Complex analysis

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Contents

Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations)**.** Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a, f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Definition 1.1.2. *Wirtinger derivatives* are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f\colon \Omega \to \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a. The function f is *holomorphic* if it is holomorphic in every point of Ω. We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with $\mathcal{C}^1$-smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w - z}\, dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z}\, dw \wedge d\overline{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\mathbb{\Delta}(z, \varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\mathbb{\Delta}(z, \varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w-z}\, dw$ on Ω_ε. As $d\omega = \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w} \wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z}\, dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z}\, d\overline{w} \wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z}\, dw = \oint_{\partial\Omega} \frac{f(w)}{w - z}\, dw - \oint_{\partial\mathbb{\Delta}(z,\varepsilon)} \frac{f(w)}{w - z}\, dw.$$

In the limit, we have

$$\lim_{\varepsilon\to 0} \oint\limits_{\partial\Delta(z,\varepsilon)} \frac{f(w)}{w-z}\,dw = \lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{f\left(z+\varepsilon e^{it}\right)}{\varepsilon e^{it}}\cdot \varepsilon i e^{it}\,dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon\to 0}\iint\limits_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega\setminus\{z\}} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw = \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,d\overline{w}\wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i}\oint\limits_{\partial\Omega}\frac{f(w)}{w-z}\,dw - f(z) = -\frac{1}{2\pi i}\iint\limits_{\Omega}\frac{\partial f}{\partial \overline{w}}\cdot\frac{1}{w-z}\,dw\wedge d\overline{w}.$$ □

Theorem 1.1.5 (Power series expansion)**.** Let $\Omega\subseteq\mathbb{C}$ be an open subset, $f\in\mathcal{O}(\Omega)$ and $a\in\Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\Delta(a,r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i}\oint\limits_{\partial\Delta(a,\rho)}\frac{f(w)}{(w-z)^{k+1}}\,dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k\cdot(z-a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k\to\infty}\sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity)**.** Let $\Omega\subseteq\mathbb{C}$ be a domain and $f\in\mathcal{O}(\Omega)$ a holomorphic function. Let $A\subseteq\Omega$ be a subset such that $f(z)=0$ for all $z\in A$. If A has an accumulation point in Ω, then $f(z)=0$ for all $z\in\Omega$.

Proof. Let $a\in\Omega$ be an accumulation point of A. By continuity, we have $f(a)=0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k(z-a)^k,$$

where we assume $c_{k_0}\neq 0$. But now $g(z)=\frac{f(z)}{(z-a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a)=0$, which is a contradiction. It follows that $c_k=0$ for all $k\in\mathbb{N}_0$. It follows that the set $\operatorname{Int}\{z\in\Omega\mid f(z)=0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to Ω. □

Lemma 1.1.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\Delta(a,r)} \subseteq \Omega$. If

$$|f(a)| < \min_{\partial\Delta(a,r)} |f|,$$

then f has a zero in $\Delta(a,r)$.

Proof. Assume otherwise. From the inequality it follows that f has no zeroes on the boundary either. By continuity, f has no zero on an open set V with $\Delta(a,r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint\limits_{\partial\Delta(a,r)} \frac{g(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g\left(a + r\cdot e^{it}\right)}{re^{it}} \cdot rie^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} g\left(a + re^{it}\right)\,dt.$$

We can therefore get a bound on $|g(a)|$ as

$$|g(a)| \leq \max_{\partial\Delta(a,r)} |g|,$$

but as the condition on f can be rewritten as

$$|g(a)| > \max_{\partial\Delta(a,r)} |g|,$$

we have reached a contradiction. $\square$

Theorem 1.1.8 (Open mapping)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If f is not constant, it is an open map.

Proof. Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\Delta(z_0, \rho) \subseteq U$ and z_0 is the only pre-image of w_0 in $\Delta(z_0, 2\rho)$.[1]

Since $\partial\Delta(z_0, \rho)$ is a compact set and

$$|f(z) - w_0| > 0$$

for all $z \in \partial\Delta(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a $w \in \Delta(w_0, \varepsilon)$. As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$

on the boundary and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

by the above lemma, $f(z_0) - w$ has a root on $\Delta(z, \rho)$. $\square$

Theorem 1.1.9 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function f is constant.

[1] If such a disk does not exist, f is constant by the identity theorem.

Proof. Suppose that f is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As f is an open map, it also attains the value $(1+\varepsilon)\cdot f(z)$, which is a contradiction as the modulus then equals $(1+\varepsilon)\cdot |f(z)| > |f(z)|$. $\square$

Theorem 1.1.10 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

Proof. As $\overline{\Omega}$ is compact, f attains a global maximum on this set. If the maximum is attained in the interior, f is constant, therefore it is also attained on the boundary. $\square$

Definition 1.1.11. A function $f\colon \Omega \setminus \{a\} \to \mathbb{C}$ is *locally bounded* near a if there exists an open neighbourhood $U \subseteq \Omega$ of a such that $f|_{U\setminus\{a\}}$ is bounded.

Theorem 1.1.12 (Riemann removable singularity theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$. If f is locally bounded near a, then there exists a unique function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega\setminus\{a\}} = f$.

Proof. Define the function $F\colon \Omega \to \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \dfrac{1}{2\pi i} \displaystyle\oint_{\partial\Delta(a,\rho)} \frac{f(w)}{w-a}\, dw & z = a. \end{cases}$$

It remains to check that F is complex differentiable at a. Indeed, for $z \in \Delta(a,\rho)$ we have

$$\begin{aligned} \lim_{z\to a} \frac{F(z)-F(a)}{z-a} &= \lim_{z\to a} \frac{1}{z-a} \oint_{\partial\Delta(a,\rho)} \left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a}\right) dw \\ &= \lim_{z\to a} \frac{1}{2\pi i} \cdot \frac{1}{z-a} \cdot \oint_{\partial\Delta(a,\rho)} f(w) \cdot \frac{z-a}{(w-z)(w-a)}\, dw \\ &= \frac{1}{2\pi i} \oint_{\partial\Delta(a,\rho)} \frac{f(w)}{(w-a)^2}\, dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem. $\square$

Theorem 1.1.13 (Schwarz lemma)**.** Let $f\colon \Delta \to \Delta$ be a holomorphic function with $f(0) = 0$. Then, $|f'(0)| \le 1$ and the inequality $|f(z)| \le |z|$ holds for all $z \in \Delta$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ holds for any $z \neq 0$, then $f(z) = \beta z$ for some $\beta \in \partial\Delta$.

Proof. We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for g on the domain $\Delta(\rho)$. We get

$$\sup_{z\in\Delta(\rho)} |g(z)| \le \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as $\rho \to 1$, it follows that

$$\sup_{z \in \mathbb{\Delta}} |g(z)| \leq 1.$$

It immediately follows that $|f'(0)| = |g(0)| \leq 1$. Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z|.$$

Suppose we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. As then $|g(z_0)| = 1$, it follows that g is constant, therefore $f(z) = \beta z$ for some $\beta \in \partial \mathbb{\Delta}$. If we have $|f'(0)| = 0$, the same argument works for $z_0 = 0$. $\square$

1.2 The $\overline{\partial}$ equation

Lemma 1.2.1. Let $g \in \mathcal{C}^\infty(\mathbb{C})$ be a function with compact support. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Let

$$f(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{g(w)}{w - z} \, dw \wedge d\overline{w}.$$

As

$$dw \wedge d\overline{w} = -2ri \, dr \wedge d\varphi$$

holds for polar coordinates centered at z, we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{rg\left(z + re^{i\varphi}\right)}{re^{i\varphi}} \, dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some R such that $g|_{\mathbb{C}\setminus\Delta(z,R)} = 0$. We get

$$f(z) = -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi,$$

which obviously converges. The function f is therefore well defined. As we are integrating a smooth function on a compact set, the function f is smooth as well.

For $u = re^{i\varphi}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}}(z) &= -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} \frac{\partial}{\partial \overline{z}} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial}{\partial \overline{z}} g(u + z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial g}{\partial \overline{u}}(u + z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w - z} \, dw \wedge d\overline{w}.
\end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint\limits_{\partial\Delta(z,R)} \frac{g(w)}{w - z} \, dw + \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w - z} \, dw \wedge d\overline{w}.$$

by the choice of R, we get

$$\frac{\partial f}{\partial \overline{z}}(z) = g(z).$$ □

Lemma 1.2.2. Given bounded domain $U \subset V \subset \mathbb{R}^n$ such that $\partial U \cap \partial V = \emptyset$, there exists a smooth function $\chi \colon \mathbb{R}^n \to [0, 1]$ such that $\chi|_U = 1$ and $\operatorname{supp} \chi \subseteq V$.

Proof. There is a partition of unity on the sets V and $\mathbb{R}^n \setminus \overline{U}$. □

Lemma 1.2.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $h_j \colon \Omega \to \mathbb{C}$ be holomorphic functions. If the sequence $(h_j)_{j\in\mathbb{N}}$ converges uniformly on compact sets, the limit is also holomorphic on Ω.

Proof. Apply Morera's theorem.[2] □

Theorem 1.2.4 (Dolbeault lemma)**.** Let $g \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ for some $R \in (0, \infty]$. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Define disks X_j as follows:

i) If $R = \infty$, set $X_j = \mathbb{\Delta}(j)$.

ii) If $R < \infty$, set $X_j = \mathbb{\Delta}\left(R - \frac{1}{j}\right)$ (for large enough j).

Applying the above lemma, define functions χ_j with $\chi_j|_{X_j} = 1$ and $\operatorname{supp} \chi_j \subseteq X_{j+1}$ and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \mathbb{\Delta}(R), \\ 0 & z \notin \mathbb{\Delta}(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function $f_j \in \mathcal{C}^\infty(\mathbb{C})$ with

$$\frac{\partial f_j}{\partial \overline{z}} = g_j.$$

We inductively construct a new sequence $\widetilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$ such that

$$\frac{\partial \widetilde{f}_j}{\partial \overline{z}} = g$$

on X_j and

$$\left\| \widetilde{f}_j - \widetilde{f}_{j-1} \right\|_{X_{j-2}} \le 2^{-j}.$$

Set $\widetilde{f}_1 = f_1$. Observe the function $F = f_{j+1} - \widetilde{f}_j$ on X_j. By construction, we have $\frac{\partial F}{\partial \overline{z}} = 0$ on X_j. It follows that F can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on X_j. As power series converge uniformly on compact sets, there exists some polynomial $p \in \mathbb{C}[z]$ such that

$$\|F - p\|_{X_{j-1}} \le 2^{-j}.$$

Now just set $\widetilde{f}_{j+1} = f_{j+1} - p$.

Let $z \in \mathbb{\Delta}(R)$ be arbitrary. By construction, it is contained in some X_{j_0}, therefore, $\widetilde{f}_j$ is defined for $j \ge j_0$. As $\left(\widetilde{f}_j(z)\right)_{j \ge j_0}$ is a Cauchy sequence, we can define

$$f(z) = \lim_{j\to\infty} \widetilde{f}_j(z).$$

[2] Analysis 2b, theorem 3.4.6.

But as

$$f - \widetilde{f}_j = \sum_{k=j}^{\infty} \left(\widetilde{f}_{j+1} - \widetilde{f}_j \right)$$

is a sum of holomorphic functions that converges uniformly, the function $f - \widetilde{f}_j$ is a holomorphic function. Therefore, f is smooth and satisfies $\frac{\partial f}{\partial \bar{z}} = g$. $\square$

1.3 Meromorphic functions

Definition 1.3.1. Let $\Omega \subset \mathbb{C}$ be an open subset. We call a function f *meromorphic* of Ω if there exists $A \subset \Omega$ such that $f \in \mathcal{O}(\Omega \setminus A)$, A has no accumulation points in Ω and for all $a \in A$ there exists some $k \in \mathbb{N}$ such that

$$\lim_{z \to a} f(z) \cdot (z-a)^k \neq 0$$

exists. We call A the set of *poles* of the function f. We denote the set of meromorphic functions on Ω with $\mathcal{M}(\Omega)$.

Theorem 1.3.2. Let $0 \leq r < R \leq \infty$. Suppose that $f \in \mathcal{O}(D_{R,r}(a))$ is a holomorphic function, where

$$D_{R,r}(a) = \{z \in \mathbb{C} \mid r < |z-a| < R\}.$$

Then there exists a uniquely determined *Laurent series*

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

that converges to f uniformly and absolutely on compact subsets of $D_{R,r}(a)$. We have

$$c_k = \frac{1}{2\pi i} \oint_{\partial \Delta(a,\rho)} \frac{f(w)}{(w-a)^k} \, dw$$

for $r < \rho < R$.

Definition 1.3.3. Let

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

be a Laurent series. The series

$$\sum_{k=-\infty}^{-1} c_k (z-a)^k$$

is called the *principle part*.

Lemma 1.3.4. Let $f \in \mathcal{O}(\Omega \setminus \{a\})$ be a holomorphic function. Then f is meromorphic on Ω if and only if f has a finite principle part in a.

Proof. Suppose that f is meromorphic on Ω. If a is a removable singularity, f is holomorphic in a, therefore the principle part is trivial. Otherwise, set $m \in \mathbb{N}$ such that

$$\lim_{z \to a} (z-a)^m f(z) \neq 0$$

exists and set $g(z) = (z-a)^m f(z)$. As g is bounded near a, we can extend it to Ω by the Riemann removable singularity theorem. The power series of g corresponds to a finite Laurent series of f.

The converse is obvious. □

Theorem 1.3.5. If $f \in \mathcal{M}(\mathbb{C})$ is a meromorphic function, there exist entire functions g and h such that $f = \frac{g}{h}$.

Definition 1.3.6. Let $\Omega \subseteq \mathbb{C}$ be an open set. An *additive Cousin problem* on Ω is an open cover $\{U_j\}_{j\in J}$ of Ω and functions $f_j \in \mathcal{M}(U_j)$ such that $f_j - f_k|_{U_j \cap U_k}$ is holomorphic for all $j, k \in J$. A function $f \in \mathcal{M}(\Omega)$ is a solution to the additive Cousin problem if $f|_{U_j} - f_j$ is holomorphic for all $j \in J$.

Definition 1.3.7. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A *generalized additive Cousin problem* is an open cover $\{U_j\}_{j\in J}$ of Ω and functions $f_{j,k} \in \mathcal{O}(U_j \cap U_k)$ for each $(j, k) \in J^2$, such that

i) $f_{j,k} = -f_{k,j}$ on $U_j \cap U_k$ for all $(j, k) \in J^2$ and

ii) $f_{j,k} + f_{k,\ell} + f_{\ell,j} = 0$ on $U_j \cap U_k \cap U_\ell$ for all $(j, k, \ell) \in J^3$.

A solution to the generalized additive Cousin problem is given by functions $f_j \in \mathcal{O}(U_j)$ for each $j \in J$ such that $F_{j,k} = f_j - f_k$ for each $(j, k) \in J^2$.

Lemma 1.3.8 (Partition of unity)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $\{U_j\}_{j\in J}$ be an open cover of Ω. Then there exists a partition of unity subordinate to $\{U_j\}_{j\in J}$.

Lemma 1.3.9. Given a generalized additive Cousin problem on $\Omega \subseteq \mathbb{C}$, there exist functions $g_j \in \mathcal{C}^\infty(U_j)$ such that $f_{j,k} = g_j - g_k$ for all $(j, k) \in J^2$.

Proof. Let $\{(V_a, \chi_a)\}_{a\in A}$ be a partition of unity, subordinate to $\{U_j\}_{j\in J}$. For all $a \in A$ choose a $j(a) \in J$ such that $V_a \subseteq U_{j(a)}$. For all $k \in J$, define

$$g_k = -\sum_{a\in A} \chi_a \cdot f_{j(a),k}.$$

This is of course a smooth function on U_k. Now note that

$$g_k - g_\ell = \sum_{a\in A} \chi_a \cdot \left(-f_{j(a),k} + f_{j(a),\ell}\right) = \sum_{a\in A} \chi_a \cdot f_{k,\ell} = f_{k,\ell}. \qquad \square$$

Proposition 1.3.10. The generalized additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

Proof. Let $f_{j,k} = g_j - g_k$ for $g_j \in \mathcal{C}^\infty(U_j)$. Note that

$$\frac{\partial g_j}{\partial \overline{z}} = \frac{\partial g_k}{\partial \overline{z}},$$

therefore,

$$h|_{U_j} = \frac{\partial g_j}{\partial \overline{z}}$$

induces a smooth function $h \colon \Omega \to \mathbb{C}$. By the Dolbeault lemma, there exists a function $g \in \mathcal{C}^\infty(\Omega)$ such that $\frac{\partial g}{\partial \overline{z}} = h$. It is clear that $f_j = g_j - g$ solves the generalized additive Cousin problem. $\square$

Proposition 1.3.11. The additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

Proof. An additive Cousin problem induces a generalized additive Cousin problem for functions $f_{j,k} = f_j - f_k$. Let g_j be a solution to the generalized problem. As $f_j - f_k = f_{j,k} = g_j - g_k$ on $U_j \cap U_k$, we can define a function $f \in \mathcal{M}(\Omega)$ with $f|_{U_j} = f_j - g_j$. This function is of course well defined. As $f|_{U_j} - f_j = g_j \in \mathcal{O}(U_j)$, this function indeed solves the additive Cousin problem. $\square$

Theorem 1.3.12 (Mittag-Leffler)**.** Let $(a_k)_{k\in\mathbb{N}}$ be a sequence without repetition and accumulation points. Let

$$f_k(z) = \sum_{\ell=-m_k}^{-1} c_{k,\ell}(z-a_k)^\ell$$

be finite principal parts. Then there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ with poles in $(a_k)_{k\in\mathbb{N}}$ such that f has principle part f_k in a_k for each $k \in \mathbb{N}$.

Proof. For each a_k choose a disk U_k containing no other a_k. Also set $U_0 = \mathbb{C}\setminus\{a_k \mid k \in \mathbb{N}\}$ and $f_0 = 0$. As $\{U_k \mid k \in \mathbb{N}_0\}$ is an open cover of $\mathbb{C}$, there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ that solves the corresponding additive Cousin problem. It is easy to see that the principle parts of f at a_k are precisely f_k. □

1.4 Sequences of holomorphic functions

Definition 1.4.1. A family of functions $\mathcal{F}$ from Ω to $\mathbb{C}$ is *locally bounded*, if for all $p \in \Omega$ there exist a $\rho > 0$ and $M > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{z \in \mathbb{\Delta}(p,\rho)} |f(z)| < M.$$

Lemma 1.4.2. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ a locally bounded family of functions. Then for all $p \in \Omega$ there exists a $\rho > 0$ such that $\mathcal{F}$ is equi-continuous on $\Omega \cap \mathbb{\Delta}(p, \rho)$.

Proof. Fix $p \in \Omega$ and choose $r > 0$ such that $D = \overline{\mathbb{\Delta}(p, 2r)} \subseteq \Omega$. For any $z, w \in D$ and $f \in \mathcal{F}$ we have

$$f((z) - f(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - w} \, d\xi = \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} \, d\xi.$$

Note that the family $\mathcal{F}$ is bounded on every compact. Therefore, we can write

$$\sup_{f \in \mathcal{F}} \sup_{z \in \partial D} |f(z)| < M.$$

Now, for $z, w \in \mathbb{\Delta}(p, r)$ we have

$$|f((z) - f(w)| = \left| \frac{z - w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} \, d\xi \right| \leq |z - w| \cdot \frac{2M}{r}.$$

□

Theorem 1.4.3 (Arzelà-Ascoli)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be an infinite family such that the following conditions hold:

i) $\mathcal{F}$ is point-wise bounded.

ii) $\mathcal{F}$ is locally equi-continuous.

Then there $\mathcal{F}$ contains a sequence that converges uniformly on compacts of Ω.

Proof. Choose a dense countable subset $A \subseteq \Omega$ and enumerate it as a sequence $(a_k)_{k \in \mathbb{N}}$. Pick any sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ with pairwise distinct terms. As $|f_n(a_1)| < M$ for all n, we can choose a subsequence $(f_{1,n})_{n \in \mathbb{N}}$ such that $f_{1,n}(a_1)$ converges by Bolzano-Weierstrass.

Similarly, for every $k \in \mathbb{N}$ there exists a subsequence $(f_{k,n})_n$ of $(f_{k-1,n})_n$ such that $(f_{k,n}(a_k))_n$ converges. Now define $F_n = f_{n,n}$. Observe that (F_n) converges at every point in A.

Fix a $p \in \Omega$. By local equi-continuity, there exists a $\rho > 0$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta < \rho$ and $|F_n(z) - F_n(w)| < \frac{\varepsilon}{3}$ for all $z, w \in \mathbb{\Delta}(p, \rho)$ such that $|z - w| < \delta$. Choose an element $a \in A \cap \mathbb{\Delta}(z, \delta)$.[3] Then,we have

$$|F_n(z) - F_m(z)| \leq |F_n(z) - F_n(a)| + |F_n(a) - F_m(a)| + |F_m(a) - F_m(z)| < 3 \cdot \frac{\varepsilon}{3}.$$

It follows that (F_n) is locally uniformly convergent, therefore it converges uniformly on compact sets. □

[3] By compactness of $\overline{\mathbb{\Delta}(p, \rho)}$ we can choose a from a finite set.

Theorem 1.4.4 (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f_n \colon \Omega \to \mathbb{C}$ be a locally bounded sequence of holomorphic functions. Then $(f_n)_n$ contains a subsequence that converges uniformly on compacts.

Proof. As the sequence is locally bounded, it is locally equi-continuous by lemma 1.4.2. By Arzelà-Ascoli, there exists a convergent subsequence. □

Definition 1.4.5. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family of functions $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is *normal* if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on compacts.

Theorem 1.4.6 (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is normal if and only if it is locally bounded.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.4.7 (Vitali)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $(f_n)_n \subseteq \mathcal{O}(\Omega)$ a locally bounded sequence of holomorphic functions. The following statements are equivalent:

i) The sequence $(f_n)_n$ converges uniformly on compact subsets of Ω.

ii) For a point $p \in \Omega$ the sequence $(f_n^{(k)}(p))_n$ converges for all $k \in \mathbb{N}_0$.

iii) The set
$$A = \left\{ z \in \Omega \;\middle|\; \lim_{n\to\infty} f_n(z) \text{ converges} \right\}$$
has an accumulation point in Ω.

Proof. Suppose that the sequence converges uniformly on compact subsets. Given a $p \in \Omega$, choose a $\delta > 0$ such that $D = \overline{\mathbb{A}(p, \delta)} \subseteq \Omega$. Note that

$$\left| g^{(k)}(p) \right| \leq \frac{k!}{\delta^k} \cdot \|g\|_D$$

holds for all holomorphic functions g. As $\|f - f_n\|$ converges to 0, the derivatives of f_n converge.

Suppose that the sequences of derivatives converge at a point $p \in \Omega$ and choose a $\delta > 0$ such that $D = \overline{\mathbb{A}(p, \delta)} \subseteq \Omega$. As the sequence is locally bounded, there exists a constant M such that $\|f_n\|_D \leq M$ holds for all $n \in \mathbb{N}$. We can now develop power series

$$f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z-p)^k.$$

They converge uniformly on compact subsets of $\mathbb{A}(p, \delta)$. Note that

$$a_{k,n} = \frac{f_n^{(k)}(p)}{k!}.$$

As derivatives converge, we can define the limit

$$a_k = \lim_{n\to\infty} a_{k,n}.$$

Now define the formal power series

$$f(z)=\sum_{k=0}^{\infty} a_k(z-p)^k.$$

The Cauchy bounds give us the inequality

$$|a_{k,n}|=\frac{\left|f^{(k)}(p)\right|}{k!}\le\frac{M}{\delta^k},$$

therefore

$$\limsup_{k\to\infty}\sqrt[k]{|a_k|}\le\limsup_{k\to\infty}\frac{\sqrt[k]{M}}{\delta}=\frac{1}{\delta}.$$

We conclude that the radius of convergence is at least δ. Consider some $\rho\in(0,\delta)$ and $z\in\mathbb{\Delta}(p,\rho)$. We have

$$\begin{aligned}
|f_n(z)-f(z)|&\le\left|\sum_{k=0}^{m}(a_{k,n}-a_k)\cdot(p-z)^k\right|+\left|\sum_{k=m+1}^{\infty}(a_{k,n}-a_k)\cdot(p-z)^k\right|\\
&\le\sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+\sum_{k=m+1}^{\infty}2M\cdot\frac{\rho^k}{\delta^k}\\
&=\sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+2M\cdot\left(\frac{\rho}{\delta}\right)^{m+1}\cdot\frac{\delta}{\delta-\rho}\\
&=2\cdot\frac{\varepsilon}{2}
\end{aligned}$$

for large enough m and n. It follows that p is an accumulation point of A.

Suppose now that A has an accumulation point in Ω. By Montel's theorem there exists a subsequence $(f_{n_m})_m$ that converges uniformly on compact subsets of Ω to a limit function f. Note that all such subsequences have the same limit by the identity principle.

Assume that the sequence $(f_n)_n$ does not converge uniformly on a compact subset $K\subseteq\Omega$. We can therefore construct another subsequence $(g_n)_n$ of $(f_n)_n$ such that

$$\|g_n-f\|_K>\varepsilon$$

for all $n\in\mathbb{N}$. But note that $(g_n)_n$ also has a convergent subsequence by Montel's theorem, which is of course a contradiction, as it cannot converge to f. $\square$

November 7, 2023

1.5 Riemann mapping theorem

Definition 1.5.1. A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if every closed path in Ω is homotopic to a constant path in Ω.

Lemma 1.5.2. Let $\Omega \subset \mathbb{C}$ be a domain and $a \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Then there exists a holomorphic injection $f\colon \Omega \to \mathbb{A}$ such that $f(a) = 0$.

Proof. Fix a point $p \in \mathbb{C} \setminus \Omega$. By our assumption, there exists a function $v \in \mathcal{O}^*(\Omega)$ such that $v(z)^2 = z - p$. Note that v is injective. Similarly, we have $v(\Omega) \cap -v(\Omega) = \emptyset$. Now choose a point $b \in -v(\Omega)$. As v is not constant, it is an open map. Therefore, there exists some $r > 0$ such that $\mathbb{A}(b, r) \cap v(\Omega) = \emptyset$. The Möbius transformation

$$h(w) = r \cdot \left(\frac{1}{w - b} - \frac{1}{v(a) - b}\right)$$

thus maps $v(\Omega)$ into $\mathbb{A}$. The map f is therefore given as $f = h \circ v$. $\square$

Definition 1.5.3. An *expansion* if a map $\kappa\colon \Omega \to \mathbb{A}$ where $0 \in \Omega \subset \mathbb{A}$ such that $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ holds for all $z \neq 0$.

Lemma 1.5.4. Let $\Omega \subset \mathbb{A}$ be a domain with $0 \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Choose $c \in \mathbb{A}$ such that $c^2 \notin \Omega$. For all $a \in \mathbb{A}$, let

$$g_a = \frac{z - a}{\overline{a}z - 1}$$

and choose $v \in \mathcal{O}(\Omega)$ such that $v(z)^2 = g_{c^2}(z)$ and $v(0) = c$. Then the map $\kappa = g_c \circ v$ is an expansion and

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = \mathrm{id}_\Omega .$$

Proof. Note that v is indeed well-defined. Also note that

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = g_{c^2} \circ (z \mapsto z^2) \circ v = g_{c^2} \circ g_{c^2} = \mathrm{id} .$$

We of course have $\kappa(0) = 0$. Denote $\psi_c = g_{c^2} \circ (z \mapsto z^2) \circ g_c$. It remains to check that $|\kappa(z)| > |z|$, which is equivalent to $|\psi_c(z)| < |z|$ for $z \neq 0$ as $\psi_c \circ \kappa = \mathrm{id}$. Note that $\psi_c\colon \mathbb{A} \to \mathbb{A}$ is holomorphic. As it is not a rotation (it is not injective), the conclusion follows from the Schwarz lemma. $\square$

Lemma 1.5.5 (Hurwitz)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f_n\colon \Omega \to \mathbb{C}$ be holomorphic functions. Suppose that the sequence $(f_n)_n$ converges uniformly on compacts of Ω to a non-constant function $f\colon \Omega \to \mathbb{C}$. Then for all points $p \in \Omega$ there exists a sequence $(p_n)_n \subseteq \Omega$ with limit p such that $f_n(p_n) = f(p)$ for all $n > N$.

Proof. Let $w = f(p)$. There exists a disk $\mathbb{A}(p, \delta)$ such that $f(z) \neq w$ for all points $z \in \overline{\mathbb{A}(p, \delta)} \setminus \{p\}$. Note that we have

$$\min_{z \in \partial \mathbb{A}(p,\delta)} |f(z) - w| > |f(p) - w| = 0.$$

As $(f_n)_n$ converges uniformly on $\overline{\mathbb{A}(p,\delta)}$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have
$$\min_{z \in \partial \mathbb{A}(p,\delta)} |f_n(z) - w| > |f_n(p) - w| .$$
By lemma 1.1.7, $f_n(z) - w$ has a root $p_n \in \mathbb{A}(p,\delta)$. For any convergent subsequence $(p_{n_k})_k$ with limit q we have
$$f(p) = \lim_{k \to \infty} f_{n_k}(p_{n_k}) = f(q),$$
therefore $p = q$. □

Corollary 1.5.5.1. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n \colon \Omega \to \mathbb{C}$ be holomorphic functions such that $(f_n)_n$ converges uniformly on compacts of Ω to $f \colon \Omega \to \mathbb{C}$. If all the f_n are nowhere vanishing and $f \neq 0$, then f is nowhere vanishing.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.5.6 (Hurwitz)**.** Let $\Omega, \Omega' \subseteq \Omega$ be domains and $f_n \colon \Omega \to \Omega'$ be holomorphic functions that converge uniformly on compacts of Ω to $f \colon \Omega \to \Omega'$. Assume that f is not constant.

i) If $f_n \colon \Omega \to \Omega'$ is injective, f is also injective.

ii) We have $f(\Omega) \subseteq \Omega'$.

Proof.

i) Let $p \in \Omega$ and observe the functions $g_n(z) = f_n(z) - f_n(p)$. This is a sequence of nowhere vanishing functions. As f is not constant, $f(z) - f(p)$ is nowhere vanishing as well. It follows that f is injective.

ii) Suppose otherwise and apply the Hurwitz lemma for a point p with $f(p) \notin \Omega'$. □

Theorem 1.5.7 (Riemann mapping)**.** For a proper domain $\Omega \subset \mathbb{C}$ the following are equivalent:

i) Ω is simply connected.

ii) Ω admits a logarithm for any $f \in \mathcal{O}^*(\Omega)$.

iii) Ω admits a square root for any $f \in \mathcal{O}^*(\Omega)$.

iv) Ω is biholomorphic to $\mathbb{A}$.

Proof. Note that if Ω is biholomorphic to $\mathbb{A}$, it is of course simply connected. Suppose that Ω is simply connected. Then
$$F(z) = a + \int_{z_0}^{z} \frac{f'(z)}{f(z)}\, dz$$
defines a logarithm for any $f \in \mathcal{O}^*(\Omega)$. Given a logarithm of a function, we can of course construct a square root with $\sqrt{f} = e^{\frac{1}{2}\ln f}$. It remains to check that all domains admitting square roots are biholomorphic to $\mathbb{A}$.

By lemma 1.5.2 we can assume that $\Omega \subseteq \mathbb{A}$ and $0 \in \Omega$. Now define the family of functions

$$\mathcal{F} = \{f : \Omega \to \mathbb{A} \mid f \in \mathcal{O}(\Omega) \wedge f(0) = 0 \wedge f \text{ is injective}\}.$$

If $\mathcal{F}$ has no biholomorphic map, it is infinite. Note that $\mathcal{F}$ is bounded, so it is normal by Montel.

Choose a point $p \in \Omega$ with $p \neq 0$. We claim that if $h \in \mathcal{F}$ and

$$|h(p)| = \sup_{f \in \mathcal{F}} |f(p)|,$$

we have $h(\Omega) = \mathbb{A}$. Indeed, if that were not the case, we'd reach a contradiction with the expansion κ of Ω as

$$|\kappa(h(p))| > |h(p)|$$

and $\kappa \circ h \in \mathcal{F}$.

Let

$$M = \sup_{f \in \mathcal{F}} |f(p)|$$

and find a sequence $(f_n)_n \subseteq \mathcal{F}$ with

$$\lim_{n \to \infty} |f_n(p)| = M.$$

As $\mathcal{F}$ is a normal family, there exists a convergent subsequence. The limit is not constant as $f(p) \neq 0$. By Hurwitz, f is injective and $f(\Omega) \subseteq \mathbb{A}$. By the above claim, we have $f(\Omega) = \mathbb{A}$. $\square$

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