point  $p' \in \Delta(p, \delta) \cap S$ . Then,  $\Delta(p', 2\delta) \subseteq S$  and therefore  $\Delta(p, \delta) \subseteq S$ , which is a contradiction and therefore proves our claim.

Now suppose that  $Z$  is the unbounded component and take a point  $d \in Z$  such that  $K \subseteq \Delta(0, |d|)$ . Then all functions  $\left(\frac{1}{z-d}\right)^n$  can be approximated uniformly on  $K$  by Taylor's polynomials about 0.  $\square$

**Definition 4.1.5.** For any set  $P \subseteq \mathbb{C}$  denote by  $\mathbb{C}_P[z]$  the family of rational functions with poles in  $P$ .

**Theorem 4.1.6** (Runge approximation). Let  $K \subseteq \mathbb{C}$  be a compact. If  $P$  intersects every bounded connected component of  $\mathbb{C} \setminus K$ , then for every function  $f$ , holomorphic in a neighbourhood of  $K$ , and every  $\varepsilon > 0$  there exists a function  $q \in \mathbb{C}_P[z]$  such that

$$\|f - q\|_K < \varepsilon.$$

*Proof.* By lemma 4.1.3, we can find a compact union  $\sigma$  of line segments such that every such  $f$  can be approximated by

$$\tilde{q} = \sum_{k=1}^m \frac{c_k}{z - w_k},$$

where  $c_k \in \mathbb{C}$  and  $w_k \in \sigma$ . Suppose then

$$\|f - \tilde{q}\|_K < \frac{\varepsilon}{2}.$$

Let  $Z_k$  be the connected component of  $\mathbb{C} \setminus K$  that contains  $w_k$ .

If  $Z_k$  is bounded, then choose  $t_k \in P \cap Z_k$ . By the pole shifting lemma, we can find a polynomial  $g_k$  such that

$$\left\| \frac{c_k}{z - w_k} - g_k(1)z - t_k \right\|_K < \frac{\varepsilon}{2m}.$$

If  $Z_k$  is unbounded, we can instead approximate  $\frac{c_k}{z - w_k}$  by a polynomial instead. Choosing

$$q = \sum_{k=1}^m g_k,$$

we find

$$\|f - q\|_K \leq \|f - \tilde{q}\|_K + \|\tilde{q} - q\|_K < \frac{\varepsilon}{2} + m \cdot \frac{\varepsilon}{2m} = \varepsilon. \quad \square$$

**Theorem 4.1.7** (Runge approximation). Let  $\Omega \subseteq \mathbb{C}$  be an open subset and let  $K \subseteq \Omega$  be a compact. If every bounded component of  $\mathbb{C} \setminus K$  intersects every bounded component of  $\mathbb{C} \setminus \Omega$ , then for every function  $f$  that is holomorphic on a neighbourhood of  $K$  and every  $\varepsilon > 0$  there exists a function  $q \in \mathcal{O}(\Omega)$  such that

$$\|f - q\|_K < \varepsilon.$$

*Proof.* Choose  $P = \mathbb{C} \setminus \Omega$  in theorem 4.1.6.  $\square$

**Corollary 4.1.7.1** (Runge's little theorem). Let  $K \subseteq \mathbb{C}$  be a compact set such that  $\mathbb{C} \setminus K$  is connected. Then for every holomorphic function on a neighbourhood of  $K$  and  $\varepsilon > 0$  there exists a polynomial  $q \in \mathbb{C}[z]$  such that

$$\|f - q\|_K < \varepsilon.$$

*Proof.* Choose  $P = \emptyset$  in theorem [4.1.6](#). □

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