Problems

Sunday, August 28, 2016 10:47 PM

Problems and Notes

These are the Other Math Club problems, a big collection scattered across the landscape. They are in categories by difficulty level and so on.

Very Important: Most of these problems will not make sense to you today. When you read them and say 'what???' you can also say 'We must discuss this further at Math Club!' We will use club time to learn how these problems work.

Final Examination

Why is the Final Examination first? Because the club moves backwards in time of course.

0. What is $e^{i\pi} + 1$?
1. Write down some mathematics, please.
2. How many times does the number 7 appear in this question together with your answer once we include these as well: 7 7 7 7 7 7 ?
3. Write down some form of infinity.

Problem Solving

If you are stuck on a problem: Here are some general guidelines that might help.

RULE ZERO: Look at yourself from the outside and give yourself some advice
RULE 1: Don't hurry! You might rush to a wrong conclusion
RULE 2: Read the problem slowly out loud... more than once until it makes sense to you
RULE 3: Ask yourself if you understand every single word in the problem.
RULE 4: Solve a simpler, related problem
RULE 5: Ask yourself to guess the answer, or what the answer looks like

Moderately Difficult Problems

E0. Prove or Disprove: Even + Even = Even, Even + Odd = Odd, Odd + Even = Odd, Odd + Odd = Odd.

E1. Prove or Disprove: Even x Even = Even, Even x Odd = Even, Odd x Odd = Odd, Odd x Even = Odd.

E2. Prove or Disprove: If a number n is evenly divisible by a number m then the number $(n + 1)$ is not divisible by that same number m.

E3. Prove or Disprove: If you square some number a you will always get one more than $(a - 1)$ times $(a + 1)$. For example 7 x 7 is one more than 6 times 8: 49 = 48 + 1.

E4. Prove or Disprove: All prime numbers bigger than 5 end in one of these four digits: 1, 3, 7 or 9.

E5. Prove or Disprove: Beyond (greater than) 3, 5 and 7 there are no triplet primes.

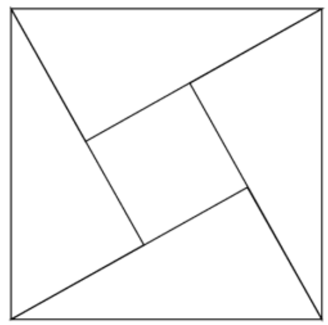
E6. Prove or Disprove: Every even number greater than two can be written as the sum of two prime numbers.

E7. A kite is stuck at the top of a tree with a kite string 130 feet long dangling downwards. You grab the loose end of that string and pull it out until it reaches a point 50 feet from the base of the tree. How high is the tree? What kind of tree is it?

E8. Color the odd and even entries of Pascal's triangle using two colors to produce a Serpienski gasket.

Medium Difficult Problems

M1. Use the area formula for right triangles $A = \frac{1}{2} \cdot b \cdot h$ to arrive at the Pythagorean theorem, from:



M2. Consider odd magic squares: 1 x 1, 3 x 3, 5 x 5 and so on; in general *n* x *n* where *n* is odd. Say we always number the cells 1, 2, 3, …, up to n^2. What is the formula for the magic number (what the rows, columns and diagonals each add up to) in terms of *n*?

M3. Determine the value of $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$

M4. Determine the value of $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}$

M5. Formulate an argument or proof of Euler's planar graph formula $V+F=E+2$.

M6. Describe the shortest distance from a line to a point not on the line.

M7. Prove or disprove: $\left(1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\cdots\right)\cdot(a-1)=a$

M8. Geometric constructions. Research the idea of constructions using a straight edge and a compass. Don't worry about the 'collapsible compass'; just use a normal one (the kind that can draw circles, not the one that registers magnetic north). Using these two tools and a pencil construct: A line, a line segment, a line segment bisected, a circle, a circle bisected, a circle quartered, a circle with a chord, a circle with tangent, an isosceles triangle, a scalene triangle, an equilateral triangles, a square, a hexagon, a bisected angle, a square inscribed in triangle, parallel lines built from a single triangle, a 12-gon, an 8-gon, the "3 centers" of a triangle, and the bisection of the supplement of an acute angle.

M9. Show how you can drag the Fibonacci series out of the Pascal triangle by summing peculiar diagonals. Why dos this work?

M10. What does this infinite series of fractions add up to: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

M11. Mister Halfway never finishes what he starts. He decides to walk on the real number line from 0 to 1 one day; but he only gets halfway there before he stops and give us. He then chooses to walk back to the origin but only makes it halfway before he stops and gives up. Then he turns around and starts walking back to where he first stopped at 1/2… but you guessed it, he only makes it half way before he stops and gives up. In this way he keeps going back and forth, each time walking a smaller and smaller distance. Where does he finally arrive on the number line?

M12. Prove or Disprove: There is a finite number of prime numbers.

Hard Problems

H1. Use Euler's Formula for Planar Graphs to determine all of the Platonic solids.

H2. This is the Demolition Derby or Zusammengestossen problem. Sometimes it is stated using little bugs instead of cars but I will use cars. A Demolition Derby is a competition in a big arena where cars drive around crashing into one another. The last car that can still move is declared the winner. In this problem four cars begin at the vertices of a square 100 meters on a side. Each car faces directly towards the next around the perimeter, say going clockwise. The cars are identical and are driven by identical quadruplets who drive exactly the same. At the Go signal all four cars start out heading directly towards the next car, the one they are facing. Clearly if all cars drive identically they will soon crash together in the center of the square. How far does each car travel and how many times does it cross the diagonal before all the cars crash together (in German '*zusammengestossen*') at the center? We solve this problem in an idealized manner: The cars are idealized as points and the trajectories are all identical and idealized. You can also extend this problem to different numbers of cars: 2 cars, 3 cars, 7 cars, n cars, even to an infinite number of cars (which is also known as I-5).

H3. Find the ratio of convergence for a general Fibonacci sequence $(a_0, a_1) \neq (0,0)$, that is find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

H4. Determine the sum of the angles of a regular n-gon.

H5. Determine the sum of the rows of the Pascal triangle by induction.

H6. Show that positive integers greater than 1 have a unique factorization. This fact looks obvious but can be a little tricky to prove. It is so important that it is called the Fundamental Theorem of Arithmetic. It is also called the Unique Factorization Theorem. You know it must be important if it has two names.

H7. Determine what this quote means: 'Logos, Pathos, Ethos' -Aristotle

H8. Prove that a graph with $2n$ vertices and n^2+1 edges contains a triangle. (Erdos to Posa when the latter was 11)

H9. Prove or disprove: $\sum_n \frac{1}{n^s} = \prod_p (1-(p^s)^{-1})^{-1}$

Impossible Problems

I1. The Sixes game: Determine the expected number of roles of *n* 6-sided dice such that each comes up a six at least once. You can play this game with a big pile of dice by rolling them all at once, removing the dice that come up 6, and then rolling the remainder until you have none left. Count how many rolls this took. This is one experimental observation. Do this a million times and take the average: That will be close to the correct answer probably. How can this number be derived mathematically?

I2. Recruiter problem: Suppose *k* people are ranked in some order *1, 2, …, k*. An organization wants to recruit them to fill a set of *N* jobs. They are invited in order according to their rank; where each person may accept or decline the offer. The offers are extended until all *N* positions are filled. The probability that any person will accept an offer is *a* (a constant independent of the person's rank). The last piece of information is that the probability that person with rank $k \cdot N$ receiving an offer is small, leaving abstractions 'What is *k* and what is *small*?'. Two questions follow: What is *a*? What is the probability that the person with rank *m* will receive a job offer?

I3. This problem is solvable only because the statements of P and S are sequential in time. Two numbers x and y are selected such that $1 < x < y$ and $x + y \leq 100$. Two people P and S know this but do not the values of x and y. P is taken aside and told the value of $x \cdot y$. S is taken aside and told the value of $x + y$. They make the following statements:

> P: I don't know { x, y }.
> S: I knew you didn't.
> P: Now I know { x, y }.
> S: So do I.

What is { x, y }?

I4. Bertrand's Conjecture: Prove or Disprove that there is always a prime on (n, 2n) for n > 1.

Team Projects

T1. Extend Euler's Formula for Planar Graphs to forests and to a torus.

T2. Make a mathematical argument that rainbows exist. The idea is to not use empirical evidence like pictures of rainbows. Two ideas may help: You can look up (Wikipedia) and use the laws of physics for refraction of light through a transparent interface. And you can assume sunlight interacts with small droplets of water suspended in the air that are perfectly spherical.

T3. Build a hexahexaflexagon and make a graph of its configurations.

T4. A baseball is thrown horizontally at 8000 meters per second. It is thrown from a height of 2 meters off the ground. In one second a baseball falls 5 meters. This is on earth, which is a sphere with a radius of 6,400,000 meters. After the baseball travels horizontally for one second and falls five meters: How high off the ground is it? You can approximate this answer using just one right triangle.

T5. A Rosie problem: Explore gallon puzzles as graphs: Is there something to be gained from this approach? Under what conditions are gallon problems unsolvable? Can Nim and Hanoi benefit from graphs?

T6. You are floating in space observing the sun when the earth happens to pass nearby. A friend of yours has built a pole from copper pipe that is 9 times the earth's radius and planted this in the ground, sticking straight up. As the earth goes by the other end of the copper pipe goes whizzing past your nose. How fast is it moving?

T7. An Egon problem: What pattern is produced by the 'dice and halfway' procedure on the Monte Carlo piazza?

T8. How many amino acids can be encoded by a sequence of three nucleotides? How many amino acids are actually available? What is a peptide? What is a neurotoxin? Can a peptide be a neurotoxin? Does the sea snail Conus textile eat vegetables? If not: How does she get dinner? How does she build her shell? How does she decorate her shell?

Computational Problems

C1. Euler wrote $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ which is very curious. What is e^0? e^1? e^2? e^{-1}? e^{-2}? How shall we look at this function? Map one line into a parallel line with bridge lines; and try cross-ways.

Unsorted Problems

Bee

Poker chips

Hole through sphere

How far do you expect to have to walk on a tape of pi to find *n* arbitrary digits. How many occurrences of this sequence do you expect to find?

Prove or Disprove: There is no Conway Life configuration that grows without bound.

Prove or Disprove: The sum $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is unbounded.

Prove or Disprove: The sum $\sum_{n=1}^{\infty} \frac{1}{n}$ is unbounded.

Prove or Disprove: The sum $\sum_{p} \frac{1}{p}$ is bounded.

Prove or Disprove: The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is bounded. If bounded: Equals...?

Should you go first or second playing 2-dot Sprouts? (First player with no moves loses.) What is your strategy?

Spider fly

Tiling by symmetric notching