Fibonacci Numbers and Binet's Formula

Roadmap (Überblick)

■ The generalized Fibonacci equation

■ 0. A remark on the historical development

In his *Liber abaci* (1202), Leonardo of Pisa (called *Fibonacci*, ca. 1170 - 1250) formulated a problem giving rise to the following famous sequence of numbers now called the "Fibonacci" numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Its most important property is that every member of the sequence is the sum of its two immediate predecessors (except for the initial values):

$$F_{k+2} = F_{k+1} + F_k$$

It took several centuries until J. P. M. Binet (1786-1856) finally presented the following formula for the Fibonacci numbers:

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k$$

■ 1. Generalisation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0 \qquad \text{(* GFE *)}$$

This equation clearly generalizes the Fibonacci equation; we will, therefore, call it the *generalized Fibonacci equation* (GFE) for short.

■ 2. Applying standard methodology: Introducing new parameters (in this case: a "tuning" parameter)

Due to this result we have to handle the decomposition in a slightly more subtle way by introducing an extra parameter called t (for "tuning") in the following way.

$$y_{k+2} + (a_1 + t) \cdot y_{k+1} = 0$$

$$-t \cdot y_{k+1} + a_0 \cdot y_k = 0$$

GFE can be thought of as being the sum of these first-order equations. If, by choosing a suitable "tuning" value for t, we can make these two first-order equations identical, then they will have the same closed form representations and we can try to combine their individual solutions into a solution for the generalized Fibonacci equation.

■ 3. The tuning process

Written in the "standard" form for geometric sequences the last two first-order equations read

$$y_{k+2} = -(a_1 + t) \cdot y_{k+1} \qquad \text{(* GS-1 *)}$$

$$y_{k+1} = \frac{a_0}{t} \cdot y_k \qquad \text{(* GS-2 *)}$$

These difference equations for geometric sequences are identical if their coefficients $-(a_1 + t)$ and $\frac{a_0}{t}$ are equal. (The "index-shift" by 1 is irrelevant, since the equations are valid for all values of k).
A necessary condition for equality, hence, is

$$-(a_1 + t) = \frac{a_0}{t}$$

■ 4. The characteristic polynomial

$$t^2 + a_1 \cdot t + a_0 = 0$$

Thus, the above geometric sequences are identical if the "tuning" parameter t satisfies the so-called *characteristic equation* of (* GFE *):

$$x^2 + a_1 \cdot x + a_0 = 0$$

We finally obtain the tuning parameters

$$t_1 = \frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}$$

$$t_2 = \frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}$$

■ 5. Solutions - by applying the results on geometric series

Thus, the geometric sequences adding up to (* GFE *) are

1. By using the root t_1:

$$y_{k+2} = -\left(a_1 + \frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}} \cdot y_k$$

2. By using the root t_2:

$$y_{k+2} = -\left(a_1 + \frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}} \cdot y_k$$

■ 6. Vieta's formulae

The tuning parameters t_1 and t_2, being the roots of the quadratic equation

$$x^2 + a_1 \cdot x + a_0 = 0$$

must satisfy Vieta's equations:

$$t_1 + t_2 = -a_1 \quad \text{and} \quad t_1 \cdot t_2 = a_0$$

Thus, by substituting either t_1 or t_2 for t and applying Vieta's formulae we can rewrite (* GS-1 *) and (GS-2 *) in the following way:

$$y_{k+2} = t_1 \cdot y_{k+1} \qquad \text{(* GS-1.1 *)}$$

$$y_{k+2} = t_2 \cdot y_{k+1} \qquad \text{(* GS-1.2 *)}$$

$$y_{k+1} = t_1 \cdot y_k \qquad \text{(* GS-2.1 *)}$$

$$y_{k+1} = t_2 \cdot y_k \qquad \text{(* GS-2.2 *)}$$

By pure combinatorics, substituting two roots into two equations formally gives four cases, but by algebra (Vieta) these melt down to two essentially different cases.

■ 7. Results

Theorem (solutions of the generalized Fibonacci equation)
(cf. [*DZ 1989*], Satz 13.1, page 90)
The generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

has the following "solutions" (i.e. closed form representations):

(a1) $\quad y_k = \left(\frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}\right)^k$

and

(a2) $\quad y_k = \left(\frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}\right)^k$

■ 8. Obtaining more solutions

Theorem (combining solutions)
If the sequences
$(u_k)_{k=0,\ldots,\infty}$ and $(v_k)_{k=0,\ldots,\infty}$
are solutions of the generalized Fibonacci equation
$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$
then their "sum"
(a1) $(u_k)_{k=0,\ldots,\infty} \oplus (v_k)_{k=0,\ldots,\infty} := (u_k + v_k)_{k=0,\ldots,\infty}$
and for any real number C the "scalar multiple"
(a2) $C \odot (u_k)_{k=0,\ldots,\infty} := (C \cdot u_k)_{k=0,\ldots,\infty}$
also are solutions of the generalized Fibonacci equation.

Henceforth we will use the simpler operation symbols $+$ and $\cdot$ instead of $\oplus$ and $\odot$.

Corollary (linear combination of solutions):
If the sequences
$(u_k)_{k=0,\ldots,\infty}$ and $(v_k)_{k=0,\ldots,\infty}$
are solutions of the generalized Fibonacci equation
$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$
then for any real numbers C_1 and C_2 the "linear combination"
$C_1 \cdot (u_k)_{k=0,\ldots,\infty} + C_2 \cdot (v_k)_{k=0,\ldots,\infty}$
also are solutions of the generalized Fibonacci equation.

Corollary
The generalized Fibonacci equation
$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$
has the following "solutions" (i.e. closed form representations):

$$C_1 \cdot \left(\frac{-a_1 + \sqrt{a_1{}^2 - 4\,a_0}}{2}\right)^k + C_2 \cdot \left(\frac{-a_1 - \sqrt{a_1{}^2 - 4\,a_0}}{2}\right)^k \qquad \text{(* S-GFE *)}$$

where C_1 and C_2 are arbitrary real (or complex) numbers.

In other words: The set of all solutions of the generalized Fibonacci equation is a *vector space* over a suitable scalar field (usually the field the coefficients are taken from); cf. [*DZ 1989*], Satz 14.1, page 91.
Furthermore, it is not difficult to see that the dimension of this vector space is 2 and that, in case the solutions t_1 and t_2 of GFE's characteristic equation do not coincide, then the sequences $(t_1)^k$ and $(t_2)^k$ are a basis of this vector space.

■ 9. Initial values

The above results were valid independent of any initial values y_0 and y_1 of the GFE. Let us now assume that the solution (* S-GFE *), additionally, is to satisfy these initial values. Then for $k = 0$ and $k = 1$ the following two linear equations will have to be satisfied by C_1 and C_2:

Corollary
The initial values of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

can be expressed in the following way:

$$C_1 \cdot \left(\frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2}\right)^0 + C_2 \cdot \left(\frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2}\right)^0 = y_0 \qquad \text{(* LE-1 *)}$$

$$C_1 \cdot \left(\frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2}\right)^1 + C_2 \cdot \left(\frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2}\right)^1 = y_1 \qquad \text{(* LE-2 *)}$$

i.e.

$$C_1 + C_2 = y_0$$

$$C_1 \cdot \frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2} + C_2 \cdot \frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2} = y_1$$

Notwithstanding the algebraic complexity of these equations, they are two simple linear equations in the two unknowns C_1 and C_2 which can be solved by straightforward algebraic procedures.

Exercise: Show that

$$C_1 = -\frac{-a_1\,y_0-\sqrt{-4\,a_0+a_1^2}\;y_0-2\,y_1}{2\,\sqrt{-4\,a_0+a_1^2}}$$

and

$$C_2 = -\frac{a_1\,y_0-\sqrt{-4\,a_0+a_1^2}\;y_0+2\,y_1}{2\,\sqrt{-4\,a_0+a_1^2}}$$

are solutions of (* LE-1 *) and (* LE-2 *).

■ 10. Applying the results to the sequence of the standard Fibonacci numbers - Binet's formula

The sequence of the standard Fibonacci numbers, equivalently either starting with index 0 or index 1 is given by

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	...
0	1	1	2	3	5	8	13	...

Specializing from GFE, its parameters are:

$$a_1 = -1 \quad \text{and} \quad a_0 = -1.$$

Hence, the homogeneous equation

$$y_{k+2} - y_{k+1} - y_k = 0$$

has the "general" solution

$$y_k = C_1 \cdot \left(\frac{-a_1+\sqrt{a_1{}^2-4\,a_0}}{2}\right)^k + C_2 \cdot \left(\frac{-a_1-\sqrt{a_1{}^2-4\,a_0}}{2}\right)^k$$

i.e.

$$y_k = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^k + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^k$$

$$\left\{\left\{C_1 \to -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}},\ C_2 \to -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}\right\}\right\}$$

$$\mathbf{C_1 = -\frac{-a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 - 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}\ \ /.\ \{a_0 \to -1,\ a_1 \to -1,\ y_0 \to 0,\ y_1 \to 1\}}$$

$$\frac{1}{\sqrt{5}}$$

$$\mathbf{C_2 = -\frac{a_1\, y_0 - \sqrt{-4\, a_0 + a_1^2}\; y_0 + 2\, y_1}{2\sqrt{-4\, a_0 + a_1^2}}\ \ /.\ \{a_0 \to -1,\ a_1 \to -1,\ y_0 \to 0,\ y_1 \to 1\}}$$

$$-\frac{1}{\sqrt{5}}$$

■ 11. Binet's Formula

Theorem (Binet)
The Fibonacci equation

$$y_{k+2} = y_{k+1} + y_k$$

has the following "solution" (i.e. closed form representations):

$$y_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

■ 12. Check

$$\mathbf{Table\left[Simplify\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k\right],\ \{k,\ 0,\ 30\}\right]}$$

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{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181,
 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040}
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