# Difference Equations

(Differenzengleichungen)

- Part 1 -

# Introduction

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## ■ References

*Dürr R. / J. Ziegenbalg*: Mathematik für Computeranwendungen: Dynamische Prozesse und ihre Mathematisierung durch Differenzengleichungen, Schöningh Verlag, Paderborn 1989
*Second edition:* Mathematik für Computeranwendungen; Ferdinand Schöningh Verlag, Paderborn 1989

*Goldberg S.*: Introduction to Difference Equations; John Wiley, New York 1958

*Rommelfanger H.*: Differenzen- und Differentialgleichungen; B.I., Zürich 1977

*J. Ziegenbalg*: Figurierte Zahlen; Springer-Spektrum, Wiesbaden 2018

## ■ Introduction

Let $(y_k)_{k=0,1,\ldots,\infty}$ be an arbitrary sequence of numbers. An equation establishing a relation between any successive $n+1$ values $y_k, y_{k+1}, \ldots, y_{k+n}$ of the sequence $(y_k)$ is called a *difference equation* of *order n*.

### ■ First examples

#### ■ 1. Arithmetic sequence (*German*: arithmetische Folge)

Let $d$ be an arbitrary real number. A sequence satisfying the difference equation (of oder 1)

$$y_{k+1} = y_k + d \qquad (\text{for } k \in \mathbb{N})$$

is called an *arithmetic sequence* (or *arithmetic progression*).

## ■ 2. Geometric sequence (*German*: geometrische Folge)

Let $q$ be an arbitrary real number. A sequence satisfying the difference equation (of order 1)

$$y_{k+1} = q \cdot y_k \qquad \text{(for } k \in \mathbb{N}\text{)}$$

is called a *geometric sequence* (or *geometric progression*).

## ■ 3. Fibonacci numbers

The sequence satisfying the difference equation (of order 2)

$$y_{k+2} = y_{k+1} + y_k \qquad \text{(with initial conditions } y_0 = 0 \text{ and } y_1 = 1\text{)}$$

is called the sequence of *Fibonacci numbers*.

## ■ Some more classifications

The above three examples represented very special types of difference equations. In general, the defining relation of a difference equation is of the form

$$F(y_k,\ y_{k+1},\ \ldots,\ y_{k+n}) = 0 \qquad \text{(where } F \text{ is an arbitrary function of } n+1 \text{ arguments)}$$

If a difference equation is given in this form (which is the most general form representing a difference equation) it is said to be given in the *implicit form*.

If the difference equation is given in the form

$$y_{k+n} = f(y_k,\ y_{k+1},\ \ldots,\ y_{k+n-1}) \qquad \text{(where } f \text{ is an arbitrary function of } n \text{ arguments)}$$

it is said to be given in the *explicit form*. Difference equations given in the explicit form are also called *recursive equations* (or *recurrence relations*).

In the following definition we take the set $M$ to be a suitable set of numbers (usually $M = \mathbb{R}$ or $M = \mathbb{C}$).
Let $n$ be a (fixed) natural number and $f_i : \mathbb{N} \to M$ $(i = 0, ..., n)$ and $g : \mathbb{N} \to M$ arbitrary functions. Then a difference equation of the form

$$f_n(k) \cdot y_{k+n} + f_{n-1}(k) \cdot y_{k+n-1} + \ldots + f_2(k) \cdot y_{k+2} + f_1(k) \cdot y_{k+1} + f_0(k) \cdot y_k = g(k)$$

is called a *linear* difference equation (because the $y$-terms appear only in the first power).

If all of the functions $f_i$ $(i = 0, \ldots, n)$ are constant, e.g.

$$f_i(k) = a_i \qquad (i = 0, \ldots, n)$$

then the difference equation

$$a_n \cdot y_{k+n} + a_{n-1} \cdot y_{k+n-1} + \ldots + a_2 \cdot y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = g(k)$$

is called a *linear difference equation with constant coefficients*.

In the last examples the function $g$ is called the *inhomogeneity function* (German: Inhomogenität).

If the inhomogeneity function is constant, e.g.

$$g(k) = b$$

then the difference equation

$$a_n \cdot y_{k+n} + a_{n-1} \cdot y_{k+n-1} + ... + a_2 \cdot y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = b$$

is called a *linear difference equation with constant coefficients and constant inhonogeneity*.

If, furthermore, $b = 0$, the difference equation is called a *homogeneous difference equation*.

*Example*: The "Fibonacci" difference equation

$$y_{k+2} - y_{k+1} - y_k = 0$$

can thus be classified as a homogeneous linear difference equation of oder 2 with constant coefficients (the coefficients being 1, -1 and -1). As is shown in Example 3 (above) it can easily be presented in an explicit form.

## ■ First examples in *Mathematica*

## ■ Arithmetic sequence

```
ArithmeticSequence[y0_, d_, k_] :=
  (For[(y = y0; i = 0), i < k, i = i + 1, y = y + d]; Return[y])
```

```
ArithmeticSequence[1, 3, 5]
```

```
16
```

```
ArithmeticSequence[1, 2, 15]
```

```
31
```

```
Table[ArithmeticSequence[1, 2, i], {i, 0, 15}]
```

```
{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31}
```

```
ArithmeticSequence[y0, d, 15]
```

```
15 d + y0
```

```
Table[ArithmeticSequence[y0, d, i], {i, 0, 15}]
```

```
{y0, d + y0, 2 d + y0, 3 d + y0, 4 d + y0, 5 d + y0, 6 d + y0, 7 d + y0,
 8 d + y0, 9 d + y0, 10 d + y0, 11 d + y0, 12 d + y0, 13 d + y0, 14 d + y0, 15 d + y0}
```

## ■ Geometric sequence

```
GeometricSequence[y0_, q_, k_] :=
 (For[(y = y0; i = 0), i < k, i = i + 1, y = q * y]; Return[y])
```

```
GeometricSequence[1, 2, 15]
```

```
32768
```

```
2^15
```

```
32768
```

```
Table[GeometricSequence[1, 2, i], {i, 0, 25}]
```

```
{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536,
 131072, 262144, 524288, 1048576, 2097152, 4194304, 8388608, 16777216, 33554432}
```

```
GeometricSequence[y0, q, 15]
```

$q^{15}$ y0

```
Table[GeometricSequence[1, q, i], {i, 0, 15}]
```

$\{1, q, q^2, q^3, q^4, q^5, q^6, q^7, q^8, q^9, q^{10}, q^{11}, q^{12}, q^{13}, q^{14}, q^{15}\}$

## ■ Fibonacci numbers

```
FibonacciSequence[k_] :=
 (For[(y0 = 0; y1 = 1; y = 0; i = 0), i < k, i = i + 1, y0 = y1; y1 = y; y = y0 + y1]; Return[y])
```

```
FibonacciSequence[5]
```

```
5
```

```
Table[FibonacciSequence[i], {i, 0, 15}]
```

```
{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610}
```

```
Table[{i, FibonacciSequence[i]}, {i, 0, 15}] // TableForm
```

| | |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
| 5 | 5 |
| 6 | 8 |
| 7 | 13 |
| 8 | 21 |
| 9 | 34 |
| 10 | 55 |
| 11 | 89 |
| 12 | 144 |
| 13 | 233 |
| 14 | 377 |
| 15 | 610 |

**Exercise**: Implement the above *Mathematica* functions ArithmeticSequence, GeometricSequence and Fibonacci-Sequence by only using the Module and While constructs of *Mathematica*.

# ■ The annuity equation

By combining the difference equations leading to the arithmetic and the geometric sequence in a straightforward way we get a sequence of the following type:

$$y_{k+1} = q \cdot y_k + d$$

It is of the form

$$a_1 \cdot y_{k+1} + a_0 \cdot y_k = b \qquad \text{(where } a_0, a_1 \text{ and } b \text{ are fixed constants; } a_1 \neq 0)$$

Hence it is a linear inhomogeneous difference equation of order 1 with constant coefficients and constant inhomogeneity.
In what follows we will rather call it the *annuity equation* (*German*: Tilgungsgleichung), for short and write it in the form

$$y_{k+1} = A \cdot y_k + B \qquad \text{(where } A = -\frac{a_0}{a_1} \text{ and } B = \frac{b}{a_1})$$

## ■ Implementation in *Mathematica*

```
Annuity[y0_, A_, B_, k_] :=
 Module[{y = y0, i = 0},
  While [i < k,
   i = i + 1;
   y = A * y + B];
  Return[y] ]
```

```
Annuity[100000 , 1.05, -10000, 4]
```

```
78449.4
```

```
Table[{i, Annuity[100000, 1.05, -10000, i]}, {i, 1, 15}]
```

```
{{1, 95000.}, {2, 89750.}, {3, 84237.5}, {4, 78449.4}, {5, 72371.8},
 {6, 65990.4}, {7, 59290.}, {8, 52254.5}, {9, 44867.2}, {10, 37110.5},
 {11, 28966.1}, {12, 20414.4}, {13, 11435.1}, {14, 2006.84}, {15, -7892.82}}
```

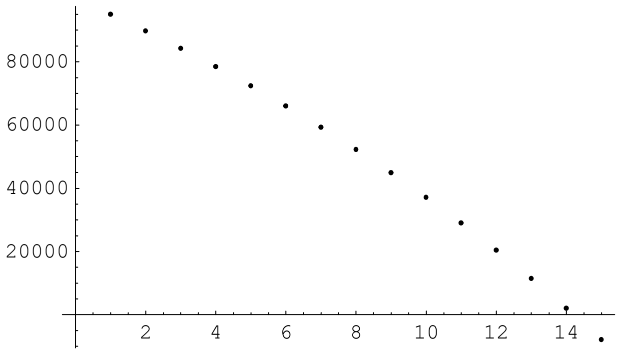
```
% // TableForm
```

| | |
|---|---|
| 1 | 95000. |
| 2 | 89750. |
| 3 | 84237.5 |
| 4 | 78449.4 |
| 5 | 72371.8 |
| 6 | 65990.4 |
| 7 | 59290. |
| 8 | 52254.5 |
| 9 | 44867.2 |
| 10 | 37110.5 |
| 11 | 28966.1 |
| 12 | 20414.4 |
| 13 | 11435.1 |
| 14 | 2006.84 |
| 15 | -7892.82 |

```
Table[{i, Annuity[100000, 1.05, -10000, i]}, {i, 1, 15}]
```

```
{{1, 95000.}, {2, 89750.}, {3, 84237.5}, {4, 78449.4}, {5, 72371.8},
 {6, 65990.4}, {7, 59290.}, {8, 52254.5}, {9, 44867.2}, {10, 37110.5},
 {11, 28966.1}, {12, 20414.4}, {13, 11435.1}, {14, 2006.84}, {15, -7892.82}}
```

```
% // ListPlot
```




- Graphics -

```
Remove[y0, y, A, B];
```

```
Table[{i, Annuity[y0, A, B, i]}, {i, 1, 10}] // TableForm
```

| | |
|---|---|
| 1 | B + A y0 |
| 2 | B + A (B + A y0) |
| 3 | B + A (B + A (B + A y0)) |
| 4 | B + A (B + A (B + A (B + A y0))) |
| 5 | B + A (B + A (B + A (B + A (B + A y0)))) |
| 6 | B + A (B + A (B + A (B + A (B + A (B + A y0))))) |
| 7 | B + A (B + A (B + A (B + A (B + A (B + A (B + A y0)))))) |
| 8 | B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A y0))))))) |
| 9 | B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A y0)))))))) |
| 10 | B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A y0))))))))) |

```
Table[{i, Annuity[y0, A, B, i]}, {i, 1, 10}] // Simplify // TableForm
```

| | |
|---|---|
| 1 | $B + A\, y0$ |
| 2 | $B + A\,(B + A\, y0)$ |
| 3 | $B + A\,(B + A\,(B + A\, y0))$ |
| 4 | $(1 + A + A^2 + A^3)\, B + A^4\, y0$ |
| 5 | $(1 + A + A^2 + A^3 + A^4)\, B + A^5\, y0$ |
| 6 | $(1 + A + A^2 + A^3 + A^4 + A^5)\, B + A^6\, y0$ |
| 7 | $(1 + A + A^2 + A^3 + A^4 + A^5 + A^6)\, B + A^7\, y0$ |
| 8 | $(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7)\, B + A^8\, y0$ |
| 9 | $(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8)\, B + A^9\, y0$ |
| 10 | $(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9)\, B + A^{10}\, y0$ |

**$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9)$ // Simplify**

$1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9$

**$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9) * (1 - A)$ // Simplify**

$1 - A^{10}$

**$(1 - A^{10}) / (1 - A)$ // Factor // Simplify**

$1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9$

**$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9) == (1 - A^{10}) / (1 - A)$ // Simplify**

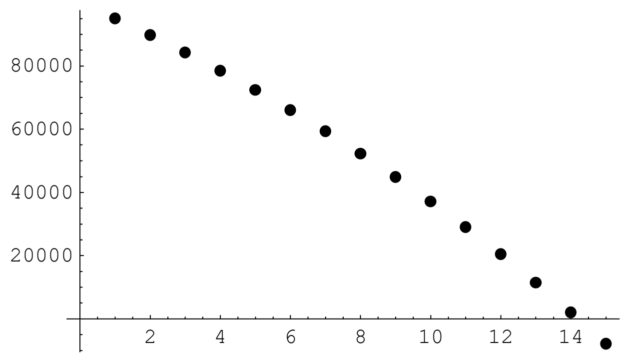
True

## ■ Graphical presentation of the annuity equation

More on the topic of graphical presentation will be covered in the *Mathematica* notebook "Difference-Equations-2-Cobweb.nb".

## ■ Timeline diagrams

```
ListPlot[Table[{i, Annuity[100000, 1.05, -10000, i]}, {i, 1, 15}],
 PlotStyle → PointSize[0.02]]
```




- Graphics -

## ■ Cobweb diagrams

```
AL = Table[Annuity[100000, 1.05, -10000, i], {i, 0, 15}]
```

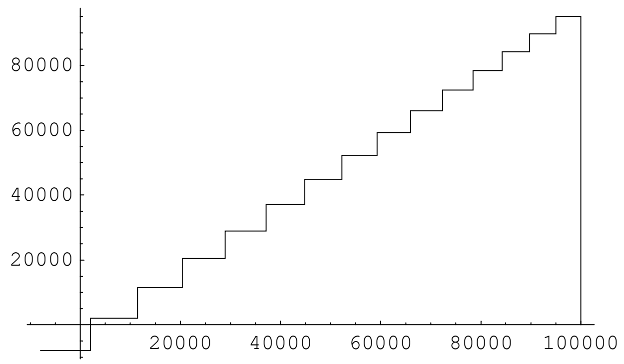
```
{100000, 95000., 89750., 84237.5, 78449.4, 72371.8, 65990.4, 59290.,
 52254.5, 44867.2, 37110.5, 28966.1, 20414.4, 11435.1, 2006.84, -7892.82}
```

```
CobList[AL_] :=        (*  AL for: Annuity List *)
 Module[{L = AL, CL = {}},
  (If[Length[L] > 0, CL = {{First[L], 0 }} ];
   While[
    Length[L] > 1,
    CL = Append[CL, {L[[1]], L[[2]]} ] ;
    CL = Append[CL, {L[[2]], L[[2]]} ] ;
    L = Delete[L, 1] ]);
  Return[CL] ]
```

```
CobList[AL]
```

```
{{100000, 0}, {100000, 95000.}, {95000., 95000.}, {95000., 89750.},
 {89750., 89750.}, {89750., 84237.5}, {84237.5, 84237.5}, {84237.5, 78449.4},
 {78449.4, 78449.4}, {78449.4, 72371.8}, {72371.8, 72371.8}, {72371.8, 65990.4},
 {65990.4, 65990.4}, {65990.4, 59290.}, {59290., 59290.}, {59290., 52254.5},
 {52254.5, 52254.5}, {52254.5, 44867.2}, {44867.2, 44867.2}, {44867.2, 37110.5},
 {37110.5, 37110.5}, {37110.5, 28966.1}, {28966.1, 28966.1}, {28966.1, 20414.4},
 {20414.4, 20414.4}, {20414.4, 11435.1}, {11435.1, 11435.1}, {11435.1, 2006.84},
 {2006.84, 2006.84}, {2006.84, -7892.82}, {-7892.82, -7892.82}}
```

```
ListPlot[CobList[AL], PlotJoined → True]
```



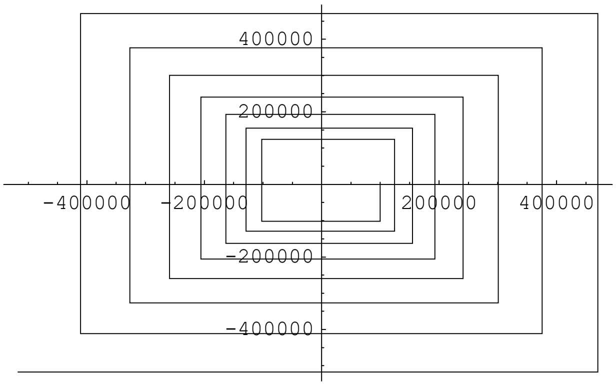

- Graphics -

## ■ Cobweb diagrams - some other parameter values

```
AL = Prepend[Table[Annuity[100000, -1.12, 10000, i], {i, 1, 15}], 100000]
```

```
{100000, -102000., 124240., -129149., 154647., -163204., 192789., -205923.,
 240634., -259510., 300652., -326730., 375937., -411050., 470376., -516821.}
```

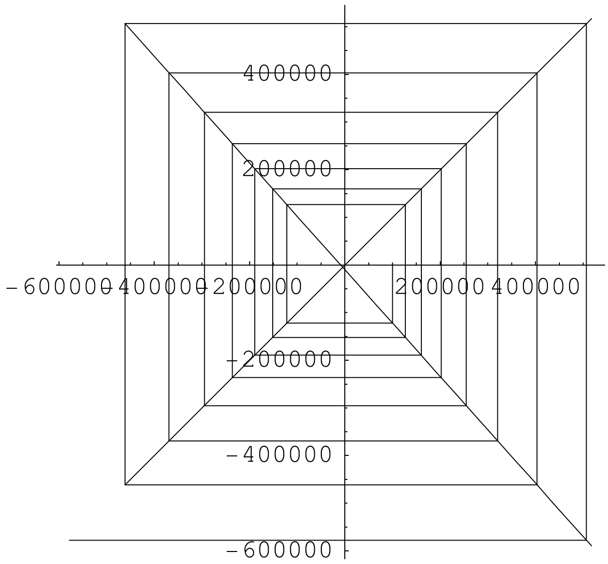
```
ListPlot[CobList[AL], PlotJoined → True]
```




- Graphics -

### ■ Cobweb diagrams - everthing bound together

If you draw the 45-degree-axis and the line given by $y = A \cdot x + B$ in this diagram, it becomes evident why these diagrams are called cobweb diagrams.




**Exercise**: Write a program for displaying "full-fledged" cobweb diagrams.

## ■ The generalized Fibonacci equation

In this section we will consider the following linear difference equation of order 2 with constant coefficients and constant inhomogeneity

$$a_2 \cdot y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = b \qquad \text{(where } a_0, a_1, a_2 \text{ and } b \text{ are fixed constants; } a_2 \neq 0)$$

This equation clearly generalizes the Fibonacci equation; we will, therefore, call it the *generalized Fibonacci equation* for short.

## ■ Auxiliary stuff