Classes of models to investigate intermittent synchronisation and the pacing of ice ages by Milankovich forcing

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1 Models of the glacial-interglacial cycles

This is a summary of different (conceptual, simple, low-dimensional) models that could reflect the glacial-interglacial cycles. By studying how different types of models synchronise or pace their glacial cycles to periodic and quasi-periodic forcing (Milankovich forcing) we may be able to classify different types of "pacing" - from "tight pacing" to "intermittent pacing" to "no pacing".

One way of pacing, non-linear phase locking, has been described by Tziperman et al. [2006].

2 Minimal models - oscillators

2.1 Van der Pol oscillator

The first model to consider is the SCW De Saedeleer et al. [2013] model

$$\tau \frac{d}{dt}x = \gamma F(t) - \beta - y \tag{1}$$

$$\tau \frac{d}{dt}y = \alpha(y - y^3/3 + x) \tag{2}$$

or the version with noise

$$\tau dx = (\gamma F(t) - \beta - y)dt + \xi_x dW_x \tag{3}$$

$$\tau dy = \alpha(y - y^3/3 + x)dt + \xi_y dW_y \tag{4}$$

where the forcing term $F(t)$ represents the astronomical forcing and can be either periodic (e.g. only obliquity component) or quasiperiodic. The parameters De Saedeleer et al. [2013] give are

$$\alpha = 36, \quad \beta = 0.75, \quad \gamma = 0.4, \quad \tau = 36. \tag{5}$$

This model is based on a modified version of the *van der Pol* oscillator. The nonlinearity is only in the damping of the free oscillation. In Crucifix [2012] it is presented as a minimal model for the glacial cycles. In terms of its behaviour under (quasi)periodic forcing it may be rather special though De Saedeleer et al. [2013].

2.2 Van der Pol-Duffing oscillator

A similar oscillator extended by an additional nonlinear term (not only in the damping) is the *van der Pol-Duffing* oscillator:

$$\tau^2 \frac{d^2}{dt^2}x + \tau\varepsilon(x^2 - 1)\frac{d}{dt}x + \omega x + f x^3 = \beta + \gamma F(t), \tag{6}$$

where the right hand side is representing the constant plus time dependent forcing.

2.3 Forced Phase oscillators

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3 Conceptual models of the glacial cycles

3.1 Saltzman et al. models

The models constructed by Saltzman et al Saltzman [2002], Saltzman and Verbitsky [1993], Saltzman and Maasch [1991], Maasch and Saltzman [1990], Saltzman and Maasch [1990, 1988] have established the idea that the (late Pleistocene) glacial cycles appear as a limit cycle in the unforced system then synchronised in some way to the orbital forcing. The dynamics (and consequently the specific form of the limit cycle and the bifurcation leading to the limit cycle) varies across the different models. In most of them, however, it is assumed that the background climate slowly varies throughout the Pliocene-Pleistocene ('tectonically driven' decline in atmospheric CO_2-concentration) and the model is formulated as an anomaly model to a (slowly evolving) background climate. Crucifix [2012] analysed the bifurcation structure of two of these models (Saltzman and Maasch [1991, 1990]) for the full (non-anomaly) equation with respect to one parameter, the slowly drifting 'tectonic' CO_2 decline $F_\mu(t)$. Much of the interesting dynamics depends on the specific form of the CO_2 equation, which at the same time is the most problematic to physically interpret.

3.1.1 Saltzman and Maasch 1988

This model Saltzman and Maasch [1988] (SM88) has three (slow) variables, the ice volume I, the North Atlantic deep water formation rate N and the atmospheric CO_2 concentration μ. All variables are assumed to be the departure from a true equilibrium state, which is not further specified. Moreover, two fast response variables are involved, the global mean sea surface temperature τ and the global extent of permanent sea ice η, which are expressed in terms of the slow response variables I, μ and N, and do not appear explicitly in the equations. Dynamical system:

$$\dot{I} = -a_0 I - a_1\mu + F_I(t) \tag{7}$$
$$\dot{\mu} = b_0 I + b_1\mu - (r_3 - b_3 N)N - b_4 N^2\mu + F_\mu(t) \tag{8}$$
$$\dot{N} = -c_0 I - c_2 N + F_N(t) \tag{9}$$

The F_X terms denote forcing terms, which are combinations of external (e.g. orbital) forcings $\mathcal{F}_I$, $\mathcal{F}_\mu$, $\mathcal{F}_N$, $\mathcal{F}_\tau$ and $\mathcal{F}_\eta$ on the dynamical variables (e.g. $F_I = \mathcal{F}_I - s_1\mathcal{F}_\tau + s_3\mathcal{F}_\eta$). A (standard) set of parameters used in Saltzman and Maasch [1988] is given in Table. 3.1.1.

Rescaling the equations by $t = [a_0^{-1}]t^*$, $I = [c_2 c_0^{-1}(a_0/b_4)^{1/2}]X$, $\mu = [c_2(a_1 c_0)^{-1}(a_0^3/b_4)^{1/2}]Y$ and $N = [(a_0/b_4)^{1/2}]Z$, and setting $b_0 = 0$ the non-dimensional system becomes:

$$\dot{X} = -X - Y \tag{10}$$
$$\dot{Y} = -pZ + rY + sZ^2 - Z^2 Y \tag{11}$$
$$\dot{Z} = -q(X + Z), \tag{12}$$

with four parameters p, q, r, and s (asymmetry parameter). With the chosen set of parameters Saltzman and Maasch [1988] were able to obtain strongly asymmetric glacial cycles. All forcing terms are set to zero. However, they also note that with the given set of parameters there exists only one true equilibrium at $(X, Y, Z) = (0, 0, 0)$, such that the system does not show a bifurcation at the Mid-Pleistocene transition.

Parameter	physical meaning
$a_0 = 10^{-4}$ yr^{-1}	Linear response time ice.
$a_1 = 4.37 \times 10^{13}$ kg (yr ppm)$^{-1}$	Effect of CO_2 on ice dynamics.
$b_0 = 0$	
$b_1 = 8 \times 10^{-5}$ yr^{-1}	Linear response time CO_2.
$b_2 = 2.84 \times 10^{-20}$ ppm (yr m^3)$^{-1}$	
$b_3 = 2.11 \times 10^{-37}$ ppm (yr m^6)$^{-1}$	
$b_4 = 6.97 \times 10^{-39}$ (yr m^6)$^{-1}$	
$c_0 = 8.7 \times 10^{-7}$ m^3 (yr kg)$^{-1}$	Effect of ice on thermohaline circulation dynamics.
$c_2 = 1.2 \times 10^{-4}$ yr^{-1}	Linear response time thermohaline circulation.
$p = 0.9$	$p = a_1 c_0 b_2 / a_0^2 c_2$
$q = 1.2$	$q = c_2/a_0$
$r = 0.8$	$r = b_1/a_0$
$s = 0.8$	$s = a_1 b_3 c_0 (b_4/a_0)^{1/2}/a_0 b_4 c_2$

Table 1: Parameters and their meaning of the model by Saltzman and Maasch [1988].

3.1.2 Saltzman and Maasch 1990

This model Saltzman and Maasch [1990] (SM90) has three (slow) variables, the ice volume I, the atmospheric CO_2 concentration μ and the deep ocean temperature (or North Atlantic deep water formation rate) θ. Equations are similar to the Saltzman 1988 model, but not a priori defined as an anomaly to an equilibrium state. A fast variable is again included, the zonal mean surface air temperature at high latitudes in summer $\tilde{\tau}$, which is expressed in terms of the three main slow variables (through a series of assumptions). Dynamical system:

$$\dot{I} = \alpha_1 - \alpha_2 \tanh(c\mu) - \alpha_3 I - k_\theta \alpha_2 \theta - k_R \alpha_2 F_I(t) + \mathcal{W}_I(t), \tag{13}$$

$$\dot{\mu} = [\beta_1 - (\beta_2 - \beta_3\theta + \beta_4\theta^2)\mu - (\beta_5 - \beta_6\theta)\theta] + F_\mu(t) + \mathcal{W}_\mu(t), \tag{14}$$

$$\dot{\theta} = \gamma_1 - \gamma_2 I - \gamma_3\theta + F_\theta(t) + \mathcal{W}_\theta(t). \tag{15}$$

Parameters are given in Table 3.1.2.

Again, the F_X terms denote forcing terms and the $\mathcal{W}_X$ terms represent potential stochastic forcing. When applying the set of equations to the conditions during the Pleistocene, where atmospheric CO_2 is assumed to vary between 150–350 ppm, it is considered safe to approximate $\tanh(c\mu) \simeq c\mu$ Saltzman and Maasch [1990]. In Crucifix [2012] (C12) the full model is analysed with respect to variations in the parameter F_μ, while the other forcings F_I and F_θ are set to zero.

Next, the model is divided into a steady state part in equilibrium with the deterministic tectonic average forcing (where in this version of the model $F_I = F_\theta = 0$) I_0, μ_0, θ_0 and the departure from that equilibrium I', μ', θ'. After scaling the departure variables by $t = a_2^{-1}t^*$, $I' = (c_2/c_1)(a_2/b_5)^{1/2}X$, $\mu' = (c_2/(a_1c_1))(a_2^3/b_5)^{1/2}Y$ and $\theta' = (a_2/b_5)^{1/2}Z$, we end up with

	SM90	C12 (full model)	units	physical meaning
α_1	1.7×10^{16}	$1.8.075 \times 10^{16}$	kg yr^{-1}	Constant ice growth rate.
α_2	1.3×10^{16}	1.275×10^{16}	kg yr^{-1}	Effect of CO_2 and θ on ice dynamics.
α_3	1.0×10^{-4}	10^{-4}	yr^{-1}	Linear response time ice.
β_1	1.2	1.355608	ppm yr^{-1}	CO_2 coefficients.
β_2	4.7×10^{-3}	5.4688×10^{-3}	yr^{-1}	CO_2 coefficients.
β_3	5.4×10^{-2}	2.213×10^{-3}	ppm (°C^2 yr)$^{-1}$	CO_2 coefficients.
β_4	2.2×10^{-4}	2.2×10^{-4}	(°C^2 yr)$^{-1}$	CO_2 coefficients.
β_5	0.5	0.541055	ppm (°C^2 yr)$^{-1}$	CO_2 coefficients.
β_6	2.1×10^{-3}	5.3×10^{-2}	ppm (°C^2 yr)$^{-1}$	CO_2 coefficients.
γ_1	1.9×10^{-3}	1.836×10^{-3}	°C yr^{-1}	Constant θ growth.
γ_2	1.2×10^{-23}	1.2×10^{18}	°C(kg yr)$^{-1}$	Effect of ice on θ.
γ_3	2.5×10^{-4}	2.4×10^{-4}	yr^{-1}	Linear response time θ.
c	4×10^{-3}	4×10^{-3}	ppm$-$1	
κ_θ		3.333×10^{-2}	(°C)$^{-1}$	

Table 2: Parameters and their meaning of the model by Saltzman and Maasch [1990] (SM90). Also given are the parameter values used by Crucifix [2012] in the analysis of the full model (C12). Still need to check for typos in non-dimensional equations and parameters, SM90.

the non-dimensional departure system:

$$\begin{aligned}\dot{X} &= -X - Y - vZ - uR(t^*) + \mathcal{W}_X(t^*) && (16)\\ \dot{Y} &= -pZ + rY + sZ^2 - wYZ - Z^2Y + \mathcal{W}_Y(t^*) && (17)\\ \dot{Z} &= -q(X+Z) + \mathcal{W}_Z(t^*), && (18)\end{aligned}$$

with seven parameters:

$$\begin{aligned}
p &= \frac{a_1 b_2 c_1}{a_2^2 c_2} = \frac{c\alpha_2(\beta_5 - \beta_3\mu_0 - 2\beta_6\theta_0 + 2\beta_4\theta_0\mu_0)\gamma_2}{\alpha_3^2\gamma_3}\\
q &= \frac{c_2}{a_2} = \frac{\gamma_3}{\alpha_3}\\
r &= \frac{b_1}{a_2} = \frac{-\beta_2 + \beta_3\theta_0 - \beta_4\theta_0^2}{\alpha_3}\\
s &= \frac{a_1 b_3 c_1}{(a_2^3 b_5)^{1/2} c_2} = \frac{c\alpha_2(\beta_6 - \beta_4\mu_0)\gamma_2}{(\alpha_3^3\beta_4)^{1/2}\gamma_3}\\
u &= \left(\frac{\kappa_{\mathcal{R}}|\mathcal{R}|}{c}\right)\left(\frac{a_1c_1}{c_2}\right)\left(\frac{b_5}{a_2^3}\right)^{1/2} = \kappa_{\mathcal{R}}|\mathcal{R}|\left(\frac{\alpha_2\gamma_2}{\gamma_3}\right)\left(\frac{\beta_4}{\alpha_3^3}\right)^{1/2}\\
v &= \left(\frac{\kappa_\theta}{c}\right)\left(\frac{a_1c_1}{a_2c_2}\right) = \kappa_\theta\left(\frac{\alpha_2\gamma_2}{\alpha_3\gamma_3}\right)\\
w &= \frac{b_4}{(a_2b_5)^{1/2}} = \frac{-\beta_3 + 2\beta_4\theta_0}{(\alpha_3\beta_4)^{1/2}}
\end{aligned}$$

The special solution discussed in Saltzman and Maasch [1990] was found with parameter values $p = 1.0$, $q = 2.5$, $r = 0.9$, $s = 1.0$, $u = 0.6$, $v = 0.2$, $w = 0.5$. u represents the strength of the orbital forcing, q is a ratio of the linear response times of ice to ocean, r and s depend on the (drifting) equilibrium solution (I_0, μ_0, θ_0).

3.1.3 Saltzman and Maasch 1991

In Saltzman and Maasch [1991] (SM91) the model described in the previous section is modified, mostly in the representation of the carbon cycle dynamics. The complete system nevertheless exhibits different dynamics Crucifix [2012]. Dynamical system:

$$\begin{aligned}\dot{I} &= \alpha_1 - \alpha_2\tanh(c\mu) - \alpha_3 I - k_\theta\alpha_2\theta - k_R\alpha_2F_I(t) + \mathcal{W}_I(t), && (19)\\ \dot{\mu} &= \beta_1 - \beta_2\mu + \beta_3\mu^2 - \beta_4\mu^3\mu - \beta_5\theta + F_\mu(t) + \mathcal{W}_\mu(t), && (20)\\ \dot{\theta} &= \gamma_1 - \gamma_2 I - \gamma_3\theta + F_\theta(t) + \mathcal{W}_\theta(t). && (21)\end{aligned}$$

In Crucifix [2012] (C12) the full model is analysed with respect to variations in the parameter F_μ, while the other forcings F_I and F_θ are set to zero.

The model is again transformed into an anomaly model, by dividing into a steady state part in equilibrium with the deterministic tectonic average forcing (including F_μ). I_0, μ_0, θ_0 and the departure from that equilibrium I', μ', θ'. After scaling the departure variables by

$t = a_2^{-1}t^*$, $I' = (a_1/(a_2b_3)^{1/2})X$, $\mu' = (a_2/b_3)^{1/2}Y$ and $\theta' = (a_1c_1/(c_2(a_2b_3)^{1/2}))Z$, we end up with the non-dimensional departure system:

$$\begin{aligned}\dot{X} &= -X - Y - vZ - uR(t^*) + \mathcal{W}_X(t^*) \quad (22)\\ \dot{Y} &= -pZ + rY - sY^2 - Y^3 + \mathcal{W}_Y(t^*) \quad (23)\\ \dot{Z} &= -q(X+Z) + \mathcal{W}_Z(t^*), \quad (24)\end{aligned}$$

and six non-dimensional parameters:

$$\begin{aligned}p &= \frac{a_1b_4c_1}{a_2^2c_2} = \frac{c\alpha_2\beta_5\gamma_2}{\alpha_3^2\gamma_3}\\ q &= \frac{c_2}{a_2} = \frac{\gamma_3}{\alpha_3}\\ r &= \frac{b_1}{a_2} = \frac{-\beta_2 + 2\beta_3\mu_0 - 3\beta_4\mu_0^2}{\alpha_3}\\ s &= \frac{b_2}{(a_2b_3)^{1/2}} = \frac{3\beta_4\mu_0 - \beta_3}{(\alpha_3\beta_4)^{1/2}}\\ u &= \left(\frac{\kappa_{\mathcal{R}}|\mathcal{R}|}{c}\right)\left(\frac{b_3}{a_2}\right)^{1/2} = \kappa_{\mathcal{R}}|\mathcal{R}|c\left(\frac{\beta_4}{\alpha_3}\right)^{1/2}\\ v &= \left(\frac{\kappa_\theta}{c}\right)\left(\frac{a_1c_1}{a_2c_2}\right) = \kappa_\theta\left(\frac{\alpha_2\gamma_2}{\alpha_3\gamma_3}\right)\end{aligned}$$

The special solution discussed in Saltzman and Maasch [1991] was found with parameter values $p = 1.0$, $q = 2.5$, $r = 1.3$, $s = 0.6$, $u = 0.5$, $v = 0.2$. u represents the strength of the

	SM91	C12 (full model)	units	physical meaning
α_1	1.4×10^{16}	1.673915×10^{16}	kg yr^{-1}	Constant ice growth rate.
α_2	9.4×10^{15}	9.52381×10^{15}	kg yr^{-1}	Effect of CO_2 and θ on ice dynamics.
α_3	1.0×10^{-4}	10^{-4}	yr^{-1}	Inverse linear response time ice.
β_1	$0.5 - \bar{F}_{\mu_0}$	0.5118377	ppm yr^{-1}	CO_2 coefficients.
β_2	6.3×10^{-3}	6.258680×10^{-3}	yr^{-1}	CO_2 coefficients.
β_3	2.6×10^{-5}	2.639456×10^{-5}	(ppm yr)$^{-1}$	CO_2 coefficients.
β_4	3.6×10^{-8}	3.628118×10^{-8}	(ppm^2 yr)$^{-1}$	CO_2 coefficients.
β_5	5.6×10^{-3}	5.833333×10^{-3}	ppm (°C yr)$^{-1}$	CO_2 coefficients.
γ_1	1.9×10^{-3}	1.85125×10^{-3}	°C yr^{-1}	Constant θ growth.
γ_2	1.2×10^{-23}	1.125×10^{-23}	°C(kg yr)$^{-1}$	Effect of ice on θ.
γ_3	2.5×10^{-4}	2.5×10^{-4}	yr^{-1}	Inverse linear response time θ.
c	4×10^{-3}	4×10^{-3}	ppm-1	
κ_θ		4.4444444×10^{-2}	(°C)$^{-1}$	

Table 3: Parameters and their meaning of the model by Saltzman and Maasch [1991] (SM91). Also given are the parameter values used by Crucifix [2012] in the analysis of the full model (C12). Still need to check for typos in non-dimensional equations and parameters, SM91.

orbital forcing, q is a ratio of the linear response times of ice to ocean, r and s depend on the (drifting) equilibrium solution (I_0, μ_0, θ_0).

3.2 Relaxation oscillators

There exist a number of models, where – as in the Saltzman models – the late Pleistocene glacial cycles appear as an internal, unforced oscillation, but where the fast variable is explicitly resolved. These are so-called relaxation oscillators, the simplest version thereof was developed by Paillard [1998]. In this model, the climate is assumed to have three distinct regimes, an interglacial i, a weak glacial g and strong glacial G. Certain transitions between the regimes ($i \to G$, $g \to G$, $G \to i$, are triggered by thresholds in insolation and ice volume. Note that the original version of the Paillard model is strictly speaking not an oscillator, because the transition from $g \to G$ is induced by a threshold in the orbital forcing. The extension used in Ashwin and Ditlevsen [2015] includes an explicit form of the slow manifold and is a real oscillator in this sense.

AH: Is this true?

As discussed in Omta et al. [2013], the fact that the glacial cycles show a saw-tooth-like, asymmetric shape seems to require a special type of oscillator - where the relaxation oscillator of slow-fast systems is a good candidate. Asymmetric oscillations in nonlinear models can be generated by (i) switches or (ii) dynamically when there is an additional variable that exhibits sharp spikes at the 'rapid' transitions (similar to a switch).

3.2.1 The Paillard-Parenin model

In the model by Paillard and Parrenin [2004], an explicit (though qualitative) physical explanation for switches between glacial and interglacial regimes is sought. The main assumption here is that deglaciations are induced by atmospheric CO_2. In this case, the (slow) transition from interglacial to glacial can be explained by Milankovitch theory, where the summer insolation in northern high latitudes play a major role. The glacial to interglacial transition (how the escape from a deep glacial) is more difficult to explain and requires a mechanism to release CO_2 from the deep ocean into the atmosphere. In this model, this is achieved by an oceanic switch, characterised by a *salty bottom water formation efficiency* parameter F. During cold periods, the southern ocean bottom water formation is strong, the deep ocean is strongly stratified (cold and salty Antarctic bottom water, AABW) and can store a lot of carbon. However, a few thousand years after a glacial maximum, the Antarctic ice sheet reaches its maximum extent making salty AABW formation through brine rejection on the continental shelf more difficult, and Southern Ocean stratification weakens/ atmospheric CO_2 starts to rise.

The model involves three variables, the global ice volume V forced by CO_2 and NH summer insolation, the Antarctic ice sheet extent A driven by sea level changes (via V) and the atmospheric CO_2 C connected to the deep water state (glacial or interglacial):

$$\frac{dV}{dt} = (-xC - yI_{65N} + z - V)/\tau_V, \tag{25}$$

$$\frac{dA}{dt} = (V - A)/\tau_A, \tag{26}$$

$$\frac{dC}{dt} = (\alpha I_{65N} - \beta V + \gamma\mathcal{H}(-F) + \delta - C)/\tau_C. \tag{27}$$

The oceanic switch parameter F is given by:

$$F = aV - bA - cI_{60S} + d, \tag{28}$$

and parameter values are summarised in Table 3.2.1.

Parameter	Value PP04	Value C12	units	Explanation
τ_V	15	15	kyr	Time scale global ice volume
τ_C	5	5	kyr	Time scale carbon cycle
τ_A	12	12	kyr	Time scale Antarctic ice sheet
x	1.3	1.3		driving V with C
y	0.5	0.5		driving V by insolation
z	0.8	0.8		
α	0.15	0.15		driving C with insolation
β	0.5	0.5		driving C with V
γ	0.5	0.5		strength of ocean switch
δ	0.4	0.4		
a	0.3	0.3		salty bottom water efficiency parameters
b	0.7	0.7		
c	0.01	0 (no forcing)		
d	0.27	0.27		

Table 4: Parameters, constants and their meaning of the model by Paillard and Parrenin [2004].

The unforced system exhibits internal oscillations at a 132 kyr period with the parameters used by Paillard and Parrenin [2004].

To simulate the change in glacial periodicities over the last 5 Myr, a drift in the F parameter is introduced, $F = aV - BA - cI_{60S} + d + kt$ (with t time and $k = 8.5 \times 10^{-5}$ kyr^{-1}). In this case the model starts in a permanent interglacial state with (linear) oscillations on the 23 kyr time scale, larger (internal) oscillations on 41 kyr time scale after 3 Myr ago finally switches to the 100 kyr oscillations around 1 Myr. The change in dominant oscillation frequency at within the region of internal oscillations is probably related to limiting the global ice volume V to positive values.

3.2.2 The sea-ice switch model

Gildor and Tziperman [2001] developed a box model of the Earth system, where glacial cycles appeared as relaxation oscillations without external forcing. The atmosphere is represented by 4 meridional boxes while the ocean component consists of two layers of 4 meridional boxes each. The model includes land ice and sea ice representations in the polar boxes. The fast sea ice-albedo feedback is responsible for the abrupt glacial-interglacial variations — the so-called sea ice switch mechanism as suggested by Gildor and Tziperman [2001]. This mechanism generates the glacial cycles in the model as self sustained relaxation oscillations because the ice volume threshold for switching sea ice cover from ‘on’ to ‘off’ differs from the one for switching from "off" to "on" [Crucifix, 2012]. When the land ice volume slowly grows (accumulation exceeds ablation), the atmospheric and surface ocean temperature decrease due to increasing albedo of the planet. Once the polar surface ocean temperature has reached a critical value cold enough to form sea ice, the polar box is rapidly covered with sea ice, which further reduces the atmospheric temperature through the ice-albedo feedback and prevents evaporation from the polar ocean box. In addition, atmospheric moisture content is reduced due to lower temperatures, which leads to decreasing land ice volume (accumulation is smaller than ablation). Temperature starts rising again both due to smaller albedo and because the ocean warms below the insulating sea-ice cover until it is warm enough to melt the polar sea ice. At this point there is a change in regime: the global temperature quickly rises, moisture content in the atmosphere increases and the land ice starts growing again (accumulation becomes larger than ablation).

In later versions of the model, both orbital and seasonal forcing Gildor and Tziperman [2000] as well as ocean biogeochemistry to dynamically simulate atmospheric CO_2 Gildor et al. [2002] have been added to the model. The full model equations are described in the appendix of von der Heydt and Ashwin [2017]. Orbital forcing is included in the model through varying incoming solar radiation averaged over each atmospheric box on seasonal and orbital time scales and modulating the Northern Hemisphere land ice ablation term by the (northern polar box averaged) summer insolation on orbital time scales [Gildor and Tziperman, 2000]. It modifies the otherwise regular 100 kyr cycles of the unforced system. Although there is some degree of synchronisation of the glaciation and deglaciations to the orbital forcing, the relation between land ice and global mean solar radiation is not trivial von der Heydt and Ashwin [2017]. The dynamic CO_2 again modifies the glacial cycles by slightly increasing their amplitude. Simulated CO_2 differences between glacial and interglacial regimes are about 75 ppmv, which here are almost completely generated by the effect of the solubility pump in the ocean.

To illustrate the behaviour of the sea-ice switch generated glacial cycles throughout the whole Pleistocene, Tziperman and Gildor [2003] analysed a simplified version of the box model, which includes only two dynamic (global) variables; the Northern Hemisphere land ice volume $V_{land-ice}$ as slow variable and a temperature T representing the atmosphere and upper ocean:

$$\frac{dV_{land-ice}}{dt} = P(T, a_{sea-ice}) - S_{abl}(T,t), \tag{29}$$

$$\frac{C_{ocn}}{a_{ocn}}\frac{dT}{dt} = -\varepsilon\sigma T^4 + H_s\left(1 - \alpha_s\frac{a_{sea-ice}}{a} - \alpha_L\frac{a_{land-ice}}{a}\right)(1-\alpha_C)\,. \tag{30}$$

The functions $P(T, a_{sea-ice})$, $S_{abl}(T,t)$, $a_{sea-ice}$ $a_{land-ice}$ and $q(T)$ are given by:

$$a_{sea-ice} = \begin{cases} I_s^0, & \text{if } T < T_f \\ 0, & \text{if } T > T_f, \end{cases} \tag{31}$$

$$a_{land-ice} = \left(L^{E-W}\right)^{1/3} \left(V_{land-ice}/\left(2\lambda^{1/2}\right)\right)^{2/3}, \tag{32}$$

$$P(T, a_{sea-ice}) = [P_0 + P_1 q(T)] \left(1 - \frac{a_{sea-ice}}{a_{ocn}}\right), \tag{33}$$

$$q(T) = q_r \epsilon_q A \exp\left(-B/T\right)/p_s, \tag{34}$$

$$S_{abl}(T,t) = S_0 + S_M M(t) + S_T(T - 273.15). \tag{35}$$

and constants and parameters are given in Table 3.2.2.

Symbol	Value, units	physical meaning
$q_r,\ \epsilon_q$	0.7, 0.622	Clausius-Clayperon parameters
$A,\ B,\ P_s$	2.53×10^{11} Pa, 5.42×10^3 K, 10^5 Pa	
$P_0,\ P_1$	0.06 Sv, 40 Sv/Pa	accumulation parameterisation
$S_0,\ S_M,\ S_T$	0.15 Sv, 0.08 Sv, 0.0015 Sv/K	ablation parameterisation
ε	0.64	emissivity
σ	5.67×10^{-8} Wm^{-2}K^{-4}	Stefan Boltzman constant
H_s	350.0 Wm^{-2}	incoming solar radiation
$\alpha_S, \alpha_L, \alpha_C$	0.65, 0.7, 0.27	albedo of sea ice, land ice and clouds
C_{ocn}	$C_p V_{ocn} \rho_0$, 9.2573×10^{22} J K^{-1}	atmosphere and upper ocean's heat capacity
$c_p,\ \rho_0$	4.1813 J (gK)$^{-1}$, 1025 kg m^{-3}	specific heat of water, density of sea water
$a_{ocn},\ a_{land}$	20×10^6 km^2, 20×10^6 km^2	area of land and ocean
a	$a_{ocn} + a_{land}$, 40×10^6 km^2	total area
V_{ocn}	21.6×10^6 km^3	volume of upper ocean
I_s^0	0.3 a_{ocn}	max ice during switch-on periods
λ	10 m	glacier model
L^{E-W}	4000 km	(fixed) width of land and glacier

Table 5: Parameters, constants and their meaning of the model by Tziperman and Gildor [2003].

3.2.3 Calcifier models - alternative to sea ice switch

In order to explain the sawtooth-shape oscillations in the atmospheric CO_2 record another relaxation mechanism has been suggested Omta et al. [2013]. So far, this has been only applied to explain the CO_2 record including the Mid-Pleistocene transition Omta et al. [2016], while it has not been coupled to ice volume or ocean dynamics. It supports the idea suggested by Wallmann [2014] that the glacial cycles could be *driven* by atmospheric CO_2, which may show self-sustained oscillations due to (nonlinear) feedbacks in the carbon and phosphorus cycles. In contrast, most of the previously described models rely on oscillations internal to the (nonlinear) land ice or sea ice/ocean dynamics.

The basic model consists of two variables, the ocean alkalinity A (mol eq m^{-3}) and a marine calcifier population C (mol m^{-3}):

$$\frac{dA}{dt} = I - kAC \tag{36}$$
$$\frac{dC}{dt} = kAC - MC, \tag{37}$$

where I is the (constant) input rate for alkalinity from river runoff and weathering, k is the effective growth rate of the calcifiers and M their sedimentation rate. The model equations describe an autocatalytic process, which leads to a slowly growing alkalinity (by input) until a spike in the calcifier population leads to a rapid alkalinity drop. On these time scales, the alkalinity in the ocean is related to the atmospheric CO_2 (alkalinity input = ocean uptake of CO_2, output = release of oceanic CO_2 into the atmopshere).

The unforced model exhibits asymmetric oscillations of around 100kyr periodicity, however, these oscillations are always slowly damped Omta et al. [2013]. In Omta et al. [2016], a periodic (and noisy) forcing is added to the model via the growth rate k:

$$k = k_0 \left(1 + \alpha \cos\left(\frac{2\pi t}{T}\right) + \epsilon\right), \tag{38}$$

where ϵ represents white noise forcing. Parameters are explained in Table 3.2.3.

Symbol	Value, units	physical meaning
I	4×10^{-6} mol eq m^{-3} yr^{-1}	Alkalinity input
k	0.05 (mol eq)$^{-1}$ m^3 yr^{-1}	Reaction rate
M	0.1 yr^{-1}	Mortality rate
α	variable	Periodic forcing amplitude
ϵ	0.005	Random forcing amplitude

Table 6: Parameters and their meaning of the model by Omta et al. [2016].

Next step: couple to ice volume, atmospheric CO_2, climate T, etc. using the Gildor-Tziperman model. Wallmann [2014] use a more complex carbon cycle representation (3 vertical boxes), where the calcifier dynamics could be built in.

4 Models of orbitally forced climate without ice

Carbon cycle box models Zeebe [2012], orbitally forced Zeebe et al. [2017].

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