1. Solve the following system of linear equations:

$$\begin{cases} x - 2y + z = 2 \\ -2x + y - z = -1 \\ -x + y + z = 0 \end{cases}$$

Solution: $\frac{1}{5}\begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix}$

2. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix}$.

Solution: $A^{-1} = \begin{bmatrix} -16 & 4 & 3 \\ -2 & 1 & 0 \\ 7 & -2 & -1 \end{bmatrix}$

3. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix}$.

Solution: $\det A = -1$

4. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 4$ find the following

$$\det \begin{bmatrix} g & h & i \\ 4d & 4e & 4f \\ a-g & b-h & c-i \end{bmatrix}$$

Solution:

$$\det \begin{bmatrix} g & h & i \\ 4d & 4e & 4f \\ a-g & b-h & c-i \end{bmatrix} \underbrace{=}_{R_3+R_1} \det \begin{bmatrix} g & h & i \\ 4d & 4e & 4f \\ a & b & c \end{bmatrix} \underbrace{=}_{R_3\leftrightarrow R_1} -\det \begin{bmatrix} a & b & c \\ 4d & 4e & 4f \\ g & h & i \end{bmatrix}$$

$$= -4\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -16$$

5. Find the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point $\theta°$ counterclockwise about the origin, where $\theta° = 5\pi/6$.

Solution : $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}x-y}{2} \\ \frac{x-\sqrt{3}y}{2} \end{bmatrix}$

6. Find the standard matrix of the linear transformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \pi x - e^2 y \\ \sqrt{\pi} y - z \\ 3x - 2y + z \end{bmatrix}$.

Solution : $[T] = \begin{bmatrix} \pi & -e^2 & 0 \\ 0 & \sqrt{\pi} & -1 \\ 3 & -2 & 1 \end{bmatrix}$

7. Find the standard matrix of a linear transformation that <u>first</u> rotates a point counterclockwise about the origin by $\pi/4$ radians and then projects the result onto the y-axis.

Solution: The standard matrix of the first linear transformation is $\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}$ and the standard matrix of the second linear transformation is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So the standard matrix of the result is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

8. A woman has a ring, bracelet, and necklace that she wants to give to three of her 5 daughters.
   a) How many ways can she do this?
   b) The woman also has 2 identical pendants which she is going to give to two of her daughters. How many ways can she do this?

Solution:

a) $P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!}$

b) $C(5,2) = \frac{5!}{2!3!}$

9. The department of Mathematics and Statistics at a university has 12 mathematicians and 5 statisticians.
   a) If 6 people are randomly selected from the department to serve on a committee, what is the probability there will be 4 mathematicians and 2 statisticians on the committee?
   b) If 6 people are randomly selected for the committee, what is the probability that at least 2 will be statisticians?

Solution:

a) There are $\binom{17}{6}$ choices totally, from which $\binom{12}{4}\binom{5}{2}$ are possible ways to choose 4 mathematicians and 2 statisticians. So, the probability is $\frac{\binom{12}{4}\binom{5}{2}}{\binom{17}{6}}$.

b) $P(\text{at least two statistician}) = 1 - P(\text{no statistician}) - P(1\text{ statistician}) = 1 - \frac{\binom{12}{6}\binom{5}{0}}{\binom{17}{6}} - \frac{\binom{12}{5}\binom{5}{1}}{\binom{17}{6}}$

10. A small company has found that there is a 10% chance that an employee will call in sick on a Friday. If there are 9 employees in the company, what is the probability that
    a) Exactly 4 people call in sick on a Friday?
    b) At most 1 person calls in sick on a Friday?

    Solution:

    a) $\binom{9}{4}(0.1)^4(0.9)^5$
    b) $\binom{9}{0}(0.1)^0(0.9)^9 + \binom{9}{1}(0.1)^1(0.9)^8$

11. At a university, a computer science major can select either a BSc. with honours or a standard BSc. 30% of graduates selected the honours program (H). 90% of graduated obtained employment in the computing science field (E). 2% of graduates have an honours degree and did not get employment in the computing science field.
    a) Find $P(H \cup E^c)$.
    b) If a student has an honours degree, what is the probability they got employment in the field?
    c) If a student is not employed in the field, what is the probability they have an honours degree?
    d) Are H and E independent? Explain why or why not mathematically.

Solution:

a) $P(H \cup E^c) = P(H) + P(E^c) - P(H \cap E^c) = 0.3 + 0.1 - 0.02 = 0.38$
b) $P(E|H) = \frac{P(H \cap E)}{P(H)} = \frac{0.28}{0.3} = 0.9333$
c) $P(H|E^c) = \frac{P(H \cap E^c)}{P(E^c)} = \frac{0.02}{0.1} = 0.2$
d) They are not independent because….??