The Pythagorean Means

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■ References

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■ Introduction

The three "classic" means A (the *arithmetic mean*), G (the *geometric mean*), and H (the *harmonic mean*), in the sense of the definitions below, are sometimes known as the *Pythagorean means*.

■ 1. The arithmetic mean and arithmetic sequences

■ 1.1 The arithmetic mean (*German*: arithmetisches Mittel) of two magnitudes

Let a and b be any two magnitudes of the same type. The *arithmetic mean* of a and b is defined by

$$A(a, b) := \frac{a+b}{2} .$$

■ 1.2 Arithmetic sequences (*German*: arithmetische Folgen)

Let d be an arbitrary real number. A sequence satisfying the equation

$$x_{k+1} = x_k + d \qquad \text{(for } k \in \mathbb{N}\text{)}$$

is called an *arithmetic sequence* (or *arithmetic progression*).

■ 1.3 Arithmetic sequences and the arithmetic mean

Let $(x_k)_{k \in \mathbb{N}}$ be an arithmetic sequence. Then any member of the sequence is the arithmetic mean of its two immediate neighbours; more formally: for any $k > 1$ x_k is the arithmetic mean of x_{k-1} and x_{k+1}.

Proof: Exercise (straightforward)

■ 1.4 The arithmetic mean in *Mathematica*

```
ArithmeticMean[a_, b_] := (a + b)/2
```

```
ArithmeticMean[2, 3]
```

$\frac{5}{2}$

```
% // N
```

2.5

■ 1.5 The arithmetic mean: A typical application

The association of mean means has 20000 members in January 2005. In January 2006 200 new members and in January 2007 300 new members join the association. What is the average increase in these two years?

```
ArithmeticMean[200, 300]
```

250

Plausibility check: The number of members after the two years is:

1. As it happened:

```
20000 + 200 + 300
```

20500

2. By applying the arithmetic mean as average value:

```
20000 + 250 + 250
```

20500

Exercise: Show that in general (like in the previous example) the arithmetic mean provides for the correct growth process.

Another example (demonstrating Mathematica's symbol manipulation facilities in connection with the task of computing with magnitudes)

Two rods of length *2 m* and *3 m* are given. Then the average length of the rods is given by

```
ArithmeticMean[ 2.2 m, 3.4 m]
```

```
2.8 m
```

■ 2. The geometric mean and geometric sequences

■ 2.1 The geometric mean (*German*: geometrisches Mittel) of two magnitudes

Let a and b be any two magnitudes of the same type. The *geometric mean* of a and b is defined by

$$G(a, b) := \sqrt{a \cdot b}\ .$$

■ 2.2 Geometric sequences (*German*: geometrische Folgen)

Let q be an arbitrary real number. A sequence satisfying the equation

$$x_{k+1} = x_k \cdot q \quad (\text{for } k \in \mathbb{N})$$

is called a *geometric sequence* (or *geometric progression*).

■ 2.3 Geometric sequences and the geometric mean

Let $(x_k)_{k \in \mathbb{N}}$ be a geometric sequence. Then any member of the sequence is the geometric mean of its two immediate neighbours; more formally: for any $k > 1$ x_k is the geometric mean of x_{k-1} and x_{k+1}.

Proof: Exercise (straightforward)

■ 2.4 The geometric mean in *Mathematica*

```
GeometricMean[a_, b_] := √(a b)
```

```
GeometricMean[2, 8]
```

```
4
```

■ 2.5 The geometric mean: A typical application

A car costs 20000 EUR in January 2005. The price was raised by 2.5 % in January 2006 and by 4 % in January 2007. What is the average price increase in these two years?

```
100 * ( GeometricMean[1 + 2.5 / 100, 1 + 4 / 100] - 1 )
```

```
3.24728
```

Plausibility check: The price after the two price increases is:

1. As it happened:

```
20000 * (1 + 2.5 / 100) * (1 + 4 / 100)
```

```
21320.
```

2. By applying the geometric mean as average value:

```
20000 * (1 + 3.24728 / 100) * (1 + 3.24728 / 100)
```

```
21320.
```

Exercise: Show that in general (like in the previous example) the geometric mean provides for the correct "growth rate".

■ 3. The harmonic mean and harmonic sequences

■ 3.1 The harmonic mean (*German*: harmonisches Mittel) of two magnitudes

Let a and b be any two magnitudes of the same type. The *harmonic mean* of a and b is defined by

$$H(a, b) := \frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}} .$$

In verbalized form: The harmonic mean of a and b is the reciprocal of the arithmetic mean of the reciprocals of a and b.

In short form: The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals.

Proposition: $H(a, b) := \frac{2\cdot a\cdot b}{a+b}$

Proof: Exercise (straightforward)

■ 3.2 The harmonic sequence (*German*: Die harmonische Folge)

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \ldots, \frac{1}{k}, \ldots$$

is called the *harmonic sequence* (or *harmonic progression*).

More generally: Let a and d be arbitrary real numbers. A sequence of the form

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2\,d}, \frac{1}{a+3\,d}, \ldots$$

is called a *harmonic sequence* (or *harmonic progression*).

■ 3.3 Harmonic sequences and the harmonic mean

Let $(x_k)_{k\in\mathbb{N}}$ be a harmonic sequence. Then any member of the sequence is the harmonic mean of its two immediate neighbours; more formally: for any $k > 1$ x_k is the harmonic mean of x_{k-1} and x_{k+1}.

Proof: Exercise (straightforward)

■ 3.4 The harmonic mean in *Mathematica*

```
HarmonicMean[a_, b_] := 2 a b / (a + b)
```

```
HarmonicMean[1 / 2, 1 / 4]
```

$\frac{1}{3}$

■ 3.5 The harmonic mean: A typical application

A car travels from A to B (lying 100 km apart) at a speed of 40 km/h and then back from B to A at a speed of 60 km/h. What is the average speed of the total trip?

```
HarmonicMean[40 km/h, 60 km/h]
```

$\frac{48\ \mathrm{km}}{\mathrm{h}}$

Plausibility check: The basic equation connecting distance, time and velocity is

$v = \frac{s}{t}$ and hence $t = \frac{s}{v}$

Thus, the total traveling time is:

1. As it happened: $t_1 + t_2 = \frac{s_1}{v_1} + \frac{s_2}{v_2}$

```
100 km / (40 km / h) + 100 km / (60 km / h)
```

$\frac{25\ \mathrm{h}}{6}$

2. By applying the harmonic mean as average value:

```
200 km / HarmonicMean[40 km/h, 60 km/h]
```

$\frac{25\ \mathrm{h}}{6}$

Exercise: Show that in general (like in the previous example) the harmonic mean provides for the correct total travelling time.

3. By applying the average value - the general case:

```
t1 = s1 / v1 ; t2 = s2 / v2 ;
```

```
t1 + t2
```

$\frac{s1}{v1} + \frac{s2}{v2}$

```
% // FullSimplify
```

$$\frac{s1}{v1}+\frac{s2}{v2}$$

```
t1 + t2   /. {s1 → s, s2 → s}
```

$$\frac{s}{v1}+\frac{s}{v2}$$

```
(s1 + s2) / HarmonicMean[v1, v2]   /. {s1 → s, s2 → s}
```

$$\frac{s\ (v1+v2)}{v1\ v2}$$

```
% // FullSimplify
```

$$\frac{s\ (v1+v2)}{v1\ v2}$$

```
t1 + t2 == (s1 + s2) / HarmonicMean[v1, v2]      /. {s1 → s, s2 → s}
```

$$\frac{s}{v1}+\frac{s}{v2}==\frac{s\ (v1+v2)}{v1\ v2}$$

```
% // FullSimplify
```

True

■ 4. The Pythagorean means put together

Proposition: $G=\sqrt{A\,H}$

Proof: Exercise (straightforward)

Proof by means of symbol manipulation in CAS systems:

```
A = ArithmeticMean[a, b]
```

$$\frac{1}{2}\ (0.475+b)$$

```
G = GeometricMean[a, b]
```

$$0.689202\ \sqrt{b}$$

```
H = HarmonicMean[a, b]
```

$$\frac{0.95\ b}{0.475+b}$$

$$\sqrt{A\ H}$$

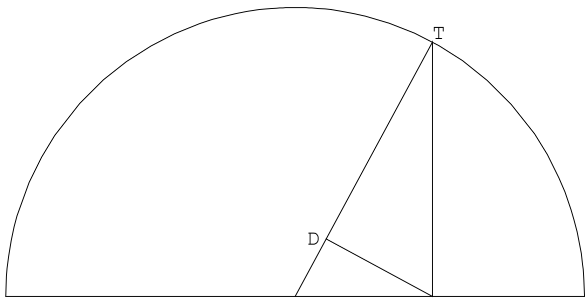
$$0.689202\ \sqrt{b}$$

$$G==\sqrt{A\ H}$$

True

The "Pythagoras diagram" of means

```
f[x_] = √(1 - x^2) ;
a = 0.475;
G1 = Plot[f[x], {x, -1, 1},
   AspectRatio → Automatic, Axes → False, DisplayFunction -> Identity];
G2 = Line[{{-1, 0}, {1, 0}}];
Points = {{0, 0}, {a, f[a]}, {a, 0} };
P2 = {x, y} /. Solve[{y == x * f[a] / a, y == x * (-a) / f[a] + a * a / f[a] }, {x, y}] ;
Points = Append[Points, First[P2]];
G3 = Line[Points];
G4 = Text["D  ", First[P2]];
G5 = Text[" T", {a, 0.03 + N[√(1 - a * a)]}];
Show[G1, Graphics[{G2, G3, G4, G5}], DisplayFunction → $DisplayFunction]
```



- Graphics -

Theorem: DT = HarmonicMean[a,b].

Proof: By the Pythagorean Theorem (in the version of the "Kathetensatz") we have $\mathrm{DT} * \frac{a+b}{2} = a * b$.

Corallary: For all $a, b \in \mathbb{R}$:

ArithmeticMean[a, b] ≥ GeometricMean[a, b] ≥ HarmonicMean[a, b]

with equality when $a = b$.

■ 5. Generalizations

■ 5.1 The arithmetic mean

The *arithmetic mean* of $a_1, \ldots, a_n$ is defined by

$$A(a_1, \ldots, a_n) := \frac{a_1 + \ldots + a_n}{n}.$$

■ 5.2 The geometric mean

The geometric *mean* of $a_1, \ldots, a_n$ is defined by

$$G(a_1, \ldots, a_n) := \sqrt[n]{a_1 \cdot \ldots \cdot a_n} .$$

■ 5.3 The harmonic mean

The *harmonic mean* of $a_1, \ldots, a_n$ is defined by

$$H(a_1, \ldots, a_n) := \frac{1}{\frac{\frac{1}{a_1} + \ldots + \frac{1}{a_n}}{n}}.$$

In verbalized form : The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals.

In other words:

$$\frac{1}{H(a_1, \ldots, a_n)} = \frac{1}{n} \cdot \left(\frac{1}{a_1} + \ldots + \frac{1}{a_n} \right).$$