

Complex analysis

Luka Horjak (luka1.horjak@gmail.com)

October 4, 2023

Contents

Introduction	3
1 Holomorphic functions	4
1.1 Properties of holomorphic functions	4
Index	8

Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f: \Omega \rightarrow \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations). Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a , f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Definition 1.1.2. Wirtinger derivatives are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f: \Omega \rightarrow \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a . The function f is *holomorphic* if it is holomorphic in every point of Ω . We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with \mathcal{C}^1 -smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\Delta(z, \varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\Delta(z, \varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w - z} dw$ on Ω_ε . As $d\omega = \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} dw = \oint_{\partial\Omega} \frac{f(w)}{w - z} dw - \oint_{\partial\Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw.$$

In the limit, we have

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial \Delta(z, \varepsilon)} \frac{f(w)}{w - z} dw = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{f(z + \varepsilon e^{it})}{\varepsilon e^{it}} \cdot \varepsilon i e^{it} dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega \setminus \{z\}} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw = \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} d\bar{w} \wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f(w)}{w - z} dw - f(z) = -\frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{w}} \cdot \frac{1}{w - z} dw \wedge d\bar{w}. \quad \square$$

Theorem 1.1.5 (Power series expansion). Let $\Omega \subseteq \mathbb{C}$ be an open subset, $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\Delta(a, r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \oint_{\partial \Delta(a, \rho)} \frac{f(w)}{(w - a)^{k+1}} dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z - a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity). Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a holomorphic function. Let $A \subseteq \Omega$ be a subset such that $f(z) = 0$ for all $z \in A$. If A has an accumulation point in Ω , then $f(z) = 0$ for all $z \in \Omega$.

Proof. Let $a \in \Omega$ be an accumulation point of A . By continuity, we have $f(a) = 0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k (z - a)^k,$$

where we assume $c_{k_0} \neq 0$. But now $g(z) = \frac{f(z)}{(z - a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a) = 0$, which is a contradiction. It follows that $c_k = 0$ for all $k \in \mathbb{N}_0$. It follows that the set $\text{Int} \{z \in \Omega \mid f(z) = 0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to Ω . \square

Theorem 1.1.7 (Open mapping). Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If f is not constant, it is an open map.

Proof. We first prove the following lemma:

Lemma. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\Delta(a, r)} \subseteq \Omega$. If

$$|f(a)| < \min_{\partial\Delta(a, r)} |f|,$$

then f has a zero in $\Delta(a, r)$.

Proof (lemma). Assume otherwise. From the inequality it follows that f has no zeroes on the boundary either. By continuity, f has no zero on an open set V with $\Delta(a, r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint_{\partial\Delta(a, r)} \frac{g(z)}{z-a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(a + r \cdot e^{it})}{e^{it}} \cdot r i e^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} g(a + r e^{it}) dt.$$

We can therefore get a bound on $|g(a)|$ as

$$|g(a)| \leq \max_{\partial\Delta(a, r)} |g|,$$

but as the condition on f can be rewritten as

$$|g(a)| > \max_{\partial\Delta(a, r)} |g|,$$

we have reached a contradiction. □

Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\Delta(z_0, \rho) \subseteq U$ and z_0 is the only pre-image of w_0 in $\Delta(z_0, 2\rho)$.¹

Since $\partial\Delta(z_0, \rho)$ is a compact set and

$$|f(z) - w_0| > 0$$

for all $z \in \partial\Delta(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a $w \in \Delta(w_0, \varepsilon)$. As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$

on the boundary and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

by the above lemma, $f(z_0) - w$ has a root on $\Delta(z, \rho)$. □

Theorem 1.1.8 (Maximum principle). Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function f is constant.

¹ If such a disk does not exist, f is constant by the identity theorem.

Proof. Suppose that f is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As f is an open map, it also attains the value $(1 + \varepsilon) \cdot f(z)$, which is a contradiction as the modulus then equals $(1 + \varepsilon) \cdot |f(z)| > |f(z)|$. \square

Theorem 1.1.9 (Maximum principle). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

Proof. As $\overline{\Omega}$ is compact, f attains a global maximum on this set. If the maximum is attained in the interior, f is constant, therefore it is also attained on the boundary. \square

Index

C

Cauchy integral formula, [4](#)

Cauchy-Riemann equations, [4](#)

F

function

 complex differentiable, [4](#)

 holomorphic, [4](#)

I

identity theorem, [5](#)

M

maximum principle, [6](#)

O

open mapping theorem, [6](#)

W

Wirtinger derivatives, [4](#)