Independent subsets of a finite set

1 Independent nontrivial subsets

```
(%i1) pairs_independent_nontrivial_subsets(n) :=
      block([a, b, d, s : 0 ],
        for a:1 thru n-1 do
          for d:1 thru a do
            ( b : n*d / a,
              if integerp(b) and b<n
                then (s : s + binomial(n,a)*binomial(a,d)*binomial(n-a,b-d)) ) ,
          s ) ;
```

$$(\%o1)\ \text{pairs\_independent\_nontrivial\_subsets}(n) := \text{block}\left([a,b,d,s:0],\ \text{for } a \text{ thru } n-1 \text{ do for } d \text{ thru } a \text{ do } \left(b:\frac{n\,d}{a},\ \text{if integerp}(b) \text{ and } b<n \text{ then } s:s+\binom{n}{a}\binom{a}{d}\binom{(n-a)}{(b-d)}\right),s\right)$$

```
(%i2) pairs_independent_nontrivial_subsets(4);
```

$$(\%o2)\ 24$$

```
(%i3) L30 : makelist(i, i, 0, 30);
```

$$(L30)\ [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30]$$

```
(%i4) L_nontrivial_30  :  map(pairs_independent_nontrivial_subsets, L30);
```

$$(L\_nontrivial\_30)\ [0,0,0,0,24,0,720,0,7000,15120,126000,0,1777776,0,23543520,55855800,274565720,0,5337775872,0,63026049424,117920013120,995265791520,0,15265486117744,14283091977000,216344919117600,240142901941800,2854493961432480,0,55689696384165720]$$

```
(%i5) a(n) :=
      sum(
         sum(
              (b : n*d / a,
              if integerp(b) and b<n then
                 binomial(n,a)*binomial(a,d)*binomial(n-a,b-d) else 0), d,1,a), a,1,n-1) ;
```

$$(\%o5)\ a(n) := \sum_{a=1}^{n-1}\left(\sum_{d=1}^{a}\left(\left(b:\frac{n\,d}{a},\ \text{if integerp}(b) \text{ and } b<n \text{ then } \binom{n}{a}\binom{a}{d}\binom{(n-a)}{(b-d)} \text{ else } 0\right)\right)\right)$$

```
(%i6) a(6);
```

$$(\%o6)\ 720$$

```
(%i7) L_a_30 : map(a, L30);
```

$$(L\_a\_30)\ [0,0,0,0,24,0,720,0,7000,15120,126000,0,1777776,0,23543520,55855800,274565720,0,5337775872,0,63026049424,117920013120,995265791520,0,15265486117744,14283091977000,216344919117600,240142901941800,2854493961432480,0,55689696384165720]$$

```
(%i8) is(L_a_30 = L_nontrivial_30);
```

$$(\%o8)\ \textbf{true}$$

```
(%i9) is(map(pairs_independent_nontrivial_subsets, L30) = map(a,L30) );
(%o9) true
```

2 Independent proper subsets

```
(%i10) pairs_independent_proper_subsets(n) :=
         if is(n=0) then 0 else a(n) + 2*(2^n -1)  - 1 ;
```

(%o10) $pairs\_independent\_proper\_subsets(n) := if\ is(n=0)\ then\ 0\ else\ a(n) + 2\left(2^n - 1\right) - 1$

```
(%i11) L_proper_30 : map(pairs_independent_proper_subsets, L30);
```

(L_proper_30) [0, 1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045, 4093, 1785965, 16381, 23576285, 55921333, 274696789, 262141, 5338300157, 1048573, 63028146573, 117924207421, 995274180125, 16777213, 15265519672173, 14283159085861, 216345053335325, 240143170377253, 2854494498303389, 1073741821, 55689698531649365]

```
(%i12) OEIS_A158345 : [ 1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045, 4093, 1785965, 16381, 2357
         55921333, 274696789, 262141, 5338300157, 1048573, 63028146573, 117924207421, 995274
         16777213, 15265519672173, 14283159085861 ];
```

(OEIS_A158345) [1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045, 4093, 1785965, 16381, 23576285, 55921333, 274696789, 262141, 5338300157, 1048573, 63028146573, 117924207421, 995274180125, 16777213, 15265519672173, 14283159085861]

```
(%i13) rest(rest(L_proper_30), -5) ;
```

(%o13) [1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045, 4093, 1785965, 16381, 23576285, 55921333, 274696789, 262141, 5338300157, 1048573, 63028146573, 117924207421, 995274180125, 16777213, 15265519672173, 14283159085861]

```
(%i14) is( rest(rest(L_proper_30), -5)  =  OEIS_A158345);
(%o14) true
```

```
(%i15) pairs_independent_proper_subsets_2(n) :=
       block([a, b, d, s : 2*(2^n -1)  - 1 ],
          if  is(n=0)  then  s : 0  else
            for a:1 thru n-1 do
              for d:1 thru a do
               (b : n*d / a,
                if integerp(b) and b<n
                    then ( s : s + binomial(n,a)*binomial(a,d)*binomial(n-a,b-d) ) )  ,
          s ) ;
```

(%o15) $pairs\_independent\_proper\_subsets_2(n) := block$

```
(%i16) map(pairs_independent_proper_subsets_2, L30) ;
```

(%o16) [0, 1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045, 4093, 1785965, 16381, 23576285, 55921333, 274696789, 262141, 5338300157, 1048573, 63028146573, 117924207421, 995274180125, 16777213, 15265519672173, 14283159085861, 216345053335325, 240143170377253, 2854494498303389, 1073741821, 55689698531649365]

```
(%i17)  is(map(pairs_independent_proper_subsets_2, L30) = L_proper_30) ;
(%o17) true
```

```
(%i18) pips(n) := if is(n=0) then 0 else
         2*(2^n -1)  - 1 +
          sum(
            sum(
                 (b : n*d / a,
                  if verbose then print(d, a, b, binomial(n,a)*binomial(a,d)*binomial(n-a,b-d)),
                  if integerp(b) and b<n
                    then binomial(n,a)*binomial(a,d)*binomial(n-a,b-d) else 0), d,1,a), a,1,n-1) ;
```

$$(\%o18)\quad pips(n):=if\ is(n=0)\ then\ 0\ else\ 2\left(2^n-1\right)-1+\sum_{a=1}^{n-1}\left(\sum_{d=1}^{a}\left(\left(b:\frac{n\,d}{a},if\ verbose\ then\ print\left(d,a,b,\binom{n}{a}\binom{a}{d}\binom{(n-a)}{(b-d)}\right),if\ integerp(b)\ and\ b<n\ then\ \binom{n}{a}\binom{a}{d}\binom{(n-}{(b-}\right.\right.$$

```
(%i20) verbose : true; pips(4);
```

(verbose) **true**

$$1\ \ 1\ \ 4\ \ 4$$

$$1\ \ 2\ \ 2\ \ 24$$

$$2\ \ 2\ \ 4\ \ 6$$

$$1\ \ 3\ \ \frac{4}{3}\ \ 12\binom{1}{\left(\frac{1}{3}\right)}$$

$$2\ \ 3\ \ \frac{8}{3}\ \ 12\binom{1}{\left(\frac{2}{3}\right)}$$

$$3\ \ 3\ \ 4\ \ 4$$

(%o20) 53

```
(%i22) verbose : false $ ;  map(pips, L30);
```

(%o22) **[**0, 1, 5, 13, 53, 61, 845, 253, 7509, 16141, 128045, 4093, 1785965, 16381, 23576285, 55921333, 274696789, 262141, 5338300157, 1048573, 63028146573, 117924207421, 995274180125, 16777213, 15265519672173, 14283159085861, 216345053335325, 240143170377253, 2854494498303389, 1073741821, 55689698531649365**]**

```
(%i23) is (map(pips, L30) = L_proper_30);
```

(%o23) **true**

3 All independent subsets

```
(%i24) pairs_independent_subsets(n) :=
         if is(n=0)  then 1 else a(n) + 4*(2^n -1);
```

$$(\%o24)\quad pairs\_independent\_subsets(n):=if\ is(n=0)\ then\ 1\ else\ a(n)+4\left(2^n-1\right)$$

```
(%i25) pairs_independent_subsets(6);
```

(%o25) 972

```
(%i26) L_independent_30  :  map(pairs_independent_subsets, L30);
```

(L_independent_30) **[**1, 4, 12, 28, 84, 124, 972, 508, 8020, 17164, 130092, 8188, 1794156, 32764, 23609052, 55986868, 274827860, 524284, 5338824444, 2097148, 63030243724, 117928401724, 995282568732, 33554428, 15265553226604, 14283226194724, 216345187553052, 240143438812708, 2854495035174300, 2147483644, 55689700679133012**]**

(%i27) OEIS_A121312 : [1, 4, 12, 28, 84, 124, 972, 508, 8020, 17164, 130092, 8188, 1794156, 32764, 23609052, 55986868, 274827860, 524284, 5338824444, 2097148, 63030243724, 117928401724, 995282568732, 33554428, 15265553226604, 14283226194724, 2163451875

(OEIS_A121312) ***[*** *1, 4, 12, 28, 84, 124, 972, 508, 8020, 17164, 130092, 8188, 1794156, 32764, 23609052, 55986868, 274827860, 524284, 5338824444, 2097148, 63030243724, 117928401724, 995282568732, 33554428, 15265553226604, 14283226194724, 216345187553052* ***]***

(%i28) is(rest(L_independent_30, -4) = OEIS_A121312);

(%o28) ***true***

(%i29) transpose(matrix(L30,L_independent_30, L_proper_30, L_nontrivial_30));

(%o29)

0	1	0	0
1	4	1	0
2	12	5	0
3	28	13	0
4	84	53	24
5	124	61	0
6	972	845	720
7	508	253	0
8	8020	7509	7000
9	17164	16141	15120
10	130092	128045	126000
11	8188	4093	0
12	1794156	1785965	1777776
13	32764	16381	0
14	23609052	23576285	23543520
15	55986868	55921333	55855800
16	274827860	274696789	274565720
17	524284	262141	0
18	5338824444	5338300157	5337775872
19	2097148	1048573	0
20	63030243724	63028146573	63026049424
21	117928401724	117924207421	117920013120
22	995282568732	995274180125	995265791520
23	33554428	16777213	0
24	15265553226604	15265519672173	15265486117744
25	14283226194724	14283159085861	14283091977000
26	216345187553052	216345053335325	216344919117600
27	240143438812708	240143170377253	240142901941800
28	2854495035174300	2854494498303389	2854493961432480
29	2147483644	1073741821	0
30	55689700679133012	55689698531649365	55689696384165720