Difference Equations

- Part 3 -

Representation in "closed form"
Formula solutions

Autor: Jochen Ziegenbalg
Email: ziegenbalg.edu@gmail.com
Internet: https://jochen-ziegenbalg.github.io/root/

■ References

Dürr R. / J. Ziegenbalg: Mathematik für Computeranwendungen: Dynamische Prozesse und ihre Mathematisierung durch Differenzengleichungen, Schöningh Verlag, Paderborn 1989
Second edition: Mathematik für Computeranwendungen; Ferdinand Schöningh Verlag, Paderborn 1989

Goldberg S.: Introduction to Difference Equations; John Wiley, New York 1958

Rommelfanger H.: Differenzen- und Differentialgleichungen; B.I., Zürich 1977

J. Ziegenbalg: Figurierte Zahlen; Springer-Spektrum, Wiesbaden 2018

■ The annuity equation

In the introduction, the *annuity equation* was defined as the difference equation

$$y_{k+1} = A \cdot y_k + B$$

■ Implementation in *Mathematica*

```
Annuity[y0_, A_, B_, k_] :=
 Module[{y = y0, i = 0},
  While [i < k,
   i = i + 1;
   y = A * y + B];
  Return[y] ]
```

```
Annuity[100000 , 1.05, -10000, 4]
```

```
78449.4
```

```
Table[{i, Annuity[100000, 1.05, -10000, i]}, {i, 1, 15}]
```

```
{{1, 95000.}, {2, 89750.}, {3, 84237.5}, {4, 78449.4}, {5, 72371.8},
 {6, 65990.4}, {7, 59290.}, {8, 52254.5}, {9, 44867.2}, {10, 37110.5},
 {11, 28966.1}, {12, 20414.4}, {13, 11435.1}, {14, 2006.84}, {15, -7892.82}}
```

```
% // TableForm
```

1	95000.
2	89750.
3	84237.5
4	78449.4
5	72371.8
6	65990.4
7	59290.
8	52254.5
9	44867.2
10	37110.5
11	28966.1
12	20414.4
13	11435.1
14	2006.84
15	-7892.82

```
Remove[y0, y, A, B];
Table[{i, Annuity[y0, A, B, i]}, {i, 1, 10}] // TableForm
```

1	B + A y0
2	B + A (B + A y0)
3	B + A (B + A (B + A y0))
4	B + A (B + A (B + A (B + A y0)))
5	B + A (B + A (B + A (B + A (B + A y0))))
6	B + A (B + A (B + A (B + A (B + A (B + A y0)))))
7	B + A (B + A (B + A (B + A (B + A (B + A (B + A y0))))))
8	B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A y0)))))))
9	B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A y0))))))))
10	B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A (B + A y0)))))))))

```
Table[{i, Annuity[y0, A, B, i]}, {i, 1, 10}] // Simplify // TableForm
```

1	B + A y0
2	B + A (B + A y0)
3	B + A (B + A (B + A y0))
4	$(1 + A + A^2 + A^3)\, B + A^4\, y0$
5	$(1 + A + A^2 + A^3 + A^4)\, B + A^5\, y0$
6	$(1 + A + A^2 + A^3 + A^4 + A^5)\, B + A^6\, y0$
7	$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6)\, B + A^7\, y0$
8	$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7)\, B + A^8\, y0$
9	$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8)\, B + A^9\, y0$
10	$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9)\, B + A^{10}\, y0$

$(1 + A + A^2 + A^3 + A^4 + A^5 + A^6 + A^7 + A^8 + A^9) * (1 - A)$ // Simplify

$1 - A^{10}$

$$\sum_{i=0}^{k} A^i$$

$$\frac{-1 + A^{1+k}}{-1 + A}$$

$$\left(\sum_{i=0}^{k} A^i\right) * (1 - A) \quad // \text{ Simplify}$$

$$1 - A^{1+k}$$

■ A more "difference equation" like style of notation

```
y[y0_, A_, B_, k_] :=
 Module[{yy = y0, i = 0},
  While [i < k,
   i = i + 1;
   yy = A * yy + B];
  Return[yy] ]
```

```
y[100000 , 1.05, -10000, 4]
```

```
78449.4
```

```
Table[{i, y[100000, 1.05, -10000, i]}, {i, 1, 15}]
```

```
{{1, 95000.}, {2, 89750.}, {3, 84237.5}, {4, 78449.4}, {5, 72371.8},
 {6, 65990.4}, {7, 59290.}, {8, 52254.5}, {9, 44867.2}, {10, 37110.5},
 {11, 28966.1}, {12, 20414.4}, {13, 11435.1}, {14, 2006.84}, {15, -7892.82}}
```

```
% // TableForm
```

1	95000.
2	89750.
3	84237.5
4	78449.4
5	72371.8
6	65990.4
7	59290.
8	52254.5
9	44867.2
10	37110.5
11	28966.1
12	20414.4
13	11435.1
14	2006.84
15	-7892.82

■ Representations / solutions in "closed form"

Theorem (Annuity equation in closed form - AECF)
(cf. [*Dürr / Ziegenbalg 1989*], Satz 7.1, page 48)
The term y_k of the annuity equation
$y_{k+1} = A \cdot y_k + B$
can be expressed in the following way
(a) $y_k = (1 + A + A^2 + A^3 + A^4 + ... + A^{k-1}) \cdot B + A^k \cdot y_0$

If $A \neq 1$ then
(a1) $y_k = \frac{A^k - 1}{A-1} \cdot B + A^k \cdot y_0$

If $A = 1$ then
(a2) $y_k = k \cdot B + A^k \cdot y_0$

Exercise: Prove the above theorem by formally applying (mathematical) induction.

Corollary
(a1) The term y_k of the *geometric sequence*
$y_{k+1} = A \cdot y_k$
can be expressed in the following way
$y_k = A^k \cdot y_0$
(a2) The term y_k of the *arithmetic sequence*
$y_{k+1} = y_k + B$
can be expressed in the following way
$y_k = k \cdot B + y_0$

■ Some remarks on methodology and terminology

In the literature on difference equations the above theorem (in particular variant a1) is often expressed in a wording according to which the closed form representation $y_k = \frac{A^k - 1}{A-1} \cdot B + A^k \cdot y_0$ is the "solution" of the difference equation $y_{k+1} = A \cdot y_k + B$.

It is, however, debatable, in which sense this is the case and for what purposes the closed form representation is more adequate than the original (recursive) version (cf. J. Ziegenbalg: "Formula versus Algorithm"; paper presented at the the conference **The Origins of Algebra: From al-Khwarizmi to Descartes**, Universitat Pompeu Fabra (UPF) and Institució Catalana de Recerca i Estudis Avançats (ICREA), Barcelona, March 27-29, 2003)

Aspects subject to discussion are:

1. Computational efficiency
2. Cognitive efficiency
3. Historical aspects

The representations (a1) and (a2) in theorem AECF are often called *representations in closed form* or *solutions in closed form*. The terminology "closed form" suggests that these representations are non-recursive. But at a closer look this turns out to be only part of the truth, because in the evaluation, for instance, of a term like A^k recursion comes into the game, again.

In spite of all this, in the following text the standard terminology is adopted and in particular the term $y_k = \frac{A^k-1}{A-1} \cdot B + A^k \cdot y_0$ will be called a "solution" of the difference equation $y_{k+1} = A \cdot y_k + B$.

Similarly, the term "solution" will also be used in this way for other (more general) difference equations.

■ Working with the *Mathematica* Package "Discrete Math"

The "add-on" package "Discrete Math" distributed with *Mathematica* contains the function RSolve (for "recurrence solve") by which solutions in closed form can conveniently be obtained for some types of recursive equations.

The next line shows how to load the RSolve function.

```
<< DiscreteMath`RSolve`
```

```
? RSolve
```

```
RSolve[eqn, a[n], n] solves a recurrence equation for a[n]. RSolve[{eqn1, eqn2, ... },
   {a1[n], a2[n], ... }, n] solves a system of recurrence equations. RSolve[eqn,
   a[n1, n2, ... ], {n1, n2, ... }] solves a partial recurrence equation. Mehr…
```

The next line shows an example of how to use the RSolve function.

```
RSolve[y[k + 1] == A * y[k] + B, y[k], k]
```

$$\left\{\left\{y[k] \to -\frac{(1 - A^k)\ B}{-1 + A} + A^{-1+k}\ C[1]\right\}\right\}$$

In the last expression C[1] is a constant which can be determined by adjusting the general form to initial values.
The next line shows an evaluation, resulting in y[0].

$$\text{ReplaceAll}\left[\frac{(-1 + A^k)\ B + (-1 + A)\ A^k\ C[1]}{-1 + A},\ k \to 0\right]$$

```
C[1]
```

The same call in a different syntax:

$$\frac{(-1 + A^k)\ B + (-1 + A)\ A^k\ C[1]}{-1 + A}\ /.\ k \to 0$$

```
C[1]
```

Above, we used the definition y[0] = y0. So the constant C[1] is just our initial value y0.

In the next example, the initial conditions are specified within RSolve.

```
RSolve[{y[k + 1] == A * y[k] + B, y[0] == y0}, y[k], k]
```

$$\left\{\left\{y[k] \to \frac{-B + A^k B - A^k y0 + A^{1+k} y0}{-1 + A}\right\}\right\}$$

■ The generalized Fibonacci equation

■ A remark on the historical development

In his *Liber abaci* (1202), Leonardo of Pisa (called *Fibonacci*, ca. 1170 - 1250) formulated a problem giving rise to the following famous sequence of numbers now called the "Fibonacci" numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Its most important property is that every member of the sequence is the sum of its two immediate predecessors (except for the initial values):

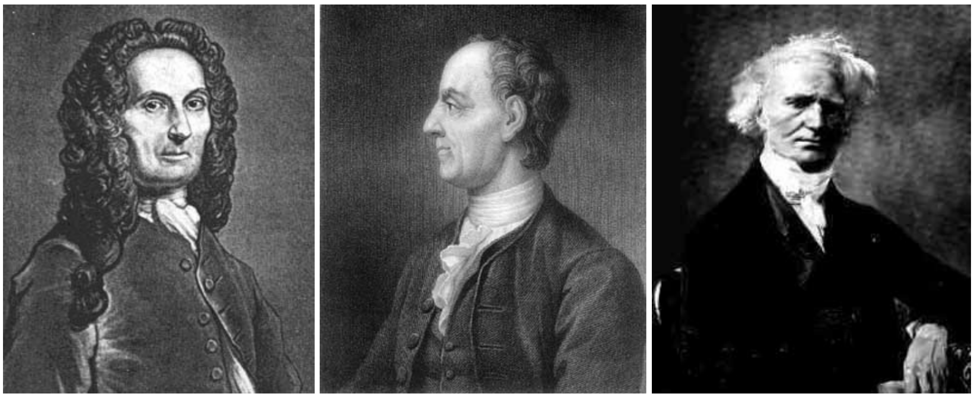
$$F_{k+2} = F_{k+1} + F_k$$

It took several centuries until J. P. M. Binet (1786-1856), based on results of L. Euler and A. de Moivre, finally presented the following formula for the Fibonacci numbers:

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k$$

The formula, today, is known as Binet's formula. Below, we will develop this formula within the framework of difference equations.

A. de Moivre (1667 – 1754) L. Euler (1707 – 1783) J. P. M. Binet (1786 – 1856)



In case that the reader is doubtful that the formula is correct (which is perfectly plausible, regarding the complexity of the formula and in particular the embedded root expressions), here is a preliminary test:

$$\text{Table}\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k, \{k, 1, 15\}\right]$$

$$\left\{-\frac{1-\sqrt{5}}{2\sqrt{5}}+\frac{1+\sqrt{5}}{2\sqrt{5}}, -\frac{(1-\sqrt{5})^2}{4\sqrt{5}}+\frac{(1+\sqrt{5})^2}{4\sqrt{5}}, -\frac{(1-\sqrt{5})^3}{8\sqrt{5}}+\frac{(1+\sqrt{5})^3}{8\sqrt{5}},\right.$$
$$-\frac{(1-\sqrt{5})^4}{16\sqrt{5}}+\frac{(1+\sqrt{5})^4}{16\sqrt{5}}, -\frac{(1-\sqrt{5})^5}{32\sqrt{5}}+\frac{(1+\sqrt{5})^5}{32\sqrt{5}}, -\frac{(1-\sqrt{5})^6}{64\sqrt{5}}+\frac{(1+\sqrt{5})^6}{64\sqrt{5}},$$
$$-\frac{(1-\sqrt{5})^7}{128\sqrt{5}}+\frac{(1+\sqrt{5})^7}{128\sqrt{5}}, -\frac{(1-\sqrt{5})^8}{256\sqrt{5}}+\frac{(1+\sqrt{5})^8}{256\sqrt{5}}, -\frac{(1-\sqrt{5})^9}{512\sqrt{5}}+\frac{(1+\sqrt{5})^9}{512\sqrt{5}},$$
$$-\frac{(1-\sqrt{5})^{10}}{1024\sqrt{5}}+\frac{(1+\sqrt{5})^{10}}{1024\sqrt{5}}, -\frac{(1-\sqrt{5})^{11}}{2048\sqrt{5}}+\frac{(1+\sqrt{5})^{11}}{2048\sqrt{5}}, -\frac{(1-\sqrt{5})^{12}}{4096\sqrt{5}}+\frac{(1+\sqrt{5})^{12}}{4096\sqrt{5}},$$
$$\left.-\frac{(1-\sqrt{5})^{13}}{8192\sqrt{5}}+\frac{(1+\sqrt{5})^{13}}{8192\sqrt{5}}, -\frac{(1-\sqrt{5})^{14}}{16384\sqrt{5}}+\frac{(1+\sqrt{5})^{14}}{16384\sqrt{5}}, -\frac{(1-\sqrt{5})^{15}}{32768\sqrt{5}}+\frac{(1+\sqrt{5})^{15}}{32768\sqrt{5}}\right\}$$

$$\text{Table}\left[\text{Simplify}\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k\right], \{k, 1, 15\}\right]$$

```
{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610}
```

$$\text{Table}\left[\text{N}\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k\right], \{k, 1, 15\}\right]$$

```
{1., 1., 2., 3., 5., 8., 13., 21., 34., 55., 89., 144., 233., 377., 610.}
```

■ Terminology

In this section we will consider the following linear difference equation of oder 2 with constant coefficients and constant inhomogeneity

$$A_2 \cdot y_{k+2} + A_1 \cdot y_{k+1} + A_0 \cdot y_k = 0 \qquad \text{(where } A_0, A_1 \text{ and } A_2 \text{ are fixed constants; } A_2 \neq 0)$$

Since $A_2 \neq 0$ we can divide this equation by A_2 and obtain the somewhat simpler but equivalent form

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0 \qquad \text{(* GFE *)}$$

This equation clearly generalizes the Fibonacci equation; we will, therefore, call it the *generalized Fibonacci equation* (GFE) for short.

■ Applying standard methodology

Usually in mathematics (and elsewhere) it is more difficult to solve a generalized form of a specific problem than to solve the specific problem itself, for the solution of the generalized problem contains the solution of the specific problem. Sometimes, however, the solution of the more general problem turns out to be easier than the solution of the specific problem – thus supporting the dictum

"*be wise, generalize*".

Obtaining a closed form representation for the Fibonacci numbers is a striking illustration of this fact. It will turn out to be simpler and more natural to solve the generalized Fibonacci equation than the original (special) Fibonacci equation.

A general strategy in mathematical problem solving is to try to reduce a new, unknown situation to well-known cases. In this way, we will try to convert the generalized Fibonacci equation (of order 2) into two annuity equations (of order 1). If we succeed, we can hope combine the closed form representations of these annuity equations into a closed form representation of the generalized Fibonacci equation.

■ First attempt

The generalized Fibonacci equation might, for instance, be thought of as the "addition" of the following two first-order equations:

$$y_{k+2} + \frac{1}{2}\, a_1 \cdot y_{k+1} = 0$$

$$\frac{1}{2}\, a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

Exercise: Convert each of these two first-order equations into the standard form for geometric sequences and show that they, in general, have no common solution (i.e. no common closed form representation).

■ Second attempt: Introducing new parameters for greater flexibility (in this case: introducing a "tuning" parameter)

Due to this result we have to handle the decomposition in a slightly more subtle way by introducing an extra parameter called t (for "tuning") in the following way.

$$y_{k+2} + (a_1 + t) \cdot y_{k+1} = 0$$

$$-t \cdot y_{k+1} + a_0 \cdot y_k = 0$$

Still, GFE can be thought of as being the sum of these first-order equations. If, by choosing a suitable "tuning" value for t, we can make these two first-order equations identical, then they will have the same closed form representations and we can try to combine their individual solutions into a solution for the generalized Fibonacci equation.

Written in the "standard" form for geometric sequences the last two first-order equations read

■ Tuning process - with the goal to make the two equations identical

$$y_{k+2} = -(a_1 + t) \cdot y_{k+1} \qquad \text{(* GS-1 *)}$$

$$y_{k+1} = \frac{a_0}{t} \cdot y_k \qquad \text{(* GS-2 *)}$$

These difference equations for geometric sequences are identical if their coefficients $-(a_1 + t)$ and $\frac{a_0}{t}$ are equal. (The "index-shift" by 1 is irrelevant, since the equations are valid for all values of k).
A necessary condition for equality, hence, is

$$-(a_1 + t) = \frac{a_0}{t}$$

i.e.

$$t^2 + a_1 \cdot t + a_0 = 0$$

■ The characteristic polynomial

Thus, the above geometric sequences are identical if the "tuning" parameter t satisfies the so-called *characteristic equation* of (* GFE *):

$$x^2 + a_1 \cdot x + a_0 = 0$$

We finally obtain the tuning parameters

$$t_1 = \frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}$$

$$t_2 = \frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}$$

In *Mathematica* notation:

```
Solve[t^2 + a1 * t + a0 == 0, t]
```

$$\left\{\left\{t \to \frac{1}{2}\left(-a1 - \sqrt{-4\ a0 + a1^2}\right)\right\}, \left\{t \to \frac{1}{2}\left(-a1 + \sqrt{-4\ a0 + a1^2}\right)\right\}\right\}$$

■ Solutions - by applying the results on geometric series

Thus, the geometric sequences adding up to (* GFE *) are

1. By using the root t_1:

$$y_{k+2} = -\left(a_1 + \frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}} \cdot y_k$$

2. By using the root t_2:

$$y_{k+2} = -\left(a_1 + \frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}} \cdot y_k$$

Exercise: Simplify these formulae.

■ Vieta's formulae

Excursion: Vieta's formulae

The tuning parameters t_1 and t_2, being the roots of the quadratic equation

$$x^2 + a_1 \cdot x + a_0 = 0$$

must satisfy Vieta's equations:

$$t_1 + t_2 = -a_1 \quad \text{and} \quad t_1 \cdot t_2 = a_0$$

Thus, by substituting either t_1 or t_2 for t and applying Vieta's formulae we can rewrite (* GS-1 *) and (GS-2 *) in the following way:

$$y_{k+2} = t_1 \cdot y_{k+1} \qquad \text{(* GS-1.1 *)}$$

$$y_{k+2} = t_2 \cdot y_{k+1} \qquad \text{(* GS-1.2 *)}$$

$$y_{k+1} = t_1 \cdot y_k \qquad \text{(* GS-2.1 *)}$$

$$y_{k+1} = t_2 \cdot y_k \qquad \text{(* GS-2.2 *)}$$

By pure combinatorics, substituting two roots into two equations formally gives four cases, but by algebra (Vieta) these melt down to two essentially different cases.

■ Results

Theorem (solutions of the generalized Fibonacci equation)
(cf. [*Dürr / Ziegenbalg 1989*], Satz 13.1, page 90)
The generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

has the following "solutions" (i.e. closed form representations):

(a1) $$y_k = \left(\frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}\right)^k$$

and

(a2) $$y_k = \left(\frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}\right)^k$$

Obtaining more solutions:

Theorem (combining solutions)
If the sequences

$$(u_k)_{k=0,\ldots,\infty} \quad \text{and} \quad (v_k)_{k=0,\ldots,\infty}$$

are solutions of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

then their "sum"

(a1) $$(u_k)_{k=0,\ldots,\infty} \oplus (v_k)_{k=0,\ldots,\infty} \;:=\; (u_k + v_k)_{k=0,\ldots,\infty}$$

and for any real number C the "scalar multiple"
(a2) $C \odot (u_k)_{k=0,\ldots,\infty} \;:=\; (C \cdot u_k)_{k=0,\ldots,\infty}$
also are solutions of the generalized Fibonacci equation.

Proof: Exercise

Henceforth we will use the simpler operation symbols $+$ and $\cdot$ instead of $\oplus$ and $\odot$.

Corollary (linear combination of solutions):
If the sequences

$$(u_k)_{k=0,\ldots,\infty} \text{ and } (v_k)_{k=0,\ldots,\infty}$$

are solutions of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

then for any real numbers C_1 and C_2 the "linear combination"

$$C_1 \cdot (u_k)_{k=0,\ldots,\infty} + C_2 \cdot (v_k)_{k=0,\ldots,\infty}$$

also are solutions of the generalized Fibonacci equation.

Corollary
The generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

has the following "solutions" (i.e. closed form representations):

$$C_1 \cdot \left(\frac{-a_1 + \sqrt{a_1^2 - 4 a_0}}{2} \right)^k + C_2 \cdot \left(\frac{-a_1 - \sqrt{a_1^2 - 4 a_0}}{2} \right)^k \qquad \text{(* S-GFE *)}$$

where C_1 and C_2 are arbitrary real (or complex) numbers.

In other words: The set of all solutions of the generalized Fibonacci equation is a *vector space* over a suitable scalar field (i.e. usually the field the coefficients are taken from); cf. [*Dürr / Ziegenbalg 1989*], Satz 14.1, page 91. Furthermore, it is not difficult to see that the dimension of this vector space is 2 and that, in case the solutions t_1 and t_2 of GFE's characteristic equation do not coincide, then the sequences $(t_1)^k$ and $(t_2)^k$ are a basis of this vector space.

■ Initial values

The above results were valid independent of any initial values y_0 and y_1 of the GFE. Let us now assume that the solution (* S-GFE *), additionally, is to satisfy these initial values. Then for $k = 0$ and $k = 1$ the following two linear equations will have to be satisfied by C_1 and C_2:

Corollary
The initial values of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

can be expressed in the following way:

$$C_1 \cdot \left(\frac{-a_1 + \sqrt{a_1^2 - 4 a_0}}{2} \right)^0 + C_2 \cdot \left(\frac{-a_1 - \sqrt{a_1^2 - 4 a_0}}{2} \right)^0 = y_0 \qquad \text{(* LE-1 *)}$$

$$C_1 \cdot \left(\frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2} \right)^1 + C_2 \cdot \left(\frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2} \right)^1 = y_1 \qquad \text{(* LE-2 *)}$$

i.e.

$$C_1 + C_2 = y_0$$

$$C_1 \cdot \frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2} + C_2 \cdot \frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2} = y_1$$

Notwithstanding the algebraic complexity of these equations, they are two simple linear equations in the two unknowns C_1 and C_2 which can be solved by straightforward algebraic procedures.

Exercise: Show that

$$C_1 = -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

and

$$C_2 = -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

are solutions of (* LE-1 *) and (* LE-2 *).

■ Applying the results to the sequence of the standard Fibonacci numbers - Binet's formula

The sequence of the standard Fibonacci numbers, equivalently either starting with index 0 or index 1 is given by

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	...
0	1	1	2	3	5	8	13	...

Specializing from GFE, its parameters are:

$a_1 = -1$ and $a_0 = -1$.

Hence, the homogeneous equation

$$y_{k+2} - y_{k+1} - y_k = 0$$

has the "general" solution

$$y_k = C_1 \cdot \left(\frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2} \right)^k + C_2 \cdot \left(\frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2} \right)^k$$

i.e.

$$y_k = C_1 \cdot \left(\frac{1+\sqrt{5}}{2} \right)^k + C_2 \cdot \left(\frac{1-\sqrt{5}}{2} \right)^k$$

■ Putting it all together using *Mathematica*'s "symbol manipulation" features

$$\text{Solve}\left[\left\{C_1 + C_2 == y_0,\ C_1 * \left(\frac{-a_1 + \sqrt{a_1{}^2 - 4\, a_0}}{2}\right) + C_2 * \left(\frac{-a_1 - \sqrt{a_1{}^2 - 4\, a_0}}{2}\right) == y_1\right\},\ \{C_1, C_2\}\right]$$

$$\text{Solve}\left[\left\{0 == y_0,\ -\frac{-a_1 - \sqrt{-4\, a_0 + a_1^2}}{2\sqrt{5}} + \frac{-a_1 + \sqrt{-4\, a_0 + a_1^2}}{2\sqrt{5}} == y_1\right\},\ \left\{\frac{1}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right\}\right]$$

$$C_1 = -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

$$-\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

$$C_1 = -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}} \quad /.\ \{a_0 \to -1,\ a_1 \to -1,\ y_0 \to 0,\ y_1 \to 1\}$$

$$\frac{1}{\sqrt{5}}$$

$$C_2 = -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

$$-\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}$$

$$C_2 = -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}} \quad /.\ \{a_0 \to -1,\ a_1 \to -1,\ y_0 \to 0,\ y_1 \to 1\}$$

$$-\frac{1}{\sqrt{5}}$$

■ Binet's Formula

Theorem (Binet)
The Fibonacci equation

$$y_{k+2} = y_{k+1} + y_k$$

has the following "solution" (i.e. closed form representations):

$$y_k = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^k$$

Check

$$y_k = \frac{1}{\sqrt{5}} * \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} * \left(\frac{1-\sqrt{5}}{2}\right)^k$$

$$-\frac{\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^k}{\sqrt{5}} + \frac{\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^k}{\sqrt{5}}$$

$$\text{Table}\left[\left\{k, \frac{1}{\sqrt{5}} * \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} * \left(\frac{1-\sqrt{5}}{2}\right)^k\right\}, \{k, 0, 5\}\right]$$

$$\left\{\{0, 0\}, \left\{1, -\frac{1-\sqrt{5}}{2\sqrt{5}} + \frac{1+\sqrt{5}}{2\sqrt{5}}\right\}, \left\{2, -\frac{\left(1-\sqrt{5}\right)^2}{4\sqrt{5}} + \frac{\left(1+\sqrt{5}\right)^2}{4\sqrt{5}}\right\}, \left\{3, -\frac{\left(1-\sqrt{5}\right)^3}{8\sqrt{5}} + \frac{\left(1+\sqrt{5}\right)^3}{8\sqrt{5}}\right\}, \left\{4, -\frac{\left(1-\sqrt{5}\right)^4}{16\sqrt{5}} + \frac{\left(1+\sqrt{5}\right)^4}{16\sqrt{5}}\right\}, \left\{5, -\frac{\left(1-\sqrt{5}\right)^5}{32\sqrt{5}} + \frac{\left(1+\sqrt{5}\right)^5}{32\sqrt{5}}\right\}\right\}$$

$$\text{Table}\left[\text{Simplify}\left[\left\{k, \frac{1}{\sqrt{5}} * \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} * \left(\frac{1-\sqrt{5}}{2}\right)^k\right\}\right], \{k, 0, 5\}\right]$$

```
{{0, 0}, {1, 1}, {2, 1}, {3, 2}, {4, 3}, {5, 5}}
```

$$\text{Table}\left[\left\{k, \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k \text{ // } N\right\}, \{k, 0, 25\}\right]$$

```
{{0, 0.}, {1, 1.}, {2, 1.}, {3, 2.}, {4, 3.}, {5, 5.}, {6, 8.}, {7, 13.},
 {8, 21.}, {9, 34.}, {10, 55.}, {11, 89.}, {12, 144.}, {13, 233.}, {14, 377.},
 {15, 610.}, {16, 987.}, {17, 1597.}, {18, 2584.}, {19, 4181.}, {20, 6765.},
 {21, 10946.}, {22, 17711.}, {23, 28657.}, {24, 46368.}, {25, 75025.}}
```

```
% // TableForm
```

0	0.
1	1.
2	1.
3	2.
4	3.
5	5.
6	8.
7	13.
8	21.
9	34.
10	55.
11	89.
12	144.
13	233.
14	377.
15	610.
16	987.
17	1597.
18	2584.
19	4181.
20	6765.
21	10946.
22	17711.
23	28657.
24	46368.
25	75025.

■ Applying genuine computeralgebra features

$$\text{Table}\left[\left\{k,\ \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k\right\},\ \{k,\ 0,\ 10\}\right]$$

$$\Big\{\{0,0\},\ \Big\{1,\ -\frac{1-\sqrt{5}}{2\sqrt{5}}+\frac{1+\sqrt{5}}{2\sqrt{5}}\Big\},\ \Big\{2,\ -\frac{(1-\sqrt{5})^2}{4\sqrt{5}}+\frac{(1+\sqrt{5})^2}{4\sqrt{5}}\Big\},$$
$$\Big\{3,\ -\frac{(1-\sqrt{5})^3}{8\sqrt{5}}+\frac{(1+\sqrt{5})^3}{8\sqrt{5}}\Big\},\ \Big\{4,\ -\frac{(1-\sqrt{5})^4}{16\sqrt{5}}+\frac{(1+\sqrt{5})^4}{16\sqrt{5}}\Big\},$$
$$\Big\{5,\ -\frac{(1-\sqrt{5})^5}{32\sqrt{5}}+\frac{(1+\sqrt{5})^5}{32\sqrt{5}}\Big\},\ \Big\{6,\ -\frac{(1-\sqrt{5})^6}{64\sqrt{5}}+\frac{(1+\sqrt{5})^6}{64\sqrt{5}}\Big\},$$
$$\Big\{7,\ -\frac{(1-\sqrt{5})^7}{128\sqrt{5}}+\frac{(1+\sqrt{5})^7}{128\sqrt{5}}\Big\},\ \Big\{8,\ -\frac{(1-\sqrt{5})^8}{256\sqrt{5}}+\frac{(1+\sqrt{5})^8}{256\sqrt{5}}\Big\},$$
$$\Big\{9,\ -\frac{(1-\sqrt{5})^9}{512\sqrt{5}}+\frac{(1+\sqrt{5})^9}{512\sqrt{5}}\Big\},\ \Big\{10,\ -\frac{(1-\sqrt{5})^{10}}{1024\sqrt{5}}+\frac{(1+\sqrt{5})^{10}}{1024\sqrt{5}}\Big\}\Big\}$$

```
% // Simplify
```

```
{{0, 0}, {1, 1}, {2, 1}, {3, 2}, {4, 3},
 {5, 5}, {6, 8}, {7, 13}, {8, 21}, {9, 34}, {10, 55}}
```

■ RSolve for GFE

```
Remove[y, y0, y1, a0, a1]
```

RSolving the GFE:

```
RSolve[{y[k + 2] + a1 * y[k + 1] + a0 * y[k] == 0, y[0] == y0, y[1] == y1}, y[k], k]
```

$$\left\{\left\{y[k] \to \frac{1}{\sqrt{-4\,a0 + a1^2}} \left(2^{-1-k} \left(-a1 \left(-a1 - \sqrt{-4\,a0 + a1^2}\right)^k y0 + \sqrt{-4\,a0 + a1^2} \left(-a1 - \sqrt{-4\,a0 + a1^2}\right)^k y0 + a1 \left(-a1 + \sqrt{-4\,a0 + a1^2}\right)^k y0 + \sqrt{-4\,a0 + a1^2} \left(-a1 + \sqrt{-4\,a0 + a1^2}\right)^k y0 - 2 \left(-a1 - \sqrt{-4\,a0 + a1^2}\right)^k y1 + 2 \left(-a1 + \sqrt{-4\,a0 + a1^2}\right)^k y1\right)\right)\right\}\right\}$$

RSolving the original Fibonacci equation:

```
RSolve[{y[k + 2] - y[k + 1] - y[k] == 0, y[0] == 0, y[1] == 1}, y[k], k]
```

$$\left\{\left\{y[k] \to -\frac{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^k - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^k}{\sqrt{5}}\right\}\right\}$$

■ Auxiliary stuff