Fibonacci Numbers and Binet's Formula

Roadmap (Überblick)

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■ The generalized Fibonacci equation

■ 0. A remark on the historical development

In his *Liber abaci* (1202), Leonardo of Pisa (called *Fibonacci*, ca. 1170 - 1250) formulated a problem giving rise to the following famous sequence of numbers now called the "Fibonacci" numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Its most important property is that every member of the sequence is the sum of its two immediate predecessors (except for the initial values):

$$F_{k+2} = F_{k+1} + F_k$$

It took several centuries until J. P. M. Binet (1786-1856) finally presented the following formula for the Fibonacci numbers:

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k$$

■ 1. Generalisation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0 \qquad \text{(* GFE *)}$$

This equation clearly generalizes the Fibonacci equation; we will, therefore, call it the *generalized Fibonacci equation* (GFE) for short.

■ 2. Applying standard methodology: Introducing new parameters (in this case: a "tuning" parameter)

Due to this result we have to handle the decomposition in a slightly more subtle way by introducing an extra parameter called t (for "tuning") in the following way.

$$y_{k+2} + (a_1 + t) \cdot y_{k+1} = 0$$

$$-t \cdot y_{k+1} + a_0 \cdot y_k = 0$$

GFE can be thought of as being the sum of these first-order equations. If, by choosing a suitable "tuning" value for t, we can make these two first-order equations identical, then they will have the same closed form representations and we can try to combine their individual solutions into a solution for the generalized Fibonacci equation.

■ 3. The tuning process

Written in the "standard" form for geometric sequences the last two first-order equations read

$$y_{k+2} = -(a_1 + t) \cdot y_{k+1} \qquad \text{(* GS-1 *)}$$

$$y_{k+1} = \frac{a_0}{t} \cdot y_k \qquad \text{(* GS-2 *)}$$

These difference equations for geometric sequences are identical if their coefficients $-(a_1 + t)$ and $\frac{a_0}{t}$ are equal. (The "index-shift" by 1 is irrelevant, since the equations are valid for all values of k).
A necessary condition for equality, hence, is

$$-(a_1 + t) = \frac{a_0}{t}$$

■ 4. The characteristic polynomial

$$t^2 + a_1 \cdot t + a_0 = 0$$

Thus, the above geometric sequences are identical if the "tuning" parameter t satisfies the so-called *characteristic equation* of (* GFE *):

$$x^2 + a_1 \cdot x + a_0 = 0$$

We finally obtain the tuning parameters

$$t_1 = \frac{-a_1 + \sqrt{a_1{}^2 - 4 a_0}}{2}$$

$$t_2 = \frac{-a_1 - \sqrt{a_1{}^2 - 4 a_0}}{2}$$

■ 5. Solutions - by applying the results on geometric series

Thus, the geometric sequences adding up to (* GFE *) are

1. By using the root t_1:

$$y_{k+2} = -\left(a_1 + \frac{-a_1 + \sqrt{a_1{}^2 - 4 a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 + \sqrt{a_1{}^2 - 4 a_0}}{2}} \cdot y_k$$

2. By using the root t_2:

$$y_{k+2} = -\left(a_1 + \frac{-a_1 - \sqrt{a_1^2 - 4\,a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 - \sqrt{a_1^2 - 4\,a_0}}{2}} \cdot y_k$$

6. Vieta's formulae

The tuning parameters t_1 and t_2, being the roots of the quadratic equation

$$x^2 + a_1 \cdot x + a_0 = 0$$

must satisfy Vieta's equations:

$$t_1 + t_2 = -a_1 \quad \text{and} \quad t_1 \cdot t_2 = a_0$$

Thus, by substituting either t_1 or t_2 for t and applying Vieta's formulae we can rewrite (* GS-1 *) and (GS-2 *) in the following way:

$$y_{k+2} = t_1 \cdot y_{k+1} \qquad \text{(* GS-1.1 *)}$$

$$y_{k+2} = t_2 \cdot y_{k+1} \qquad \text{(* GS-1.2 *)}$$

$$y_{k+1} = t_1 \cdot y_k \qquad \text{(* GS-2.1 *)}$$

$$y_{k+1} = t_2 \cdot y_k \qquad \text{(* GS-2.2 *)}$$

By pure combinatorics, substituting two roots into two equations formally gives four cases, but by algebra (Vieta) these melt down to two essentially different cases.

7. Results

Theorem (solutions of the generalized Fibonacci equation)
(cf. [*DZ 1989*], Satz 13.1, page 90)
The generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

has the following "solutions" (i.e. closed form representations):

(a1) $$y_k = \left(\frac{-a_1 + \sqrt{a_1^2 - 4\,a_0}}{2}\right)^k$$

and

(a2) $$y_k = \left(\frac{-a_1 - \sqrt{a_1^2 - 4\,a_0}}{2}\right)^k$$

■ 8. Obtaining more solutions

Theorem (combining solutions)
If the sequences

$(u_k)_{k=0,\ldots,\infty}$ and $(v_k)_{k=0,\ldots,\infty}$

are solutions of the generalized Fibonacci equation

$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$

then their "sum"

(a1) $(u_k)_{k=0,\ldots,\infty} \oplus (v_k)_{k=0,\ldots,\infty} \;:=\; (u_k + v_k)_{k=0,\ldots,\infty}$

and for any real number C the "scalar multiple"

(a2) $C \odot (u_k)_{k=0,\ldots,\infty} \;:=\; (C \cdot u_k)_{k=0,\ldots,\infty}$

also are solutions of the generalized Fibonacci equation.

Henceforth we will use the simpler operation symbols $+$ and $\cdot$ instead of $\oplus$ and $\odot$.

Corollary (linear combination of solutions):
If the sequences

$(u_k)_{k=0,\ldots,\infty}$ and $(v_k)_{k=0,\ldots,\infty}$

are solutions of the generalized Fibonacci equation

$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$

then for any real numbers C_1 and C_2 the "linear combination"

$C_1 \cdot (u_k)_{k=0,\ldots,\infty} + C_2 \cdot (v_k)_{k=0,\ldots,\infty}$

also are solutions of the generalized Fibonacci equation.

Corollary
The generalized Fibonacci equation

$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$

has the following "solutions" (i.e. closed form representations):

$$C_1 \cdot \left(\frac{-a_1 + \sqrt{a_1{}^2 - 4\, a_0}}{2} \right)^k + C_2 \cdot \left(\frac{-a_1 - \sqrt{a_1{}^2 - 4\, a_0}}{2} \right)^k \qquad \text{(* S-GFE *)}$$

where C_1 and C_2 are arbitrary real (or complex) numbers.

In other words: The set of all solutions of the generalized Fibonacci equation is a *vector space* over a suitable scalar field (usually the field the coefficients are taken from); cf. [*DZ 1989*], Satz 14.1, page 91.
Furthermore, it is not difficult to see that the dimension of this vector space is 2 and that, in case the solutions t_1 and t_2 of GFE's characteristic equation do not coincide, then the sequences $(t_1)^k$ and $(t_2)^k$ are a basis of this vector space.

■ 9. Initial values

The above results were valid independent of any initial values y_0 and y_1 of the GFE. Let us now assume that the solution (* S-GFE *), additionally, is to satisfy these initial values. Then for $k = 0$ and $k = 1$ the following two linear equations will have to be satisfied by C_1 and C_2:

Corollary
The initial values of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

can be expressed in the following way:

$$C_1 \cdot \left(\frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2}\right)^0 + C_2 \cdot \left(\frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2}\right)^0 = y_0 \qquad \text{(* LE-1 *)}$$

$$C_1 \cdot \left(\frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2}\right)^1 + C_2 \cdot \left(\frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2}\right)^1 = y_1 \qquad \text{(* LE-2 *)}$$

i.e.

$$C_1 + C_2 = y_0$$

$$C_1 \cdot \frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2} + C_2 \cdot \frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2} = y_1$$

Notwithstanding the algebraic complexity of these equations, they are two simple linear equations in the two unknowns C_1 and C_2 which can be solved by straightforward algebraic procedures.

Exercise: Show that

$$C_1 = -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

and

$$C_2 = -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

are solutions of (* LE-1 *) and (* LE-2 *).

■ 10. Applying the results to the sequence of the standard Fibonacci numbers - Binet's formula

The sequence of the standard Fibonacci numbers, equivalently either starting with index 0 or index 1 is given by

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	...
0	1	1	2	3	5	8	13	...

Specializing from GFE, its parameters are:

$a_1 = -1$ and $a_0 = -1$.

Hence, the homogeneous equation

$$y_{k+2} - y_{k+1} - y_k = 0$$

has the "general" solution

$$y_k = C_1 \cdot \left(\frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2}\right)^k + C_2 \cdot \left(\frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2}\right)^k$$

i.e.

$$y_k = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^k + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^k$$

$$\left\{\left\{C_1 \to -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}},\; C_2 \to -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}\right\}\right\}$$

$$\mathbf{C_1 = -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}\quad /.\ \{a_0 \to -1,\ a_1 \to -1,\ y_0 \to 0,\ y_1 \to 1\}}$$

$$\frac{1}{\sqrt{5}}$$

$$\mathbf{C_2 = -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}\quad /.\ \{a_0 \to -1,\ a_1 \to -1,\ y_0 \to 0,\ y_1 \to 1\}}$$

$$-\frac{1}{\sqrt{5}}$$

■ 11. Binet's Formula

Theorem (Binet)

The Fibonacci equation

$$y_{k+2} = y_{k+1} + y_k$$

has the following "solution" (i.e. closed form representations):

$$y_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

■ 12. Check

$$\mathbf{Table\left[Simplify\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k\right],\ \{k,\ 0,\ 30\}\right]}$$

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{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181,
 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040}
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