Complex analysis

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Contents

Introduction

These are my lecture notes on the course Complex analysis in the year 2023/24. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Holomorphic functions

1.1 Properties of holomorphic functions

Definition 1.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f \colon \Omega \to \mathbb{C}$ is *complex differentiable* in a point $a \in \Omega$ if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists.

Remark 1.1.1.1 (Cauchy-Riemann equations)**.** Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where f is real differentiable in a, f is complex differentiable in a if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Definition 1.1.2. *Wirtinger derivatives* are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Remark 1.1.2.1. A function f is complex differentiable in a if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

In that case, we also have

$$\frac{\partial f}{\partial z}(a) = f'(a).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function $f \colon \Omega \to \mathbb{C}$ is *holomorphic in a* if it is complex differentiable in an open neighbourhood of a. The function f is *holomorphic* if it is holomorphic in every point of Ω. We denote the set of holomorphic functions in Ω as $\mathcal{O}(\Omega)$.

Theorem 1.1.4 (Inhomogeneous Cauchy integral formula)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with $\mathcal{C}^1$-smooth boundary and $f \in \mathcal{C}^1(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w - z} \, dw + \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z} \, dw \wedge d\overline{w}$$

Proof. As Ω is an open set, there exists an $\varepsilon > 0$ such that $\overline{\Delta(z, \varepsilon)} \subseteq \Omega$. Define a new domain $\Omega_\varepsilon = \Omega \setminus \overline{\Delta(z, \varepsilon)}$.

We now apply Stokes' theorem to $\omega = \frac{f(w)}{w-z} \, dw$ on Ω_ε. As $d\omega = \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z} \, d\overline{w} \wedge dw$, we have

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} \, dw = \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w - z} \, d\overline{w} \wedge dw.$$

Note that

$$\oint_{\partial\Omega_\varepsilon} \frac{f(w)}{w - z} \, dw = \oint_{\partial\Omega} \frac{f(w)}{w - z} \, dw - \oint_{\partial\Delta(z,\varepsilon)} \frac{f(w)}{w - z} \, dw.$$

In the limit, we have

$$\lim_{\varepsilon \to 0} \oint\limits_{\partial \mathbb{A}(z,\varepsilon)} \frac{f(w)}{w-z}\, dw = \lim_{\varepsilon \to 0} \int_0^{2\pi} \frac{f\left(z + \varepsilon e^{it}\right)}{\varepsilon e^{it}} \cdot \varepsilon i e^{it}\, dt = 2\pi i f(z)$$

by continuity. Also note that

$$\lim_{\varepsilon \to 0} \iint\limits_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w} \wedge dw = \iint\limits_{\Omega \setminus \{z\}} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w} \wedge dw = \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, d\overline{w} \wedge dw.$$

Applying the limit to the Stokes' theorem equation, it follows that

$$\frac{1}{2\pi i} \oint\limits_{\partial \Omega} \frac{f(w)}{w-z}\, dw - f(z) = -\frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\partial f}{\partial \overline{w}} \cdot \frac{1}{w-z}\, dw \wedge d\overline{w}. \qquad \square$$

Theorem 1.1.5 (Power series expansion)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset, $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$. The function f can be developed into a power series about a that converges absolutely and uniformly to f in compacts inside $\mathbb{A}(a, r)$, where r is the radius of convergence. For

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \oint\limits_{\partial \mathbb{A}(a,\rho)} \frac{f(w)}{(w-z)^{k+1}}\, dw$$

we have

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k.$$

Remark 1.1.5.1. The converse is also true – any complex power series defines a holomorphic function inside its radius of convergence.

Remark 1.1.5.2. The radius of convergence is given by the formula

$$\frac{1}{r} = \limsup_{k \to \infty} \sqrt[k]{|c_k|}.$$

Theorem 1.1.6 (Identity)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a holomorphic function. Let $A \subseteq \Omega$ be a subset such that $f(z) = 0$ for all $z \in A$. If A has an accumulation point in Ω, then $f(z) = 0$ for all $z \in \Omega$.

Proof. Let $a \in \Omega$ be an accumulation point of A. By continuity, we have $f(a) = 0$. We can now write

$$f(z) = \sum_{k=k_0}^{\infty} c_k (z-a)^k,$$

where we assume $c_{k_0} \neq 0$. But now $g(z) = \frac{f(z)}{(z-a)^{k_0}}$ is also holomorphic. Again, by continuity, we must have $g(a) = 0$, which is a contradiction. It follows that $c_k = 0$ for all $k \in \mathbb{N}_0$. It follows that the set $\operatorname{Int} \{z \in \Omega \mid f(z) = 0\}$ is non-empty. By the same argument as above, it has an empty boundary and is therefore equal to Ω. $\square$

Lemma 1.1.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose that for $a \in \Omega$ and $r > 0$ we have $\overline{\mathbb{A}(a,r)} \subseteq \Omega$. If
$$|f(a)| < \min_{\partial\mathbb{A}(a,r)} |f|,$$
then f has a zero in $\mathbb{A}(a,r)$.

Proof. Assume otherwise. From the inequality it follows that f has no zeroes on the boundary either. By continuity, f has no zero on an open set V with $\mathbb{A}(a,r) \subseteq V$. We can therefore define $g \in \mathcal{O}(V)$ with $g(z) = \frac{1}{f(z)}$. We now have

$$g(a) = \frac{1}{2\pi i} \oint_{\partial\mathbb{A}(a,r)} \frac{g(z)}{z-a}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g\left(a + r \cdot e^{it}\right)}{re^{it}} \cdot rie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} g\left(a + re^{it}\right) dt.$$

We can therefore get a bound on $|g(a)|$ as

$$|g(a)| \leq \max_{\partial\mathbb{A}(a,r)} |g|,$$

but as the condition on f can be rewritten as

$$|g(a)| > \max_{\partial\mathbb{A}(a,r)} |g|,$$

we have reached a contradiction. $\square$

Theorem 1.1.8 (Open mapping)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ a function. If f is not constant, it is an open map.

Proof. Let $U \subseteq \Omega$ be an open set and $w_0 \in f(U)$. Choose a $z_0 \in U$ such that $f(z_0) = w_0$. Choose a $\rho > 0$ such that $\mathbb{A}(z_0, \rho) \subseteq U$ and z_0 is the only pre-image of w_0 in $\mathbb{A}(z_0, 2\rho)$.[1]

Since $\partial\mathbb{A}(z_0, \rho)$ is a compact set and

$$|f(z) - w_0| > 0$$

for all $z \in \partial\mathbb{A}(z_0, \rho)$, we can choose some $\varepsilon > 0$ such that

$$|f(z) - w_0| > 2\varepsilon$$

holds on the boundary of the disk. Choose a $w \in \mathbb{A}(w_0, \varepsilon)$. As we have

$$|f(z) - w| > |f(z) - w_0| - |w_0 - w| \geq \varepsilon$$

on the boundary and
$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$
by the above lemma, $f(z_0) - w$ has a root on $\mathbb{A}(z, \rho)$. $\square$

Theorem 1.1.9 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain. If the modulus $|f|$ of a function $f \in \mathcal{O}(\Omega)$ attains a local maximum, the function f is constant.

[1] If such a disk does not exist, f is constant by the identity theorem.

Proof. Suppose that f is non-constant and that its modulus attains a local maximum at $z \in \Omega$. As f is an open map, it also attains the value $(1+\varepsilon) \cdot f(z)$, which is a contradiction as the modulus then equals $(1+\varepsilon) \cdot |f(z)| > |f(z)|$. $\square$

Theorem 1.1.10 (Maximum principle)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and assume that $f \in \mathcal{O}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$. Then, the maximum of $|f|$ is attained in the boundary $\partial\Omega$.

Proof. As $\overline{\Omega}$ is compact, f attains a global maximum on this set. If the maximum is attained in the interior, f is constant, therefore it is also attained on the boundary. $\square$

Definition 1.1.11. A function $f \colon \Omega \setminus \{a\} \to \mathbb{C}$ is *locally bounded* near a if there exists an open neighbourhood $U \subseteq \Omega$ of a such that $f|_{U\setminus\{a\}}$ is bounded.

Theorem 1.1.12 (Riemann removable singularity theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$. If f is locally bounded near a, then there exists a unique function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega\setminus\{a\}} = f$.

Proof. Define the function $F \colon \Omega \to \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\}, \\ \dfrac{1}{2\pi i} \displaystyle\oint_{\partial\mathbb{\Delta}(a,\rho)} \dfrac{f(w)}{w-a}\, dw & z = a. \end{cases}$$

It remains to check that F is complex differentiable at a. Indeed, for $z \in \mathbb{\Delta}(a,\rho)$ we have

$$\begin{aligned} \lim_{z\to a} \frac{F(z)-F(a)}{z-a} &= \lim_{z\to a} \frac{1}{z-a} \oint_{\partial\mathbb{\Delta}(a,\rho)} \left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a}\right) dw \\ &= \lim_{z\to a} \frac{1}{2\pi i} \cdot \frac{1}{z-a} \cdot \oint_{\partial\mathbb{\Delta}(a,\rho)} f(w) \cdot \frac{z-a}{(w-z)(w-a)}\, dw \\ &= \frac{1}{2\pi i} \oint_{\partial\mathbb{\Delta}(a,\rho)} \frac{f(w)}{(w-a)^2}\, dw, \end{aligned}$$

which exists. Uniqueness follows from the identity theorem. $\square$

Theorem 1.1.13 (Schwarz lemma)**.** Let $f \colon \mathbb{\Delta} \to \mathbb{\Delta}$ be a holomorphic function with $f(0) = 0$. Then, $|f'(0)| \le 1$ and the inequality $|f(z)| \le |z|$ holds for all $z \in \mathbb{\Delta}$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ holds for any $z \neq 0$, then $f(z) = \beta z$ for some $\beta \in \partial\mathbb{\Delta}$.

Proof. We can write

$$f(z) = \sum_{k=1}^{\infty} c_k z^k.$$

We define

$$g(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

The radius of convergence for both series is at least 1. Now apply the maximum principle for g on the domain $\mathbb{\Delta}(\rho)$. We get

$$\sup_{z\in\mathbb{\Delta}(\rho)} |g(z)| \le \max_{|z|=\rho} |g(z)| = \frac{1}{\rho} \max_{|z|=\rho} |f(z)| < \frac{1}{\rho}.$$

In the limit as $\rho \to 1$, it follows that

$$\sup_{z \in \mathbb{\Delta}} |g(z)| \leq 1.$$

It immediately follows that $|f'(0)| = |g(0)| \leq 1$. Also note that

$$\frac{|f(z)|}{|z|} \leq \frac{1}{\rho},$$

which in the limit gives

$$|f(z)| \leq |z| .$$

Suppose we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. As then $|g(z_0)| = 1$, it follows that g is constant, therefore $f(z) = \beta z$ for some $\beta \in \partial \mathbb{\Delta}$. If we have $|f'(0)| = 0$, the same argument works for $z_0 = 0$. $\square$

1.2 The $\overline{\partial}$ equation

Lemma 1.2.1. Let $g \in \mathcal{C}^\infty(\mathbb{C})$ be a function with compact support. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Let

$$f(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{g(w)}{w - z} \, dw \wedge d\overline{w}.$$

As

$$dw \wedge d\overline{w} = -2ri \, dr \wedge d\varphi$$

holds for polar coordinates centered at z, we can express the integral as

$$f(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{rg\left(z + re^{i\varphi}\right)}{re^{i\varphi}} \, dr \wedge d\varphi.$$

We can further simplify the integral, as there exists some R such that $g|_{\mathbb{C}\setminus\Delta(z,R)} = 0$. We get

$$f(z) = -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi,$$

which obviously converges. The function f is therefore well defined. As we are integrating a smooth function on a compact set, the function f is smooth as well.

For $u = re^{i\varphi}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}}(z) &= -\frac{1}{\pi} \iint\limits_{\Delta(z,R)} \frac{\partial}{\partial \overline{z}} g\left(z + re^{i\varphi}\right) e^{-i\varphi} \, dr \wedge d\varphi \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial}{\partial \overline{z}} g(u+z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(0,R)} \frac{\partial g}{\partial \overline{u}}(u+z) \frac{1}{u} \, du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w-z} \, dw \wedge d\overline{w}.
\end{aligned}$$

Now we can apply the inhomogeneous Cauchy integral formula. We get

$$g(z) = \frac{1}{2\pi i} \oint\limits_{\partial\Delta(z,R)} \frac{g(w)}{w-z} \, dw + \frac{1}{2\pi i} \iint\limits_{\Delta(z,R)} \frac{\partial g}{\partial \overline{w}}(w) \frac{1}{w-z} \, dw \wedge d\overline{w}.$$

by the choice of R, we get

$$\frac{\partial f}{\partial \overline{z}}(z) = g(z).$$

□

Lemma 1.2.2. Given bounded domain $U \subset V \subset \mathbb{R}^n$ such that $\partial U \cap \partial V = \emptyset$, there exists a smooth function $\chi \colon \mathbb{R}^n \to [0,1]$ such that $\chi|_U = 1$ and $\operatorname{supp} \chi \subseteq V$.

Proof. There is a partition of unity on the sets V and $\mathbb{R}^n \setminus \overline{U}$. □

Lemma 1.2.3. Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $h_j \colon \Omega \to \mathbb{C}$ be holomorphic functions. If the sequence $(h_j)_{j\in\mathbb{N}}$ converges uniformly on compact sets, the limit is also holomorphic on Ω.

Proof. Apply Morera's theorem.[2] □

Theorem 1.2.4 (Dolbeault lemma)**.** Let $g \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ for some $R \in (0, \infty]$. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{\Delta}(R))$ such that $\frac{\partial f}{\partial \overline{z}} = g$.

Proof. Define disks X_j as follows:

i) If $R = \infty$, set $X_j = \mathbb{\Delta}(j)$.

ii) If $R < \infty$, set $X_j = \mathbb{\Delta}\left(R - \frac{1}{j}\right)$ (for large enough j).

Applying the above lemma, define functions χ_j with $\chi_j|_{X_j} = 1$ and $\operatorname{supp} \chi_j \subseteq X_{j+1}$ and set

$$g_j = \begin{cases} \chi_j \cdot g & z \in \mathbb{\Delta}(R), \\ 0 & z \notin \mathbb{\Delta}(R). \end{cases}$$

This is of course a smooth function, so by lemma 1.2.1 there exists a function $f_j \in \mathcal{C}^\infty(\mathbb{C})$ with

$$\frac{\partial f_j}{\partial \overline{z}} = g_j.$$

We inductively construct a new sequence $\widetilde{f}_j \in \mathcal{C}^\infty(\mathbb{C})$ such that

$$\frac{\partial \widetilde{f}_j}{\partial \overline{z}} = g$$

on X_j and

$$\left\| \widetilde{f}_j - \widetilde{f}_{j-1} \right\|_{X_{j-2}} \leq 2^{-j}.$$

Set $\widetilde{f}_1 = f_1$. Observe the function $F = f_{j+1} - \widetilde{f}_j$ on X_j. By construction, we have $\frac{\partial F}{\partial \overline{z}} = 0$ on X_j. It follows that F can be developed into a power series

$$F = \sum_{k=0}^{\infty} c_k z^k$$

on X_j. As power series converge uniformly on compact sets, there exists some polynomial $p \in \mathbb{C}[z]$ such that

$$\|F - p\|_{X_{j-1}} \leq 2^{-j}.$$

Now just set $\widetilde{f}_{j+1} = f_{j+1} - p$.

Let $z \in \mathbb{\Delta}(R)$ be arbitrary. By construction, it is contained in some X_{j_0}, therefore, $\widetilde{f}_j$ is defined for $j \geq j_0$. As $\left(\widetilde{f}_j(z)\right)_{j \geq j_0}$ is a Cauchy sequence, we can define

$$f(z) = \lim_{j\to\infty} \widetilde{f}_j(z).$$

[2] Analysis 2b, theorem 3.4.6.

But as

$$f - \widetilde{f}_j = \sum_{k=j}^{\infty} \left(\widetilde{f}_{j+1} - \widetilde{f}_j \right)$$

is a sum of holomorphic functions that converges uniformly, the function $f - \widetilde{f}_j$ is a holomorphic function. Therefore, f is smooth and satisfies $\frac{\partial f}{\partial \bar{z}} = g$. $\square$

1.3 Meromorphic functions

Definition 1.3.1. Let $\Omega \subset \mathbb{C}$ be an open subset. We call a function f *meromorphic* of Ω if there exists $A \subset \Omega$ such that $f \in \mathcal{O}(\Omega \setminus A)$, A has no accumulation points in Ω and for all $a \in A$ there exists some $k \in \mathbb{N}$ such that

$$\lim_{z \to a} f(z) \cdot (z-a)^k \neq 0$$

exists. We call A the set of *poles* of the function f. We denote the set of meromorphic functions on Ω with $\mathcal{M}(\Omega)$.

Theorem 1.3.2. Let $0 \leq r < R \leq \infty$. Suppose that $f \in \mathcal{O}(D_{R,r}(a))$ is a holomorphic function, where

$$D_{R,r}(a) = \{z \in \mathbb{C} \mid r < |z-a| < R\}.$$

Then there exists a uniquely determined *Laurent series*

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

that converges to f uniformly and absolutely on compact subsets of $D_{R,r}(a)$. We have

$$c_k = \frac{1}{2\pi i} \oint_{\partial \Delta(a,\rho)} \frac{f(w)}{(w-a)^k} \, dw$$

for $r < \rho < R$.

Definition 1.3.3. Let

$$\sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

be a Laurent series. The series

$$\sum_{k=-\infty}^{-1} c_k (z-a)^k$$

is called the *principle part*.

Lemma 1.3.4. Let $f \in \mathcal{O}(\Omega \setminus \{a\})$ be a holomorphic function. Then f is meromorphic on Ω if and only if f has a finite principle part in a.

Proof. Suppose that f is meromorphic on Ω. If a is a removable singularity, f is holomorphic in a, therefore the principle part is trivial. Otherwise, set $m \in \mathbb{N}$ such that

$$\lim_{z \to a} (z-a)^m f(z) \neq 0$$

exists and set $g(z) = (z-a)^m f(z)$. As g is bounded near a, we can extend it to Ω by the Riemann removable singularity theorem. The power series of g corresponds to a finite Laurent series of f.

The converse is obvious. $\square$

Theorem 1.3.5. If $f \in \mathcal{M}(\mathbb{C})$ is a meromorphic function, there exist entire functions g and h such that $f = \frac{g}{h}$.

Definition 1.3.6. Let $\Omega \subseteq \mathbb{C}$ be an open set. An *additive Cousin problem* on Ω is an open cover $\{U_j\}_{j\in J}$ of Ω and functions $f_j \in \mathcal{M}(U_j)$ such that $f_j - f_k|_{U_j\cap U_k}$ is holomorphic for all $j,k \in J$. A function $f \in \mathcal{M}(\Omega)$ is a solution to the additive Cousin problem if $f|_{U_j} - f_j$ is holomorphic for all $j \in J$.

Definition 1.3.7. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A *generalized additive Cousin problem* is an open cover $\{U_j\}_{j\in J}$ of Ω and functions $f_{j,k} \in \mathcal{O}(U_j \cap U_k)$ for each $(j,k) \in J^2$, such that

i) $f_{j,k} = -f_{k,j}$ on $U_j \cap U_k$ for all $(j,k) \in J^2$ and

ii) $f_{j,k} + f_{k,\ell} + f_{\ell,j} = 0$ on $U_j \cap U_k \cap U_\ell$ for all $(j,k,\ell) \in J^3$.

A solution to the generalized additive Cousin problem is given by functions $f_j \in \mathcal{O}(U_j)$ for each $j \in J$ such that $f_{j,k} = f_j - f_k$ for each $(j,k) \in J^2$.

Lemma 1.3.8 (Partition of unity)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $\{U_j\}_{j\in J}$ be an open cover of Ω. Then there exists a partition of unity subordinate to $\{U_j\}_{j\in J}$.

Lemma 1.3.9. Given a generalized additive Cousin problem on $\Omega \subseteq \mathbb{C}$, there exist functions $g_j \in \mathcal{C}^\infty(U_j)$ such that $f_{j,k} = g_j - g_k$ for all $(j,k) \in J^2$.

Proof. Let $\{(V_a, \chi_a)\}_{a\in A}$ be a partition of unity, subordinate to $\{U_j\}_{j\in J}$. For all $a \in A$ choose a $j(a) \in J$ such that $V_a \subseteq U_{j(a)}$. For all $k \in J$, define

$$g_k = -\sum_{a\in A} \chi_a \cdot f_{j(a),k}.$$

This is of course a smooth function on U_k. Now note that

$$g_k - g_\ell = \sum_{a\in A} \chi_a \cdot \left(-f_{j(a),k} + f_{j(a),\ell}\right) = \sum_{a\in A} \chi_a \cdot f_{k,\ell} = f_{k,\ell}.$$ □

Proposition 1.3.10. The generalized additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

Proof. Let $f_{j,k} = g_j - g_k$ for $g_j \in \mathcal{C}^\infty(U_j)$. Note that

$$\frac{\partial g_j}{\partial \overline{z}} = \frac{\partial g_k}{\partial \overline{z}},$$

therefore,

$$h|_{U_j} = \frac{\partial g_j}{\partial \overline{z}}$$

induces a smooth function $h \colon \Omega \to \mathbb{C}$. By the Dolbeault lemma, there exists a function $g \in \mathcal{C}^\infty(\Omega)$ such that $\frac{\partial g}{\partial \overline{z}} = h$. It is clear that $f_j = g_j - g$ solves the generalized additive Cousin problem. □

Proposition 1.3.11. The additive Cousin problem is solvable for $\Omega = \mathbb{\Delta}(r)$ and $\Omega = \mathbb{C}$.

Proof. An additive Cousin problem induces a generalized additive Cousin problem for functions $f_{j,k} = f_j - f_k$. Let g_j be a solution to the generalized problem. As $f_j - f_k = f_{j,k} = g_j - g_k$ on $U_j \cap U_k$, we can define a function $f \in \mathcal{M}(\Omega)$ with $f|_{U_j} = f_j - g_j$. This function is of course well defined. As $f|_{U_j} - f_j = g_j \in \mathcal{O}(U_j)$, this function indeed solves the additive Cousin problem. □

Theorem 1.3.12 (Mittag-Leffler)**.** Let $(a_k)_{k\in\mathbb{N}}$ be a sequence without repetition and accumulation points. Let

$$f_k(z) = \sum_{\ell=-m_k}^{-1} c_{k,\ell}(z-a_k)^\ell$$

be finite principal parts. Then there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ with poles in $(a_k)_{k\in\mathbb{N}}$ such that f has principle part f_k in a_k for each $k \in \mathbb{N}$.

Proof. For each a_k choose a disk U_k containing no other a_k. Also set $U_0 = \mathbb{C} \setminus \{a_k \mid k \in \mathbb{N}\}$ and $f_0 = 0$. As $\{U_k \mid k \in \mathbb{N}_0\}$ is an open cover of $\mathbb{C}$, there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ that solves the corresponding additive Cousin problem. It is easy to see that the principle parts of f at a_k are precisely f_k. $\square$

1.4 Sequences of holomorphic functions

Definition 1.4.1. A family of functions $\mathcal{F}$ from Ω to $\mathbb{C}$ is *locally bounded*, if for all $p \in \Omega$ there exist a $\rho > 0$ and $M > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{z \in \Delta(p,\rho)} |f(z)| < M.$$

Lemma 1.4.2. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ a locally bounded family of functions. Then for all $p \in \Omega$ there exists a $\rho > 0$ such that $\mathcal{F}$ is equi-continuous on $\Omega \cap \Delta(p, \rho)$.

Proof. Fix $p \in \Omega$ and choose $r > 0$ such that $D = \overline{\Delta(p, 2r)} \subseteq \Omega$. For any $z, w \in D$ and $f \in \mathcal{F}$ we have

$$f((z) - f(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{\xi - w} \, d\xi = \frac{z-w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} \, d\xi.$$

Note that the family $\mathcal{F}$ is bounded on every compact. Therefore, we can write

$$\sup_{f \in \mathcal{F}} \sup_{z \in \partial D} |f(z)| < M.$$

Now, for $z, w \in \Delta(p, r)$ we have

$$|f((z) - f(w)| = \left| \frac{z-w}{2\pi i} \oint_{\partial D} \frac{f(\xi)}{(\xi - z)(\xi - w)} \, d\xi \right| \leq |z - w| \cdot \frac{2M}{r}.$$

□

Theorem 1.4.3 (Arzelà-Ascoli)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be an infinite family such that the following conditions hold:

i) $\mathcal{F}$ is point-wise bounded.

ii) $\mathcal{F}$ is locally equi-continuous.

Then there $\mathcal{F}$ contains a sequence that converges uniformly on compacts of Ω.

Proof. Choose a dense countable subset $A \subseteq \Omega$ and enumerate it as a sequence $(a_k)_{k \in \mathbb{N}}$. Pick any sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ with pairwise distinct terms. As $|f_n(a_1)| < M$ for all n, we can choose a subsequence $(f_{1,n})_{n \in \mathbb{N}}$ such that $f_{1,n}(a_1)$ converges by Bolzano-Weierstraß.

Similarly, for every $k \in \mathbb{N}$ there exists a subsequence $(f_{k,n})_n$ of $(f_{k-1,n})_n$ such that $(f_{k,n}(a_k))_n$ converges. Now define $F_n = f_{n,n}$. Observe that (F_n) converges at every point in A.

Fix a $p \in \Omega$. By local equi-continuity, there exists a $\rho > 0$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta < \rho$ and $|F_n(z) - F_n(w)| < \frac{\varepsilon}{3}$ for all $z, w \in \Delta(p, \rho)$ such that $|z - w| < \delta$. Choose an element $a \in A \cap \Delta(z, \delta)$.[3] Then,we have

$$|F_n(z) - F_m(z)| \leq |F_n(z) - F_n(a)| + |F_n(a) - F_m(a)| + |F_m(a) - F_m(z)| < 3 \cdot \frac{\varepsilon}{3}.$$

It follows that (F_n) is locally uniformly convergent, therefore it converges uniformly on compact sets. □

[3] By compactness of $\overline{\Delta(p, \rho)}$ we can choose a from a finite set.

Theorem 1.4.4 (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f_n \colon \Omega \to \mathbb{C}$ be a locally bounded sequence of holomorphic functions. Then $(f_n)_n$ contains a subsequence that converges uniformly on compacts.

Proof. As the sequence is locally bounded, it is locally equi-continuous by lemma 1.4.2. By Arzelà-Ascoli, there exists a convergent subsequence. □

Definition 1.4.5. Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family of functions $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is *normal* if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on compacts.

Theorem 1.4.6 (Montel)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is normal if and only if it is locally bounded.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.4.7 (Vitali)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $(f_n)_n \subseteq \mathcal{O}(\Omega)$ a locally bounded sequence of holomorphic functions. The following statements are equivalent:

i) The sequence $(f_n)_n$ converges uniformly on compact subsets of Ω.

ii) For a point $p \in \Omega$ the sequence $(f_n^{(k)}(p))_n$ converges for all $k \in \mathbb{N}_0$.

iii) The set
$$A = \left\{z \in \Omega \;\middle|\; \lim_{n\to\infty} f_n(z) \text{ converges}\right\}$$
has an accumulation point in Ω.

Proof. Suppose that the sequence converges uniformly on compact subsets. Given a $p \in \Omega$, choose a $\delta > 0$ such that $D = \overline{\mathbb{\Delta}(p, \delta)} \subseteq \Omega$. Note that

$$\left|g^{(k)}(p)\right| \le \frac{k!}{\delta^k} \cdot \|g\|_D$$

holds for all holomorphic functions g. As $\|f - f_n\|$ converges to 0, the derivatives of f_n converge.

Suppose that the sequences of derivatives converge at a point $p \in \Omega$ and choose a $\delta > 0$ such that $D = \overline{\mathbb{\Delta}(p, \delta)} \subseteq \Omega$. As the sequence is locally bounded, there exists a constant M such that $\|f_n\|_D \le M$ holds for all $n \in \mathbb{N}$. We can now develop power series

$$f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z - p)^k.$$

They converge uniformly on compact subsets of $\mathbb{\Delta}(p, \delta)$. Note that

$$a_{k,n} = \frac{f_n^{(k)}(p)}{k!}.$$

As derivatives converge, we can define the limit

$$a_k = \lim_{n\to\infty} a_{k,n}.$$

Now define the formal power series

$$f(z)=\sum_{k=0}^{\infty} a_k(z-p)^k.$$

The Cauchy bounds give us the inequality

$$|a_{k,n}|=\frac{\left|f^{(k)}(p)\right|}{k!}\le \frac{M}{\delta^k},$$

therefore

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|}\le \limsup_{k\to\infty} \frac{\sqrt[k]{M}}{\delta}=\frac{1}{\delta}.$$

We conclude that the radius of convergence is at least δ. Consider some $\rho\in(0,\delta)$ and $z\in \mathbb{\Delta}(p,\rho)$. We have

$$\begin{aligned}
|f_n(z)-f(z)| &\le \left|\sum_{k=0}^{m}(a_{k,n}-a_k)\cdot(p-z)^k\right|+\left|\sum_{k=m+1}^{\infty}(a_{k,n}-a_k)\cdot(p-z)^k\right| \\
&\le \sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+\sum_{k=m+1}^{\infty}2M\cdot\frac{\rho^k}{\delta^k} \\
&= \sum_{k=0}^{m}|a_{n,k}-a_k|\,\rho^k+2M\cdot\left(\frac{\rho}{\delta}\right)^{m+1}\cdot\frac{\delta}{\delta-\rho} \\
&= 2\cdot\frac{\varepsilon}{2}
\end{aligned}$$

for large enough m and n. It follows that p is an accumulation point of A.

Suppose now that A has an accumulation point in Ω. By Montel's theorem there exists a subsequence $(f_{n_m})_m$ that converges uniformly on compact subsets of Ω to a limit function f. Note that all such subsequences have the same limit by the identity principle.

Assume that the sequence $(f_n)_n$ does not converge uniformly on a compact subset $K\subseteq\Omega$. We can therefore construct another subsequence $(g_n)_n$ of $(f_n)_n$ such that

$$\|g_n-f\|_K>\varepsilon$$

for all $n\in\mathbb{N}$. But note that $(g_n)_n$ also has a convergent subsequence by Montel's theorem, which is of course a contradiction, as it cannot converge to f. $\square$

2 Theorems about holomorphic functions

2.1 Riemann mapping theorem

Definition 2.1.1. A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if every closed path in Ω is homotopic to a constant path in Ω.

Lemma 2.1.2. Let $\Omega \subset \mathbb{C}$ be a domain and $a \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Then there exists a holomorphic injection $f \colon \Omega \to \mathbb{\Delta}$ such that $f(a) = 0$.

Proof. Fix a point $p \in \mathbb{C} \setminus \Omega$. By our assumption, there exists a function $v \in \mathcal{O}^*(\Omega)$ such that $v(z)^2 = z - p$. Note that v is injective. Similarly, we have $v(\Omega) \cap -v(\Omega) = \emptyset$. Now choose a point $b \in -v(\Omega)$. As v is not constant, it is an open map. Therefore, there exists some $r > 0$ such that $\mathbb{\Delta}(b, r) \cap v(\Omega) = \emptyset$. The Möbius transformation

$$h(w) = r \cdot \left(\frac{1}{w - b} - \frac{1}{v(a) - b} \right)$$

thus maps $v(\Omega)$ into $\mathbb{\Delta}$. The map f is therefore given as $f = h \circ v$. $\square$

Definition 2.1.3. An *expansion* if a map $\kappa \colon \Omega \to \mathbb{\Delta}$ where $0 \in \Omega \subset \mathbb{\Delta}$ such that $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ holds for all $z \neq 0$.

Lemma 2.1.4. Let $\Omega \subset \mathbb{\Delta}$ be a domain with $0 \in \Omega$. Assume that Ω admits a square root for all function $g \in \mathcal{O}^*(\Omega)$. Choose $c \in \mathbb{\Delta}$ such that $c^2 \notin \Omega$. For all $a \in \mathbb{\Delta}$, let

$$g_a = \frac{z - a}{\overline{a} z - 1}$$

and choose $v \in \mathcal{O}(\Omega)$ such that $v(z)^2 = g_{c^2}(z)$ and $v(0) = c$. Then the map $\kappa = g_c \circ v$ is an expansion and

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = \mathrm{id}_\Omega .$$

Proof. Note that v is indeed well-defined. Also note that

$$g_{c^2} \circ (z \mapsto z^2) \circ g_c \circ \kappa = g_{c^2} \circ (z \mapsto z^2) \circ v = g_{c^2} \circ g_{c^2} = \mathrm{id} .$$

We of course have $\kappa(0) = 0$. Denote $\psi_c = g_{c^2} \circ (z \mapsto z^2) \circ g_c$. It remains to check that $|\kappa(z)| > |z|$, which is equivalent to $|\psi_c(z)| < |z|$ for $z \neq 0$ as $\psi_c \circ \kappa = \mathrm{id}$. Note that $\psi_c \colon \mathbb{\Delta} \to \mathbb{\Delta}$ is holomorphic. As it is not a rotation (it is not injective), the conclusion follows from the Schwarz lemma. $\square$

Lemma 2.1.5 (Hurwitz)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f_n \colon \Omega \to \mathbb{C}$ be holomorphic functions. Suppose that the sequence $(f_n)_n$ converges uniformly on compacts of Ω to a non-constant function $f \colon \Omega \to \mathbb{C}$. Then for all points $p \in \Omega$ there exists a sequence $(p_n)_n \subseteq \Omega$ with limit p such that $f_n(p_n) = f(p)$ for all $n > N$.

Proof. Let $w = f(p)$. There exists a disk $\mathbb{\Delta}(p, \delta)$ such that $f(z) \neq w$ for all points $z \in \overline{\mathbb{\Delta}(p, \delta)} \setminus \{p\}$. Note that we have

$$\min_{z \in \partial \mathbb{\Delta}(p, \delta)} |f(z) - w| > |f(p) - w| = 0.$$

As $(f_n)_n$ converges uniformly on $\overline{\mathbb{\Delta}(p, \delta)}$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\min_{z \in \partial \mathbb{\Delta}(p,\delta)} |f_n(z) - w| > |f_n(p) - w| .$$

By lemma 1.1.7, $f_n(z) - w$ has a root $p_n \in \mathbb{\Delta}(p, \delta)$. For any convergent subsequence $(p_{n_k})_k$ with limit q we have

$$f(p) = \lim_{k \to \infty} f_{n_k}(p_{n_k}) = f(q),$$

therefore $p = q$. □

Corollary 2.1.5.1. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n \colon \Omega \to \mathbb{C}$ be holomorphic functions such that $(f_n)_n$ converges uniformly on compacts of Ω to $f \colon \Omega \to \mathbb{C}$. If all the f_n are nowhere vanishing and $f \neq 0$, then f is nowhere vanishing.

Proof. The proof is obvious and need not be mentioned. □

Theorem 2.1.6 (Hurwitz)**.** Let $\Omega, \Omega' \subseteq \Omega$ be domains and $f_n \colon \Omega \to \Omega'$ be holomorphic functions that converge uniformly on compacts of Ω to $f \colon \Omega \to \Omega'$. Assume that f is not constant.

i) If $f_n \colon \Omega \to \Omega'$ is injective, f is also injective.

ii) We have $f(\Omega) \subseteq \Omega'$.

Proof.

i) Let $p \in \Omega$ and observe the functions $g_n(z) = f_n(z) - f_n(p)$. This is a sequence of nowhere vanishing functions. As f is not constant, $f(z) - f(p)$ is nowhere vanishing as well. It follows that f is injective.

ii) Suppose otherwise and apply the Hurwitz lemma for a point p with $f(p) \notin \Omega'$. □

Theorem 2.1.7 (Riemann mapping)**.** For a proper domain $\Omega \subset \mathbb{C}$ the following are equivalent:

i) Ω is simply connected.

ii) Ω admits a logarithm for any $f \in \mathcal{O}^*(\Omega)$.

iii) Ω admits a square root for any $f \in \mathcal{O}^*(\Omega)$.

iv) Ω is biholomorphic to $\mathbb{\Delta}$.

Proof. Note that if Ω is biholomorphic to $\mathbb{\Delta}$, it is of course simply connected. Suppose that Ω is simply connected. Then

$$F(z) = a + \int_{z_0}^{z} \frac{f'(z)}{f(z)} \, dz$$

defines a logarithm for any $f \in \mathcal{O}^*(\Omega)$. Given a logarithm of a function, we can of course construct a square root with $\sqrt{f} = e^{\frac{1}{2} \ln f}$. It remains to check that all domains admitting square roots are biholomorphic to $\mathbb{\Delta}$.

By lemma 2.1.2 we can assume that $\Omega \subseteq \mathbb{A}$ and $0 \in \Omega$. Now define the family of functions

$$\mathcal{F} = \{f \colon \Omega \to \mathbb{A} \mid f \in \mathcal{O}(\Omega) \wedge f(0) = 0 \wedge f \text{ is injective}\}.$$

If $\mathcal{F}$ has no biholomorphic map, it is infinite. Note that $\mathcal{F}$ is bounded, so it is normal by Montel.

Choose a point $p \in \Omega$ with $p \neq 0$. We claim that if $h \in \mathcal{F}$ and

$$|h(p)| = \sup_{f \in \mathcal{F}} |f(p)|,$$

we have $h(\Omega) = \mathbb{A}$. Indeed, if that were not the case, we'd reach a contradiction with the expansion κ of Ω as

$$|\kappa(h(p))| > |h(p)|$$

and $\kappa \circ h \in \mathcal{F}$.

Let

$$M = \sup_{f \in \mathcal{F}} |f(p)|$$

and find a sequence $(f_n)_n \subseteq \mathcal{F}$ with

$$\lim_{n \to \infty} |f_n(p)| = M.$$

As $\mathcal{F}$ is a normal family, there exists a convergent subsequence. The limit is not constant as $f(p) \neq 0$. By Hurwitz, f is injective and $f(\Omega) \subseteq \mathbb{A}$. By the above claim, we have $f(\Omega) = \mathbb{A}$. $\square$

2.2 Bloch's theorem

Lemma 2.2.1. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $f \colon \overline{\Omega} \to \mathbb{C}$ a continuous map such that $f|_{\Omega}$ is an open map. Let $a \in \Omega$ be a point such that

$$s = \min_{z \in \partial\Omega} |f(z) - f(a)| > 0.$$

Then $f(\Omega)$ contains the disk $\mathbb{\Delta}(f(a), s)$.

Proof. By compactness, there exists a $w_0 \in \partial f(\Omega)$ such that $d(\partial f(\Omega), f(a)) = |w_0 - f(a)|$. Let $(z_k)_k \subseteq \Omega$ be a sequence, convergent to z_0, such that

$$\lim_{k \to \infty} f(z_k) = w_0.$$

Of course $f(z_0) = w_0$. Note that, as $f|_{\Omega}$ is open, we have $z_0 \in \partial\Omega$. But then

$$d(\partial f(\Omega), f(a)) = |f(z_0) - f(a)| \geq s. \qquad \square$$

Lemma 2.2.2. Let f be a non-constant function, holomorphic in a neighbourhood of $\overline{\mathbb{\Delta}(a, r)}$. Assume that

$$\sup_{z \in \overline{\mathbb{\Delta}(a,r)}} |f'(z)| \leq 2 \left| f'(a) \right|.$$

Then $\mathbb{\Delta}(f(a), R) \subseteq f(\mathbb{\Delta}(a, r))$, where

$$R = \left(3 - 2\sqrt{2}\right) \cdot r \cdot \left| f'(a) \right|.$$

Proof. Without loss of generality assume that $a = f(a) = 0$. Define

$$A(z) = f(z) - f'(0)z = \int_0^1 \left(f'(tz) - f'(0)\right) z \, dt.$$

Note that

$$f'(v) - f'(0) = \frac{1}{2\pi i} \oint_{\partial\mathbb{\Delta}(a,r)} f'(\xi) \cdot \left(\frac{1}{\xi - v} - \frac{1}{\zeta}\right) d\xi,$$

therefore

$$|f'(v) - f'(0)| \leq \frac{1}{2\pi} \cdot |v| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r \cdot (r - |v|)} \cdot 2\pi r = |v| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r - |v|}.$$

It follows that

$$\begin{aligned}
|A(z)| &\leq \int_0^1 |z| \cdot |f'(tz) - f'(0)| \, dt \\
&\leq |z| \cdot \int_0^1 |tz| \cdot \frac{\|f'\|_{\mathbb{\Delta}(a,r)}}{r - |tz|} \, dt \\
&\leq |z|^2 \cdot \|f'\|_{\mathbb{\Delta}(a,r)} \cdot \int_0^1 t \cdot \frac{1}{r - |z|} \\
&= |z|^2 \cdot 2 \frac{|f'(0)|}{r - |z|}.
\end{aligned}$$

Now, using the triangle inequality, we get

$$|f(z)| \geq |z| \cdot |f'(0)| - |A(z)|.$$

Let $|z| = \rho \in (0, r)$. We get

$$|f(z)| \geq \rho \cdot |f'(0)| - |A(z)| \geq \rho \cdot |f'(0)| - \frac{\rho^2}{r-\rho} \cdot |f'(0)| \geq |f'(0)| \cdot \left(\rho - \frac{\rho^2}{r-\rho}\right).$$

Note that there exists a ρ_0 such that

$$\rho_0 - \frac{\rho_0^2}{r-\rho_0} = r \cdot \left(3 - 2\sqrt{2}\right).$$

Therefore, we get

$$|f(z)| \geq |f'(0)| \cdot r \cdot \left(3 - 2\sqrt{2}\right).$$

Now just apply the previous lemma to the disk $\mathbb{\Delta}(0, \rho_0)$. □

Theorem 2.2.3 (Bloch)**.** Let f be a function, holomorphic in a neighbourhood of $\overline{\mathbb{\Delta}}$, with $f'(0) = 1$. Then $f(\mathbb{\Delta})$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.

Proof. Define $h(z) = |f'(z)|\,(1 - |z|) \geq 0$. Not that $h \not\equiv 0$ as f is not constant. Therefore h attains a maximum in a point $p \in \overline{\mathbb{\Delta}}$. In particular, as $h|_{\partial\mathbb{\Delta}} = 0$, we have $p \in \mathbb{\Delta}$. Observe $\Omega = \mathbb{\Delta}\,(p, t)$ for $t = \frac{1}{2} \cdot (1 - |p|)$. For all $z \in \Omega$, we have $1 - |z| \geq t$ and

$$|f'(z)| \cdot (1 - |z|) \leq |f'(p)| \cdot (1 - |p|) = |f'(p)| \cdot 2t \leq |f'(p)| \cdot 2 \cdot (1 - |z|).$$

Now, applying lemma 2.2.2, we have $\mathbb{\Delta}(f(p), R) \subseteq f(\mathbb{\Delta})$ with

$$R = \left(3 - 2\sqrt{2}\right) \cdot \frac{1}{2} \cdot (1 - |p|) \cdot |f'(p)| \geq \frac{3}{2} - \sqrt{2}$$

by choice of p. □

Remark 2.2.3.1. Let

$$\mathcal{F} = \left\{ f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{\Delta}} \;\middle|\; f'(0) = 1 \right\}.$$

For $f \in \mathcal{F}$, denote by L_f the supremum of radii of disks contained in $f(\mathbb{\Delta})$, and by B_f the supremum of radii of disks contained in $f(\mathbb{\Delta})$ that is a biholomorphic image of some subdomain of $\mathbb{\Delta}$. We then define the *Landau's constant*

$$L = \inf_{f \in \mathcal{F}} L_f$$

and the *Bloch's constant*

$$B = \inf_{f \in \mathcal{F}} B_f.$$

The current known bounds for the constants are

$$0.5 < L < 0.544 \quad \text{and} \quad \frac{\sqrt{3}}{4} + 10^{-14} < B \leq \sqrt{\frac{\sqrt{3}-1}{2}} \cdot \frac{\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

Corollary 2.2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{O}(\Omega)$ a function and $p \in \Omega$. Let $r = d(p, \partial\Omega)$. Then $f(\Omega)$ contains a disk of radius

$$\left(\frac{3}{2} - \sqrt{2}\right) \cdot r \cdot |f'(p)|.$$

Proof. The proof is obvious and need not be mentioned. □

Remark 2.2.3.3. Liouville's theorem follows from this corollary.

Lemma 2.2.4. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $1, -1 \notin f(\Omega)$. Then there exists a function $F \in \mathcal{O}(\Omega)$ such that $f = \cos(F)$.

Proof. Note that, as Ω is simply connected, we can define

$$F(z) = \frac{1}{i} \cdot \ln\left(f(z) + \sqrt{f(z)^2 - 1}\right).$$ □

Theorem 2.2.5. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let $f \in \mathcal{O}(\Omega)$. Suppose that $0, 1 \notin f(\Omega)$. Then the following statements are true:

i) There exists a function $g \in \mathcal{O}(\Omega)$ such that

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right).$$

ii) If any $g \in \mathcal{O}(\Omega)$ satisfies the above equality, then $g(\Omega)$ contains no disk of radius 1.

Proof.

i) Apply the previous lemma twice.

ii) Define

$$A = \left\{m \pm \frac{i}{\pi} \ln\left(n + \sqrt{n^2 - 1}\right) \;\middle|\; m \in \mathbb{Z} \wedge n \in \mathbb{N}\right\}.$$

We claim that $g(\Omega) \cap A = \emptyset$. Indeed, for $a \in A$ we have

$$f(a) = \frac{1}{2}\left(1 + \cos(\pm\pi \cdot n)\right) \in \{0, 1\}.$$

Now note that

$$\begin{aligned} \ln\left(n + 1 + \sqrt{n^2 + 2n}\right) - \ln\left(n + \sqrt{n^2 - 1}\right) &= \ln\left(\frac{n + 1 + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 - 1}}\right) \\ &\le \ln\left(\frac{2n + 2}{n}\right) \\ &\le \ln(4) \\ &< \pi. \end{aligned}$$

It's straightforward to check that every disk of radius 1 intersects A. □

Theorem 2.2.6 (Picard's little theorem)**.** Every non-constant entire function omits at most one complex value.

Proof. Without loss of generality assume that f omits 0 and 1. Applying the above theorem, we can write

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right).$$

Recall that $g(\mathbb{C})$ contains no disk of radius 1. If g is not constant, $g(\mathbb{C})$ contains arbitrarily large disks by corollary 2.2.3.2, which is a contradiction. □

Corollary 2.2.6.1. Suppose that $f \in \mathcal{M}(\mathbb{C})$ is a non-constant function. Then f omits at most 2 values.

Proof. Suppose that f omits distinct values a, b and c. Then

$$g(z) = \frac{1}{f(z) - a}$$

is an entire function that omits values $\frac{1}{b-a}$ and $\frac{1}{c-a}$, therefore it is constant. $\square$

Theorem 2.2.7. Let $f \in \mathcal{O}(\mathbb{C})$ be an entire function. Then either $f \circ f$ has a fixed point of $f(z) = z + c$.

Proof. If $f \circ f$ has no fixed point, the same holds for f. We can therefore define an entire holomorphic function g with

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}.$$

Note that g omits both 0 and 1, therefore it is constant. But then

$$f(f(z)) - z = \lambda(f(z) - z)$$

for some $\lambda \notin \{0, 1\}$ by Picard's little theorem. Taking the derivative, we get

$$f'(f(z)) \cdot f'(z) - 1 = \lambda(f'(z) - 1),$$

or equivalently

$$f'(z) \cdot (f'(f(z)) - \lambda) = 1 - \lambda \neq 0.$$

Note that $f' \circ f$ omits both λ and 0, therefore it is constant. But then f' is constant as well. The only option is $f'(z) = 1$. $\square$

Lemma 2.2.8. For all $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos(\pi v) = w$ and $|v| \leq 1 + |w|$.

Proof. Let $v = \alpha + i\beta$ and note that

$$|w|^2 = \cos(\pi\alpha)^2 + \sinh(\pi\beta)^2 \geq \pi^2\beta^2.$$

Observe that we can choose some α such that $|\alpha| \leq 1$, therefore

$$1 + |w| \geq 1 + \pi \cdot |\beta| \geq |\alpha| + |\beta| \geq |v|. \quad \square$$

Theorem 2.2.9. Let f be a function, holomorphic on a neighbourhood of $\overline{\mathbb{A}}$, such that $0, 1 \notin f(\Omega)$. There exists a function g, holomorphic on a neighbouhood of $\overline{\mathbb{A}}$, such that

i) the equality

$$f = \frac{1}{2}\left(1 + \cos(\pi \cdot \cos(\pi \cdot g))\right)$$

holds with $|g(0)| \leq 3 + 2\,|f(0)|$, and

ii) the inequality

$$|g(z)| \leq |g(0)| + \frac{\theta}{\gamma(1 - \theta)}$$

holds for all $|z| \leq \theta$.

Proof. Again, apply lemma 2.2.4 and let

$$2f - 1 = \cos(\pi \cdot F).$$

Using the above lemma, we can transform F such that $|F(0)| \le 1 + |2f(0) - 1|$. Applying lemma 2.2.4 again, we define g such that

$$F = \cos(\pi g).$$

Again, using the above lemma, set $|g(0)| \le 1 + |F(0)|$. We therefore have

$$|g(0)| \le 1 + |F(0)| \le 2 + |2f(0) - 1| \le 3 + 2\,|f(0)|\,.$$

Recall that $g(\mathbb{\Delta})$ does not contain a disk of radius 1. Let $z \in \overline{\mathbb{\Delta}(\theta)}$. Then, by Bloch's theorem, $g(\mathbb{\Delta})$ contains a disk of radius $R = \gamma \cdot |g'(z)| \cdot (1 - \theta)$. Therefore, we must have

$$|g'(z)| < \frac{1}{\gamma(1 - \theta)}.$$

It follows that

$$|g(z)| = \left|g(0) + \int_0^z g'(\xi)\, d\xi\right| \le |g(0)| + \int_0^z |g'(\xi)|\; d\xi \le |g(0)| + |z| \cdot \frac{1}{\gamma(1 - \theta)}. \qquad \square$$

Definition 2.2.10. For $r \ge 0$, let

$$S(r) = \left\{f \text{ holomorphic on a neighbourhood of } \overline{\mathbb{\Delta}} \;\middle|\; 0, 1 \notin f\left(\overline{\mathbb{\Delta}}\right) \wedge |f(0)| \le r\right\}.$$

For $\theta \in (0, 1)$ and $r > 0$, let

$$L(\theta, r) = \exp\left(\pi \cdot \exp\left(3 + 2r + \frac{\theta}{\gamma(1 - \theta)}\right)\right),$$

where γ is any constant such that Bloch's theorem holds, e.g. $\gamma = \frac{3}{2} - \sqrt{2}$.

Theorem 2.2.11 (Schottky)**.** Let $f \in S(r)$. Then for all $z \in \mathbb{\Delta}$ such that $|z| < \theta$ we have

$$|f(z)| \le L(\theta, r).$$

Proof. Let g be a holomorphic function as in the previous theorem. Note that $|\cos(w)| \le e^{|w|}$. We must therefore also have

$$\frac{1}{2} \cdot |1 + \cos(w)| \le e^{|w|}.$$

Using this inequality, we get

$$|f(z)| \le \exp\left(\pi \cdot \exp\left(\pi \cdot |g(z)|\right)\right) \le L(\theta, r). \qquad \square$$

2.3 The great Picard theorem

Lemma 2.3.1. Let $\Omega \subseteq \mathbb{C}$ be a domain, $\omega \in \Omega$ and $r \in (0, \infty)$. Let

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

and $\mathcal{F}_{\omega,r} \subseteq \mathcal{F}$ a subfamily with $|f(\omega)| \leq r$ for all $f \in \mathcal{F}_{\omega,r}$.

i) There exists some $t > 0$ such that $\mathcal{F}_{\omega,r}|_{\Delta(\omega,t)}$ is bounded.

ii) The family $\mathcal{F}_{\omega,1}$ is locally bounded in Ω.

Proof.

i) Choose a $t > 0$ such that $\overline{\Delta(\omega, 2t)} \subseteq \Omega$ and set $\varphi(z) = 2tz + \omega$. By Schottky's theorem, we have

$$|f \circ \varphi(z)| \leq L\left(\frac{1}{2}, r\right)$$

for $|z| < \frac{1}{2}$, or equivalently

$$\sup_{v \in \Delta(w,t)} |f(v)| \leq L\left(\frac{1}{2}, r\right).$$

The family $\mathcal{F}_{\omega,r}$ is therefore bounded.

ii) Let

$$\mathcal{U} = \{u \in \Omega \mid \mathcal{F}_{\omega,1} \text{ is bounded in a neighbourhood of } u\}.$$

Note that $\omega \in \mathcal{U}$, therefore the set is non-empty. Also observe that $\mathcal{U}$ is open. Suppose that $\mathcal{U} \neq \Omega$ and let $v \in \partial\mathcal{U} \cap \Omega$. Then there exists a sequence $(f_n)_n \subseteq \mathcal{F}_{\omega,1}$ such that

$$\lim_{n\to\infty} |f_n(v)| = \infty.$$

Define $g_n = \frac{1}{f_n}$. These functions are holomorphic and omit both 0 and 1 by definition, therefore $g_n \in \mathcal{F}$. Applying the item i) for the sequence $(g_n)_n$ at point v, the sequence is bounded in a neighbourhood of v. By Montel's theorem, there exists a subsequence $(g_{n_k})_k$ that converges to a function g uniformly on compacts of $\Delta(v, s)$. By corollary 2.1.5.1, the function g is constant. But then

$$\lim_{k\to\infty} |f_{n_k}(z)| = \infty$$

for all $z \in \Delta(v, s)$, which is not possible as v is a boundary point. It follows that $\mathcal{U} = \Omega$. □

Definition 2.3.2. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f_n \colon \Omega \to \mathbb{C}$ a sequence of functions. We say that f_n *converges to* ∞ if

$$\lim_{n\to\infty} \|f_n\|_K = \infty$$

for every compact $K \subset \Omega$.

Theorem 2.3.3 (Montel – sharp)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \mid 0, 1 \notin f(\Omega)\}.$$

Then $\mathcal{F}$ is normal in Ω where we also allow convergence to ∞.

Proof. Let $\Omega \subseteq \mathbb{C}$ be a domain and $p \in \Omega$. Consider the family $\mathcal{F}_{p,1}$. Let $(f_n)_n \subseteq \mathcal{F}$ be a sequence. If there exists a subsequence $(f_{n_k})_k \subseteq \mathcal{F}_{p,1}$, we can apply the above lemma. By the classical Montel's theorem, this subsequence has a convergent subsequence.

Suppose now that no such subsequence exists, that is $(f_n)_n$ has only finitely many terms in $\mathcal{F}_{p,1}$. But then there exists a subsequence $\left(\frac{1}{f_{n_k}}\right)_k \subseteq \mathcal{F}_{p,1}$. As before, this sequence has a convergent subsequence with limit g. If g is nowhere-vanishing, then $\frac{1}{g}$ is the limit of a subsequence of $(f_n)_n$. Otherwise, by corollary 2.1.5.1, we have $g = 0$ and therefore $(f_n)_n$ converges to ∞. $\square$

Definition 2.3.4. Let $\Omega \subseteq \mathbb{C}$ be an open set and $p \in \Omega$. A function $f \in \mathcal{O}(\Omega \setminus \{p\})$ has an *essential singularity* in p if the limit

$$\lim_{z \to p} f(z)$$

does not exist and

$$\lim_{z \to p} |f(z)| \neq \infty.$$

Theorem 2.3.5 (Picard's great theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set $p \in \Omega$ a point and $f \in \mathcal{O}(\Omega \setminus \{p\})$ a function. If f has an essential singularity at p, then f assumes every complex number as a value infinitely many times with at most one exception.

Proof. Without loss of generality assume that $p = 0$ and consider $\Omega = \mathbb{\Delta}(\varepsilon)$. Suppose that f omits two values on $\mathbb{\Delta}(\varepsilon)$, without loss of generality 0 and 1.

We now claim that f or $\frac{1}{f}$ is bounded in a neighbourhood of 0. Define the sequence of holomorphic functions $(f_n)_n$ with $f_n(z) = f\left(\frac{z}{n}\right)$. This sequence also omits 0 and 1, therefore either $(f_n)_n$ or $\left(\frac{1}{f_n}\right)_n$ has a convergent subsequence that converges uniformly on compacts by the sharp version of Montel's theorem. Denote the subsequence by $(g_{n_k})_k$ and set $g = f$ or $g = \frac{1}{f}$ accordingly.

Observe that there exists a constant M such that

$$\|g_{n_k}\|_{\partial\mathbb{\Delta}\left(\frac{\varepsilon}{2}\right)} \leq M$$

holds for all $k \in \mathbb{N}$. This is equivalent to

$$|g(z)| \leq M$$

for $|z| = \frac{1}{n_k} \cdot \frac{\varepsilon}{2}$. By the maximum principle, we have

$$|g(z)| \leq M$$

for all z such that

$$\frac{\varepsilon}{2} \cdot \frac{1}{n_k} \leq |z| \leq \frac{\varepsilon}{2}.$$

But as $(n_k)_k$ diverges, the inequality $g(z) \leq M$ holds for all z such that $|z| \leq \frac{\varepsilon}{2}$, therefore f or $\frac{1}{f}$ is bounded near 0.

Observe that f is not bounded in a neighbourhood of 0, as otherwise 0 is a removable singularity, which is not possible. Similarly, if $\frac{1}{f}$ is bounded, then f has either a removable singularity or a pole at 0, which is again a contradiction. $\square$

3 Infinite products

3.1 Definition and convergence

Definition 3.1.1. Let $(a_k)_k$ be a sequence of complex numbers. The sequence

$$n \mapsto \prod_{k=1}^{n} a_k$$

is called the *sequence of partial products* with factors a_k. We denote

$$p_{m,n} = \prod_{k=m}^{n} a_k.$$

We say that the infinite product is *convergent* if there exists an index $m \in \mathbb{N}$ such that the limit

$$\widehat{a}_m = \lim_{n \to \infty} p_{m,n}$$

exists and is non-zero. We then define

$$\prod_{k=1}^{\infty} a_k = p_{1,m-1} \cdot \widehat{a}_m.$$

as the limit of the infinite product.

Remark 3.1.1.1. The limit is uniquely defined.

Remark 3.1.1.2. An infinite product is convergent if and only if the product of all its non-zero factors has a non-zero limit and only finitely many factors are non-zero.

Lemma 3.1.2. Let $(a_k)_k \subseteq \mathbb{R}_{\geq 0}$ be a sequence such that

$$\sum_{k=1}^{\infty} (1 - a_k) = \infty.$$

Then

$$\lim_{n \to \infty} \prod_{k=p}^{n} a_k = 0$$

for all $p \in \mathbb{N}$. In particular, the infinite product is divergent.

Proof. Observe that

$$0 \leq \prod_{k=p}^{n} a_k \leq \prod_{k=p}^{n} e^{a_k - 1},$$

which converges to 0. □

Definition 3.1.3. Let $X \subseteq \mathbb{C}$ be a set.

i) A series

$$\sum_{k=1}^{\infty} g_k$$

of continuous functions $g_k \in \mathcal{C}(X)$ is *normally convergent* if for every compact $K \subseteq X$ the series

$$\sum_{k=1}^{\infty} \|g_k\|_K$$

converges.

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ii) A product
$$\prod_{k=1}^{\infty} f_k$$
of continuous functions $f_k = 1 + g_k \in \mathcal{C}(X)$ is *normally convergent* if the series
$$\sum_{k=1}^{\infty} g_k$$
is normally convergent.

Definition 3.1.4. Let $X \subseteq \mathbb{C}$ be a set and $f_k \in \mathcal{C}(X)$ be continuous functions. Denote
$$p_{m,n} = \prod_{k=m}^{n} f_k.$$
We say that the infinite product
$$\prod_{k=1}^{\infty} f_k$$
converges *uniformly* on a set $L \subseteq X$ if there exists an index $m \in \mathbb{N}$ such that $f_k|_L$ has no zeroes for $k \geq m$ and
$$\lim_{n\to\infty} p_{m,n} = \widehat{f}_k$$
exists, is uniform on L and has no zeroes on L. We define
$$\prod_{k=1}^{\infty} f_k = p_{1,m-1} \cdot \widehat{f}_m$$
on L.

Theorem 3.1.5 (Reordering of infinite products)**.** Let
$$\prod_{k=1}^{\infty} f_k$$
be a normally convergent product in $X \subseteq \mathbb{C}$. Then there exists a functions $f \colon X \to \mathbb{C}$ such that for all bijections $\tau \colon \mathbb{N} \to \mathbb{N}$ the product
$$\prod_{k=1}^{\infty} f_{\tau(k)}$$
converges to f uniformly on compacts of X. In particular, the infinite product converges uniformly on compacts.

Proof. Recall that, for $w \in \mathbb{\Delta}$, we can define
$$\log(1+w) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} w^k.$$
Then,
$$|\log(1+w)| \leq |w| \cdot \sum_{k=0}^{\infty} |w|^k = \frac{|w|}{1-|w|}.$$

In particular, if $|w| \leq \frac{1}{2}$, we have

$$|\log(1+w)| \leq 2\,|w|\,.$$

Let $L \subseteq X$ be a compact and write $f_k = 1 + g_k$. For all $k > N$ we have $\|g_k\|_L \leq \frac{1}{2}$, therefore we can write

$$\log f_k = \log(1+g_k) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} g_k^{\ell}.$$

But then

$$\|\log f_k\|_L \leq 2\,\|g_k\|_L\,.$$

It follows that the series

$$\sum_{k=N}^{\infty} \|\log f_k\|_L$$

converges. But then the series

$$h_N = \sum_{k=N}^{\infty} \log f_k$$

converges absolutely, and therefore all reorderings of the series converge as well to the same limit h_N.

Observe that

$$e^{h_N} = \prod_{k=N}^{\infty} e^{\log f_k} = \prod_{k=N}^{\infty} f_k.$$

This product therefore converges uniformly on L, independently of reorderings. We now define

$$f = \prod_{k=1}^{N-1} f_k \cdot e^{h_N}.$$

Note that this holds for all reorderings, as they differ from a suitable one by only finitely many transpositions. $\square$

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3.2 Zeroes of infinite products

Definition 3.2.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{O}(\Omega)$. The *zero set* of f is the set

$$Z(f) = \{z \in \Omega \mid f(z) = 0\}.$$

For all $c \in \Omega$, define the *zero order* of f in c as follows: if

$$f(z) = (z-c)^k \cdot g(z)$$

where $g(c) \neq 0$ is a holomorphic function, then $\operatorname{ord}_c(f) = k$.

Remark 3.2.1.1. For non-zero $f \in \mathcal{O}(\Omega)$, the set $Z(f)$ is discrete in Ω.

Remark 3.2.1.2. We have

$$\operatorname{ord}_c\left(\prod_{k=1}^n f_k\right) = \sum_{k=1}^n \operatorname{ord}_c(f_k).$$

Lemma 3.2.2. Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^\infty f_k$$

be a normally convergent product in Ω, where $f_k \in \mathcal{O}(\Omega)$ are non-zero holomorphic functions. Then f is a non-zero function with

$$Z(f) = \bigcup_{k=1}^\infty Z(f_k)$$

and

$$\operatorname{ord}_c(f) = \sum_{k=1}^\infty \operatorname{ord}_c(f_k).$$

Proof. Recall that normally convergent products converge uniformly on compacts of Ω. In particular, f is a holomorphic function.

Pick a point $c \in \Omega$. By definition of convergence, there exists some $m \in \mathbb{N}$ such that $\widehat{f}_m(c) \neq 0$. As $\widehat{f}_m$ is holomorphic as well, we have

$$f(c) = \left(p_{1,m-1} \cdot \widehat{f}_m\right)(c),$$

but then

$$\operatorname{ord}_c(f) = \sum_{k=1}^{m-1} \operatorname{ord}_c(f_k) = \sum_{k=1}^\infty \operatorname{ord}_c(f_k). \qquad \square$$

Lemma 3.2.3. Let $\Omega \subseteq \mathbb{C}$ be a domain. If

$$f = \prod_{k=1}^\infty f_k$$

is a normally convergent product in Ω, where $f_k \in \mathcal{O}(\Omega)$ are holomorphic functions, then the sequence $\left(\widehat{f}_n\right)_n$ converges to 1 uniformly on compacts.

Proof. Choose $m \in \mathbb{N}$ such that $\widehat{f}_m \neq 0$. Then the set $Z\left(\widehat{f}_m\right)$ has no accumulation points in Ω. We can therefore write

$$\widehat{f}_n = \frac{\widehat{f}_m}{p_{m,n-1}}$$

on $\Omega \setminus Z\left(\widehat{f}_m\right)$. As $p_{m,n-1}$ converges to $\widehat{f}_m$ on compacts of Ω,

$$\lim_{n\to\infty} \widehat{f}_n = 1$$

uniformly on compacts of $\Omega \setminus Z\left(\widehat{f}_m\right)$. For any compact set $K \subseteq \Omega$, taking m large enough, we have $Z\left(\widehat{f}_m\right) \cap K = \emptyset$. The conclusion follows. $\square$

Definition 3.2.4. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. The meromorphic function $\frac{f'}{f}$ is called the *logarithmic derivative* of f.

Remark 3.2.4.1. For holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(\Omega)$ we have

$$\left(\prod_{k=1}^n f_k\right)' \cdot \left(\prod_{k=1}^n f_k\right)^{-1} = \sum_{k=1}^n \frac{f_k'}{f_k}.$$

Definition 3.2.5. Let $g_k \in \mathcal{M}(\Omega)$ be meromorphic functions. The series

$$\sum_{k=1}^\infty g_k$$

is *normally convergent* in Ω if for every compact $L \subseteq \Omega$ there exists some $m \in \mathbb{N}$ such that

$$\sum_{k=m}^\infty \|g_k\|_L$$

converges.

Theorem 3.2.6 (Logarithmic differentiation)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and

$$f = \prod_{k=1}^\infty f_k$$

be a normally convergent product in Ω, where $f_k \in \mathcal{O}(\Omega)$ are non-zero functions. Then

$$\sum_{k=1}^\infty \frac{f_k'}{f_k}$$

is normally convergent in Ω and

$$\sum_{k=1}^\infty \frac{f_k'}{f_k} = \frac{f'}{f}.$$

Proof. As $\widehat{f}_n$ converges to 1 uniformly on compacts, the sequence $(f_n')_n$ converges to 0 uniformly on compacts by Cauchy estimates. Then for any compact L, $\frac{\widehat{f}_n'}{\widehat{f}_n}$ converges to 0 as $\widehat{f}_n$ has no zeroes in L for n large enough. It follows that

$$\lim_{n\to\infty} \frac{f'}{f} - \sum_{k=1}^n \frac{f_k'}{f_k} = \lim_{n\to\infty} \frac{\widehat{f}_{n+1}'}{\widehat{f}_{n+1}} = 0.$$

Write $f_k = 1 + g_k$ and fix a compact set $L \subseteq \Omega$. Choose an index m such that we have $Z\left(\widehat{f}_m\right) \cap L = \emptyset$ and

$$\min_{z \in L} |f_k(z)| \geq \frac{1}{2}.$$

Choose $\varepsilon > 0$ such that

$$L_\varepsilon = \{z \in \mathbb{C} \mid d(z, L) \leq \varepsilon\} \subseteq \Omega.$$

By the Cauchy estimates, we have $\|g_k'\|_L \leq \frac{1}{\varepsilon} \|g_k\|_L$. But then

$$\sum_{k=m}^{\infty} \left\| \frac{f_k'}{f_k} \right\|_L = \sum_{k=m}^{\infty} \left\| \frac{g_k'}{f_k} \right\|_L \leq 2 \cdot \sum_{k=m}^{\infty} \|g_k'\|_L \leq \frac{2}{\varepsilon} \cdot \sum_{k=m}^{\infty} \|g_k\|,$$

which is convergent by our assumptions. □

Lemma 3.2.7. Let g be meromorphic on $\mathbb{C}$ with poles in $\mathbb{Z}$ with principal parts $\frac{1}{z-m}$. Moreover, assume that g is an odd function that satisfies

$$2g(2z) = g(z) + g\left(z + \frac{1}{2}\right).$$

Then $g(z) = \pi \cdot \cot(\pi z)$.

Proof. Simple calculations show that $\pi \cdot \cot(\pi z)$ is indeed a solution of the functional equation. Define $h(z) = g(z) - \pi \cdot \cot(\pi z)$. This another solution of the functional equation, and an odd function. In particular, $h(0) = 0$. Observe that the principal parts of h are 0, therefore $h \in \mathcal{O}(\mathbb{C})$ is an entire function.

Suppose that h is not constant. In particular, there exists some $c \in \partial \mathbb{\Delta}(2)$ such that

$$|h(z)| < |h(c)|$$

for all $z \in \mathbb{\Delta}(2)$. As $\frac{c}{2}, \frac{c+1}{2} \in \mathbb{\Delta}(2)$, we can write

$$2\,|h(c)| = \left| h\left(\frac{c}{2}\right) + h\left(\frac{c+1}{2}\right) \right| \leq \left| h\left(\frac{c}{2}\right) \right| + \left| h\left(\frac{c+1}{2}\right) \right| < 2\,|h(c)|,$$

which is a contradiction. It follows that $h = 0$. □

Corollary 3.2.7.1. We have

$$\pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

Proof. Note that

$$\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2} = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k} \right),$$

therefore the series has poles in $\mathbb{Z}$ with principal parts $\frac{1}{z-m}$. It is also an odd function. A calculation shows that, for

$$r_n(z) = \frac{1}{z} + \sum_{k=1}^{n} \frac{2z}{z^2 - k^2},$$

we have

$$r_n(z) + r_n\left(z + \frac{1}{2}\right) = 2r_{2n}(2z) + \frac{2}{2z + 2n + 1}.$$

Taking $n \to \infty$, the conclusion follows. □

Theorem 3.2.8. We have

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Proof. The above product is obviously normally convergent, therefore we can take its logarithmic derivative. A simple calculation shows that it is equal to $\pi \cot(\pi z)$. As logarithmic derivatives are equal only for scalar multiples, we only have to check equality in one point. $\square$

3.3 The Euler gamma function

Lemma 3.3.1. The infinite product

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-\frac{z}{k}}$$

in normally convergent in $\mathbb{C}$.

Proof. Write

$$\begin{aligned} |1 - (1-\omega)e^{\omega}| &= |1 - e^{\omega} + \omega e^{\omega}| \\ &= \left| -\sum_{k=1}^{\infty} \frac{\omega^k}{k!} + \sum_{k=0}^{\infty} \frac{\omega^{k+1}}{k!} \right| \\ &= \left| \omega^2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \omega^{k-1} \right| \\ &\le |\omega|^2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \\ &= |\omega|^2 \end{aligned}$$

for $|\omega| \le 1$. But then the sum

$$\sum_{k=\lceil |z| \rceil}^{\infty} \left| 1 - \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right| \le \sum_{k=\lceil |z| \rceil}^{\infty} \left| \frac{z^2}{k^2} \right|$$

converges normally. The infinite product must then converge normally in $\mathbb{C}$ as well. $\square$

Lemma 3.3.2. Let

$$H(z) = z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Then $H(1) = e^{-\gamma}$, where γ is the *Euler-Mascheroni constant*, that is

$$\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \log(n).$$

Proof. First note that

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = \prod_{k=1}^{n} \frac{k+1}{k} = n + 1.$$

We therefore have

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) e^{-\frac{1}{k}} = \exp\left(\log(n+1) - \sum_{k=1}^{n} \frac{1}{k} \right),$$

therefore

$$H(1) = \lim_{n \to \infty} \exp\left(\log(n+1) - \sum_{k=1}^{n} \frac{1}{k} \right) = e^{-\gamma}.$$ $\square$

Lemma 3.3.3. Let $\Delta(z) = e^{\gamma z} H(z)$.

i) We have $\Delta(1) = 1$ and $\Delta(z) = z\Delta(z+1)$.

ii) We have $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$.

Proof. Note that $\Delta(1) = 1$ by the previous lemma. Rewrite the partial products as

$$z \cdot \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} = \frac{z}{n!} \cdot \prod_{k=1}^{n} (z+k) \cdot \exp\left(-z \sum_{k=1}^{n} \frac{1}{k}\right).$$

We therefore have

$$\begin{aligned} \Delta(z) &= \lim_{n\to\infty} \frac{e^{\gamma z}}{n!} \cdot \prod_{k=0}^{n} (z+k) \cdot \exp\left(-z \sum_{k=1}^{n} \frac{1}{k}\right) \\ &= \lim_{n\to\infty} \frac{e^{\gamma z}}{n! \cdot n^z} \cdot \prod_{k=0}^{n} (z+k) \cdot \exp\left(z \log(n) - z \sum_{k=1}^{n} \frac{1}{k}\right) \\ &= \lim_{n\to\infty} \frac{1}{n! \cdot n^z} \cdot \prod_{k=0}^{n} (z+k). \end{aligned}$$

We can now calculate

$$z \cdot \Delta(z+1) = \lim_{n\to\infty} z \cdot \frac{1}{n! \cdot n^{z+1}} \cdot \prod_{k=1}^{n+1} (z+k) = \Delta(z) \cdot \lim_{n\to\infty} \frac{z+n+1}{n} = \Delta(z).$$

It remains to check the equality $\pi \cdot \Delta(z)\Delta(1-z) = \sin(\pi z)$. We have

$$\begin{aligned} \pi \cdot \Delta(z)\Delta(1-z) &= \pi \cdot \Delta(z) \cdot \frac{\Delta(-z)}{-z} \\ &= \pi e^{\gamma z} \cdot z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \cdot e^{-\gamma z} \cdot \frac{-z}{-z} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{\frac{z}{k}} \\ &= \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \\ &= \sin(\pi z). \end{aligned}$$

□

Definition 3.3.4. The *Euler gamma function* is defined as

$$\Gamma(z) = \frac{1}{\Delta(z)}.$$

Theorem 3.3.5. The Γ function satisfies the following properties:

1. The function Γ is meromorphic with simple poles in $-\mathbb{N}_0$.
2. We have $\Gamma(1) = 1$.
3. The function Γ satisfies $\Gamma(z+1) = z\Gamma(z)$.
4. The function Γ satisfies
$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$
5. We have
$$\Gamma(z) = \lim_{n\to\infty} n! \cdot n^z \cdot \left(\prod_{k=0}^{n} (z+k)\right)^{-1}.$$

Proof. The proof is obvious and need not be mentioned. $\square$

Theorem 3.3.6. Let F be holomorphic in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ and assume $F(z+1) = z \cdot F(z)$. Furthermore, assume that F is bounded on the strip $1 \leq \operatorname{Re}(z) < 2$ and $F(1) = 1$. Then $F = \Gamma$.

3.4 Weierstraß factors

Definition 3.4.1. The *Weierstraß factors* are functions

$$E_n(z) = (1 - z) \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^n}{n}\right).$$

Lemma 3.4.2. The Weierstraß factors satisfy the following:

i) For $n \geq 1$ we have
$$E_n'(z) = -z^n \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^n}{n}\right).$$

ii) For $n \geq 0$ we have
$$E_n(z) = 1 + \sum_{k=n+1}^{\infty} a_k z^k,$$
where
$$\sum_{k=n+1}^{\infty} |a_k| = 1.$$

iii) For $n \geq 0$ and $|z| \leq 1$ we have
$$|E_n(z) - 1| \leq |z|^{n+1}.$$

Proof.

i) Evident.

ii) Observing the derivative, we see that $a_1 = a_2 = \cdots = a_n = 0$, and $a_k \leq 0$ for $k > n$. But then
$$\sum_{k=n+1}^{\infty} |a_k| = -\sum_{k=n+1}^{\infty} a_k = 1 - E_n(1) = 1.$$

iii) We have
$$|E_n(z) - 1| = \left|\sum_{k=n+1}^{\infty} a_k z^k\right| \leq \sum_{k=n+1}^{\infty} |a_k| \cdot |z|^k \leq |z|^{n+1}. \qquad \square$$

Lemma 3.4.3. Let $(a_k)_k \subset \mathbb{C}^*$ be a sequence of complex numbers with no accumulation point and let $(p_k)_k \subseteq \mathbb{N}_0$ be non-negative integers with

$$\sum_{k=1}^{\infty} \left|\frac{r}{a_k}\right|^{p_k+1}$$

converges for every $r > 0$. Then the *Weierstraß product*

$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$$

converges normally on $\mathbb{C}$.

Proof. Note that $|a_k| > |z|$ for all but finitely many k. Now just apply the previous lemma. $\square$

Theorem 3.4.4 (Weierstraß factorization theorem)**.** For any sequence $(a_k)_k \subset \mathbb{C}$ with no accumulation point there exists a Weierstraß product

$$z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$$

that converges normally on $\mathbb{C}$.

Proof. Set $p_k = k - 1$. For any $r > 0$ choose $m \in \mathbb{N}_0$ such that $|a_k| > 2r$ for all $k \geq m$. We then have

$$\sum_{k=m}^{\infty} \left|\frac{r}{a_k}\right|^{p_k+1} \leq \sum_{k=m}^{\infty} \frac{1}{2^k} \leq 2.$$ $\square$

Theorem 3.4.5 (Weierstraß product theorem)**.** Let $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$ be a holomorphic function. Then there exists a function $g \in \mathcal{O}(\mathbb{C})$ such that

$$f = e^g \cdot z^q \cdot \prod_{\substack{k=1 \\ a_k \neq 0}}^{\infty} E_{k-1}\left(\frac{z}{a_k}\right),$$

where a_k are zeroes of f on $\mathbb{C} \setminus \{0\}$, counted with multiplicities, and $q = \operatorname{ord}_0(f)$.

Proof. The proof is obvious and need not be mentioned. $\square$

Lemma 3.4.6. Let $\Omega \subset \mathbb{C}$ be an open subset, $(a_k)_k \subset \Omega$ a sequence with no accumulation point in Ω and $A = \{a_k \mid k \in \mathbb{N}\}$. Let $(b_k)_k \subset \mathbb{C} \setminus \Omega$ and $(p_k)_k \subseteq \mathbb{N}$ be sequences such that the series

$$\sum_{k=1}^{\infty} |r(a_k - b_k)|^{p_k+1}$$

converges for all $r > 0$ and denote $B = \{b_k \mid k \in \mathbb{N}\}$. Then the infinite product

$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{a_k - b_k}{z - b_k}\right)$$

converges normally on $\mathbb{C} \setminus \overline{B}$.

Proof. Let $L \subseteq \mathbb{C} \setminus \overline{B}$ be a compact set and let $\ell = d\left(L, \overline{B}\right) > 0$. We then have $|z - b_k| \geq \ell$ for all $z \in L$ and $k \in \mathbb{N}$.

We can now bound

$$\left\|\frac{a_k - b_k}{z - b_k}\right\|_L \leq \frac{|a_k - b_k|}{\ell}.$$

By the assumption of convergence for $r = \frac{1}{\ell}$, we must have

$$|r \cdot (a_k - b_k)| < 1$$

for all $k \geq n(L)$, but then

$$\sum_{k=n(L)}^{\infty} \left\| E_{p_k}\left(\frac{a_k - b_k}{z - b_k}\right) - 1 \right\|_L \leq \sum_{k=n(L)}^{\infty} \left\| \frac{a_k - b_k}{z - b_k} \right\|_L^{p_k+1} \leq \sum_{k=n(L)}^{\infty} |r \cdot (a_k - b_k)|^{p_k+1},$$

which converges. □

Remark 3.4.6.1. The Weierstraß factor $E_{p_k}\left(\frac{a_k-b_k}{z-b_k}\right)$ is zero if and only if $z = a_k$.

Lemma 3.4.7. Let $A \subset \mathbb{C}$ be a discrete set and define $A' = \overline{A} \setminus A$. Suppose that $A' \neq \emptyset$ and let

$$A_1 = \{z \in A \mid |z| \cdot d(z, A') \geq 1\}$$

and $A_2 = A \setminus A_1$. Now let

$$A_2(\varepsilon) = \{z \in A_2 \mid d(z, A') \geq \varepsilon\}.$$

Then A_1 is a closed set and $A_2(\varepsilon)$ is finite for any $\varepsilon > 0$.

Proof. Assume A_1 has an accumulation point a and let $(a_k)_k \subseteq A$ be a sequence, converging to a. But then

$$\lim_{k\to\infty} |a_k| \cdot d(a_k, A') = 0,$$

which is a contradiction.

Note that, for all $z \in A_2(\varepsilon)$, we have $|z| < \frac{1}{\varepsilon}$. If the set is infinite, it has an accumulation point, which is impossible as $d(z, A') \geq \varepsilon$. □

Remark 3.4.7.1. If $A \subset \mathbb{C}$ is a discrete set, then A' is a closed set in $\mathbb{C}$.

Theorem 3.4.8 (Weierstraß product theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k)_k \subset \Omega$ be a sequence without accumulation points in Ω and denote $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$. Then there exists a Weierstraß product for $(a_k)_k$ that converges normally in $\mathbb{C} \setminus A'$. This product has zeros precisely in $(a_k)_k$, counted with multiplicities.

Proof. Assume that $\Omega \neq \mathbb{C}$ and $A' \neq \emptyset$.[4] Write $A = A_1 \cup A_2$ as in the above lemma. Recall that A_1 has no accumulation points, therefore we can apply theorem 3.4.5 for A_1. It remains to construct a Weierstraß product for A_2.

Observe that $A' = A_2'$. As this is a closed space, for all $a_k \in A_2$ there exists some $b_k \in A_2'$ such that

$$|a_k - b_k| = d(a_k, A_2').$$

Observe that

$$\lim_{\substack{k\to\infty \\ a_k \in A_2}} |a_k - b_k| = 0,$$

as the sets $A_2(\varepsilon)$ are finite. Now set $p_k = k$ and apply lemma 3.4.6. □

[4] Otherwise just apply theorem 3.4.5.

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Corollary 3.4.8.1 (Blaschke products)**.** Let $(a_k)_k \subset \mathbb{\Delta} \setminus \{0\}$ be a sequence without accumulation points in $\mathbb{\Delta}$. If the series

$$\sum_{k=1}^{\infty} (1 - |a_k|)$$

converges, then the product

$$\prod_{k=1}^{\infty} E_0 \left(\frac{a_k - \frac{1}{\overline{a}_k}}{z - \frac{1}{\overline{a}_k}} \right)$$

converges normally in $\mathbb{\Delta}$ and has zeros precisely in $(a_k)_k$, counted with multiplicities.

Proof. Note that

$$|a_k - b_k| = \left| a_k - \frac{1}{\overline{a}_k} \right| = \left| \frac{1}{\overline{a}_k} \right| \cdot \left| |a_k|^2 - 1 \right| = \left| \frac{1}{\overline{a}_k} \right| \cdot (1 - |a_k|)(1 + |a_k|) \leq \frac{2}{m} \cdot (1 - |a_k|) \, ,$$

where

$$m = \min \left\{ |a_k| \mid k \in \mathbb{N} \right\} .$$

It follows that the series

$$\sum_{k=1}^{\infty} r \cdot |a_k - b_k|$$

converges, therefore we can apply lemma 3.4.6. □

Theorem 3.4.9. Let $\Omega \subseteq \Omega$ be a domain and $f \in \mathcal{O}(\Omega) \setminus \{0\}$. Then we can write

$$f = g \cdot \prod_{k=1}^{\infty} f_k,$$

where $g \in \mathcal{O}^*(\Omega)$ and f_k are Weierstraß factors.

Proof. The proof is obvious and need not be mentioned. □

Theorem 3.4.10. Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{M}(\Omega)$. Then we can write $f = \frac{g}{h}$, where $g, h \in \mathcal{O}(\Omega)$.

Proof. Define h as the Weierstraß product of the poles of f. □

Remark 3.4.10.1. Let $\Omega \subseteq \mathbb{C}$ be a domain. Then $\mathcal{O}(\Omega)$ is not a factorial ring,[5] but $\gcd(f, g) \in \mathcal{O}(\Omega)$ exists.

Definition 3.4.11. Let Ω be an open subset and $\{a_k\}_k$ be a sequence without accumulation pints and without repetition. Let

$$q_k(z) = \sum_{n=1}^{\infty} c_{k,m}(z - a)^{-m}$$

be a principal part in a_k for each k.

[5] "Kolobar z enolično faktorizacijo."

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If there exist functions $g_k \in \mathcal{O}(\Omega)$ for each k such that

$$\sum_{k=1}^{\infty} q_k - g_k$$

converges normally in Ω, we call it the *Mittag-Leffler series* for the distribution of principal part (a_k, q_k).

Remark 3.4.11.1. We adopt the following conventions: If $0 \in \{a_k\}_k$, then $a_1 = 0$.

Theorem 3.4.12 (Mittag-Leffler for $\mathbb{C}$)**.** For every distribution of principal parts in $\mathbb{C}$ there exists a corresponding Mittag-Leffler series.

Proof. Let g_k be the Taylor series of q_k about 0 in the disk $\mathbb{\Delta}(|a_k|)$ such that the inequality $\|q_k - g_k\| < 2^{-k}$ holds for each $k \geq 2$. Note that

$$\lim_{k\to\infty} |a_k| = \infty$$

as the points don't accumulate. For each $r > 0$ we can therefore find an integer n such that $r < \frac{1}{2}|a_k|$ for all $k \geq n$. Then

$$\sum_{k=n}^{\infty} \|q_k - g_k\|_{\overline{\mathbb{\Delta}(r)}} \leq 1.$$

□

Remark 3.4.12.1. The above series $f \in \mathcal{O}(\mathbb{C} \setminus \{a_1, a_2, \dots\})$ with principal parts q_k in a_k for each $k \in \mathbb{N}$. If the principal part are finite, then $f \in \mathcal{M}$.

Lemma 3.4.13. Let $a \in \mathbb{C}$, $q \in \mathcal{O}(\Omega \setminus \{a\})$ be a principal part and $b \in \mathbb{C} \setminus \{a\}$. Then q has a Laurent series expansion about b in the annulus $\{z \in \mathbb{C} \mid |z-b| > |a-b|\}$ of the form

$$q(z) = \sum_{m=1}^{\infty} c_m (z-b)^{-m}$$

that converges uniformly for $|z-b| > r > |a-b|$.

Proof. Choose a path γ_r that goes around the circle centered at b of radius r. We claim that

$$c_m = \frac{1}{2\pi} \int_{\gamma_r} \frac{q(z)}{(z-b)^{-m+1}}\, dz$$

for $m \in \mathbb{Z}$ suffice. We can estimate

$$|c_m| \geq \frac{1}{2\pi} \cdot 2\pi \frac{\|q\|_{\gamma_r}}{r^{-m}} = \frac{\|q\|_{\gamma_r}}{r^{-m}}.$$

We know that $q(z)$ is of the form

$$q(z) = \sum_{m=1}^{\infty} d_m (z-a)^{-m}$$

for some $d_m \in \mathbb{C}$ when developed into a Laurent series around a. It is trivial to show that

$$\lim_{|z|\to\infty} q(z) = 0.$$

Thus, $\|q\|_{\gamma_r}$ approaches zero as r goes to infinity. If $m \leq 0$, then

$$\lim_{r\to\infty} \|q\|_{\gamma_r} r^m = 0.$$

Therefore, $c_m = 0$ for $m \leq 0$ and

$$q(z) = \sum_{m=1}^{\infty} c_m (z-b)^{-m}$$

is indeed a power series in $z - b$ which converges uniformly for $|z-b|^{-1} \leq r$. $\square$

Definition 3.4.14. The partial sums of

$$q_\ell(z) = \sum_{m=1}^{\ell} c_m (z-b)^{-m}$$

are called the *ℓ-th Laurent terms* of q about b.

Lemma 3.4.15. Let $(a_k, q_k)_k$ be a distribution of principal parts in an open set $\Omega \subseteq \mathbb{C}$, $A = \{a_k \mid k \in \mathbb{N}\}$ and $A' = \overline{A} \setminus A$.[6] Assume there exists a sequence $(b_k)_k \subseteq A'$ with

$$\lim_{k\to\infty} |a_k - b_k| = 0.$$

Let $q_{k,\ell}$ be the ℓ-th Laurent term of q_k about b_k. Then there exists a sequence $(\ell_k)_k \subseteq \mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty} (q_k - q_{k,\ell_k})$$

is a Mittag-Leffler series for $(a_k, q_k)_k$.

Proof. For a principal part q_k the Laurent series converges uniformly on $|z - b_k| > r$ for any $r > |a_k - b_k|$ by the previous lemma. Thus, we can choose ℓ_k large enough such that

$$|q_k(z) - q_{k,\ell}(z)| < 2^{-k}$$

for all z such that $|z - b_k| \geq 2\,|a_k - b_k|$.

For any compact set $L \subseteq \mathbb{C} \setminus A'$, the distance to A' is strictly positive. Since

$$\lim_{k\to\infty} |a_k - b_k| = 0,$$

the point b_k must lie outside L for large enough k. Thus, there exists some $n(L) \in \mathbb{N}$ such that

$$L \subseteq \bigcap_{k \geq n(L)} \{z \in \mathbb{C} \mid |z - b_k| \geq 2\,|a_k - b_k|\}.$$

Therefore, we can use the previous estimate on L, to get

$$\sum_{k \geq n(L)} \|q_k - q_{k,\ell_k}\|_L \leq \sum_{k \geq n(L)} 2^{-k} \leq 2.$$ $\square$

[6] The closure is taken in $\mathbb{C}$.

Theorem 3.4.16 (Mittag-Leffler for open subsets)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k, q_k)_k$ be a distribution of principal parts in $\Omega \subseteq$ and $A = \{a_k \mid k \in \mathbb{N}\}$. Then there exists a Mittag-Leffler series for (a_k, q_k) that converges normally in $\mathbb{C} \subsetneq A' = \overline{A} \setminus A$.

Proof. By lemma 3.4.7, $(A_1)'$ is empty and $(A_2)' = A'$. If A' is empty, then

$$\lim_{k\to\infty} |a_k| = \infty$$

and we can apply theorem 3.4.12. Similarly, we can assume $\Omega \neq \mathbb{C}$. Again by the lemma, $A_2(\epsilon)$ is finite. Hence, there exist $(b_k)_k \in A'$ such that

$$\lim_{k\to\infty} |a_k - b_k| = 0.$$

We can apply lemma 3.4.15 to obtain a Mittag-Leffler series. Now we apply theorem 3.4.12 for $\mathbb{C}$ to A_1. Sum up this two series to get the series from the statement. □

Theorem 3.4.17 (Mittag-Leffler osculation theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and $(a_k)_k \subseteq \Omega$ be a sequence without accumulation points and without repetition. Furthermore, let

$$f_k(z) = \sum_{\ell=-\infty}^{n(k)} c_{k,\ell}(z - a_k)^\ell,$$

where $n(k) \in \mathbb{N}_0$, be normally convergent on $\mathbb{C} \setminus A$, where A is the set of a_k. Then there exists a function $f \in \mathcal{O}(\Omega \setminus A)$ such that $\operatorname{ord}_{a_k}(f - f_k) > n(k)$ for all $k \in \mathbb{N}$.

Proof. By the Weierstraß product theorem, there exists a function $h \in \mathcal{O}(\Omega)$ such that $\operatorname{ord}_{a_k}(h) > n(k)$ and h has no zeroes on $\mathbb{C} \setminus A$. Then $\left(a_k, \frac{f_k}{h}\right)_k$ is a distribution of principal parts. By theorem 3.4.16, there exists a $g \in \mathcal{O}(\Omega \setminus A)$ with these principal parts.

Now define $f = g \cdot h$. Then

$$f - f_k = g \cdot h - f_k = h \cdot \left(g - \frac{f_k}{h}\right),$$

which vanishes to order larger than $n(k)$ in a_k. □

Corollary 3.4.17.1. For every sequence $(a_k)_k \subseteq \Omega$ without accumulation points and without repetition and every sequence $(c_k)_k \subseteq \mathbb{C}$ there exists a function $f \in \mathcal{O}(\Omega)$ such that $f(a_k) = c_k$ for each $k \in \mathbb{N}$.

3.5 Ring structure of holomorphic functions

Definition 3.5.1. Let $\Omega \subseteq \mathbb{C}$ be an open set. A *divisor* of a meromorphic function $f \in \mathcal{M}^*(\Omega)$ is the function $(f) \colon \Omega \to \mathbb{Z}$, given by

$$(f)(z) = \begin{cases} n & f \text{ has a zero of order } n \text{ in } z \\ -n & f \text{ has a pole of order } n \text{ in } z \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.5.1.1. The divisor of a product is the sum of divisors, i.e. $(f \cdot g) = (f) + (g)$.

Definition 3.5.2. Let $S \subseteq \mathcal{O}(\Omega)$ be a subset that contains a non-zero holomorphic function.

Define

$$d(z) = \min_{f \in S \setminus \{0\}} (f)(z) \in \mathbb{N}_0.$$

By Weierstraß product theorem there exists a function $g \in \mathcal{O}(\Omega)$ such that $(g) = d$. We define $\gcd(S) = g$.[7]

Lemma 3.5.3 (Wedderburn)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and $f, g \in \mathcal{O}(\Omega)$ be functions with $\gcd(f, g) = 1$. Then there exist functions $a, b \in \mathcal{O}(\Omega)$ so that $af + bg = 1$. Moreover, we can choose a to be nonwhere vanishing.

Proof. If $g = 0$ and $f \neq 0$ then f cannot vanish by assumption on gcd, therefore $a = \frac{1}{f}$ and $b = 1$ suffice. Therefore, we can assume both f, g and are nonzero. Note that $Z(f) \cap Z(g)$ is empty, since $(z - p)$ divides $\gcd(f, g)$ for any $p \in Z(f) \cap Z(g)$. The set $Z(f) \cup Z(g)$ is thus discrete. Further, for each zero p of g, there exists a disk of radius ε and a holomorphic function $f_p \in \mathcal{O}(\mathbb{\Delta}(p, \varepsilon))$ such that

$$f = e^{f_p}.$$

By the Mittag-Leffler osculation theorem there exists a function $h \in \mathcal{O}(\Omega)$ such that $\operatorname{ord}_p(h - f_p) > \operatorname{ord}_p(g)$.

Here we stop for a short observation. Developing into the power series, we get that $e^{w^n} - 1 = w^n + O(w^{2n})$. Then,

$$\operatorname{ord}_p\left(f - e^h\right) = \operatorname{ord}_p\left(e^h \cdot \left(e^{f_p - h} - 1\right)\right) = \operatorname{ord}_p\left(\left(e^{f_p - h} - 1\right)\right) = \operatorname{ord}_p\left(f_p - h\right) > \operatorname{ord}_p(g).$$

Define $k = \frac{f - e^h}{g} \in \mathcal{O}(\Omega)$. We claim that $a = e^{-h}$ and $b = -ke^{-h}$ satisfy the conditions. Clearly, a doesn't vanish, and

$$af + bg = e^{-h}f - ke^{-h}g = e^{-h}(f - kg) = e^{-h}\left(f - \frac{f - e^h}{g}g\right) = e^{-h}e^h = 1. \qquad \square$$

Corollary 3.5.3.1. For holomorphic functions $f_j \in \mathcal{O}(\Omega)$, where $j \leq n$, we can write $f = \gcd(f_1, f_2, \ldots, f_n)$ as

$$f = \sum_{j=1}^n a_j f_j$$

[7] There are of course multiple possible functions that satisfy this condition, but their quotients are invertible.

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Proof. We proceed by induction. The base case is just Wedderburn's lemma. Now let $\hat{f} = \gcd(f_2, f_3, \dots f_n)$, which can be written as

$$\hat{f} = \sum_{j=2}^{n} \hat{a}_j f_j$$

by the induction hypothesis. Then $\frac{f_1}{f}, \frac{\hat{f}}{f} \in \mathcal{O}(\Omega)$ are holomorphic functions with gcd equal to 1. We can therefore apply Wedderburn' lemma to get functions a and b such that

$$a\frac{f_1}{f} + b\frac{\hat{f}}{f} = 1.$$

The conclusion follows □

Theorem 3.5.4. Let $I \triangleleft \mathcal{O}(\Omega)$ be the ideal generated by holomorphic functions $f_1, f_2, \dots f_n$ on Ω. Then there exists a holomorphic function f such that $I = (f)$.

Proof. Take $f = \gcd(f_j)$. This function is an element of I by the previous corollary. Since $f \mid f_j$, this implies that $I = (f)$. □

Definition 3.5.5. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal.

i) We call I *closed* if for every sequence $(f_n)_n \subseteq I$ that converges uniformly on compacts of Ω to some function f, we also have $f \in I$.
ii) We call $p \in \Omega$ a *zero* of I if $f(p) = 0$ for every $f \in I$.

Lemma 3.5.6. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ and ideal. Let $p \in \Omega$ be a point that is not a zero of I. Let $f, g \in \mathcal{O}(\Omega)$ be functions such that $f(z) \neq 0$ for all $z \neq p$. If $fg \in I$, then $g \in I$.

Proof. Since p is not a zero of I, then there exists a function $h \in I$ such that $h(p) \neq 0$. Let $n = \operatorname{ord}_p(f)$. If $n = 0$, then f is a unit, so $g \in I$. Otherwise, we have

$$\frac{f(z)}{z-p} g = -\frac{1}{h(p)} \cdot \left(\frac{h - h(p)}{z - p} fg - \frac{fg}{z-p} h \right) \in I$$

since $\frac{f}{z-p}$ is holomorphic.

We can iterate this process to find $\frac{f}{(z-p)^n} g \in I$. Since $\frac{f}{(z-p)^n}$ is a unit, g must be an element of I. □

Theorem 3.5.7. Let $\Omega \subseteq \mathbb{C}$ be a domain and $I \triangleleft \mathcal{O}(\Omega)$ an ideal. If I has no zeroes and is closed, then $I = \mathcal{O}(\Omega)$.

Proof. Let f be an arbitrary nonzero element of I. By the Weierstraß product theorem, we can write

$$f = \prod_{k=1}^{\infty} f_k,$$

where each f_k has exactly one zero in Ω, and the tails

$$\widehat{f_n} = \prod_{k=n}^{\infty} f_k$$

converge to 1 uniformly on compacts of Ω. As $f = \widehat{f}_1 = f_1 \widehat{f}_2$, we can apply the preivous lemma to find $\widehat{f}_2 \in I$. Inductively, $\widehat{f}_n \in I$ and since the ideal I is assumed to be closed, we have

$$1 = \lim_{k \to \infty} \widehat{f}_k \in I.$$

□

4 Approximation of holomorphic functions

4.1 Runge's little theorem

Lemma 4.1.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $K \subseteq \Omega$ a non-empty compact. Then there exist finitely many horizontal or vertical line segments $\sigma_1, \dots, \sigma_n$ of equal length in $\Omega \setminus K$ such that for all $f \in \mathcal{O}(\Omega)$ we have

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\sigma_k} \frac{f(\xi)}{\xi - z} \, d\xi$$

for all $z \in K$.

Proof. Define

$$\delta = \begin{cases} 1, & \Omega = \mathbb{C}, \\ d(K, \partial\Omega), & \Omega \neq \mathbb{C}. \end{cases}$$

Let Q be a grid of squares, parallel to the coordinate axes, with side length $d < \frac{\delta}{\sqrt{2}}$. As K is compact, it only intersects finitely many of them. Now just choose the boundary of the union of those squares. It is clear that all those segments are subsets of $\Omega \setminus K$.

Let Q_k denote the above squares. Then,

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \oint_{\partial Q_k} \frac{f(\xi)}{\xi - z} \, d\xi = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\sigma_k} \frac{f(\xi)}{\xi - z} \, d\xi.$$

If $z \in \operatorname{Int} Q_\ell$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q_\ell} \frac{f(\xi)}{\xi - z} \, d\xi$$

and

$$\frac{1}{2\pi i} \int_{\partial Q_k} \frac{f(\xi)}{\xi - z} \, d\xi = 0$$

for all $k \neq \ell$, therefore the lemma holds for all such z. As both sides of the equations are continuous functions, they must agree on the whole set K. $\square$

Lemma 4.1.2. Let σ be a compact line segment and $K \subseteq \mathbb{C}$ be a compact set such that $K \cap \sigma = \emptyset$. Let $h \in \mathcal{C}(\sigma)$ be a function. Then for all $\varepsilon > 0$ there exist points $c_1, \dots, c_m \in \mathbb{C}$ and $w_1, \dots, w_m \in \sigma$ such that

$$\left\| \int_{\sigma} \frac{h(\xi)}{\xi - z} \, d\xi - \sum_{k=1}^{m} \frac{c_k}{z - w_k} \right\|_K < \varepsilon.$$

Proof. Let ℓ be the length of σ and define the function $v \colon \sigma \times K \to \mathbb{C}$ with

$$v(\xi, z) = \frac{h(\xi)}{\xi - z}.$$

It is clearly continuous, therefore it is uniformly continuous. In particular, there exists a $\delta > 0$ such that

$$|v(\xi, z) - v(\xi', z)| < \frac{\varepsilon}{\ell}$$

for all $z \in \mathbb{C}$ and $|\xi - \xi'| < \delta$.

Let $\tau_1, \ldots, \tau_m$ be the partition of σ into line segments of length $d < \sigma$. Choose points $w_k \in \tau_k$ and set $c_k = -h(w_k)d$. We therefore have

$$\left| \int_{\tau_k} v(\xi, z)\, d\xi - \frac{c_k}{z - w_k} \right| = \left| \int_{\tau_k} v(\xi, z)\, d\xi - d \cdot v(w_k, z) \right| < d \cdot \frac{\varepsilon}{\ell}.$$

Summing up, we get the desired inequality. □

Lemma 4.1.3. Let $\Omega \subseteq \mathbb{C}$ be an open set and $K \subseteq \Omega$ be a non-empty compact. Then there exist finitely many line segments $\sigma_1, \ldots, \sigma_n$ in $\Omega \setminus K$ such that for any holomorphic function $f \in \mathcal{O}(\Omega)$ and $\varepsilon > 0$ there exists a rational function q of the form

$$q(z) = \sum_{k=1}^{m} \frac{c_k}{z - w_k},$$

where $c_k \in \mathbb{C}$ and $w_k \in \sigma$, such that

$$\|f - q\|_K < \varepsilon.$$

Proof. By lemma 4.1.1 there exist line segments σ_k such that

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\sigma_k} \frac{f(\xi)}{\xi - z}\, d\xi$$

for all $z \in K$. Now just apply lemma 4.1.2 to each line segment separately. □

Lemma 4.1.4 (Shifting poles)**.** Let $K \subseteq \mathbb{C}$ be a compact set, Z a connected component of $\mathbb{C} \setminus K$ and $a, b \in Z$. Then, for all $\varepsilon > 0$ there exists a polynomial q such that

$$\left\| \frac{1}{z - a} - q\left(\frac{1}{z - b}\right) \right\|_K < \varepsilon.$$

If Z is the unbounded component, we can approximate $\frac{1}{z-a}$ by $q(z)$ instead.

Proof. Let L_ω be the family of all functions that are holomorphic in a neighbourhood of K that can be approximated uniformly on K by polynomials in $\frac{1}{z-\omega}$. Note that, if $\frac{1}{z-p} \in L_q$, then $L_p \subseteq L_q$.

Consider the set

$$S = \left\{ s \in Z \;\middle|\; \frac{1}{z - s} \in L_b \right\}.$$

We claim that $S = Z$. First note that $b \in S$, therefore $S \neq \emptyset$. Take any point $p \in S$ and $\delta > 0$ such that $\mathbb{\Delta}(p, \delta) \subseteq Z$. For any $s \in \mathbb{\Delta}(p, \delta)$, we can write

$$\frac{1}{z - s} = \frac{1}{z - p} \cdot \frac{1}{1 - \frac{s-p}{z-p}} = \frac{1}{z - p} \cdot \sum_{k=0}^{\infty} \left(\frac{s - p}{z - p} \right)^k,$$

which converges as $|s - p| < |z - p|$ by choice of δ. We conclude that $\mathbb{\Delta}(p, \delta) \subseteq S$. In particular, S is an open set. Take any point $p \in \partial S \cap Z$. Suppose that $\mathbb{\Delta}(p, 3\delta) \subseteq Z$ and

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choose a point $p' \in \mathbb{A}(p, \delta) \cap S$. Then, $\mathbb{A}(p', 2\delta) \subseteq S$ and therefore $\mathbb{A}(p, \delta) \subseteq S$, which is a contradiction and therefore proves our claim.

Now suppose that Z is the unbounded component and take a point $d \in Z$ such that $K \subseteq \mathbb{A}(0, |d|)$. Then all functions $\left(\frac{1}{z-d}\right)^n$ can be approximated uniformly on K by Taylor's polynomials about 0. □

Definition 4.1.5. For any set $P \subseteq \mathbb{C}$ denote by $\mathbb{C}_P[z]$ the family of rational functions with poles in P.

Theorem 4.1.6 (Runge approximation)**.** Let $K \subseteq \mathbb{C}$ be a compact. If P intersects every bounded connected component of $\mathbb{C} \setminus K$, then for every function f, holomorphic in a neighbourhood of K, and every $\varepsilon > 0$ there exists a function $q \in \mathbb{C}_P[z]$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Be lemma 4.1.3, we can find a compact union σ of line segments such that every such f can be approximated by

$$\widetilde{q} = \sum_{k=1}^{m} \frac{c_k}{z - w_k},$$

where $c_k \in \mathbb{C}$ and $w_k \in \sigma$. Suppose then

$$\|f - \widetilde{q}\|_K < \frac{\varepsilon}{2}.$$

Let Z_k be the connected component of $\mathbb{C} \setminus K$ that contains w_k.

If Z_k is bounded, then choose $t_k \in P \cap Z_k$. By the pole shifting lemma, we can find a polynomial g_k such that

$$\left\|\frac{c_k}{z - w_k} - g_k\left(1\right)z - t_k\right\|_K < \frac{\varepsilon}{2m}.$$

If Z_k is unbounded, we can instead approximate $\frac{c_k}{z-w_k}$ by a polynomial instead. Choosing

$$q = \sum_{k=1}^{m} g_k,$$

we find

$$\|f - q\|_K \le \|f - \widetilde{q}\|_K + \|\widetilde{q} - q\|_K < \frac{\varepsilon}{2} + m \cdot \frac{\varepsilon}{2m} = \varepsilon.$$ □

Theorem 4.1.7 (Runge approximation)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $K \subseteq \Omega$ be a compact. If every bounded component of $\mathbb{C} \setminus K$ intersects $\mathbb{C} \setminus \Omega$, then for every function f that is holomorphic on a neighbourhood of K and every $\varepsilon > 0$ there exists a function $q \in \mathcal{O}(\Omega)$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Choose $P = \mathbb{C} \setminus \Omega$ in theorem 4.1.6. □

Corollary 4.1.7.1 (Runge's little theorem)**.** Let $K \subseteq \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. Then for every holomorphic function on a neighbourhood of K and $\varepsilon > 0$ there exists a polynomial $q \in \mathbb{C}[z]$ such that

$$\|f - q\|_K < \varepsilon.$$

Proof. Choose $P = \emptyset$ in theorem 4.1.6. $\square$

Definition 4.1.8. Let V be a vector space over $\mathbb{C}$, equipped with a topology. Let $T\colon V \to V$ be a linear map.

i) We call T *cyclic* if there exists some $f \in V$, called a *cyclic vector*, such that
$$\operatorname{span}_{\mathbb{C}} \{T^n(f) \mid n \in \mathbb{N}_0\} = V.$$

ii) We call T *hypercyclic* if there exists some $f \in V$, such that
$$\overline{\{T^n(f) \mid n \in \mathbb{N}_0\}} = V.$$

Theorem 4.1.9 (Birkhoff)**.** Let $\tau\colon \mathbb{C} \to \mathbb{C}$ be given by $\tau(z) = z + a$ for some $a \neq 0$. Then the map $T\colon \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$, given by $T(f) = f \circ \tau$, is hypercyclic.

Proof. Set $K_n = \overline{\mathbb{\Delta}(\ell_n \cdot a, n)}$ for a sequence $(\ell_n)_n \subseteq \mathbb{N}$, such that all K_n are pairwise disjoint and $K_n \cap \mathbb{\Delta}(0, n) = \emptyset$ for all $n \in \mathbb{N}$. Choose a sequence $(\varepsilon_n)_n \subseteq \mathbb{R}^+$, converging to 0. Furthermore, let $(p_n)_n$ be a sequence of all polynomials $(\mathbb{Q} \oplus i\mathbb{Q})[z]$.

We first construct a sequence of holomorphic functions $(f_n)_n$, such that the following conditions hold:

i) For all $n \in \mathbb{N}$, we have
$$\left\| \sum_{k=1}^{n} f_k - \tau^{-\ell_n} \circ p_n \right\|_{K_n} < \frac{\varepsilon_n}{2^n}.$$

ii) For all $m < n$, we have
$$\|f_n\|_{K_m} < \frac{\varepsilon_m}{2^n}.$$

iii) For all $n \in \mathbb{N}$, we have
$$\|f_n\|_{\mathbb{\Delta}(0,n)} < \frac{1}{2^n}.$$

Choose ℓ_n such that $\mathbb{\Delta}(0, n) \cap K_m = \emptyset$ for all $m < n$. Then the set
$$\mathbb{C} \setminus \left(\bigcup_{m=1}^{n} K_m \cup \mathbb{\Delta}(0, n) \right)$$
is obviously connected. We can now just apply Runge's little theorem by choosing the function 0 on $\mathbb{\Delta}(0, n)$ and K_m for $m < n$, and
$$\tau^{-\ell_n} \circ p_n - \sum_{k=1}^{n-1}$$
on K_n.

Note that the series
$$f = \sum_{n=1}^{\infty} f_n$$

is uniformly convergent by the third condition. We now compute

$$\begin{aligned}\left\|f\circ\tau^{\ell_n}-p_n\right\|_{\tau^{-\ell_n}(K_n)} &= \left\|f-\tau^{-\ell_n}\circ p_n\right\|_{K_n}\\ &\le \sum_{k>n}\|f_k\|_{K_n}+\left\|\sum_{k=1}^n f_k-\tau^{-\ell_n}\circ p_n\right\|_{K_n}\\ &< \varepsilon_n.\end{aligned}$$

It follows that we can approximate every rational polynomial with iterations $T^n(f)$. □

Lemma 4.1.10. Let $\Omega \subseteq \mathbb{C}$ be an open subset $K \subseteq \Omega$ a compact.

1. For every component Z of $\Omega \setminus K$ we have $\Omega \cap \partial Z \subseteq K$. If $Z \subseteq \Omega$ is relatively compact, then $\|f\|_Z \le \|f\|_K$ for all $f \in \mathcal{O}(\Omega)$.
2. Every of Z_0 be a component of $\mathbb{C}\setminus K$ that is contained in Ω is a component of $\mathbb{C}\setminus K$. If Z_0 is bounded, then $Z_0 \subseteq \Omega$ is relatively compact.

Proof. Let p be a point in $\Omega\cap\partial Z$ that is not in K. Let D be a disk centered at p contained in $\Omega \setminus K$. Then every point of D is in the same connected components as p. But then $p \notin \partial Z$, since $D \subseteq Z$.

Suppose now that Z is additionally relatively compact. By the above $\partial Z \subseteq K$ and so $\|f\|\, Z = \|f\|\, \partial Z \le \|f\|_K$ by the maximum modulus princinple.

For the second item, let Z_0 be a component of $\mathbb{C} \setminus K$ contained in Ω. There must exist a $Z_1 \subseteq \Omega \setminus K$ a connected component that contains Z_0. By maximality of connected component $Z_0 = Z_1$.

Suppose it is additionally bounded. The closure of Z_0 in $\mathbb{C}$ is contained in Ω by the first item. But then it is compact as it is bounded and closed in Ω. □

Theorem 4.1.11. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $K \subseteq \Omega$ a compact. The following are equivalent:

1. $\Omega \setminus K$ has no relatively compact components,
2. every bounded component of $\mathbb{C} \setminus K$ intersects $\mathbb{C} \setminus \Omega$
3. every function holomorphic in a neighbourhood of K ca be approximated uniformly on K by rational functions without poles in Ω,
4. every function holomorphic in a neighbourhood of K can be approximated by a function that is holomorphic in Ω,
5. for every $p \in \Omega \setminus K$ there exists a $f \in \mathcal{O}(\Omega)$ such that $|f(p)| > \|f\|\, K$
6. $\hat{K}\mathcal{O}(\Omega) = K$

Proof. From Approximation lemma we immediately get that 2. implies 3. Trivially, we have that 3. implies 4. Items 5. and 6. are the same by definition.

Next we prove that 1. implies 2. If Z_0 is a component of $\mathbb{C} \setminus K$ that is contained in Ω, then it is a component of $\Omega \setminus K$ and relatively compact in Ω by the previous lemma. By 1., this cannot happen so no bounded component of $\mathbb{C} \setminus K$ lies in Ω which is precisely 2.

Suppose 4. holds and $\Omega \setminus K$ has a relatively compact component Z. Choose $a \in Z$ and consider $f(z) = \frac{1}{z-a}$. By assumption this function can be approximated by $g \in \mathcal{O}(\Omega)$ so that for $\delta = \|z-a\|_K$ we have $\left\|\frac{1}{z-a} - g\right\|_K < \frac{1}{\delta}$. Then $\|1-(z-a)g\|_K < 1$. By the first item of the previous lemma, we have $\|1-(z-a)g\|_Z \leq \|1-(z-a)g\|_K < 1$. For $z = a$, this is a contradicition. This implies 1.

Suppose 5. holds and Z is a relatively compact component. As in the previous paragraph, we have that $\|f\|_Z \leq \|f\|_K$ for all $f \in \mathcal{O}(\Omega)$, in particular this holds for f from the assumption of 5., which is again a contradiction. □

this is the remainder of the proof that we did not finish last time, TODO merge

Proof. Suppose $\Omega \setminus K$ has no relatively compact components. Let $p \in \Omega \setminus K$. Consider $\Omega \setminus (K \cup \{p\})$. This has the same connnected components not containing p which are still not relatively compact. Removing the point p does not disconnect and open connected component, so the connected component that contained p is still relatively compact. We can use the fourth item from the statement on the function $g|_K = 0$ and $g(p) = 1$, which is holomorphic in a neighbourhood of $K \cup p$. Thus there exist h, a holomorphic function on Ω such that $\|g-h\|_{K\cup p} < \frac{1}{2}$. Then $\|h\|_K < \frac{1}{2} < |h(p)|$ which is exactly the fifth item. □

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Lemma 4.1.12. Let $\Omega \subseteq \mathbb{C}$ is an open subset and $K \subseteq \Omega$ a compact. Then there exists a compact K' such that $K \subseteq K' \subseteq \Omega$ and every bounded component of $\mathbb{C} \setminus K'$ intersects $\mathbb{C} \setminus \Omega$ and contains a compact component of $\mathbb{C} \setminus \Omega$.

Proof. For the case $\Omega = \mathbb{C}$, we can take $K' = \overline{D}_r(0)$, where r is large enough so that $K \subseteq K'$. So suppose $\Omega \neq \mathbb{C}$. Choose $0 < \rho < d(K, \partial\Omega) = d(K, \mathbb{C} \setminus \Omega)$. Let $M = \{z \in \Omega \mid d(z\mathbb{C} \setminus \Omega) \geq \rho\}$. Choose $r > 0$ such that $K \subseteq \overline{D}_r(0)$. Let $K' = M \cap \overline{D}_r(0)$, which is clearly compact, contained inside Ω, and contains K. We are left to prove the statement about the bounded components.

Let Z be a bounded component of $\mathbb{C} \setminus K'$. Note that $\mathbb{C} \setminus \overline{D}_r(0)$ is connected and unbounded and contains Z. If Z were also not in $C \setminus M$, then it would be in the unbounded component of $\mathbb{C} \setminus K'$. Thus $Z \subseteq \mathbb{C} \setminus M$.

We can rewrite $\mathbb{C} \setminus M = \bigcup_{w \in \mathbb{C} \setminus \Omega} D_\rho(w)$ by the definition of M. We have that $D_\rho(w) \subseteq Z$ or $D_\rho(w) \cap Z = \emptyset$. Thus $Z = \bigcup_{w \in Z \setminus \Omega} D_\rho(w)$ so Z intersects $\mathbb{C} \setminus \Omega$. Let S be the component of $\mathbb{C} \setminus \Omega$ that Z intersects $\mathbb{C} \setminus \Omega \subseteq \mathbb{C} \setminus K'$. Then $S \subseteq Z$ since Z is a maximal connected set in $\mathbb{C} \setminus K'$. Since Z is bounded, S is bounded. □

Definition 4.1.13. Let $\Omega \subseteq \mathbb{C}$ be an open subset. We call every bounded component of $\mathbb{C} \setminus \Omega$ a *hole* of Ω.

Remark 4.1.13.1. If K is the Cantor set, then $\Omega = \mathbb{C} \setminus K$ has uncountably many holes.

Theorem 4.1.14 (Runge's Theorem for rational apporximations)**.** Let $\Omega \subseteq \mathbb{C}$ be an open subset $P \subseteq \mathbb{C} \setminus \Omega$ such that $\overline{P}$ intersects every hole of Ω. Then $\mathbb{C}_P[z]$ is dense in $\mathcal{O}(\Omega)$.

Proof. Let $K \subseteq \Omega$ be compact. Let K' be from the previous lemma. Then $\overline{P}$ intersects every bounded componont of $\mathbb{C} \setminus K'$. We can thus apply the lemma **??** on K'. □

Corollary 4.1.14.1 (Runge's theorem for polynomial approximation)**.** If Ω has no holes then $\mathbb{C}[z]$ is dense in $\mathcal{O}(\Omega)$.

Proof. Take $P = \emptyset$. □

Remark 4.1.14.2. This gives us another proof of the Mittag-Leffler theorem 3.4.16.

Proof. Let $A = \bigcup_{k\in\mathbb{N}} \{a_k\} \subseteq \partial\Omega$. Let

$$K_j = \overline{\left\{ z \in \Omega \setminus A \;\middle|\; |z| \le j, d(z, \partial\Omega \setminus A) \ge \frac{1}{j} \right\}},$$

which form an exhaustion by compacts. Let K_j' be the compact furnished by the previous lemma for K_j. Choose a subsequence $(L_j)_{j\in\mathbb{N}}$ of $(K_j')_{j\in\mathbb{N}}$ that is an exhaustion by compacts, which can be done by the properties of exhaustion, i. e. for every K_j' there exists $k \in \mathbb{N}$ so that $K_j' \subseteq K_k \subseteq K_k'$ and apply induction.

We may assume $K_1 \cap \overline{A} = \emptyset$. Let $A_n = A \cap (L_{n+1} \setminus L_n)$. For each $a \in A_n$ the principal part of q is holomorphic in L_n. By the approximation with holomorphic functions on Ω, there exists $g \in \mathcal{O}(\Omega)$ such that $\|q - g\|_{L_n} < \max(1, |A_n|)2^{-n}$. Then

$$\sum_{n=1}^{\infty} \|q_n - g_n\|_{L_n} \le \sum_{n=1}^{\infty} 2^{-n} - 1.$$

Thus the series converges uniformly on compacts. □

Definition 4.1.15. (Ω, Ω') is a *Runge pair* if for every $f \in \mathcal{O}(\Omega)$ and each $\epsilon > 0$ and compact $K \subseteq K$ there exists $F\mathcal{O}(\Omega')$ such that $\|f - F\|_K < \epsilon$.

Theorem 4.1.16. Let $\Omega \subseteq \Omega' \subseteq \mathbb{C}$ be open sets. The following statements are equivalent:

1. $\Omega' \setminus \Omega$ has no compact components,
2. $\mathbb{C}_P[z]$ with $P \subseteq \mathbb{C} \setminus \Omega'$ is dense in $\mathcal{O}(\Omega)$,
3. (Ω, Ω') is a Runge pair,
4. $\Omega' \setminus \Omega$ contains no open compact.

Proof. □

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