Functional analysis

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Introduction

These are my lecture notes on the course Functional analysis in the year 2023/24. The lecturer that year was prof. dr. Igor Klep.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Convexity

1.1 Locally convex spaces

Definition 1.1.1. *A topological vector space*[1] V is an $\mathbb{F}$-vector space that is also a topological space, such both addition and scalar multiplication are continuous.

Definition 1.1.2. Let V be an $\mathbb{F}$-vectors pace. A map $p\colon V \to \mathbb{R}$ is a *seminorm* if the following holds:

i) $\forall x \in V : p(x) \geq 0$,

ii) $\forall \lambda \in \mathbb{F}, x \in V : p(\lambda x) = |\lambda|\, p(x)$,

iii) $\forall x, y \in V : p(x+y) \leq p(x) + p(y)$.

Definition 1.1.3. Let V be an $\mathbb{F}$-vector space and $\mathcal{P}$ a family of seminorms on V. We define a topology $\mathcal{T}$ on V with the sets

$$U(x_0, p, \varepsilon) = \{x \in V \mid p(x - x_0) < \varepsilon\}$$

as a subbasis.

Definition 1.1.4. A topological vector space X is a *locally convex space* if its topology is generated by a family of seminorms $\mathcal{P}$ satisfying

$$\bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\} = \{0\}\,.$$

Proposition 1.1.5. A locally convex space X is Hausdorff.

Proof. Let $x, y \in X$ be distinct points. Let $p \in \mathcal{P}$ be a seminorm such that $p(x - y) \neq 0$. Then the sets

$$U = \left\{z \in X \;\middle|\; p(z - x) < \frac{\varepsilon}{2}\right\} \quad \text{and} \quad V = \left\{z \in X \;\middle|\; p(z - y) < \frac{\varepsilon}{2}\right\}$$

split the points x and y. $\square$

Remark 1.1.5.1. The converse is also true.

Definition 1.1.6. A partially ordered set I is *upward directed* if for all $i', i'' \in I$ there exists some $i \in I$ such that $i \geq i'$ and $i \geq i''$.

Definition 1.1.7. A *net* is a pair $((I, \leq), x)$, where $(I, \leq)$ is an upward directed set and $x\colon I \to X$ is a function. We usually write $(x_i)_{i \in I}$.

Remark 1.1.7.1. Let $(X, \mathcal{T})$ be a topological space and $x_0 \in X$. Partially order the set

$$\mathcal{U} = \{U \subseteq X \mid x_0 \in U \wedge U \text{ is open}\}$$

with reverse inclusion. Then any choice function defines a net $(x_U)_{U \in \mathcal{U}}$.

[1] Also linear topological space.

Definition 1.1.8. Let X be a topological space. A net $(x_i)_{i\in I}$ *converges* so $x \in X$ if for all open sets $U \subseteq X$ with $x \in U$ there exists some index $i_0 \in I$ such that for all $i \geq i_0$ we have $x_i \in U$. We write
$$\lim_{i\in I} x_i = x.$$

Definition 1.1.9. A point $x \in X$ is a *cluster point* of a net $(x_i)_{i\in I}$ if for all open sets $U \subseteq X$ with $x \in U$ and index $i_0 \in I$ there exists some index $i \geq i_0$ such that $x_i \in U$.

Proposition 1.1.10. Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net $(a_i)_{i\in I}$ in A such that
$$\lim_{i\in I} a_i = x.$$

Proof. Suppose a net $(a_i)_{i\in I}$ converges to x. For any neighbourhood U of x and some $i_0 \in I$ we have $a_{i_0} \in U$. Therefore, $U \cap A \neq \emptyset$.

Assume now that $x \in \overline{A}$. Again, define
$$\mathcal{U} = \{U \subseteq X \mid x_0 \in U \land U \text{ is open}\}.$$

There is a choice function a such that $a_U \in A$ for all U. The net $(a_U)_{U\in\mathcal{U}}$ then converges to x. □

Proposition 1.1.11. Let X and Y be topological spaces and $f : X \to Y$ a map. Then, f is continuous in $x_0 \in X$ if and only if
$$\lim_{i\in I} f(x_i) = f(x_0)$$
for all nets $(x_i)_{i\in I}$ that converge to x_0.

Proof. Suppose that f is continuous at x_0. Take an open neighbourhood U of $f(x_0)$. Then there must exist some $i_0 \in I$ such that for all $i \geq i_0$ we have $x_i \in f^{-1}(U)$, therefore $f(x_i) \in U$.

Now suppose f is discontinuous at x_0. Let
$$\mathcal{U} = \{U \subseteq X \mid x_0 \in U \land U \text{ is open}\}$$
and $V \subseteq Y$ be an open set such that $f(x_0) \in V$ and x_0 is not an interior point of $f^{-1}(V)$. Now using the discontinuity of f, for all $U \in \mathcal{U}$ choose $x_U \in U$ such that $f(x_U) \notin V$. Trivially the net $(x_V)_{V\in\mathcal{V}}$ converges to x_0, but
$$\lim_{V\in\mathcal{V}} f(x_V) \neq f(x_0).$$
□

Proposition 1.1.12. The following statements are true:

i) A net $(x_i)_{i\in I}$ in a locally convex space converges to x_0 if and only if the net $(p(x_i - x_0))_{i\in I}$ converges to 0 for all $p \in \mathcal{P}$.

ii) The topology in a locally convex space X is the coarsest topology in which all the maps $x \mapsto p(x - x_0)$ are continuous for all $x_0 \in X$ and $p \in \mathcal{P}$.

Proof.

i) If $(x_i)_{i \in I}$ converges to x_0, just apply the proposition 1.1.11. Suppose that all the nets $(p(x_i - x_0))_{i \in I}$ converge to 0. Choose an open set from the local basis of x_0. It is given by
$$U = \{x \in X \mid \forall k \leq n\colon p_k(x - x_0) < \varepsilon\}.$$
But as all nets $(p_k(x_i - x_0))_{i \in I}$ converge to 0, there is some index $i_k \in I$ such that for all $i \geq i_k$ we have $p_k(x_i - x_0) < \varepsilon$. Now just take i_0 to be an upper bound of i_k. For all $i \geq i_0$ we then have $x_i \in U$.

ii) Obvious. □

Definition 1.1.13. For all $f \in X^*$ define a seminorm $p_f\colon X \to \mathbb{R}$ as $p_f(x) = |f(x)|$. The family $\mathcal{P} = \{p_f \mid f \in X^*\}$ induces the *weak topology* on X. We denote the weak topology with $\sigma(X, X^*)$.

Remark 1.1.13.1. The space X with the topology $\sigma(X, X^*)$ is a locally compact space by the Hahn-Banach theorem.[2]

Definition 1.1.14. Let X be a normed space. For all $x \in X$ we define a seminorm $p_x\colon X^* \to \mathbb{R}$ as $p_x(f) = |f(x)|$. The family $\mathcal{P} = \{p_x \mid x \in X\}$ induces the *weak-* topology* on X^*. We denote the weak-* topology with $\sigma(X^*, X)$.

Remark 1.1.14.1. The weak topology on X^* is finer than the weak-* topology, as X can be isometrically mapped into X^{**} with the map $x \mapsto (f \mapsto f(x))$.

[2] Introduction to functional analysis, corollary 2.2.5.2.

1.2 Banach-Alaoglu theorem

Theorem 1.2.1 (Banach-Alaoglu)**.** Let X be a normed space. Then the closed unit ball in X^*

$$(X^*)_1 = \{f \in X^* \mid \|f\| \leq 1\}$$

is compact in the weak-* topology on X^*.

Proof. Assign a disk to all $x \in X$ as $D_x = \{z \in \mathbb{F} \mid |z| \leq \|x\|\}$ with the euclidean topology. Define

$$P = \prod_{x \in X} D_x$$

with the product topology. The space P is then compact by Tychonoff's theorem. Now define the map $\Phi \colon (X^*)_1 \to P$ with $\Phi(f) = (f(x))_{x \in X}$. This map is injective.

Let $(f_i)_{i \in I}$ be a net in $(X^*)_1$ that weak-* converges to $f \in X^*$. Equivalently, we have

$$\lim_{i \in I} f_i(x) = f(x)$$

for all $x \in X$. By the definition of the product topology we have

$$\lim_{i \in I} \Phi(f_i) = \Phi(f).$$

Therefore, Φ is continuous. Analogously, $\Phi^{-1} \colon \operatorname{im} \Phi \to (X^*)_1$ is continuous.

Suppose that $(\Phi(f_i))_{i \in I}$ converges to some $p \in P$. By the definition of the product topology this means that $f_i(x)$ converges to p_x for all $x \in X$. Define a map $f \colon X \to \mathbb{F}$ given by $f(x) = p_x$. Then, f is linear and bounded with $\|f\| \leq 1$. Thus $p = \Phi(f) \in \operatorname{im}(\Phi)$, therefore, $\Phi((X^*)_1)$ is closed. As $(X^*)_1$ is homeomorphic to its image which is compact, it is also compact. $\square$

Corollary 1.2.1.1. Every Banach space X is isometrically isomorphic to a closed subspace $\mathcal{C}(K)$ for some compact Hausdorff space K.

Proof. Choose $K = (X^*)_1$ with the weak-* topology. By Banach-Alaoglu, K is compact and Hausdorff. Now define the map $\Delta \colon X \to K$ with $\Delta(x) = (f \mapsto f(x))$. Now observe that

$$\|\Delta(x)\|_\infty = \sup_{g \in K} |\Delta(x)(g)| = \sup_{g \in K} |g(x)| = \|x\|$$

by Hahn-Banach.[3] $\square$

[3] Introduction to functional analysis, corollary 2.2.5.1.

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