Validation of Coulomb's Law and determination of the vacuum permittivity

Jochem Langen (with labpartner Kit Lingwood),
L1 Discovery Labs, Lab Group B, Thursday
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Coulomb's Law is contains the essential formula stating the relation between the charge of objects, the distance between them and the resultant force. In the experiment described within this work, the relation between these quantities has been validated and the value of the constant present within this formula, the vacuum permittivity has been determined through the method of a Faraday's Cage and earthed plate-charge set-up.

I Introduction

In 1785 physicist Charles-Augustin de Coulomb devised the well-known Coulomb's Law, vital for the theories of electromagnetism. [1] The Law describes the dependence of the electric force on charge and distance. Looking solely at the magnitudes of the quantities, the formula is stated as:

$$|F_{el}| = \frac{1}{4\pi\epsilon_0} * \frac{|q_1 * q_2|}{r^2}. \tag{1}$$

In this work, the relations present in this law will be validated and the value of the vacuum permittivity or permittivity of free space ϵ_0 will be determined were this law to be taken as correct. Rather than using two charges, a single charged sphere will be used and an earthed plate. This plate will represent a similar sphere of opposite charge at twice the distance the plate is at. As the charges thus have a similar magnitude the first relation to be validated is:

$$|F_{el}| \propto q^2. \tag{2}$$

Secondly, the following relation will be validated:

$$|F_{el}| \propto \frac{1}{r^2}. \tag{3}$$

In order to achieve this the charge on said sphere needs to be continuously measured as the induced force is. This is done by correcting the initial charge by its exponential decay over time. The charge used is thus given by:

$$Q^2 = (Q_{initial} * e^{\gamma * t})^2 = (V * C * e^{\gamma * t})^2, \tag{4}$$

where γ is the decay factor, V the measured voltage and C the capacitance of the circuit. Finally, as a force meter is used, this needs to be calibrated. The force of a weighted mass was calculated using Newton's simplified law of gravity on earth:

$$F_{calc} = g * m, \tag{5}$$

with g being the gravitational acceleration taken as $9.80665 ms^{-2}$ [2]. The resultant calibration factor used is thus given by:

$$f = \frac{F_{calc}}{F_{meas}}. \tag{6}$$

II Methods

The experiment consisted of various parts within the set-up to determine the decay factor, the calibration factor and finally the vacuum permittivity. To get the initial charge a circuit with a Faraday cage connected to a voltage-meter was used. As the charged sphere moves within the conductive cylinder the charges on the cylinder redistribute and a voltage is induced depending on the circuit's capacitance. The electric force generated by the sphere is measured by a meter in which the sphere's connecting rod is inserted; gauging the force on rod within. This meter is connected to a display and a computer programme making measurements every 0.1 seconds. The height earthed plate the sphere is suspended above can be adjusted after which this height can read off of a Vernier scale, see Figure 1.

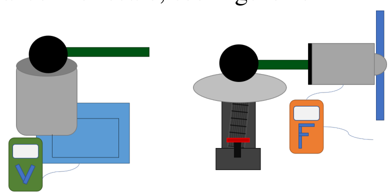

Figure 1: On the left the circuit with Faraday's Cage connected to a voltage meter with on the right the earthed plate with Vernier Scale and the Force meter.

As the initial voltage, therewith the initial charge, is metered a timer is set which is then stopped as the force measurements start. This start time is then added onto all the measurement times. Using the same timer, the changing voltage is timed as the charge decays within the set-up.

III Results

By taking the weighted mean of the discrepancy between the measured and calculated weight for six measurements of assorted paperclips the calibration factor for the force-meter was found to be 0.989 ± 0.003. Additionally, the weighted mean was taken of the decay factors obtained by two independent data-sets both of thirteen measurements of the varying logarithmic charge on the sphere over time; the resultant value being $-0.00223 \pm 0.00003\ s^{-1}$.

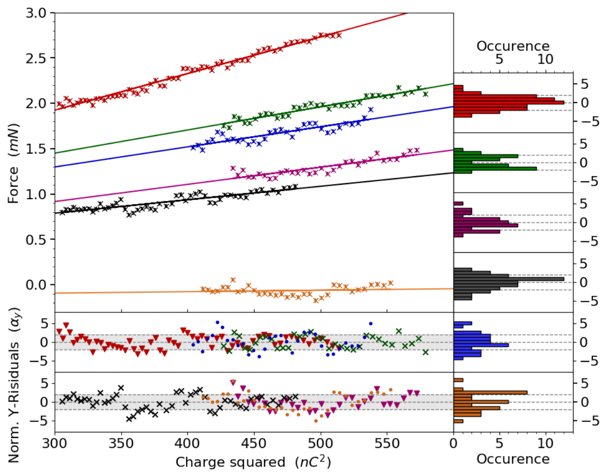

Figure 2: The force measured by the force-meter (mN) over the charge on the conductive sphere squared (nC^2). The different data-sets correspond to different charge-to-charge distances given in table 1. Shown below are the normalised residuals in the y-direction and their distributions. Within the latter two graphs, lines are drawn for the interval of two errors (α), in which 95% of the points should lie for a normal distribution, and the mean (the best fit value).

Distance (mm) ±0.5	52.2	54.8	56.0	60.0	64.0	73.2
Colour	Red	Blue	Orange	Green	Purple	Black

Table 1

For all the measurements more than 99% of the data falls within the two deviation area, 95% would give for a good normally distributed fit, combined with the clear visual accordance the dependence of the electromagnetic force on the charge squared is conclusively linear; in agreement with Coulomb's Law. The percentage of data that falls within this area becomes smaller as larger weighted means are taken for the residuals are normalised and the error diminishes quicker that the resultant residual.

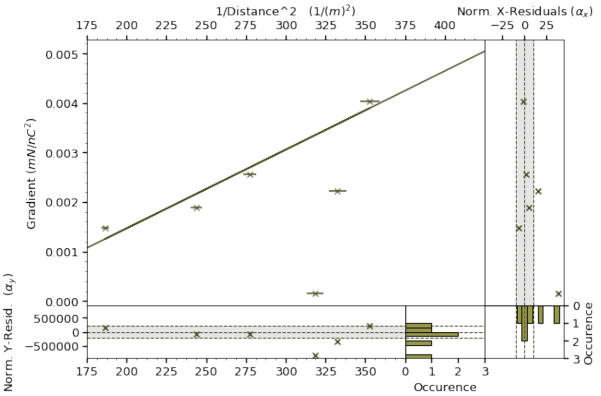

Figure 3: The gradients from the data-sets in Figure 1 over the inverse squared charge-to-charge distance $(1/m^2)$. On the bottom, the normalised Y-residual graph and distribution is plotted with $\pm 200,000$ errors as area with the X-residual graph and distribution with an area of ± 10 errors on the right.

The y over x gradient of Figure 2, to be taken as the double partial derivative of the force over the charge squared and the inverse distance squared respectively, is constant according to Coulomb's Law.

$$((F_{el})_{\partial C^2})_{\partial \frac{1}{r^2}} = k_{constant} = \frac{1}{4\pi\epsilon_0}, \tag{7}$$

where ϵ_0 is the permittivity of free space; using Python, determined to be:
$5.008028 \pm 0.000007 * 10^{-12} \quad m^{-3}kg^{-1}C^2s^2$
and using Excel: $1 \pm 2 * 10^{-11} \quad m^{-3}kg^{-1}C^2s^2$.
None of the Y-norm. residuals and only $1/8$ of the X-norm. residuals lie within the two deviations area, thus the inverse square relation as described in Coulomb's Law cannot be safely validated with the given data.

IV Discussion

The accepted value for the vacuum permittivity is determined as: $8.8541878128(13)x10 - 12Fm^{-1}$ [2]. The foregoing obtained values lie within a factor of 0.56 and 1.28 of the accepted one for Python and Excel respectively. The accepted value lies within the error obtained using Excel calculations ($0.16x$ the error), but conversely, lies within $526465x$ the error obtained with Python. Conclusively it can be said that the error on the Python method is underestimated.
Reason for general underestimation of the error is the difficult quantifiability of certain induced errors. Namely the fluctuating force measurements caused by interference of outside forces. These range from systematic errors from the objects present in the room interfering with the measured electric field to random errors caused by vibrations reaching the force meter due to hitting the equipment or that upon which it is positioned. This has lead to the discarding of set of 13 data points as anomalies within the "Black" data-set in accordance with Chauvenet's criterion [4] and by reason that data-points with a negative force value cannot be descriptive of the positive electromagnetic force.
Furthermore, several approximations have been made throughout the experiment which cause the result to be less accurate. Firstly, we assumed the charge on the conducting sphere to be uniformly distributed hence to be estimated as a point charge. However, when the dimensions of the point charges are non-negligible to the distance between them this changes and a correction factor is needed [3]. No correct factor for this experiment could be found, but this does show a systematic error in the result.
Moreover, to represent the opposite sphere the earthed plate should have been infinitely large, this was not the case. By this cause, not all of the electric field correctly hit said plate and thus the induced force will have been smaller than what it would have been for two actual oppositely charged spheres. Additionally, as the force measured for the "Orange" data set fluctuates around zero it seems as if though the sphere was not actually charged anymore. This may very well have been the case by an accidental (near-)collision of the sphere with any other object causing it to discharge. This would explain why the corresponding force-charge gradient is an outlier with a normalised residual of -852399 errors.

V Conclusions

In conclusion, although the squared charge relation within Coulomb's law as been validated, the inverse square relation with the distance has not been sufficiently confirmed. The values obtained for the vacuum permittivity are however, within considerable proximity of the accepted value which would still argue in favour of mentioned relations. The values obtained were those of:
$5.008028 \pm 0.000007 * 10^{-12}$ and $1 \pm 2 * 10^{-11}$ $m^{-3}kg^{-1}C^2s^2$.
The discrepancy that can still be observed with the accepted value of $8.8541878128(13)x10 - 12Fm^{-1}$ can largely be attributed to the unstable environment in which the experiment has had to take place.

References

[1] C.A. Coulomb, "Premier mémoire sur l'électricité et le magnétisme," Histoire de l'Académie Royale des Sciences, pp. 569–577, (1785)

[2] National Institute for Standards in Technology (NIST), Fundemental Physics Constants, "2018 CODATA recommended values", (20/05/2019), accessed 23/03/2020

[3] NISER, Institute under DAE, government India, Verification of Coulomb's law using Coulomb balance

[4] I. G. Hughes, T. P. A. Hase, Measurements and their Uncertainties, A practical guide to modern error analysis, Oxford university press, New York, United States (2010), p.6, p. nulla (inside of covers).

VI Error Appendix

The calibration factor of the force meter was calculated as the ratio of the calculated weight and the more accurate, measured weight. This ratio is the gradient of the calculated weight over the measured weight and thus can be approximated using the least square fitting method. The accompanying error was given by Excel within this method. Logically, the measured and calculated weight themselves had an error too. For the measured weight the value of one unit on the last digit was taken, conforming with the general consensus on reading from digital apparatus [4]. The error on calculated weight, consisting of the product of the measured mass and the gravitational acceleration on earth, has been determined as follows: [4]

$$\alpha_{Force} = g * \alpha_{mass}. \tag{8}$$

The approximation was made to take the gravitational acceleration as a constant, causing the corresponding error to be disregarded. However, due to the inability to take errors into account whilst determining the line of best fit in Excel, the errors on the measured and calculated force have not been taken into account in the end.

The final error on the measured electromagnetic force is consequently imposed by the same error (α_{Fmeas}) as on the aforementioned measured weight and the error on the calibration factor. The due error was calculated using the functional method: [4]

$$(\alpha_{actual})^2 = ((f + \alpha_f) * F_{meas}) - (f * F_{meas}))^2 + ((f * (F_{meas} + \alpha_{Fmeas}) - (f * F_{meas}))^2, \tag{9}$$

where f is the calibration factor and F_{meas} the originally measured force.

The decay factor γ being defined as the gradient of the logarithmic charge over time has likewise been determined using the least square fitting method. Therefore, the error (α_γ) is calculated within that method and again is solely based on the degree of fit the data has with the line. To determine the initial charge of each data-set the initial voltage (V) and capacitance (C) of the circuit in which it is measured were metered, because the charge (Q) is given by:

$$Q_{initial} = C * V. \tag{10}$$

As such the induced error is calculated as: [4]

$$\alpha_{Q_{initial}} = Q_{initial} * \sqrt{(\frac{\alpha_V}{V})^2 + (\frac{\alpha_C}{C})^2}. \tag{11}$$

For the uncertainty on the voltage a single unit on the last digit was taken whilst the error on the capacitance was given to be 5% of the value of the capacitance.

The time (t) over which the force is measured from the moment the initial charge is evaluated is the sum of the start time, the time taken from the measurement of the charge to the first measurement of the force, and the increasing time points at which each force measurement is made. As the time points are generated by the computer programme these errors (α_{point}) are relatively sufficiently small to be neglected for the error on the start time (α_{start}) was chosen to be twice the error of $0.5s$; once for starting the timer and once for stopping it, taking into account the human error being more than a single unit of the smallest digit. [1]

$$\alpha_t = \sqrt{(\alpha_{start})^2 + (\alpha_{point})^2} = \alpha_{start}. \tag{12}$$

The charge squared which was plotted against the force was calculated by:

$$Q^2 = (Q_{initial} * e^{\gamma * t})^2. \tag{13}$$

The corresponding error was calculated using the functional method, in this case manifesting as: [4]

$$\begin{aligned}(\alpha_{actual})^2 = \\ (((Q_{initial} + \alpha_{Q_{initial}}) * e^{\gamma * t})^2 - (Q_{initial} * e^{\gamma * t})^2)^2 \\ + ((Q_{initial} * e^{(\gamma + \alpha_\gamma) * t})^2 - (Q_{initial} * e^{\gamma * t})^2)^2 \\ + ((Q_{initial} * e^{(\gamma * (t + \alpha_t)})^2 - (Q_{initial} * e^{\gamma * t})^2)^2.\end{aligned} \tag{14}$$

The gradient of the force over this charge squared later used to determine the vacuum permittivity was determined using the least square fitting method, this time with Excel and Python, the latter of which produced a weighted line of best fit which did take all the previous errors into account. Ergo these two methods produced an error ($\alpha_{\partial Q^2}$).

To get to the value of the permittivity of free space first the double partial derivative had to calculated, the gradient of the aforementioned gradient over the distance. To do so, again the method of least square fitting was used with Excel and Python. Producing a non-weighted and weighted line respectively, with gradient error ($\alpha_{\partial Q^2, \frac{1}{r^2}}$).

The error on the distance ($\alpha_{\frac{1}{r^2}}$) taken into account for the weighted line of best fit consists of the various errors of the lengths making up the final distance r. Since the inverse distance square is given by:

$$\frac{1}{r^2} = \frac{1}{(2 * (r_{meas} - r_{initial}) + d_{sphere}/2) + d_{plate})}, \tag{15}$$

where r_{meas} is the measured distance the plate is at whilst $r_{initial}$ is the original plate distance at which the plate just about touches the sphere; d_{plate} is the thickness of the plate and finally, d_{sphere} is the diameter of the sphere. The corresponding error has been determined in two steps: first, using the calculus method, calculating the error on the distance without the plate thickness (r), (α_R), followed by computing the final error, with the plate distance included, using the functional method. [4]

$$\alpha_R = \sqrt{(\alpha_{Rmeas})^2 + (\alpha_{Rinitial})^2 + (\alpha_{sphere}/2)^2}, \tag{16}$$

where α_{Rmeas} is the error on the measured distance, $\alpha_{Rinitial}$ the error on the original distance and α_{sphere} the error on the diameter of the sphere. Hence, the final error is computed by: [4]

$$\begin{aligned}(\alpha\frac{1}{r^2})^2 = (\frac{1}{2 * (R + \alpha_R) + d_{plate}} - \frac{1}{2 * R + d_{plate}})^2 \\ + (\frac{1}{2 * R + (d_{plate} + \alpha_{plate})} - \frac{1}{2 * R + d_{plate}})^2,\end{aligned} \tag{17}$$

with α_{plate} being the uncertainty on the plate thickness.

Although not explicitly stated, throughout the error propagation there have been numerous unit conversions. However, all of these simply regard conversions of order of magnitude (millimeter to meter etc.). Moreover, as the error of a quantity converted as such scales exactly therewith the propagation is trivially defined as: [4]

$$\alpha_m = \alpha_{mm} * 10^{-3} \tag{18}$$

in the case of the foregoing order conversion.