Integer Partitions

Humboldt Universität Berlin
Sommerschule Lust auf Mathematik
Blossin, Juni 2024

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`https://jochen-ziegenbalg.github.io/materialien/`

Contents

Preliminary remarks:

This manuscript is part of the project Sommerschule "Lust auf Mathematik" Humboldt Universität, Berlin June 2024

The project is documented in the following files:

LaTeX-generated file on Fibonacci numbers, powerseries and generating functions:
    `https://jochen-ziegenbalg.github.io/materialien/Manuskripte/Fib-PotReih-erzFkt.pdf`

LaTeX-generated file on integer partitions:
    `https://jochen-ziegenbalg.github.io/materialien/Manuskripte/Integer-Partitions.pdf`

Maxima-generated file on integer partitions:
    `https://jochen-ziegenbalg.github.io/materialien/Manuskripte/Partitions-CAS-Maxima.pdf`

1 Some combinatorics

1.1 Changing money, stamping letters and an ingenious idee of Leonhard Euler

In the following example, we demonstrate an idea that goes back to L. Euler[1]. The *problem*: A letter has to be stamped. The postage is 85 cents and we have the following stock of stamps:

- 6 stamps at 1 Ct each,
- 5 stamps at 5 Ct each,
- 3 stamps at 10 Ct each,
- 2 stamps at 20 Ct each,
- 1 stamp at 50 Ct each

In how many ways can the letter be stamped, if there is no difference between the stamps of the same value (and if it does not matter where on the letter the stamps are affixed).

With a little (systematic) trial and error, one can get the following 14 possibilities[2]:

```
 1:  50+20+10+5
 2:  50+20+10+1+1+1+1+1
 3:  50+20 +5+5+5
 4:  50+20+5+5+1+1+1+1+1
 5:  50+10+10+10+5
 6:  50+10+10+10+1+1+1+1+1
 7:  50+10+10+5+5+5
 8:  50+10+10+5+5+1+1+1+1+1
 9:  50+10+5+5+5+5
10:  50+10+5+5+5+5+1+1+1+1+1
11:  20+20+10+10+10+5+5+5
12:  20+20+10+10+10+5+5+1+1+1+1+1
13:  20+20+10+10+5+5+5+5+5
14:  20+20+10+10+5+5+5+5+1+1+1+1+1
```

These are exactly the possibilities to sum up the number 85 with the given stock. With regard to the general procedure, we start by coding the stampings. Let the stamping: a ones, b fives, c tens, d twenties and e fifty stamps be denoted by the term[3] [4]

$$(X^1)^a \cdot (X^5)^b \cdot (X^{10})^c \cdot (X^{20})^d \cdot (X^{50})^e \tag{1}$$

Obviously, concerning (1) we have

$$X^{1 \cdot a} \cdot X^{5 \cdot b} \cdot X^{10 \cdot c} \cdot X^{20 \cdot d} \cdot X^{50 \cdot e} = X^{a \cdot 1 + b \cdot 5 + c \cdot 10 + d \cdot 20 + e \cdot 50} = X^{85}$$

with $a \cdot 1 + b \cdot 5 + c \cdot 10 + d \cdot 20 + e \cdot 50 = 85$

---

[1]Leonhard Euler, Swiss mathematician, 1707–1783

[2]Sorting criterion by the "greedy strategy" *large values first*.

[3]Even if this coding may seem strange at first, it is in any case unambiguous and will prove extremely useful in connection with polynomial calculations.

[4]It is self-evident that the stamps in the stock were only used in so far as it made sense from the point of view of the problem.

The stock of stamps is coded in the following way:

$$
\begin{array}{llll}
\text{6 stamps at} & \text{1 Ct} & \longleftrightarrow & (1 + X + X^2 + X^3 + X^4 + X^5 + X^6) \\
\text{5 stamps at} & \text{5 Ct} & \longleftrightarrow & (1 + X^5 + (X^5)^2 + (X^5)^3 + (X^5)^4 + (X^5)^5) \\
\text{3 stamps at} & \text{10 Ct} & \longleftrightarrow & (1 + X^{10} + (X^{10})^2 + (X^{10})^3) \\
\text{2 stamps at} & \text{20 Ct} & \longleftrightarrow & (1 + X^{20} + (X^{20})^2) \\
\text{1 stamp at} & \text{50 Ct} & \longleftrightarrow & (1 + X^{50})
\end{array}
\tag{2}
$$

Euler was a virtuoso at calculating with polynomials. He formed the following expression to solve the stamping problem:

$$
\begin{aligned}
&\left(1 + X^1 + X^2 + X^3 + X^4 + X^5 + X^6\right) \cdot \\
&\left(1 + X^5 + X^{5 \cdot 2} + X^{5 \cdot 3} + X^{5 \cdot 4} + X^{5 \cdot 5}\right) \cdot \\
&\left(1 + X^{10} + X^{10 \cdot 2} + X^{10 \cdot 3}\right) \cdot \\
&\left(1 + X^{20} + X^{20 \cdot 2}\right) \cdot \\
&\left(1 + X^{50}\right)
\end{aligned}
\tag{3}
$$

In order to expand the expression (3), one has to take one summand from each of the five parenthesized expressions, form their product and add up all the resulting terms[5]. This results in the expression:

$$
\begin{aligned}
& 1 + X + X^2 + X^3 + X^4 + 2 \cdot X^5 + 2 \cdot X^6 + X^7 + X^8 + X^9 + 3 \cdot X^{10} \\
& + 3 \cdot X^{11} + 2 \cdot X^{12} + 2 \cdot X^{13} + 2 \cdot X^{14} + 4 \cdot X^{15} + 4 \cdot X^{16} + 2 \cdot X^{17} + 2 \cdot X^{18} + 2 \cdot X^{19} + 6 \cdot X^{20} \\
& + 6 \cdot X^{21} + 4 \cdot X^{22} + 4 \cdot X^{23} + 4 \cdot X^{24} + 8 \cdot X^{25} + 8 \cdot X^{26} + 4 \cdot X^{27} + 4 \cdot X^{28} + 4 \cdot X^{29} + 9 \cdot X^{30} \\
& + 9 \cdot X^{31} + 5 \cdot X^{32} + 5 \cdot X^{33} + 5 \cdot X^{34} + 10 \cdot X^{35} + 10 \cdot X^{36} + 5 \cdot X^{37} + 5 \cdot X^{38} + 5 \cdot X^{39} + 11 \cdot X^{40} \\
& + 11 \cdot X^{41} + 6 \cdot X^{42} + 6 \cdot X^{43} + 6 \cdot X^{44} + 12 \cdot X^{45} + 12 \cdot X^{46} + 6 \cdot X^{47} + 6 \cdot X^{48} + 6 \cdot X^{49} + 13 \cdot X^{50} \\
& + 13 \cdot X^{51} + 7 \cdot X^{52} + 7 \cdot X^{53} + 7 \cdot X^{54} + 14 \cdot X^{55} + 14 \cdot X^{56} + 7 \cdot X^{57} + 7 \cdot X^{58} + 7 \cdot X^{59} + 14 \cdot X^{60} \\
& + 14 \cdot X^{61} + 7 \cdot X^{62} + 7 \cdot X^{63} + 7 \cdot X^{64} + 14 \cdot X^{65} + 14 \cdot X^{66} + 7 \cdot X^{67} + 7 \cdot X^{68} + 7 \cdot X^{69} + 15 \cdot X^{70} \\
& + 15 \cdot X^{71} + 8 \cdot X^{72} + 8 \cdot X^{73} + 8 \cdot X^{74} + 16 \cdot X^{75} + 16 \cdot X^{76} + 8 \cdot X^{77} + 8 \cdot X^{78} + 8 \cdot X^{79} + 15 \cdot X^{80} \\
& + 15 \cdot X^{81} + 7 \cdot X^{82} + 7 \cdot X^{83} + 7 \cdot X^{84} + 14 \cdot X^{85} + 14 \cdot X^{86} + 7 \cdot X^{87} + 7 \cdot X^{88} + 7 \cdot X^{89} + 14 \cdot X^{90} \\
& \ldots \quad + 3 \cdot X^{141} + X^{142} + X^{143} + X^{144} + 2 \cdot X^{145} + 2 \cdot X^{146} + X^{147} + X^{148} + X^{149} + X^{150} + X^{151}
\end{aligned}
\tag{4}
$$

The coefficient 14 in front of X^{85} is equal to the number of possibilities of building all combinations for $X^r \cdot X^s \cdot X^t \cdot X^u \cdot X^v$ from the expression (3) such that $r + s + t + u + v = 85$. In greater detail, the coefficient 14 is obtained as follows:

---

[5]Nowadays, a computer algebra system is usually used for this. The corresponding command for multiplication in *Maxima* is e.g. "expand"

$$
\begin{aligned}
1: &\quad (X^1)^0 \cdot (X^5)^1 \cdot (X^{10})^1 \cdot (X^{20})^1 \cdot (X^{50})^1 = 1 \cdot X^5 \cdot X^{10} \cdot X^{20} \cdot X^{50} \\
2: &\quad (X^1)^5 \cdot (X^5)^0 \cdot (X^{10})^1 \cdot (X^{20})^1 \cdot (X^{50})^1 = X^5 \cdot 1 \cdot X^{10} \cdot X^{20} \cdot X^{50} \\
3: &\quad (X^1)^0 \cdot (X^5)^3 \cdot (X^{10})^0 \cdot (X^{20})^1 \cdot (X^{50})^1 = 1 \cdot X^{15} \cdot 1 \cdot X^{20} \cdot X^{50} \\
4: &\quad (X^1)^5 \cdot (X^5)^2 \cdot (X^{10})^0 \cdot (X^{20})^1 \cdot (X^{50})^1 = X^5 \cdot X^{10} \cdot 1 \cdot X^{20} \cdot X^{50} \\
5: &\quad (X^1)^0 \cdot (X^5)^1 \cdot (X^{10})^3 \cdot (X^{20})^0 \cdot (X^{50})^1 = 1 \cdot X^5 \cdot X^{30} \cdot 1 \cdot X^{50} \\
6: &\quad (X^1)^5 \cdot (X^5)^0 \cdot (X^{10})^3 \cdot (X^{20})^0 \cdot (X^{50})^1 = X^5 \cdot 1 \cdot X^{30} \cdot 1 \cdot X^{50} \\
7: &\quad (X^1)^0 \cdot (X^5)^3 \cdot (X^{10})^2 \cdot (X^{20})^0 \cdot (X^{50})^1 = 1 \cdot X^{15} \cdot X^{20} \cdot 1 \cdot X^{50} \\
8: &\quad (X^1)^5 \cdot (X^5)^2 \cdot (X^{10})^2 \cdot (X^{20})^0 \cdot (X^{50})^1 = X^5 \cdot X^{10} \cdot X^{20} \cdot 1 \cdot X^{50} \\
9: &\quad (X^1)^0 \cdot (X^5)^5 \cdot (X^{10})^1 \cdot (X^{20})^0 \cdot (X^{50})^1 = 1 \cdot X^{25} \cdot X^{10} \cdot 1 \cdot X^{50} \\
10: &\quad (X^1)^5 \cdot (X^5)^4 \cdot (X^{10})^1 \cdot (X^{20})^0 \cdot (X^{50})^1 = X^5 \cdot X^{20} \cdot X^{10} \cdot 1 \cdot X^{50} \\
11: &\quad (X^1)^0 \cdot (X^5)^3 \cdot (X^{10})^3 \cdot (X^{20})^2 \cdot (X^{50})^0 = 1 \cdot X^{15} \cdot X^{30} \cdot X^{40} \cdot 1 \\
12: &\quad (X^1)^5 \cdot (X^5)^2 \cdot (X^{10})^3 \cdot (X^{20})^2 \cdot (X^{50})^0 = X^5 \cdot X^{10} \cdot X^{30} \cdot X^{40} \cdot 1 \\
13: &\quad (X^1)^0 \cdot (X^5)^5 \cdot (X^{10})^2 \cdot (X^{20})^2 \cdot (X^{50})^0 = 1 \cdot X^{25} \cdot X^{20} \cdot X^{40} \cdot 1 \\
14: &\quad (X^1)^5 \cdot (X^5)^4 \cdot (X^{10})^2 \cdot (X^{20})^2 \cdot (X^{50})^0 = X^5 \cdot X^{20} \cdot X^{20} \cdot X^{40} \cdot 1
\end{aligned}
\tag{5}
$$

Table (5) lists all the possibilities of stamping a letter with the stock of stamps given at the beginning of this text.

Question: What does stamping letters have to do with changing money?

1.1.1 Stamping with distinct stamps

Question (modification of the problem): What if the postage were 8 Ct and if you had exactly one stamp each of 1 Ct, 2 Ct, 3 Ct, 4 Ct, 5 Ct, 6 Ct, 7 Ct and 8 Ct available?

The analog to expression (3) in this case would be

$$
(1+X) \cdot (1+X^2) \cdot (1+X^3) \cdot (1+X^4) \cdot (1+X^5) \cdot (1+X^6) \cdot (1+X^7) \cdot (1+X^8)
\tag{6}
$$

or in the expanded form

$$
\begin{aligned}
&1 + X + X^2 + 2\,X^3 + 2\,X^4 + 3\,X^5 + 4\,X^6 + 5\,X^7 + 6\,X^8 + 7\,X^9 + 8\,X^{10} \\
&+ 9\,X^{11} + 10\,X^{12} + 11\,X^{13} + 12\,X^{14} + 13\,X^{15} + 13\,X^{16} + 13\,X^{17} + 14\,X^{18} + 13\,X^{19} + 13\,X^{20} \\
&+ 13\,X^{21} + 12\,X^{22} + 11\,X^{23} + 10\,X^{24} + 9\,X^{25} + 8\,X^{26} + 7\,X^{27} + 6\,X^{28} + 5\,X^{29} + 4\,X^{30} \\
&+ 3\,X^{31} + 2\,X^{32} + 2\,X^{33} + X^{34} + X^{35} + X^{36}
\end{aligned}
\tag{7}
$$

The coefficient of X^8 in the expanded form obviously answers the question "How many ways are there to decompose the number 8 into distinct integer parts?"[6]

Here are the 6 possibilities (the number 8 itself counts as one of these "decomposition possibilities"):

$$
8 = 7 + 1 = 6 + 2 = 5 + 3 = 5 + 2 + 1 = 4 + 3 + 1
\tag{8}
$$

---

[6]Explain in which way the word "different" is relevant in the context of equation (6).

1.1.2 Stamping with (a limited number of) multiple stamps

If you have two stamps (instead of one) of each kind in the stock, the answer to the last stamping problem would be the coefficient of X^8, i.e. 13, in the following expression:

$$(1 + X^1 + (X^1)^2) \cdot (1 + X^2 + (X^2)^2) \cdot (1 + X^3 + (X^3)^2) \cdot (1 + X^4 + (X^4)^2)$$
$$\cdot (1 + X^5 + (X^5)^2) \cdot (1 + X^6 + (X^6)^2) \cdot (1 + X^7 + (X^7)^2) \cdot (1 + X^8 + (X^8)^2) \tag{9}$$

Translated into the language of partitions, this means counting the partitions with at most two equal parts. Here, they are:

1: [[8]]
2: [[7,1], [6,2], [5,3], [4,4]]
3: [[6,1,1], [5,2,1], [4,3,1], [4,2,2], [3,3,2]]
4: [[5,1,1,1], [4,2,1,1], [3,3,1,1], [3,2,2,1], [2,2,2,2]]
5: [[4,1,1,1,1], [3,2,1,1,1], [2,2,2,1,1]]
6: [[3,1,1,1,1,1], [2,2,1,1,1,1]]
7: [[2,1,1,1,1,1,1]]
8: [[1,1,1,1,1,1,1,1]]

So, there are 13 cases with at most 2 equal parts. With a maximum of 3 equal parts, the following 3 are added: [5,1,1,1], [3,2,1,1,1], [2,2,2,1,1].

By Euler's method, we have to expand the following products
For the case "2 stamps of each kind in the stock":

$$
\begin{aligned}
P2 = & (1 + X^1 + (X^1)^2 \ ) \cdot \\
& (1 + X^2 + (X^2)^2 \ ) \cdot \\
& (1 + X^3 + (X^3)^2 \ ) \cdot \\
& (1 + X^4 + (X^4)^2 \ ) \cdot \\
& (1 + X^5 + (X^5)^2 \ ) \cdot \\
& (1 + X^6 + (X^6)^2 \ ) \cdot \\
& (1 + X^7 + (X^7)^2 \ ) \cdot \\
& (1 + X^8 + (X^8)^2 \ ) \ )
\end{aligned}
\tag{10}
$$

and for the case "3 stamps of each kind in the stock":

$$
\begin{aligned}
P3 = & (1 + X^1 + (X^1)^2 + (X^1)^3 \ ) \cdot \\
& (1 + X^2 + (X^2)^2 + (X^2)^3 \ ) \cdot \\
& (1 + X^3 + (X^3)^2 + (X^3)^3 \ ) \cdot \\
& (1 + X^4 + (X^4)^2 + (X^4)^3 \ ) \cdot \\
& (1 + X^5 + (X^5)^2 + (X^5)^3 \ ) \cdot \\
& (1 + X^6 + (X^6)^2 + (X^6)^3 \ ) \cdot \\
& (1 + X^7 + (X^7)^2 + (X^7)^3 \ ) \cdot \\
& (1 + X^8 + (X^8)^2 + (X^8)^3 \ ) \ )
\end{aligned}
\tag{11}
$$

The coefficients of X^8 are: 13 in the case of $P2$ and   16 in the case of $P3$.

Exercise: Check the previous assertion, e.g. using a computer algebra system[7].

---

[7] The corresponding function in *Maxima* is: `coeff(P2, X, 8)` resp. `coeff(P3, X, 8)`.

2 Integer partitions

It should be clear, now, how to proceed: To calculate the number of partitions of n, we have to expand the expression

$$
\begin{aligned}
Pn = {} & (1 + X^1 + (X^1)^2 + \ldots + (X^1)^n) \cdot \\
& (1 + X^2 + (X^2)^2 + \ldots + (X^2)^n) \cdot \ldots \\
& (1 + (X^j)^1 + (X^j)^2 + \ldots + (X^j)^n) \cdot \ldots \\
& (1 + (X^n)^1 + (X^n)^2 + \ldots + (X^n)^n)
\end{aligned}
\tag{12}
$$

For answering the problem: "What is the coefficient of X^{85}?" it is irrelevant[8] if exponents are generated which are too large (i.e. greater than 85). These exponents are simply disregarded.

Instead of (12) we can also use the following expression:

$$
\begin{aligned}
P = {} & (1 + X^1 + (X^1)^2 + \ldots + (X^1)^\infty) \cdot \\
& (1 + X^2 + (X^2)^2 + \ldots + (X^2)^\infty) \cdot \ldots \\
& (1 + (X^j)^1 + (X^j)^2 + \ldots + (X^j)^\infty) \cdot \ldots \\
& (1 + (X^n)^1 + (X^n)^2 + \ldots + (X^n)^\infty)
\end{aligned}
\tag{13}
$$

Using the "sum-function" symbol $\sum$ this can be expressed as:

$$
P = \sum_{n_1=0}^{\infty} (X^1)^{n_1} \cdot \sum_{n_2=0}^{\infty} (X^2)^{n_2} \cdot \ldots \cdot \sum_{n_j=0}^{\infty} (X^j)^{n_j} \cdot \ldots \cdot \sum_{n_l=0}^{\infty} (X^n)^{n_l}
\tag{14}
$$

What should this be good for? Well, as we have seen, for formal power series the following equation holds[9]:

$$
\sum_{k=0}^{\infty} Y^k = \frac{1}{1-Y}
$$

Therefore, (14) (with $X, X^2, X^j, \ldots, X^n$ instead of Y) can be written as:

$$
P = \frac{1}{1-X} \cdot \frac{1}{1-X^2} \cdot \ldots \cdot \frac{1}{1-X^j} \cdot \ldots \cdot \frac{1}{1-X^n}
\tag{15}
$$

respectively

$$
P = \prod_{j=1}^{n} \frac{1}{1-X^j}
\tag{16}
$$

The coefficients of the term (16) can be extracted by application of the *Taylor-expansion*[10] of this term.

Exercise: Check the case of $n = 85$ by using the Taylor expansion of (16)

---

[8] as for the result . . . not for the runtime

[9] see: https://jochen-ziegenbalg.github.io/materialien/Manuskripte/Fib-Pot-erzFkt.pdf

[10] see: https://jochen-ziegenbalg.github.io/materialien/Manuskripte/Fib-Pot-erzFkt.pdf
and https://jochen-ziegenbalg.github.io/materialien/Manuskripte/Partitions-CAS-Maxima.pdf

2.1 Integer partitions by explicit construction

Repetition: From elementary school we know that $12 = 7 + 5$. The right-hand side $7 + 5$ of this equation is called a *partition* of 12. The numbers 7 and 5 are called the *parts* of the partition and since they are integers, this is called an *integer partition*. In the following text we will, however, only speak about partitions, because all the partitions we will consider, will be integer partitions. There are other partitions of 12, like $4 + 3 + 2 + 2 + 1$ or $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$. For systematic reasons, just 12 is also considered a partition of 12. The number of summands (or equally, the number of parts), will also be called the *length* of the partition.

Since it is always a good idea to have some examples at hand, we first give some examples of "small" and some not quite so small cases.

Examples: Each partition is written as a *list* – wrapped in brackets.

partitions of 0 : [ ]   (the number 0 has no partition)
partitions of 1 : [1]
partitions of 2 : [1, 1],   [2]
partitions of 3 : [1, 1, 1],   [1, 2],   [3]
partitions of 4 : [1, 1, 1, 1],   [1, 1, 2],   [1, 3],   [2, 2],   [4]
partitions of 5 : [1, 1, 1, 1, 1],   [1, 1, 1, 2],   [1, 1, 3],   [1, 2, 2],   [1, 4],   [2, 3],   [5]
partitions of 6 :   [1, 1, 1, 1, 1, 1],   [1, 1, 1, 1, 2],   [1, 1, 1, 3],   [1, 1, 2, 2],   [1, 1, 4],   [1, 2, 3],   [1, 5],
    [2, 2, 2],   [2, 4],   [3, 3],   [6]
partitions of 7 :   [1, 1, 1, 1, 1, 1, 1],   [1, 1, 1, 1, 1, 2],   [1, 1, 1, 1, 3],   [1, 1, 1, 2, 2],   [1, 1, 1, 4],
    [1, 1, 2, 3],   [1, 1, 5],   [1, 2, 2, 2],   [1, 2, 4],   [1, 3, 3],   [1, 6],   [2, 2, 3],   [2, 5],   [3, 4],
    [7]
partitions of 8 :   [1, 1, 1, 1, 1, 1, 1, 1],   [1, 1, 1, 1, 1, 1, 2],   [1, 1, 1, 1, 1, 3],   [1, 1, 1, 1, 2, 2],
    [1, 1, 1, 1, 4],   [1, 1, 1, 2, 3],   [1, 1, 1, 5],   [1, 1, 2, 2, 2],   [1, 1, 2, 4],   [1, 1, 3, 3],
    [1, 1, 6],   [1, 2, 2, 3],   [1, 2, 5],   [1, 3, 4],   [1, 7],   [2, 2, 2, 2],   [2, 2, 4],   [2, 3, 3],
    [2, 6],   [3, 5],   [4, 4],   [8]
partitions of 9 :   [1, 1, 1, 1, 1, 1, 1, 1, 1],   [1, 1, 1, 1, 1, 1, 1, 2],   [1, 1, 1, 1, 1, 1, 3],
    [1, 1, 1, 1, 1, 2, 2],   [1, 1, 1, 1, 1, 4],   [1, 1, 1, 1, 2, 3],   [1, 1, 1, 1, 5],   [1, 1, 1, 2, 2, 2],
    [1, 1, 1, 2, 4],   [1, 1, 1, 3, 3],   [1, 1, 1, 6],   [1, 1, 2, 2, 3],   [1, 1, 2, 5],   [1, 1, 3, 4],
    [1, 1, 7],   [1, 2, 2, 2, 2],   [1, 2, 2, 4],   [1, 2, 3, 3],   [1, 2, 6],   [1, 3, 5],   [1, 4, 4],
    [1, 8],   [2, 2, 2, 3],   [2, 2, 5],   [2, 3, 4],   [2, 7],   [3, 3, 3],   [3, 6],   [4, 5],   [9]
partitions of 10 :   [1, 1, 1, 1, 1, 1, 1, 1, 1, 1],   [1, 1, 1, 1, 1, 1, 1, 1, 2],   [1, 1, 1, 1, 1, 1, 1, 3],
    [1, 1, 1, 1, 1, 1, 2, 2],   [1, 1, 1, 1, 1, 1, 4],   [1, 1, 1, 1, 1, 2, 3],   [1, 1, 1, 1, 1, 5],
    [1, 1, 1, 1, 2, 2, 2],   [1, 1, 1, 1, 2, 4],   [1, 1, 1, 1, 3, 3],   [1, 1, 1, 1, 6],   [1, 1, 1, 2, 2, 3],
    [1, 1, 1, 2, 5],   [1, 1, 1, 3, 4],   [1, 1, 1, 7],   [1, 1, 2, 2, 2, 2],   [1, 1, 2, 2, 4],   [1, 1, 2, 3, 3],
    [1, 1, 2, 6],   [1, 1, 3, 5],   [1, 1, 4, 4],   [1, 1, 8],   [1, 2, 2, 2, 3],   [1, 2, 2, 5],   [1, 2, 3, 4],
    [1, 2, 7],   [1, 3, 3, 3],   [1, 3, 6],   [1, 4, 5],   [1, 9],   [2, 2, 2, 2, 2],   [2, 2, 2, 4],   [2, 2, 3, 3],
    [2, 2, 6],   [2, 3, 5],   [2, 4, 4],   [2, 8],   [3, 3, 4],   [3, 7],   [4, 6],   [5, 5],   [10]

In a more structured way of dealing with the problem, we sort the partitions by the number of parts. By $PL(n, k)$ we denote the list of the partitions of n with exactly k parts[11]. For $n = 8$

---

[11]As for notation: $PL(n)$ and $PL(n, k)$ return the partitions as *lists*; $P(n)$ and $P(n, k)$ return the *number* of partitions.

we have:

$$PL(8,1) = [[8]] \hspace{4cm} |PL(8,1) = 1| \quad (17)$$
$$PL(8,2) = [[7,1],[6,2],[5,3],[4,4]] \hspace{2cm} |PL(8,2) = 4|$$
$$PL(8,3) = [[6,1,1],[5,2,1],[4,3,1],[4,2,2],[3,3,2]] \hspace{1cm} |PL(8,3) = 5|$$
$$PL(8,4) = [[5,1,1,1],[4,2,1,1],[3,3,1,1],[3,2,2,1],[2,2,2,2]] \hspace{0.5cm} |PL(8,4) = 5|$$
$$PL(8,5) = [[4,1,1,1,1],[3,2,1,1,1],[2,2,2,1,1]] \hspace{1cm} |PL(8,5) = 3|$$
$$PL(8,6) = [[3,1,1,1,1,1],[2,2,1,1,1,1]] \hspace{1.5cm} |PL(8,6) = 2|$$
$$PL(8,7) = [[2,1,1,1,1,1,1]] \hspace{2.5cm} |PL(8,7) = 1|$$
$$PL(8,8) = [[1,1,1,1,1,1,1,1]] \hspace{2.3cm} |PL(8,8) = 1|$$

So, for $PL(8)$ (the list of all integer partitions of 8) we have:

$$PL(8) = \bigcup_{k=1}^{8} PL(8,k) \qquad (18)$$

Because this is a "disjoint" union, we have $|PL(8)| = 22$.

2.1.1 An algorithm for generating the partitions of an integer

"What I cannot create, I do not understand."
Richard Feynman, amerikanischer Physiker,
1918–1988, Nobel Prize 1965

A general strategy of problem solving is to reduce an existing (initially unsolved) problem to "smaller" cases of the same type. This often leads to "recursive" approaches. In the case of integer partitions, the solutions for the small cases $n = 1, 2, 3$ are "visible to the naked eye"; but for values of n greater than 10 it is advisable to look out for a more systematic strategy for generating the partitions.

When it comes to an algorithm or a program for generating the partitions we have to decide if the program is to

1. generate the partitions themselves, or just
2. generate the *number* of the partitions.

In this article, we will concentrate on the second objective. The first objective is extensively treated in the "companion" file `partitions.wxmx` with all programs implemented in the the open source computeralgebra system *Maxima*.

For both objectives, recursion will prove extremely helpful.

First, some notation: The functions $PL(n)$ and $PL(n,k)$ will return the partitions as *lists*; $P(n)$ and $P(n,k)$ will return the *number* of partitions.

Let us take a closer look at a not too small example: $PL(n,k)$ with $n = 12$ and $k = 4$.

$$PL(12,4) = [\,[9,1,1,1],[8,2,1,1],[7,3,1,1],[6,4,1,1],[5,5,1,1],[7,2,2,1],[6,3,2,1],[5,4,2,1],$$
$$[5,3,3,1],[4,4,3,1], \hspace{3cm} (19)$$
$$[6,2,2,2],[5,3,2,2],[4,4,2,2],[4,3,3,2],[3,3,3,3]\,]$$

The first 10 of these sublists have "1" as their last element. If we delete the 1, we get the partitions of 11 with 3 elements:

$$PL(11,3) = [\ [9,1,1],[8,2,1],[7,3,1],[6,4,1],[5,5,1],[7,2,2],[6,3,2],[5,4,2], \tag{20}$$
$$[5,3,3],[4,4,3]\ ]$$

As for the last 5 sublists in (19), if we reduce each integer by 1, we get the partitions of $8\ (=12-4)$ with 4 parts: $PL(12-4,4)$

$$PL(8,4) = [\ [5,1,1,1],[4,2,1,1],[3,3,1,1],[3,2,2,1],[2,2,2,2]\ ] \tag{21}$$

If we raise each number on the right-hand side of (21) by 1 we get the partitions in (19) again.

Both of these "manipulations" are cleary reversible.

General strategy, construction, algorithm and program:

Let $PL(n,k)$ be the list of the integer patitions of n consisting of exactly k parts. Then in order to construct the list $PL(n,k)$:

1. Construct $PL(n-1,k-1)$ and insert a "1" as the last element into each of the subsets. Call the result $PL_{ins}(n-1,k-1)$ (*ins* for *"inserted"*).

2. Construct $PL(n-k,k)$ and add 1 to each of the thus generated numbers. Call the result $PL_{enh}(n-k,k)$ (*enh* for *"enhanced"*).

Clearly $PL_{ins}(n)$ and $PL_{enh}(n)$ are partitions of n with k parts. In general we get:

$$PL(n,k) = PL_{ins}(n-1,k-1)\ \cup\ PL_{enh}(n-k,k) \tag{22}$$

By construction, the lists $PL_{ins}(n-1,k-1)$ and $PL_{enh}(n-k,k)$ are disjoint, since every sublist of $PL_{ins}(n-1,k-1)$ ends with a "1" and no sublist of $PL_{enh}(n-k,k)$ does so.

Furthermore, by construction

- $PL(n-1,k-1)$ and $PL_{ins}(n-1,k-1)$ have the same number of elements, and
- $PL(n-k,k)$ and $PL_{enh}(n-k,k)$ have the same number of elements.

Thus the *number* $P(n,k)$ of partitions of n with k parts is:

$$P(n,k) = P(n-1,k-1) + P(n-k,k) \tag{23}$$

Exercise: Construct $PL(15,5)$, $P(15,5)$ and P(15) in this way.

2.1.2   A *Maxima* program

As for the problem of generating the list $PL(n,k)$ of partitions, this requires some specific list processing commands and it is done in the *Maxima* file `partitions.wxmx`. Here, we will concentrate on the problem of generating the *number* of partitions. The following *Maxima* program is more or less a direct translation of the fundamental recursion (23) into *Maxima* code.[12]

---

[12]Here, it is not the primary objective to give a very elegant or efficient version of the program, but to make it transparent. *Maxima* comments are denoted by /* ... */ in the program text.

```
        Ps(n, k) :=                              \* notation: Ps for "P stepwise" *\
             if n < 1 then 0 else
                if k < 1 then 0 else
                   if k > n then 0 else
                      if k = 1 then 1 else
                         Ps(n-1, k-1) + Ps(n-k, k)


        P(n) := sum(Ps(n,k), k, 1, n);
```

Calling `P(n)` gives the results:

```
P(45) = 89134
P(85) = 30167357
```
 (by using the "memoized" version from the file `partitions.wxmx`)

Comment: In the "fully recursive" form (above), the program is logically transparent, but not runtime-efficient. A more efficient version, which makes use of the "memoizing functions" technique[13], is given in the file `partitions.wxmx`.

The following figure gives an impression of how quickly the number of partitions grows.
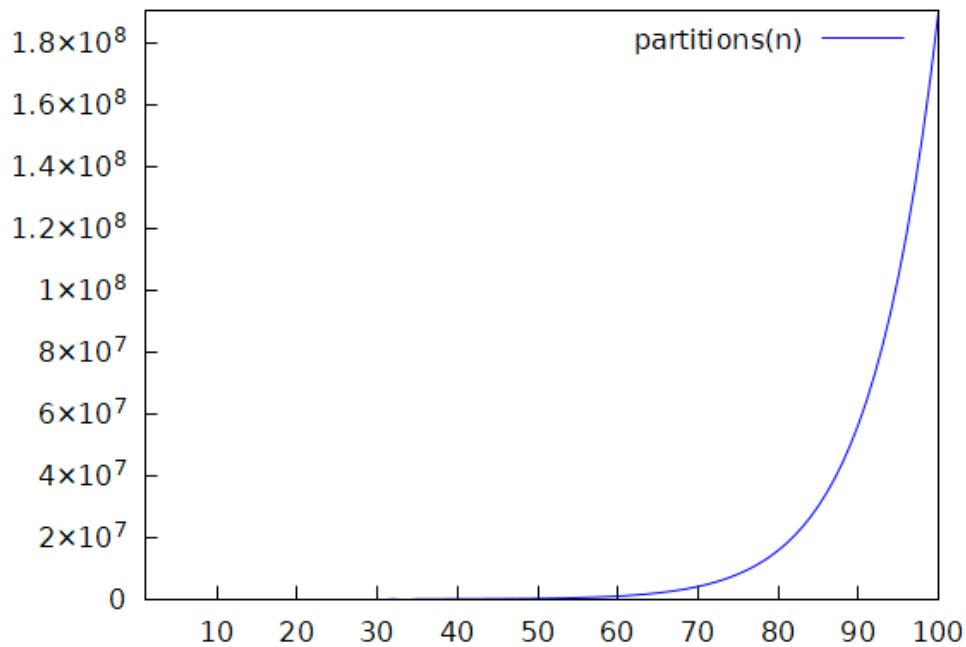

Figure 1: number of partitions of a number n

---

[13]sometimes also called "array function" technique

3 The triangle of partition numbers

The following "triangle" shows the partiton numbers $P(n,k)$ with n varying in the rows and k in the columns. It is halpful in understanding the "stepwise" version of the function for the partition numbers.

```
          k -->
n =  1:  1
n =  2:  1    1
n =  3:  1    1    1
n =  4:  1    2    1    1
n =  5:  1    2    2    1    1
n =  6:  1    3    3    2    1    1
n =  7:  1    3    4    3    2    1    1
n =  8:  1    4    5    5    3    2    1    1
n =  9:  1    4    7    6    5    3    2    1    1
n = 10:  1    5    8    9    7    5    3    2    1    1
n = 11:  1    5   10   11   10    7    5    3    2    1    1
n = 12:  1    6   12   15   13   11    7    5    3    2    1    1
n = 13:  1    6   14   18   18   14   11    7    5    3    2    1    1
n = 14:  1    7   16   23   23   20   15   11    7    5    3    2    1    1
n = 15:  1    7   19   27   30   26   21   15   11    7    5    3    2    1    1
n = 16:  1    8   21   34   37   35   28   22   15   11    7    5    3    2    1    1
n = 17:  1    8   24   39   47   44   38   29   22   15   11    7    5    3    2    1    1
n = 18:  1    9   27   47   57   58   49   40   30   22   15   11    7    5    3    2    1    1
```

Exercise: Visualize the recursive descripton (23) for $n = 18$ and $k = 5$ by inserting arrows in the "partitions triangle" above.
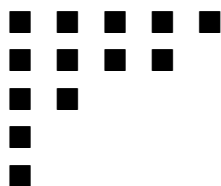
4 Ferrers diagrams

Partitions can be graphically visualized by use of "Ferrers diagrams"[14]. They occur in a number of branches of mathematics and physics, including the study of symmetric polynomials and groups, see:

   https://en.wikipedia.org/wiki/Integer_partition
      https://www.britannica.com/science/combinatorics/The-Ferrer-diagram

Examples:

The partition $5 + 4 + 2 + 1 + 1$ of the number 13 can be represented by the following diagram:

---

[14]N. M. Ferrers (1829–1903) was a British mathematician and university administrator.

The following diagrams are the Ferrers diagrams for the 5 partitions of the number 4:

$$4 \qquad 3+1 \qquad 2+2 \qquad 2+1+1 \quad 1+1+1+1$$

■ ■ ■ ■   ■ ■ ■   ■ ■   ■ ■   ■
        ■   ■ ■   ■   ■
                  ■   ■
                       ■
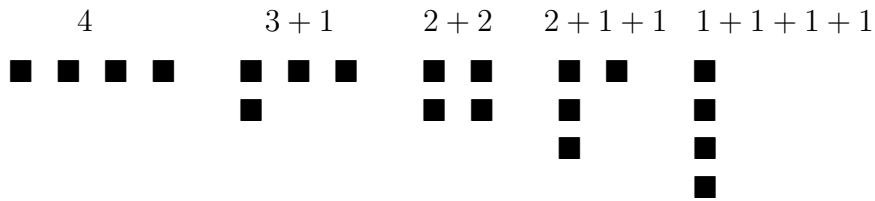
Comment: Nice visualization. ... But to what end?

Challenge: Show that there are as many partitions of a number n with greatest part g as there are partitions of n with g parts (i.e. with g summands).

Heuristics: Let $n = 17$ and $g = 6$

    One partition is $17 = 6 + 4 + 3 + 3 + 1$ (P-1)
    Another one is $17 = 5 + 4 + 4 + 2 + 1 + 1$ (P-2)
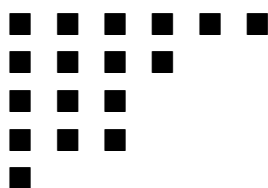
What about the challenge? ... Any Ideas?

Enter: Ferrers

Draw the Ferrers diagram of (P-1):
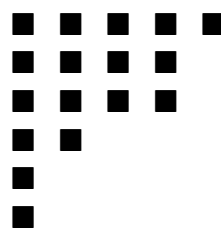*playful idea*: flipping the diagram along its "main" diagonal

    Ferrers diagram of (P-1):   Ferrers diagram of (P-2):
    $17 = 6 + 4 + 3 + 3 + 1$ $17 = 5 + 4 + 4 + 2 + 1 + 1$

■ ■ ■ ■ ■ ■   ■ ■ ■ ■ ■
■ ■ ■   ■ ■ ■ ■
■ ■ ■   ■ ■ ■ ■
■ ■ ■   ■ ■
■   ■
                                  ■

In the example:
    (P-1): $6 + 4 + 3 + 3 + 1$ (*greatest part* $= 6$)
    (P-2): $5 + 4 + 4 + 2 + 1 + 1$ (*number of parts* $= 6$)

Obvious facts:
1. To every partition of n with greatest part g there is (by flipping) a partition with g parts.

2. When two partitions are equal their flippings are equal.

3. When two flippings are equal, the originals were equal.
   What does double-flipping do?

Result: There are as many partitions of a number n with greatest part g as there are partitions of that number with g parts.

Proof: Flipping the Ferrers diagrams

4.1   Some properties of Ferrers diagrams

- partitions with odd parts
- partitions with distinct parts
- conjugate and self-conjugate partitions

Example: $n = 19$

- partition with odd parts: $9 + 5 + 3 + 1 + 1$
- partition with distinct parts: $7 + 6 + 4 + 2$
- Draw the Ferrers diagrams!

We have seen: The Ferrers diagrams $(P - 1)$ and $(P - 2)$ are converted into each other reciprocally by flipping.

Definition: If the Ferrers diagrams P and Q are converted into each other reciprocally by flipping, they are called *conjugate* diagrams.

Exercise: What if for the Ferrers diagram P: $P = flipped(P)$?   Give an example.

Definition: If for the Ferrers diagram P we have $P = flipped(P)$ then P is called *self-conjugate*.

Theorem: The number of self-conjugate partitions is the same as the number of partitions with distinct odd parts.

Proof: see Wikipedia:
`https://en.wikipedia.org/wiki/Integer_partition#Ferrers_diagram`

Some more results: Partitions with odd parts and with distinct parts

Exercise: How many partitions has the number 8? Set up a list of all the partitions.

Among the partitions of the number 8, there are 6 partitions with only *odd* parts:

- $7 + 1$
- $5 + 3$
- $5 + 1 + 1 + 1$
- $3 + 3 + 1 + 1$
- $3 + 1 + 1 + 1 + 1 + 1$
- $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

Alternatively, we could count partitions in which no number occurs more than once. Such a partition is called a partition with *distinct* parts. If we count the partitions of 8 with distinct parts, we obtain the following 6 partitions:

- 8
- $7 + 1$
- $6 + 2$
- $5 + 3$
- $5 + 2 + 1$
- $4 + 3 + 1$

This is a general property. We cite the following

Theorem: For each positive number, the number of partitions with odd parts equals the number of partitions with distinct parts.

This result was proved by Leonhard Euler in 1748 and later was generalized as Glaisher's theorem. (see: Wikipedia `https://en.wikipedia.org/wiki/Integer_partition#Ferrers_diagram`)

5 Ramanujan and Hardy

Srinivasa Ramanujan (1887–1920) was one of India's greatest mathematical geniuses and one of the most fascinating mathematicians of all times. Though he had almost no formal training in pure mathematics, he made substantial contributions to mathematical analysis, number theory, infinite series, and continued fractions, including solutions to mathematical problems considered unsolvable at his time.

Ramanujan also worked on partitions intensely and in 1918, in cooperation with the great British mathematician G.H. Hardy (1877–1947), found the following approximating function for the number of partitions $P(n)$
(see: `https://en.wikipedia.org/wiki/Partition_function_(number_theory)`.

$$P_{approx}(n) := \frac{1}{4n\sqrt{3}} \cdot e^{\left(\pi \cdot \sqrt{\frac{2n}{3}}\right)} \tag{24}$$
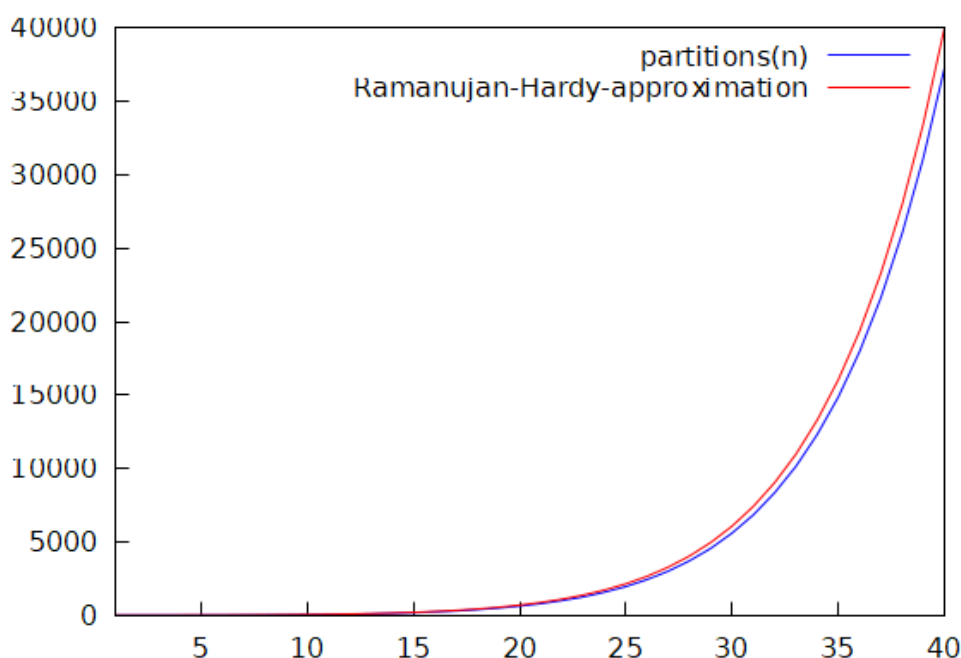

Figure 2: Partitions: approximation by Ramanujan and Hardy

Srinivasa Ramanujan discovered that the partition function has nontrivial patterns in modular arithmetic, now known as *Ramanujan's congruences.*

Literature and media on Ramanujan:

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- Wikipedia (German): `https://de.wikipedia.org/wiki/Srinivasa_Ramanujan`

- McTutor, *History of Mathematics*: `https://mathshistory.st-andrews.ac.uk/Biographies/Ramanujan/`

- Movie: Die Poesie des Unendlichen: *"trailer"*: `https://www.imdb.com/title/tt0787524/`

- Video with A. Beutelspacher: `https://www.youtube.com/watch?v=4x3qXRgnYgw&list=PLfdMKJMGPPtw7kmxHU1e4F-GRiqsJi3L3&index=1`

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