Algebraic topology 1

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Introduction Luka Horjak

Introduction

These are my lecture notes on the course Algebraic topology 1 in the year 2023/24. The lecturer that year was prof. dr. Petar Pavešić.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Basic homotopy theory

1.1 Definition

Definition 1.1.1. Continuous maps $f, g: X \to Y$ of topological spaces are *homotopic*, if there is a continuous map $H: X \times I \to Y$, such that H(x, 0) = f(x) and H(x, 1) = g(x). Such H is called a *homotopy*. We write $H: f \simeq g$.

Remark 1.1.1.1. If X is a locally compact and Hausdorff space, homotopies coincide with paths in the space C(X,Y).

Proposition 1.1.2. Homotopy is an equivalence relation on C(X,Y).

Proof. The proof is obvious and need not be mentioned.

Definition 1.1.3. We denote the set of equivalence classes of the homotopy relation on C(X,Y) by [X,Y].

Remark 1.1.3.1. If X is a locally compact and Hausdorff space, [X, Y] is the set of path components of C(X, Y).

Definition 1.1.4. With $f:(X,A) \to (Y,B)$ we denote maps $f:X \to Y$ such that $f(A) \subseteq f(B)$. Similarly, we define $\mathcal{C}((X,A),(Y,B))$ and [(X,A),(Y,B)].

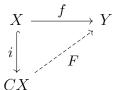
Definition 1.1.5. Let $A \subseteq X$ and $f, g: X \to Y$ be maps, such that $f|_A = g|_A$. $G: X \times I \to Y$ is a homotopy relative to A if $H: f \simeq g$ and $H_t|_A = f|_A$ for all $t \in I$.

Definition 1.1.6. A map $f: X \to Y$ is *null-homotopic* if it is homotopic to a constant.

Definition 1.1.7. Let X be a topological space. The *cone* on X is the space

$$CX = X \times I / X \times \{1\}$$
.

Proposition 1.1.8. A map $f: X \to Y$ is null-homotopic if and only if it extends to the cone CX.



Proof. In the following diagram, H exists if and only if F exists.

$$X \xrightarrow{f} Y$$

$$i_0 \downarrow H \qquad \uparrow F$$

$$X \times I \xrightarrow{g} CX$$

1.2 Homotopy equivalence

Proposition 1.2.1. Let $f, f': X \to Y$ and $g, g': Y \to Z$ be maps. If $f \simeq g$ and $f' \simeq g'$, we also have $g \circ f \simeq g' \circ f'$.

Proof. Let $H\colon X\times I\to Y$ and $K\colon Y\times I\to Z$ be the two homotopies. It is trivial to check that

$$L(x,t) = K(H(x,t),t)$$

is a homotopy of the compositions.

Definition 1.2.2. The *homotopy category* HoTop is the category with topological spaces as objects and homotopies as morphisms. Operations are induced by the compositions of maps.

Definition 1.2.3. The category $\underline{\text{Top}}^2$ has pairs of spaces (X, A) with $A \subseteq X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category HoTop^2 .

Definition 1.2.4. The category $\underline{\text{Top}}_{\bullet}$ has pairs (X, x_0) with $x_0 \in X$ as spaces, maps of pairs as morphisms and composition as the operation. Analogously as above, this category induces the category HoTop_{\bullet} .

Definition 1.2.5. Homotopy equivalence is an isomorphism in the category <u>HoTop</u>. If spaces X and Y are homotopy equivalent, we write $X \simeq Y$.

Remark 1.2.5.1. A map $f: X \to Y$ is a homotopy equivalence if there exists a map $g: Y \to X$ such that $g \circ f \simeq \operatorname{id}_X$ and $f \circ g \simeq \operatorname{id}_Y$. The map g is a homotopy inverse of f.

Definition 1.2.6. A space X is *contractible* if it is homotopy equivalent to a point.

Remark 1.2.6.1. Every cone is contractible.

Proposition 1.2.7. Let X be a topological space. The following statements are equivalent:

- i) The space X is contractible.
- ii) The map id_X is homotopy equivalent to a constant map.
- iii) The space X is a retract of CX.

Proof. The proof is obvious and need not be mentioned.

Theorem 1.2.8. Let X and Y be closed surfaces. If $X \simeq Y$, then $X \approx Y$.

Definition 1.2.9. A subspace $A \subseteq X$ is a deformation retract of X if it is a retract and the retraction is a homotopy inverse of the inclusion.

Definition 1.2.10. A subspace $A \subseteq X$ is a *strong deformation retract* is a retract and the retraction is a homotopy inverse of the inclusion relative to A.

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