Independent subsets of a finite set

1 Independent nontrivial subsets

```
(%i52) pairs_independent_nontrivial_subsets(n) :=
       block([a, b, d, s : 0 ],
         for a:1 thru n-1 do
           for d:1 thru a do
              ( b : n*d / a,
                if integerp(b) and b<n
                   then (s : s + binomial(n,a)*binomial(a,d)*binomial(n-a,b-d)) ) ,
         s)  $
```

```
(%i53) pairs_independent_nontrivial_subsets(4);
(%o53)  24
```

```
(%i54) L30 : makelist(i, i, 0, 30);
(L30)  [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,
       28,29,30]
```

```
(%i55) L_nontrivial_30  :  map(pairs_independent_nontrivial_subsets, L30);
(L_nontrivial_30)  [0,0,0,0,24,0,720,0,7000,15120,126000,0,1777776,0,23543520,55855800,
       274565720,0,5337775872,0,63026049424,117920013120,995265791520,0,15265486117744,
       14283091977000,216344919117600,2401429019418 00,2854493961432480,0,
       55689696384165720]
```

```
(%i56) a(n) :=
       sum(
          sum(
               (b : n*d / a,
               if integerp(b) and b<n then
                  binomial(n,a)*binomial(a,d)*binomial(n-a,b-d) else 0),
          d,1,a),
       a,1,n-1) $
```

```
(%i57) a(6);
(%o57)  720
```

```
(%i58) L_a_30 : map(a, L30);
(L_a_30)  [0,0,0,0,24,0,720,0,7000,15120,126000,0,1777776,0,23543520,55855800,
       274565720,0,5337775872,0,63026049424,117920013120,995265791520,0,15265486117744,
       14283091977000,216344919117600,2401429019418 00,2854493961432480,0,
       55689696384165720]
```

```
(%i60) is(L_a_30 = L_nontrivial_30);
(%o60)  true
```

```
(%i61) is(map(pairs_independent_nontrivial_subsets, L30) = map(a,L30) );
(%o61)  true
```

2 Independent proper subsets

```
(%i63) pairs_independent_proper_subsets(n) :=
          if is(n=0) then 0 else a(n) + 2*(2^n -1)  - 1 $
```

```
(%i64) L_proper_30  :  map(pairs_independent_proper_subsets, L30);
(L_proper_30) [0,1,5,13,53,61,845,253,7509,16141,128045,4093,1785965,16381,23576285,
       55921333,274696789,262141,5338300157,1048573,63028146573,117924207421,
       995274180125,16777213,15265519672173,14283159085861,216345053335325,
       240143170377253,2854494498303389,1073741821,55689698531649365]
```

```
(%i66) OEIS_A158345  :  [ 1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045,
          4093, 1785965, 16381, 23576285, 55921333, 274696789,
          262141, 5338300157, 1048573, 63028146573, 117924207421,
          995274180125, 16777213, 15265519672173, 14283159085861 ];
(OEIS_A158345) [1,5,13,53,61,845,253,7509,16141,128045,4093,1785965,16381,23576285,
       55921333,274696789,262141,5338300157,1048573,63028146573,117924207421,
       995274180125,16777213,15265519672173,14283159085861]
```

```
(%i67) rest(rest(L_proper_30), -5) ;
(%o67) [1,5,13,53,61,845,253,7509,16141,128045,4093,1785965,16381,23576285,
       55921333,274696789,262141,5338300157,1048573,63028146573,117924207421,
       995274180125,16777213,15265519672173,14283159085861]
```

```
(%i68) is( rest(rest(L_proper_30), -5)  =  OEIS_A158345);
(%o68) true
```

```
(%i69) pairs_independent_proper_subsets_2(n) :=
       block([a, b, d, s : 2*(2^n -1)  - 1 ],
          if  is(n=0)  then  s : 0  else
            for a:1 thru n-1 do
              for d:1 thru a do
               (b : n*d / a,
                if integerp(b) and b<n
                    then ( s : s + binomial(n,a)*binomial(a,d)*binomial(n-a,b-d) ) )  ,
          s ) $
```

```
(%i70) map(pairs_independent_proper_subsets_2, L30) ;
(%o70) [0,1,5,13,53,61,845,253,7509,16141,128045,4093,1785965,16381,23576285,
       55921333,274696789,262141,5338300157,1048573,63028146573,117924207421,
       995274180125,16777213,15265519672173,14283159085861,216345053335325,
       240143170377253,2854494498303389,1073741821,55689698531649365]
```

```
(%i71)  is(map(pairs_independent_proper_subsets_2, L30) = L_proper_30) ;
(%o71) true
```

```
(%i73) pips(n) := if is(n=0) then 0 else
          2*(2^n -1)  - 1 +
           sum(
             sum(
               (b : n*d / a,
                if verbose then
                     print(d, a, b, binomial(n,a)*binomial(a,d)*binomial(n-a,b-d)),
                if integerp(b) and b<n  then
                     binomial(n,a)*binomial(a,d)*binomial(n-a,b-d) else 0),
              d,1,a),
            a,1,n-1) $
```

```
(%i75) verbose : true; pips(4);
(verbose) true
          1  1  4  4
          1  2  2  24
          2  2  4  6
          1  3  4/3  12 (1/(1/3))
          2  3  8/3  12 (1/(2/3))
          3  3  4  4
(%o75)  53
```

```
(%i77) verbose : false $ ;  map(pips, L30);
(%o77) [0, 1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045, 4093, 1785965, 16381, 23576285,
        55921333, 274696789, 262141, 5338300157, 1048573, 63028146573, 117924207421,
        995274180125, 16777213, 15265519672173, 14283159085861, 216345053335325,
        240143170377253, 2854494498303389, 1073741821, 55689698531649365]
```

```
(%i78) is (map(pips, L30) = L_proper_30);
(%o78) true
```

3 All independent subsets

```
(%i79) pairs_independent_subsets(n) :=
          if is(n=0)  then 1 else a(n) + 4*(2^n -1) $
```

```
(%i80) pairs_independent_subsets(6);
(%o80) 972
```

```
(%i81) L_independent_30  :  map(pairs_independent_subsets, L30);
(L_independent_30) [1, 4, 12, 28, 84, 124, 972, 508, 8020, 17164, 130092, 8188, 1794156, 32764, 23609052,
        55986868, 274827860, 524284, 5338824444, 2097148, 63030243724, 117928401724,
        995282568732, 33554428, 15265553226604, 14283226194724, 216345187553052,
        240143438812708, 2854495035174300, 2147483644, 55689700679133012]
```

(%i82)
```
OEIS_A121312 : [ 1, 4, 12, 28, 84, 124, 972, 508, 8020, 17164,
    130092, 8188, 1794156, 32764, 23609052, 55986868,
    274827860, 524284, 5338824444, 2097148, 63030243724,
    117928401724, 995282568732, 33554428, 15265553226604,
    14283226194724, 216345187553052 ];
```

(OEIS_A121312) ***[*** *1, 4, 12, 28, 84, 124, 972, 508, 8020, 17164, 130092, 8188, 1794156, 32764, 23609052, 55986868, 274827860, 524284, 5338824444, 2097148, 63030243724, 117928401724, 995282568732, 33554428, 15265553226604, 14283226194724, 216345187553052* ***]***

(%i83)
```
is(rest(L_independent_30, -4) = OEIS_A121312);
```

(%o83) ***true***

(%i40)
```
transpose(matrix(L30,L_independent_30, L_proper_30, L_nontrivial_30) );
```

(%o40)

0	1	0	0
1	4	1	0
2	12	5	0
3	28	13	0
4	84	53	24
5	124	61	0
6	972	845	720
7	508	253	0
8	8020	7509	7000
9	17164	16141	15120
10	130092	128045	126000
11	8188	4093	0
12	1794156	1785965	1777776
13	32764	16381	0
14	23609052	23576285	23543520
15	55986868	55921333	55855800
16	274827860	274696789	274565720
17	524284	262141	0
18	5338824444	5338300157	5337775872
19	2097148	1048573	0
20	63030243724	63028146573	63026049424
21	117928401724	117924207421	117920013120
22	995282568732	995274180125	995265791520
23	33554428	16777213	0
24	15265553226604	15265519672173	15265486117744
25	14283226194724	14283159085861	14283091977000
26	216345187553052	216345053335325	216344919117600
27	240143438812708	240143170377253	240142901941800
28	2854495035174300	2854494498303389	2854493961432480
29	2147483644	1073741821	0
30	55689700679133012	55689698531649365	55689696384165720