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Leveraged Exchange-Traded Funds with Market Closure and Frictions

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Abstract. Although leveraged exchange-traded funds (ETFs) are popular products for retail investors, how to hedge them poses a great challenge to financial institutions. We develop an optimal rebalancing (hedging) model for leveraged ETFs in a comprehensive setting, including overnight market closure and market frictions. The model allows for an analytical optimal rebalancing strategy. The result extends the principle of "aiming in front of target" introduced by Gârleanu and Pedersen (2013) from a constant weight between current and future positions to a time-varying weight because the rebalancing performance is monitored only at discrete time points, but the rebalancing takes place continuously. Empirical findings and implications for the weekend effect and the intraday trading volume are also presented.

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1. Introduction

Leveraged exchange-traded funds (LETFs) aim to achieve a daily target return equal to a stated multiple of the daily return of an underlying asset, where the multiple β can be + 2, + 3 (for bull LETFs), or –1, –2, –3 (for bear LETFs), fixed for each fund. In 2006, ProShares introduced the first 12 LETFs on four U.S. equity indices (S&P 500, NASDAQ 100, Dow Jones Industrial Average, and S&P MidCap 400), each with three multiples $+2, -1, -2$. Since then, they have become popular, especially for retail investors. Indeed, LETFs provide investors with leveraged (both long and short) opportunities without directly accessing margin accounts or financial derivatives. As of March 2021, there were a total number of 108 LETFs on U.S. equity indices, with a total of 52.7 billion USD assets under management.

However, one important property of LETFs often misunderstood by retail investors is that the stated daily multiple cannot be translated directly to the exact multiple over a multiday period due to the discrete compounding effect; see, for example, Avellaneda and Zhang (2010) and Jarrow (2010). For example, assume the underlying asset has a constant daily return of 0.1%, and the LETF hits the target multiple $\beta = 2$ every day. Then, over 252 days, the LETF achieves a return of $(1+0.001\times 2)^{252}-1 = 65.45\%$, which is 2.285 (not 2) times of $(1+0.001)^{252}-1 = 28.64\%$, the annual return of the underlying asset.

Even daily, the actual daily returns of the fund's net asset value tend to deviate from the target returns, a phenomenon called (daily) *slippage*. The LETF slippage has been documented in the empirical literature. Taking LETFs on the S&P 500 index as an example, Tang and Xu (2013) find that from 2006 to 2010, the LETFs with stated $\beta = 2, -1, -2$ achieved an average daily actual net asset value return of 0.0326%, −0.0097%, and −0.0247%, as compared with the average target return of 0.0471%, −0.0236%, and −0.0471%, respectively.

The existing literature suggests several factors to explain the LETF slippage. Tang and Xu (2013) and Henderson and Buetow (2014) attribute the slippage to the interest cost; more precisely, to achieve a target leveraged exposure to the underlying asset, a bull (respectively bear) fund has to borrow (respectively lend) at the London Interbank Offered Rate (LIBOR) rate, which brings down (respectively pushes up) the daily fund asset return and causes the slippage. Avellaneda and Dobi (2012), Wagalath (2014), and Guasoni and Mayerhofer (2017) instead ascribe to transaction costs incurred during the daily LETF rebalancing.

In this paper, we develop a model for the optimal daily rebalancing of LETFs under market frictions and overnight jump risk, which yields an explanation for the LETF daily slippage. More precisely, (i) an LETF fund usually rebalances near the market closing time, when it can better estimate the position needed for the target exposure for the next day; (ii) adjusting the fund position rapidly during a short period may incur a high cost due to the market price impact. Both can cause suboptimal exposure at the market closing, which leads to daily slippage. Additionally, in our model most intraday rebalancing activities occur right after market opening and before market closure, and the slippage tends to be the largest on Mondays, thus providing an explanation of the U-shape intraday trading volume and the "weekend effect" of the slippage of LETFs.

1.1. Intuition

Financial institutions face the challenge of hedging when issuing LETFs in the presence of market frictions and overnight market closure. To illustrate this, assume that the interest rate and transaction costs are zero and consider the following simple example. To reach the targeted daily return today, which is β times the daily return of the asset, an LETF fund has to achieve today's targeted position that corresponds to a leverage ratio of β at the previous market closure. Assume $\beta = 2$, and at the previous market closure, the underlying asset price and total asset are \$100 and \$100 million, respectively, and the fund has an ideal position of two million shares in the underlying asset. Then, by keeping this targeted position unchanged throughout today until market closure, the fund will reach the targeted return today. Assume the underlying asset value drops by 1% to \$99 at today's market closure, then the fund's total asset would drop by 2% to \$98 million and achieve the targeted return, which is ideal. However, the preparation for tomorrow's targeted position becomes a problem. To achieve the target, the fund would need to have tomorrow's position of $2 \times \$98$ million/\$99 = 1.9798 million shares at today's market closure by selling 0.0202 million shares right before today's market closure.

This selling is unrealistic and leads to daily slippage for at least two reasons: (1) In the presence of market frictions, selling a large number of shares right before the market closure is costly, as a sudden portfolio adjustment within an infinitesimal period would yield a high cost due to price impact and other transaction costs, leading to a suboptimal leverage ratio and potentially large slippage. (2) The slippage caused by a suboptimal position at market closure can be amplified by the risk of overnight jumps in prices, as such jumps may move the fund's opening position further away and require larger rebalancing of positions.

In view of the two challenges, we show that the fund should start moving its aim continuously toward tomorrow's target much earlier than the market closure, in anticipation of tomorrow's target. In other words, the fund's aim is "in front of today's target." The proposed strategy can reduce the slippage significantly, as confirmed by our numerical results later.

1.2. Our Contribution and Literature Review

The contribution of this paper is fourfold. First, to the best of our knowledge, we are the first to study the daily rebalancing problem in a comprehensive setting, including market frictions and the overnight market closure. Empirical studies (e.g., Lockwood and Linn 1990, Stoll and Whaley 1990) show that the volatility is much lower during market closure as compared with trading hours, although the expected return is on a similar level. Also, the trading strategy during trading hours needs to adjust for market closure when no trading is allowed (cf. Dai et al. 2015). This is especially important for LETFs because if the target leverage ratio is not exactly met at market closing time, the fund bears the risk that an overnight price jump may lead to a large return deviation right at the next market opening.[1] We find that to reduce the overnight jump risk, it is optimal to perform a larger portion of rebalancing before market closure. Additionally, adjusting the aim for the overnight risk can lead to a slippage reduction of as much as 24%.

Without the risk of market closure, several researchers focused on modeling the slippage of LETFs under price impact either in a pure discrete setting or a pure continuous setting. Wagalath (2014) proposed a model with an endogenous price impact and derived an analytic formula for the rebalancing slippage, assuming that rebalancing happens once per day and the return deviation is measured daily at the market closing time. Based on an approximate continuous model, Guasoni and Mayerhofer (2017) studied the fund's trade-off between the short-term and long-term deviation. With proportional transaction costs, they derived the optimal trading boundary, which has an explicit form in terms of an asymptotic expansion. In contrast, our model incorporates several other additional factors, including the risk of market closure due to overnight price jump.

To help unveil the optimal intraday rebalancing strategy, we use a delicate mixture of continuous and discrete-time modeling: The intraday trading strategies and market frictions are modeled in a continuous-time setting that allows gradual adjustment across the trading hours, whereas the performance, that is, the return deviation, is monitored daily in a discrete-time manner. This mixture modeling is consistent with the LETFs in reality and complements the existing literature either in pure discrete-time or continuous-time settings. Furthermore, whereas the two papers mentioned earlier assume a linear price impact in the trading amount (i.e., proportional transaction costs), we consider a quadratic price impact in the trading amount.

Second, our rebalancing strategy contributes to the empirical study of LETF slippage. The slippage of LETFs was studied empirically in Tang and Xu (2013) and Henderson and Buetow (2014), both of which found that LETFs' realized return deviated significantly from the target return and explained the slippage via the interest cost incurred due to borrowing and lending. Several papers also linked the LETF slippage to the market friction costs. Using the 2008 financial crisis period, Shum and Kang (2013) observed a significant slippage for LETFs on international indices that tend to have lower liquidity. Avellaneda and Dobi (2012) found that the slippage tends to be prominent during volatile periods.

However, the majority of empirical literature studies the slippage by assuming the fund follows a simple suboptimal rebalancing strategy. For example, Tang and Xu (2013) and Henderson and Buetow (2014) assume that the fund achieves the target leverage ratio exactly at market closing and keeps the position until the subsequent market closing; however, this strategy is suboptimal if the interest rate is positive, even in the absence of market frictions (cf. Theorem 1). Our study complements these papers by studying the slippage based on the optimal rebalancing strategy in a comprehensive setting, including nonzero interest rate, market frictions, and market closure. Furthermore, to the best of our knowledge, we are the first to find the "weekend effect" of LETF slippage empirically; that is, the slippage tends to be the largest on Monday. By incorporating market closure, our model can produce implications consistent with the empirical finding. Our model can also suggest a testable implication of a U-shaped intraday rebalancing volume of LETFs.

Third, we demonstrate the principle of aiming in front of the target and moving gradually towards the aim, proposed in Gârleanu and Pedersen (2013, 2016), under the setting of hedging periodic cash flows. In the mean-variance optimization setting of Gârleanu and Pedersen (2013, 2016), one immediately moves to the optimal Markowitz portfolio and stays on it by continuous rebalancing in the absence of market frictions. However, in the presence of frictions, they showed that it is optimal to gradually adjust the portfolio toward an aim, which is not the current Markowitz portfolio but rather a weighted average of the current and future Markowitz portfolios. In our hedging setting, the optimal position also moves toward the aim gradually in a mean-reverting manner, where the aim stays close to the current target that minimizes today's return deviation in the morning hours and moves in front of the current target, converging to tomorrow's target in the afternoon hours, so as to prepare for the future.

Whereas the weight between current and future targets remains constant in Gârleanu and Pedersen (2013, 2016), the weight in our paper is time varying. This distinct feature arises from the special structure of our model involving the end-of-day discrete monitoring and intraday continuous rebalancing. It provides an extra economic insight on how much the aim should be in front of target in the presence of competing goals. Moreover, we report two benefits of applying this principle in LETFs' daily rebalancing: (1) By aiming in front of the current target and looking into the future, our optimal rebalancing strategy reduces the average daily slippage significantly compared with the one-day strategy that only focuses on minimizing today's deviation. (2) Moving gradually toward the target smooths out the daily rebalancing and results in a large decrease in the end-of-day trading volume, as compared with the one-shot strategy that only adjusts immediately before market closure.

Fourth, from a technical viewpoint, we obtain an analytic solution for the rebalancing strategy whose coefficients are determined by a system of periodic ordinary differential equations (ODEs).[2] This is made possible because the quadratic structure of the model reduces the high-dimensional problem into a system of nonlinear ODEs, which leads to analytic tractability.[3] Specifically, one only needs to solve the system once to determine the coefficients, which can then be used any time in the future. Because the periodic ODEs are nonlinear, we verify carefully that the solution exists and does not blow up and show that an iterative algorithm solving the periodic ODEs (along with the endogenous terminal conditions) has a unique fixed point solution.

The goal of LETFs is to achieve a fund return that is β times the underlying index return on a daily basis. Thus, even if they accomplish this goal perfectly every day, the cumulative returns are unlikely β times the underlying index returns over multiday periods due to the compounding effect. However, this goal of LETFs was misunderstood by many investors, and the compounding effect was ignored. Indeed, both the U.S. Securities and Exchange Commission and Financial Regulatory Industry Authority warned investors of the significance of the multiperiod compounding effect for a buy-and-hold strategy over an extended period. There are analytical and approximated solutions for the multiperiod compounding effect, first proposed in Cheng and Madhavan (2009) and extended to a more general setting in Avellaneda and Zhang (2010) and Haugh (2011). Unlike this strand of literature, our primary focus is on the daily slippage rather than deviation for multiday returns, and our objective is to minimize such daily slippages over intraday rebalancing rather than the compounding effect. We provide a discussion in online Supplement H for a direct link to the compounding effect in the above literature.

The remainder of this paper is organized as follows. Section 2 introduces the background and notations of the problem. Section 3 presents the main result of this paper, an optimal LETF rebalancing model with market frictions, via numerical examples. Section 4 shows LETF's position under the optimal rebalancing strategy. Section 5 explains the usage of the principle of aiming in front of target in LETF rebalancing, as well as its benefits. Sections 6 and 7 show the empirical results on slippage, intraday rebalancing pattern, and compounding effect. Section 8 concludes. All technical results and related proofs are given in the online supplement.

2. Basic Setting and Notations

We consider an infinite horizon with $0 = t_0 < t_1 < \ldots < t_{2i} < t_{2i+1} < \ldots$, where t_{2i} and t_{2i+1} are, respectively, the market opening and closing of day i, $i = 0, 1, \ldots$. Hence, the market is open during time intervals $[t_{2i}, t_{2i+1}]$, when trading is allowed; the market is closed during time intervals (t_{2i+1}, t_{2i+2}), when no trading is allowed. Denote the length of daytime (i.e., trading hours) for each day as T, and the length of nighttime (i.e., market closure) as δT. Therefore, the length of each day is $(1+\delta)T$, and $t_{2i} = i(1+\delta)T$, $t_{2i+1} = T + i(1+\delta)T$ for $i = 0, 1, \ldots$. Denote the total daytime as $\bigcup_{i=0}^{\infty}[t_{2i}, t_{2i+1}]$, and the total nighttime as $\bigcup_{i=0}^{\infty}(t_{2i+1}, t_{2i+2})$. The underlying asset price S evolves as a geometric Brownian motion with different constants during daytime and nighttime:

$$dS_u = \mu(u) S_u du + \sigma(u) S_u dW_u, \tag{1}$$

where $\mu(u) = \mu_d$ and $\sigma(u) = \sigma_d$ for daytime, $\mu(u) = \mu_n$ and $\sigma(u) = \sigma_n$ for nighttime, and the parameters μ_d, μ_n, σ_d, and σ_n are all positive constants.[4] The fund net asset value (NAV) is invested in θ shares of S, and the remaining in a risk-free account with a constant interest rate $r > 0$.[5]

Thus, the dynamics of the LETF's total NAV, X, is given by

$$dX_u = \theta_u dS_u + (X_u - \theta_u S_u) r du, \quad u \in [t_{2i-1}, t_{2i+1}), \ \forall i \geq 0, \tag{2}$$

$$X_{t_{2i+1}} = (1-\gamma) X_{t_{2i+1}-}. \tag{3}$$

Here, (3) corresponds to the industry practice of deduction of management fee from the NAV at market closing $t_{2i+1}, i \geq 0$, where γ is the daily management fee rate (γ typically has an annualized value of about 1%, although it varies across funds). Furthermore, we require that the position θ is adapted, and constant during nighttime when no trading is allowed.

The LETF's daily return on ith day is calculated using the NAV immediately before the market closing t_{2i+1} and the NAV at the market closing time t_{2i-1} on the previous day[6]: $R_i^X = (X_{t_{2i+1}-} - X_{t_{2i-1}})/X_{t_{2i-1}}$. To emphasize the role of a daily management fee, $X_{t_{2i+1}-}$ and $X_{t_{2i+1}}$ denote the before-fee NAV and the after-fee NAV (that is, the NAV publicly announced after market closing). Therefore, R_i^X denotes the daily before-fee NAV return. Denote the daily underlying asset return as $R_i^S = (S_{t_{2i+1}} - S_{t_{2i-1}})/S_{t_{2i-1}}$, so that the target return is βR_i^S. For the current day (i.e., $i = 0$), the notation $X_{t_{2i-1}}$ in R_0^X (respectively $S_{t_{2i-1}}$ in R_0^S) is the NAV (respectively underlying value) observed at the last market closing before the current day, and it will be denoted as $\bar{x}$ (respectively $\bar{s}$) throughout the paper.

Definition 1 (Slippage). The slippage on ith day is the distance between LETF's daily before-fee NAV return and target return:

$$D_i = |R_i^X - \beta R_i^S|. \tag{4}$$

Note that to achieve the investment objective[7] exactly, D_i should be zero on every trading day.

Trading in the market is costly; trading at an instantaneous speed φ_u incurs a temporary price impact of $C(\varphi_u, S_u)du = \frac{1}{2}\tilde{\Lambda} S_u^2 \varphi_u^2 du$, where $\tilde{\Lambda}$ is a nonnegative constant, and the speed φ_u is such that $d\theta_u = \varphi_u du$. This type of quadratic price impact cost is also used in Obizhaeva and Wang (2013), Rogers and Singh (2010), Gârleanu and Pedersen (2016), and Moreau et al. (2017), and supported empirically by Breen et al. (2002) (see also Grossman and Miller 1988 and Greenwood 2005 for the justification in the multiasset case). In the classic Kyle (1985) model, the equilibrium price set by the market maker increases linearly in the trader's amount of order; therefore, to the trader, the cost of price impact is quadratic in the trading amount. In our model, this price impact can be interpreted as the liquidity cost incurred during trading, for example, from the presence of front runners, especially immediately before the market closing when other market participants have a good estimation of LETFs' direction of rebalancing.[8] We also assume that C is quadratic in the underlying price S, which can be understood as that the price impact is higher when the underlying price is higher. This is especially important in the current problem, because the influence of asset price on its price impact is nonnegligible over an infinite horizon. Also, the multiplicative factor S^2 gives a natural scaling of the price impact and brings a technical convenience allowing for an analytical solution to the problem.

The fund's objective is to minimize the difference between the daily simple return of X and the target return at market closing of each trading day over an infinite horizon, with an additional penalty on the trading speed φ. Specifically, the fund minimizes the total cost

$$\sum_{i=0}^{\infty} e^{-\rho t_{2i}}\left(\frac{1}{2}X_{t_{2i-1}}^2 D_i^2 + \nu\int_{t_{2i}}^{t_{2i+1}} \frac{1}{2}\tilde{\Lambda} S_u^2 \varphi_u^2 du\right). \tag{5}$$

Here, $\frac{1}{2}X_{t_{2i-1}}^2 D_i^2$ represents the daily slippage cost, which is larger for a larger slippage.[9] The factor $X_{t_{2i-1}}^2$ provides a natural scaling for the deviation cost; indeed, a larger fund will hold a larger position and thus have a larger price impact, and therefore such scaling is required to keep the relative importance between the two types of costs. The parameter $\rho > 0$ is a subjective discount rate, representing the weight in the trade-off between optimizing short-term performance or long-term performance: the larger ρ is, the greater is the emphasis on short-term costs as compared with the long-term costs. As a special case, $\rho = +\infty$ means that the manager only cares about today. The summation of the first and second terms in the parenthesis corresponds to the daily total cost. In particular, the first term represents the daily return deviation cost from the fund's slippage, and the second term represents the daily accumulated cost from the market frictions incurred by the intraday fund rebalancing. Note that the treatment of incorporating friction cost as a separate penalty term has been used extensively in the finance literature, for example, Gârleanu and Pedersen (2013, 2016), or in macroeconomy literature, for example, Hansen and Sargent (2013).[10] The parameter $\nu \geq 0$ serves the role of prioritizing the goals of minimizing the deviation cost or minimizing the market friction costs on each day. For example, a small ν can reflect the case where, when using the futures and swaps, only a small fraction of price impact cost is transferred from the counterparty to the fund. Taking $\nu = 0$ means that the manager ignores the market friction costs and focuses solely on minimizing the slippage. To simplify notations, in the following we denote $\Lambda = \nu\tilde{\Lambda}$.

3. Theoretical Results

In this section, we solve the cost and deviation minimization problem introduced in Section 2, in the case without market frictions ($\Lambda = 0$) and with market frictions ($\Lambda > 0$). In either case, we derive the explicit value function whose coefficients are the unique solution to a system of ordinary differential equations, and the resulting explicit optimal rebalancing strategy.

3.1. The Case Without Market Frictions

Without market frictions, that is, $\Lambda = 0$, the total cost (5) reduces to the aggregated daily slippage. At the current time $0 \leq t < t_1$, the cost minimization problem becomes

$$\begin{aligned} V(t,s,x,\bar{s},\bar{x}) &= \inf_{\theta} E\left[\frac{1}{2}\bar{x}^2\left(\beta\left(\frac{S_{t_1}}{\bar{s}}-1\right)-\left(\frac{X_{t_1-}}{\bar{x}}-1\right)\right)^2\right. \\ &\quad \left. + \sum_{i=1}^{\infty} e^{-\rho t_{2i}} \frac{1}{2} X_{t_{2i-1}}^2 \left(\beta\left(\frac{S_{t_{2i+1}}}{S_{t_{2i-1}}}-1\right)-\left(\frac{X_{t_{2i+1}-}}{X_{t_{2i-1}}}-1\right)\right)^2\right], \end{aligned} \tag{6}$$

where $S_t = s$ and $X_t = x$, and recall that $\bar{s}$ and $\bar{x}$ are the *reference values* of the underlying asset and NAV observed at the last market closing before today, respectively.[11] The first and second terms inside the expectation are the deviation on the first day and the aggregated deviation from the second day, respectively. Note that at $t = 0$, this minimizes the total cost as defined in (5).

Besides keeping track of X and S at any time, we also need to keep track of their reference values $\bar{x}$ and $\bar{s}$, in order to calculate the daily return at today's market closing. On each day, the current value of S and X, together with their reference values, provide an expectation of the daily return and are hence important in the rebalancing decision. Note that $\bar{s}$ and $\bar{x}$ are not necessarily equal to the time-t underlying value s and NAV x, respectively, even at $t = 0$ due to the overnight jump.

Theorem 1 (Minimal Cost and Optimal Rebalancing Strategy Without Market Frictions). *The minimal cost* (6) *is given as* $V(t,s,x,\bar{s},\bar{x}) = \bar{x}^2 V(t, \frac{s}{\bar{s}}, \frac{x}{\bar{x}}, 1, 1)$, *where* $V(t,s,x,1,1) = a(t)x^2 + b(t)xs + c(t)s^2 + d(t)x + e(t)s + f(t)$, $t \in [0, t_1)$, *and the vector of coefficients* (a,b,c,d,e,f) *is the unique solution to a system of periodic ODEs* (F.3)–(F.8) *with endogenous terminal conditions* (F.10) *(see online supplement). Furthermore, the optimal position level is*

$$\theta_t^{fl} = \begin{cases} -\dfrac{1}{2\sigma^2 a(t)}\left[b(t)\sigma^2\dfrac{\bar{x}}{\bar{s}} + (\mu - r)\left(b(t)\dfrac{\bar{x}}{\bar{s}} + d(t)\dfrac{\bar{x}}{S_t} + 2a(t)\dfrac{X_t}{S_t}\right)\right] & t \in [0, t_1) \\ -\left[M\left(\dfrac{d(0)}{2a(0)} + e^{r\delta T}\right) + \dfrac{b(0)}{2a(0)}(1 + e^{r\delta T}M)\right]\dfrac{X_{t_1}}{S_{t_1}} & t \in [t_1, t_2), \end{cases} \tag{7}$$

where $M = \frac{e^{\mu_n \delta T} - e^{r\delta T}}{(e^{\mu_n \delta T} - e^{r\delta T})^2 + e^{2\mu_n \delta T}(e^{\sigma_n^2 \delta T} - 1)}$. *In particular, if* $r = 0$, *then*

$$\theta_t^{fl} = \frac{\beta\bar{x}}{\bar{s}}, \quad \textit{if } t \in [0, t_1) \textit{ and } \frac{\beta X_{t_1}}{S_{t_1}} \textit{ if } t \in [t_1, t_2). \tag{8}$$

Theorem 1 shows that the minimum cost is a quadratic function of the normalized stock price $s/\bar{s}$ and portfolio value $x/\bar{x}$. The optimal position level is determined by the expression of θ^{fl}, and (2)–(3) with θ replaced by θ^{fl}. During the daytime (i.e., $t \in [0, t_1)$), the current optimal position level θ_t^{fl} is continuous with respect to t, depending on the coefficients of the minimal cost function, the current underlying and fund values $S_t = s$ and $X_t = x$, as well as the reference values $\bar{s}$ and $\bar{x}$ for $[0, t_1)$. Right at the market closing time t_1, it is optimal to perform a lump-sum trade so as to adjust the position to $\theta_{t_1}^{fl}$. The position is then kept at this level during nighttime. At the next market opening, the current time t reset to 0, and the trading continues following (7), based on updated reference values $\bar{x} = X_{t_1}$ and $\bar{s} = S_{t_1}$.

The special case with $r = 0$ gives a much simpler solution, given by (8). That is, it is optimal to perform a one-shot rebalancing at every market closing time based on the underlying and fund value at market closing, and then keep the position unchanged until the next market closing, so on and so forth. If the interest rate is zero in the market, then this one-shot strategy leads to zero slippage. Indeed, this agrees with the common practice by fund managers to adjust the portfolio near market closing on each day to keep the target leverage ratio (cf. Avellaneda and Dobi 2012, Wagalath 2014).

It is worth noting that even without market frictions, when $r > 0$, ignoring the interest rate and sticking to this one-shot rebalancing strategy (8) may lead to a nonzero daily slippage $D_i = |1 - \beta|(e^{rT} - 1)$.[12] Indeed, the positive interest rate means that the fund pays the funding cost for doing leverage and receives interest for short-selling. Actually, this is the motivation for Tang and Xu (2013) and Henderson and Buetow (2014) to explain slippage in terms of interest rate: the fund ignores the interest cost when rebalancing, and therefore the interest cost will affect the fund's return and result in the slippage. In contrast, Theorem 1 considers not just the interest rate but also the market closure. Nevertheless, because the daily interest rate is typically small, the amount of rebalancing required before market closing is small without market frictions, as will be illustrated numerically in Section 4.

3.2. The Case with Market Frictions

Now we study the optimal daily rebalancing strategy that takes market frictions into account. For example, with a quadratic trading cost, it is no longer feasible to do lump-sum trading, because it incurs an infinite cost. To incorporate market frictions, we impose a mild technical requirement that θ_u is absolutely continuous with trading speed φ_u, that is,

$$d\theta_u = \varphi_u du, \quad \theta_t = z. \tag{9}$$

Here, the adapted admissible control variable $\varphi \in \mathcal{A}$, where the set of admissible strategies $\mathcal{A}$ consists of control variable φ that results in a finite expected market frictions cost and equals 0 during nighttime.[13]

For $0 \le t < t_1$, $\bar{s}, s \in \mathbb{R}^+$, $\bar{x}, x, z \in \mathbb{R}$, $\Lambda > 0$, the value function is defined in a similar way as (6):

$$\begin{aligned} V(t, s, x, z, \bar{s}, \bar{x}) &= \inf_{\varphi \in \mathcal{A}} E\Bigg[\frac{1}{2}\bar{x}^2\left(\beta \cdot \left(\frac{S_{t_1}}{\bar{s}} - 1\right) - \left(\frac{X_{t_1-}}{\bar{x}} - 1\right)\right)^2 + \int_t^{t_1} \frac{1}{2}\Lambda S_u^2 \varphi_u^2 du \\ &\quad + \sum_{i=1}^{\infty} e^{-\rho t_{2i}}\Bigg(\frac{1}{2}X_{t_{2i-1}}^2\left(\beta \cdot \left(\frac{S_{t_{2i+1}}}{S_{t_{2i-1}}} - 1\right) - \left(\frac{X_{t_{2i+1}-}}{X_{t_{2i-1}}} - 1\right)\right)^2 \\ &\quad + \int_{t_{2i}}^{t_{2i+1}} \frac{1}{2}\Lambda S_u^2 \varphi_u^2 du\Bigg)\Bigg], \end{aligned} \tag{10}$$

where the expectation is computed under the initial value $S_t = s$, $X_t = x$, and $\theta_t = z$.

Compared with (6), the value function (10) has the additional terms for the market friction costs. Because it is now only feasible to adjust the position θ at a finite speed φ, the initial position z becomes relevant for the calculation of the market friction costs and is hence required as a state variable. To see why the initial position matters, starting from a position that is farther away from the optimal position, the manager needs to perform a larger rebalancing to push the position toward the optimal one, which in turn triggers a higher cost from the market frictions.

The following theorem gives an explicit form of the value function $V(t, s, x, z, \bar{s}, \bar{x})$ in the case with market friction, as well as the optimal rebalancing strategy.

Theorem 2 (Optimal Value Function and the Optimal Rebalancing Strategy with Market Frictions). *The value function is given as* $V(t, s, x, z, \bar{s}, \bar{x}) = \bar{x}^2 V(t, \frac{s}{\bar{s}}, \frac{x}{\bar{x}}, \frac{\bar{s}}{\bar{x}}z, 1, 1)$, *where*

$$\begin{aligned} V(t, s, x, z, 1, 1) &= a(t)x^2 + (b_0(t) + b_1(t)z)xs + (c_0(t) \\ &\quad + c_1(t)z + c_2(t)z^2)s^2 + d(t)x \\ &\quad + (e_0(t) + e_1(t)z)s + f(t), \end{aligned} \tag{11}$$

where the coefficients are determined as the unique solution to a system of periodic ODEs (A.9)–(A.18) *subject to endogenous terminal condition* (D.1) *(see online supplement). Furthermore, the optimal rebalancing strategy* φ^* *equals*

$$\varphi_t^* = -\frac{1}{\Lambda}\left(b_1(t)\frac{X_t}{S_t} + c_1(t)\frac{\bar{x}}{\bar{s}} + 2c_2(t)\theta_t^* + e_1(t)\frac{\bar{x}}{S_t}\right), \quad t \in [0, t_1), \tag{12}$$

where the optimal position θ^* *satisfies* $d\theta_t^* = \varphi_t^* dt$.

The normalized value function (11) for $\bar{s} = \bar{x} = 1$ is a polynomial of degree 4 with respect to (s, x, z), which is different from the quadratic value function derived in Gârleanu and Pedersen (2013). Specifically, the underlying value s appears in (11), and z appears as a multiplicative factor in front of terms involving s (i.e., s^2, s, xs) with the highest order matching the order of s. In contrast, z also does not appear in the value function in Gârleanu and Pedersen (2013) because it does not involve the underlying value s.

The optimal rebalancing strategy φ^* depends on the current underlying and fund values $S_t = s$ and $X_t = x$ as well as the reference values $\bar{s}$ and $\bar{x}$. In addition, φ^* depends on the current position level, so that the optimal position and optimal trading speed form a feedback system. During nighttime, $\varphi^* = 0$ because no trading is allowed, and at the next market opening, the reference values are updated to $\bar{s} = S_{t_1}$ and $\bar{x} = X_{t_1}$ by definition, t reset to 0, and the optimal trading speed is again determined by (12). Because one only needs to solve the ODE system once to determine the coefficients, this optimal strategy can be implemented very efficiently.

From a technical point of view, the ODE system (A.9)–(A.18) for the coefficients of (11) (and also the system (F.3)–(F.8) for the frictionless case; see online supplement) are nonlinear and have a periodic terminal condition. Even in the one-period subproblem, the global existence of a solution is not guaranteed by the classic theory, because it may blow up in finite time. In Lemma 1 of the online supplement, we verify carefully that the solution indeed exists in one period, given a suitable terminal value. Also, because the periodic ODEs are solved in iteration, one needs to show that this iteration indeed has a unique fixed point to guarantee convergence. Proposition 3 in the online supplement shows that, as long as the discount rate is positive, we indeed have a fixed point. Finally, the uniqueness of the solution is guaranteed by the verification theorem, Proposition 4, in the online supplement. In the remaining of the paper, we shall discuss the financial implications of our rebalancing strategy mainly via numerical illustration.

4. Optimal Daily Position

This section shows the position under the optimal rebalancing strategy, where the rebalancing starts every day before market closing and finishes after the next day's market opening.

Definition 2 (Strategies for Comparison). We define the following three rebalancing strategies, all of which are given via (12) but with different coefficients b_1, c_1, c_2, and e_1:

Periodic-DN: The *periodic day-and-night* strategy is the optimal strategy that considers an infinite horizon and market closure, given as (12) whose coefficients are calculated under $\delta = 17.5/6.5$ (noting the total trading hours is 6.5) and $\rho = 0.6$.

One-day: The *one-day* strategy is the myopic suboptimal strategy that only aims at minimizing today's cost and ignores the presence of market closure, given as (12) whose coefficients are calculated under $\rho = +\infty$.

Periodic-D: The *periodic day-only* strategy is the suboptimal strategy that considers an infinite horizon but ignores market closure, given as (12) whose coefficients are calculated under $\delta = 0$ and $\rho = 0.6$.[14]

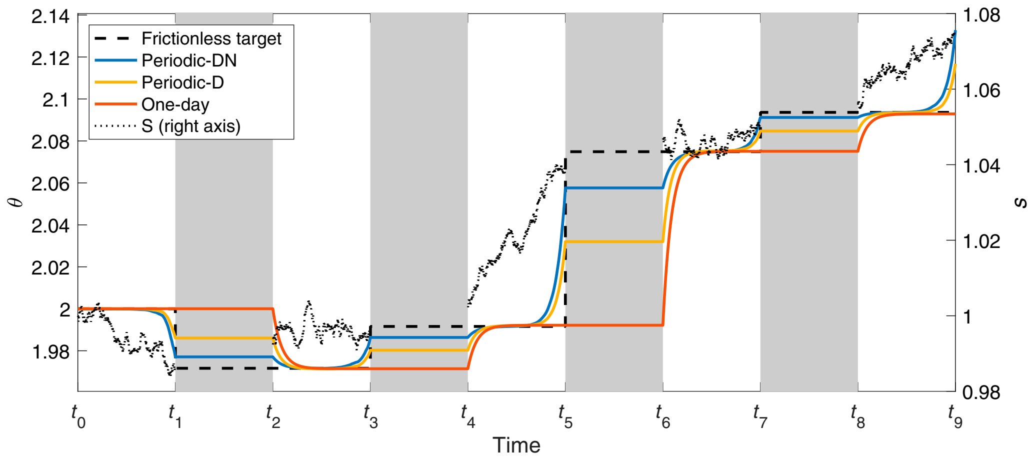
In the following, we use Monte Carlo simulation to estimate the average slippage. For this purpose, we simulate the underlying asset price S via (1) during daytime and its jumps overnight, using the default parameter values to be specified. Along the simulated path of S, one can calculate the net asset value X of the LETF for each day via (2), (3), (9), and (12), and then calculate the daily slippage using (4). Note that the reference values $X_{t_{2i-1}}$ and $S_{t_{2i-1}}$ in (12) are given as model input on day $i = 0$ (i.e., $t \in [0, t_1)$).

Because the daytime volatility σ_d is k times the nighttime volatility σ_n, we have $\sigma_d = k\sigma \cdot \sqrt{\frac{T+\delta T}{k^2T+\delta T}}$ and $\sigma_n = \sigma \cdot \sqrt{\frac{T+\delta T}{k^2T+\delta T}}$, where σ is the average volatility. Here we choose $k = 3$ and $\sigma = 0.2$. On the other hand, we use the same expected return for daytime and nighttime as suggested by empirical studies such as Stoll and Whaley (1990) and Lockwood and Linn (1990) and take $\mu_d = \mu_n = 0.1$. The default values for other parameters are[15] $\delta = 17.5/6.5$, $\nu = 1$, $\beta = 2$, $\rho = 0.6$, and $r = \gamma = 0.01$. All these parameter values are annualized.[16] Furthermore, we take $\Lambda = \nu\tilde{\Lambda} = 10^{-6}$ to match the level of magnitude of price impact documented in Robert et al. (2012).[17] Finally, we consider $s = x = \bar{s} = \bar{x} = 1, z = 2$ (so the position is optimal at t_0, and no overnight jump occurs before t_0).

We first look at the position level under the optimal strategy, which directly determines the performance. To this end, we simulate a sample path of the underlying asset value over a five-day period using the default parameter values. Due to the overnight jumps, the underlying value at the market closing time is not necessarily equal to the value at the next market opening. Then, via (9) and (12), we calculate the optimal position θ^{PDN} based on the periodic-DN strategy, θ^{O} based on the one-day strategy, and θ^{PD} based on the periodic-D strategy, as shown in Figure 1. We also plot the frictionless target θ^{fl}, that is, the optimal position without market frictions as given by (7) in Theorem 1. Finally, we overlay the five-day trajectory of the underlying asset as the dotted line corresponding to the right-hand side axis.

Figure 1 shows that, for all three strategies, the fund needs to increase (respectively decrease) the position θ by market closing if the underlying asset return is positive (respectively negative) during that day. The magnitude of such an increase or decrease in θ is also proportional to the magnitude of daily asset price return. This is consistent with the changes in the frictionless target. To understand this, when the interest rate is low and there is no friction, the fund would ideally hold $\theta^{fl}_{t_{2i+1}}$ share of the risky asset at market closing t_{2i+1}, which provides a leverage ratio of β approximately. If the underlying asset has a positive

Figure 1. (Color online) Comparison of Intraday Positions from Three Strategies



Notes. The dotted line with right-hand side axis represents the a simulated asset price trajectory for five days. The dashed line denotes the frictionless target (technically speaking, the five-day trajectories of frictionless targets under the three strategies differ, as they depend on the realized trajectories of X. However, because such differences are negligible over the five-day period in this figure, only the frictionless target corresponding to the periodic-DN strategy is presented for a better illustration). Time periods with white and gray backgrounds denote daytime and nighttime, respectively.

(respectively negative) daily return, the magnitude of return of the fund value X is greater than that of the underlying asset. If θ remains unchanged, the fund would be under-leveraged (respectively over-leveraged) at market closing. Thus, to keep the leverage ratio close to β at market closing, the fund has to increase (respectively decrease) θ.

Figure 1 shows two advantages of the periodic-DN strategy. First, it results in a smaller overall distance of the position level to the frictionless target. Indeed, for the periodic-DN and periodic-D strategies, during daytime, the position first moves toward and then stays close to the target after the market opening. Then, right before market closing, it moves away from today's target and tries to shoot for the predicted target for the next day. Due to the presence of market frictions, the position level does not hit tomorrow's target. As a result, further rebalancing is required after the next day's market opening. In contrast, the position level of the one-day strategy does not change significantly in the late trading hours and has a much larger distance from tomorrow's target before the market closing time and during nighttime. As a result of this myopic strategy, the position is significantly off the target level right after tomorrow's market opening, and one requires a greater effort to finish the rebalancing in the next day.

Second, by comparing the position levels of the periodic-DN and periodic-D strategies in Figure 1, we can observe that accounting for overnight risk leads to a better end-of-day position. Indeed, by accounting for the overnight risk, the fund adjusts its position at a faster pace late in daytime, so that the fund has a better position at the market closing and during nighttime. Consequently, the fund is exposed to a smaller overnight jump risk, and a smaller rebalancing is required for the position to reach the target after tomorrow's market opening. The discussion in Section 6.1 confirms that this indeed leads to a smaller slippage as compared with the periodic-D strategy that ignores the overnight jump risk.

Figure 1 also illustrates that the frictionless target position remains largely constant over daytime and jumps to the target level for the next day at the market closing. Therefore, although continuous rebalancing is optimal during daytime for $r > 0$ as suggested by Theorem 1, the amount of rebalancing is negligible if there is no friction.

5. Aiming in Front of Target

What is the underlying reason for the difference in the position levels as depicted in Figure 1, especially before the market closing time? To answer this question, we show in this section that the optimal strategy follows the principle of aiming in front of target and moving gradually toward aim. To this end, we rewrite the optimal rebalancing strategy (12) as

$$d\theta_t^* = \kappa_t^*(\bar{\theta}_t^* - \theta_t^*)dt, \tag{13}$$

for $t \in [0, t_1)$, where

$$\bar{\theta}_t^* = -\frac{b_1(t)}{2c_2(t)}\frac{X_t}{S_t} - \frac{c_1(t)}{2c_2(t)}\frac{\bar{x}}{\bar{s}} - \frac{e_1(t)}{2c_2(t)}\frac{\bar{x}}{S_t}, \quad \kappa_t^* = \frac{2c_2(t)}{\Lambda}. \tag{14}$$

Equation (13) shows that the optimal position exhibits a mean-reverting pattern, with speed $0 \le \kappa^* < \infty$, and aim $\bar{\theta}^*$. The finite speed κ^* is consistent with the principle of moving gradually toward aim studied in Gârleanu and Pedersen (2013), due to the existence of market friction. The speed is lower for larger friction; if there is no friction ($\Lambda = 0$), the speed κ^* is infinite, meaning that the position follows the aim exactly. The distance between the current position level and the aim $\bar{\theta}^*$, together with κ^*, decides the optimal trading speed.

Also, from the analytic form (11) of V, the aim can be represented as

$$\bar{\theta}_t^* = \arg\min_z V(t, S_t, X_t, z, \bar{s}, \bar{x}).$$

Thus, the aim is the position that minimizes the expected optimal total future costs given the current information. This interpretation is consistent with Gârleanu and Pedersen (2016). More precisely, the aim portfolio (given by equation (11) in Gârleanu and Pedersen 2016) is exactly the minimizer of the value function (given by equation (6) in their paper). Their aim can be represented as a weighted average of the future expected frictionless targets. However, a similar representation for $\bar{\theta}_t^*$ here is difficult because our frictionless target depends on the past strategy via the current fund value, creating a feedback loop because the position tries to follow an aim that is itself affected by the position. This is in stark contrast to their model, where the frictionless Markowitz target does not depend on the past strategy. The coefficients b_1, c_1, c_2, and e_1 appearing in $\bar{\theta}_t^*$ are determined by the system of Riccati ODEs (A.9)–(A.18) in the online supplement, which is in general difficult to solve explicitly, although numerical solutions are readily available. Also, for $\Lambda \in (0, \infty)$, b_1, c_1, c_2, and e_1 can be decoupled from other coefficients only in the special case $\mu - r = \sigma = 0$; however, this special case provides little extra economic insight as both S and X will have a deterministic growth at the risk-free rate.

Next, we illustrate the relationship between the aim $\bar{\theta}_t^*$ for the periodic-DN strategy with the frictionless target via a special example without market closure (that is, $\delta = 0$). From the terminal condition (A.19) of the ODE system in online Supplement A.2, we know that

$$b_1(t_1-) = e^{-\rho t_1}(1-\gamma)(b_1(0) + c_1(0) + e_1(0)),$$
$$c_1(t_1-) = e_1(t_1-) = 0, \text{ and } c_2(t_1-) = e^{\rho t_1} c_2(0).$$

As a result,

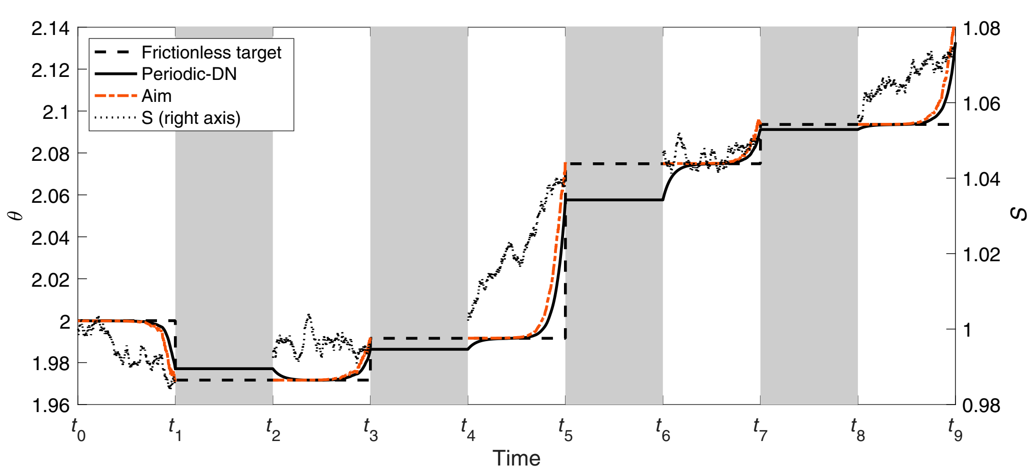
$$\bar{\theta}_0^* = \pi \frac{\beta \bar{x}}{\bar{s}} \text{ and } \bar{\theta}_{t_1-}^* = \pi \frac{\beta X_{t_1}}{S_{t_1}},$$
$$\text{where } \pi = -\frac{b_1(0) + c_1(0) + e_1(0)}{2\beta c_2(0)}.$$

Numerically, we find that π is close to 1. This means that, for each day, the aim $\bar{\theta}_t^*$ is near the current day's frictionless target at market opening, and near the next day's frictionless target $\beta X_{t_1}/S_{t_1}$ at market closing. Furthermore, the transition of aim through daytime is continuous thanks to the continuity of the coefficients in (14). In the following, we show via numerical illustrations that the previous observation also holds in the general case with market closure.

To illustrate how the position level moves toward the aim, Figure 2 plots the aim for the periodic-DN strategy over the position and stock price trajectory in Figure 1. Figure 2 shows that, consistent with (13), the position θ^{PDN} is guided by the moving aim $\bar{\theta}^{PDN}$ and moves toward the aim. However, due to the market frictions, the position can only do so with a finite speed. As a result, it does not reach the aim at market closing (e.g., at t_5), but only reaches it halfway on the next day (between t_6 and t_7).

Because the position level is guided by the aim due to (13) and illustrated in Figure 2, to explain the differences of the position levels in Figure 1 from the three strategies, it is worth comparing their respective aims. Again, we calculate the coefficients b_1, c_1, c_2, and e_1 based on the periodic-DN, periodic-D, and one-day strategies. For illustration, we use the five-day stock

Figure 2. (Color online) Position and Aim



price sample path $\tilde{S}$ as shown in Figure 1 (right-hand side axis), and calculate the path of $\tilde{X}$ under the periodic-DN strategy using θ^{PDN} and (2)–(3). We then calculate the path for the aims $\bar{\theta}^{PDN}$, $\bar{\theta}^{O}$, and $\bar{\theta}^{PD}$ via (14), using the coefficients from the corresponding strategy, $\tilde{X}$ and $\tilde{S}$, as shown in Figure 3.[18] We also include the frictionless target for comparison.

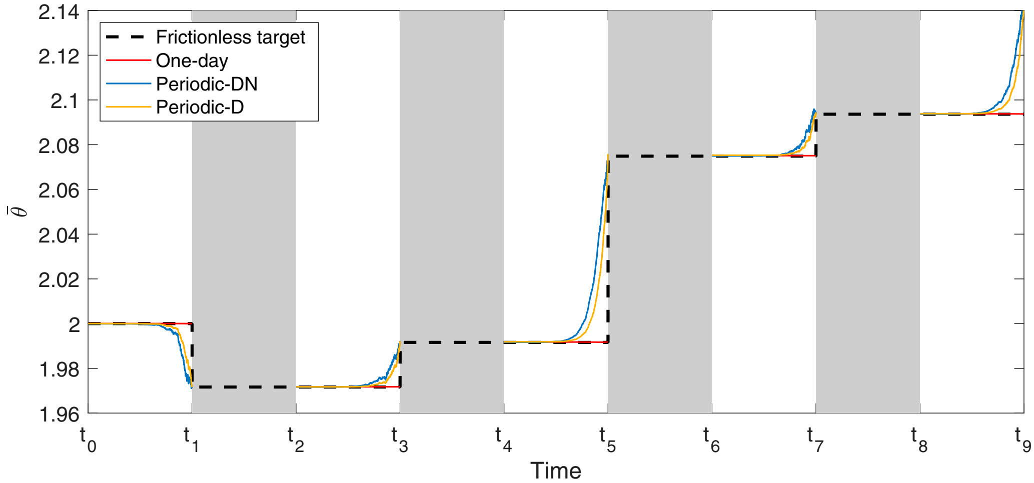
We now look at $\bar{\theta}^{O}$ for the one-day strategy. Figure 3 shows its one key characteristic: it stays on the frictionless target θ^{fl}, which is the optimal position level without market frictions, every day before market closing. As a result, $\bar{\theta}^{O}$ is updated discontinuously at the transition between today at t_{2i-1} and tomorrow t_{2i}, when the target position changes. Such behavior of the aim is because the one-day strategy only focuses on today and does not prepare for tomorrow. Were there no market frictions, such a myopic aim would not be a problem, because the position could always catch up with the aim even if the aim jumps (by rebalancing at an infinite speed). However, due to the presence of market frictions, the actual position level has to move toward this aim gradually after the market opening to catch up with the aim, as demonstrated in Figure 1. In other words, the myopic position level of the one-day strategy illustrated in Figure 1 is rooted in its myopic aim. This results in a large slippage, as we will see in Section 6.

Next, we look at the aim $\bar{\theta}^{PDN}$ for the periodic-DN strategy. In the early trading hours of each day, it overlaps with the frictionless target, similar to that of the one-day strategy. This suggests that, in the early hours, the periodic-DN strategy mainly focuses on reaching the optimal position that minimizes today's slippage. However, $\bar{\theta}^{PDN}$ behaves dramatically differently from $\bar{\theta}^{O}$ in the late hours, when $\bar{\theta}^{PDN}$ moves away from today's frictionless target and toward tomorrow's target; in other words, the aim $\bar{\theta}^{PDN}$ is in front of the today's frictionless target. Furthermore, in stark contrast to the one-day strategy whose aim is updated discontinuously, $\bar{\theta}^{PDN}$ is updated continuously and hits tomorrow's target exactly at market closing time. Therefore, as the market closing draws near, the periodic-DN strategy starts to prepare for tomorrow by looking forward into the future and moving its aim continuously toward the position level that optimizes tomorrow's performance. It should be noted, however, that although the aim $\bar{\theta}^{PDN}$ hits tomorrow's target, the position level θ^{PDN} does not, as illustrated in Figure 1, because the position can only move at a finite speed due to the presence of market frictions. Nevertheless, θ^{PDN} is much closer to tomorrow's target than θ^{O}.

Finally, the target for the periodic-D strategy $\bar{\theta}^{PD}$ behaves similarly to $\bar{\theta}^{PDN}$: it also starts the continuous update in the late hours and hits tomorrow's target exactly at market closing time. However, ignoring the overnight jump risk, $\bar{\theta}^{PD}$ moves toward tomorrow's target at a slower pace as compared with $\bar{\theta}^{PDN}$, which is shown as the difference between the blue and yellow curves in the second half of daytime of each day. Although the difference between $\bar{\theta}^{PD}$ and $\bar{\theta}^{PDN}$ does not appear to be large and eventually vanishes at market closing, it still results in a less optimal position at market closing and a larger exposure of the fund to the overnight jump risk as shown in Section 4, and a larger slippage (see Section 6.1).

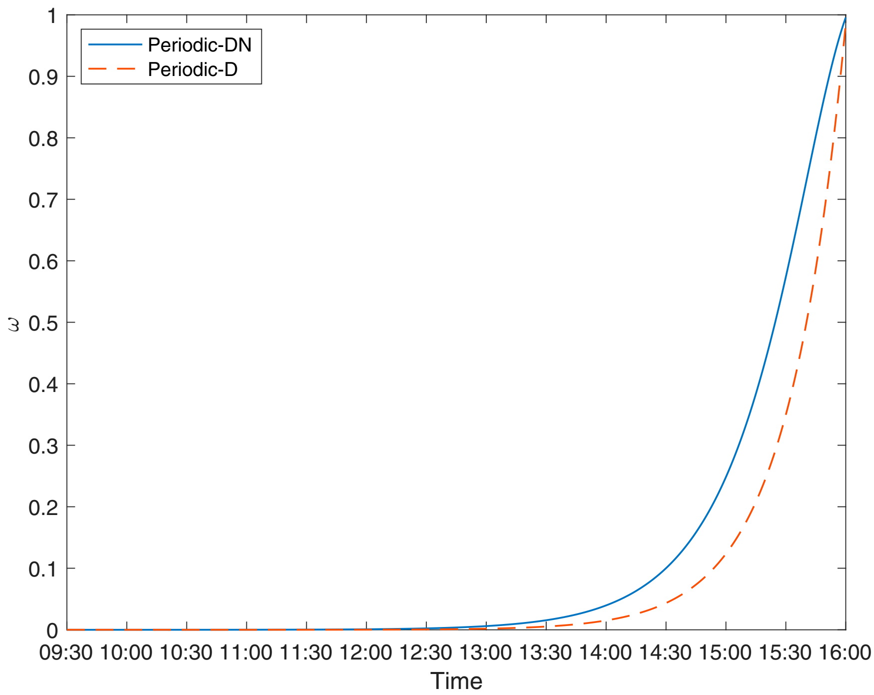
The intraday transition of aim not only holds during the five-day sample path in Figure 3, but also in general. To see this, we calculate ω_t such that $\bar{\theta}_t^{PDN} = (1-\omega_t)\bar{\theta}^{O}_{t_{2i-1}} + \omega_t\bar{\theta}^{O}_{t_{2i+1}}$, for any $t \in [t_{2i}, t_{2i+1}), i \geq 0$. In other

Figure 3. (Color online) Comparison of Aim $\bar{\theta}^{*}$



Note. Note that the aims are only defined for the daytime.

Figure 4. (Color online) Weight Between the Current Target and Future Target, ω, During Daytime



Notes. We simulate 20,000 sample paths based on the default parameters, each containing a 10-year period (2,520 days including daytime and nighttime). On each sample path, the average ω at each time point during daytime is then calculated as the 1% trimmed mean over the 10-year period.

words, ω is the weight of the aim between the current target and the future target; a value of ω close to 0 (respectively 1) means the aim is close to the current (respectively future) target. Figure 4 shows that the weight ω for both the periodic-DN and periodic-D strategies transits from 0 to 1 continuously during daytime, consistent with the observation in Figure 3. In particular, the aim stays essentially on the current target during the first half of daytime, and moves toward the future target in the second half. This is a result of the trade-off between the performance today and in the future. Indeed, because the current day's performance is only measured at market closing, moving the aim away from the current target too early would be detrimental to the performance, whereas moving too late would potentially cause a large deviation from tomorrow's target position and trigger a high friction cost. As a result, the transition in aim of the periodic-DN strategy starts earlier than that of the periodic-D strategy, by considering the overnight jump risk.

The intraday variation of weight ω is a distinct feature of our framework that was not observed in Gârleanu and Pedersen (2013, 2016), in which the weight of the aim remains constant. The fundamental reason is that in the Markowitz-type problem they considered, the performance monitoring and rebalancing occur at the same frequency: both are either at discrete time points or at any instant in the continuous-time version. Therefore, the current target in their model is also continuously updated, resulting in the constant relative importance of the current target and future target, and hence, a constant weight. In contrast, our framework features discrete monitoring at daily market closing with a continuous-time rebalancing. As a result, the current target jumps at each market closing, a set of periodic and discrete time points. This results in the time-varying relative importance of current and future targets, and subsequently, a time-varying weight.

6. Statistics of Daily Slippage

In this section, we first extend the empirical findings on daily slippage in Tang and Xu (2013) by using the updated data till 2020. Then we show the overall slippage of different trading strategies and discuss the impact of market closure on slippage.

6.1. Daily Slippage

Due to the existence of market frictions, it is costly for LETFs to achieve the target leverage ratio exactly. Therefore, one can expect nonnegligible slippage. To check this, we calculate the empirical statistics of slippage (defined as in (4)) for the same set of 12 LETFs as in Tang and Xu (2013),[19] and update the time range to 2006–2020. For comparison, we also included 2011–2020, which is entirely uncovered in Tang and Xu (2013). Launched in 2006, these 12 LETFs were among the first-ever batch of LETFs introduced to the market. They have four underlying indices, S&P 500 (SPX), Dow Jones Industrial Average (INDU), NASDAQ 100 (NDX), and S&P MidCap 400 (MID), and for each index, there is one bull LETF with multiple $\beta = +2$ and two bear LETFs with multiples $\beta = -1, -2$. The results are reported in panel A of Table 1, which confirms that the slippage is significant. For instance, during 2006–2020, the (−1x) fund on SPX has a mean slippage of 1.6917 bps (basis points). In comparison, the absolute values of daily target return (i.e., two times daily S&P 500) have an average of 69.72 bps. This suggests that the slippage accounts for 1.94% of the daily target return, which cannot be ignored.

To quantify the slippage under the periodic-DN, periodic-D, and one-day strategies, we estimate the average daily slippage D_i via simulation. Consistent with panel A, the (−2x) funds tend to have a much higher simulated slippage than the (2x) and (−1x) funds. Furthermore, the magnitude of slippage generated from the simulation result is in a similar range as that from historical data.

The periodic-DN strategy results in the lowest slippage among three models for all four values of Λ and β, followed by the periodic-D strategy, and the one-day strategy has the highest slippage. For instance, for $\Lambda = 10^{-6}$ and (−2x) fund, the periodic-DN strategy has an average slippage about 24% smaller than that of the periodic-D strategy, and less than half of that of the one-day strategy. This indicates that aiming in front of target and accounting for the overnight risk indeed leads to a smaller slippage. In particular, the one-day strategy's slippage is even higher compared with the periodic-DN and periodic-D strategy when Λ is larger. As discussed in Section 5, this can be attributed to the fact that the one-day strategy has a myopic aim, so that it does not perform rebalancing in the latter half of daytime and leads to an inferior position at the next market opening. However, the amount of rebalancing after the next market opening is restricted by the presence of market frictions, especially for large Λ, causing a large slippage. Note that the difference between the slippage of the periodic-D and periodic-DN strategy diminishes as Λ becomes larger. Indeed, as will be shown in Section 7.2, when Λ is large (e.g., $\Lambda = 10^{-3}$), the U-shape for the intraday trading volume of periodic-DN strategy becomes flatter and less asymmetric as observed in Table 5. As a result, the fund puts less emphasis on the preparation for the overnight risk, and therefore the periodic-DN and periodic-D strategy have more similar rebalancing and slippage.

Table 1. Statistics for Slippage

Panel A: Historical data

Index	Statistics	2006–2020 (2x)	2006–2020 (−1x)	2006–2020 (−2x)	2011–2020 (2x)	2011–2020 (−1x)	2011–2020 (−2x)
SPX	Mean	1.5624	1.6917	2.6820	1.4414	1.3503	2.3599
	Standard deviation	1.6581	1.6953	2.9280	1.6073	1.1008	2.5737
INDU	Mean	1.6429	1.7418	2.8077	1.5141	1.4084	2.4833
	Standard deviation	1.9110	1.8095	3.0932	1.7301	1.1693	2.4129
NDX	Mean	1.5846	1.7046	2.6196	1.4336	1.3381	2.1260
	Standard deviation	1.8105	1.6862	2.8148	1.7238	1.1794	2.3432
MID	Mean	1.7790	2.0298	2.7072	1.6594	1.8346	2.3208
	Standard deviation	2.3143	1.8698	3.3925	2.3639	1.5600	3.2562

Panel B: Simulation results

Λ	Statistics	Periodic-DN (2x)	Periodic-DN (−1x)	Periodic-DN (−2x)	Periodic-D (2x)	Periodic-D (−1x)	Periodic-D (−2x)	One-day (2x)	One-day (−1x)	One-day (−2x)
10^{-6}	Mean	0.5590	0.8332	1.5702	0.6943	0.9512	2.0698	1.1037	1.3094	3.3194
	Standard deviation	0.2505	0.5818	1.6114	0.4634	0.6524	2.2348	0.9917	1.0394	3.1840
10^{-5}	Mean	0.7628	0.9568	2.1332	0.8194	1.0321	2.3934	1.2960	1.4195	3.7130
	Standard deviation	0.7171	0.6776	1.6809	0.9415	1.1309	2.4016	1.2503	1.7798	5.9288
10^{-4}	Mean	1.1308	1.2068	3.0859	1.1541	1.2460	3.2017	4.4654	2.1652	6.1326
	Standard deviation	1.1404	1.3398	3.7767	1.3626	1.3839	5.1612	2.8479	2.7922	7.2067
10^{-3}	Mean	1.9925	2.0507	5.8098	2.0027	2.0655	5.8542	5.6674	6.0283	18.8835
	Standard deviation	3.5890	3.6416	11.4236	1.7818	2.5431	8.7676	31.5373	28.1014	62.4603

Notes. Historical LETF NAV returns are adjusted for dividends and the daily management fees. The statistics for simulation results are calculated over the daily slippage for 2,520 × 20,000 day (20,000 sample paths, 2,520 days per sample path), and the estimator is constructed as the 1% trimmed mean of the daily slippages.

Table 2. Statistics for Slippage: Real Index Data

Λ	Statistics	Periodic-DN (2x)	Periodic-DN (−1x)	Periodic-DN (−2x)	Periodic-D (2x)	Periodic-D (−1x)	Periodic-D (−2x)	One-Day (2x)	One-Day (−1x)	One-Day (−2x)
					SPX					
10^{-6}	Mean	0.3654	0.4124	1.0875	0.5034	0.5651	1.5506	0.9019	0.9580	2.7706
	Standard deviation	0.5260	0.4937	1.5211	0.6947	0.6629	2.0798	1.3043	1.3097	4.0467
10^{-5}	Mean	0.5498	0.5877	1.6329	0.6170	0.6745	1.8680	1.0383	1.0890	3.1715
	Standard deviation	0.8124	0.7868	2.4232	0.9078	0.8809	2.7414	1.5392	1.5601	4.7982
10^{-4}	Mean	0.8123	0.8322	2.3872	0.8432	0.8676	2.4920	1.4321	1.5058	4.4594
	Standard deviation	1.1891	1.1958	3.6399	1.2260	1.2344	3.7661	2.1041	2.1318	6.5470
10^{-3}	Mean	1.3015	1.3226	3.8783	1.3113	1.3334	3.9100	3.0570	3.5243	11.0877
	Standard deviation	1.7822	1.8208	5.5189	1.7982	1.8375	5.5707	2.8732	3.3166	10.4958
					INDU					
10^{-6}	Mean	0.3687	0.4180	1.0888	0.5078	0.5387	1.4881	0.8972	0.9294	2.7095
	Standard deviation	0.6739	0.6247	1.9273	0.7097	0.6674	2.0845	1.2380	1.2169	3.7311
10^{-5}	Mean	0.5309	0.5673	1.5764	0.6091	0.6526	1.8419	0.9932	1.0533	3.0659
	Standard deviation	0.9372	0.8832	2.7302	0.9479	0.8970	2.7841	1.4831	1.4535	4.4740
10^{-4}	Mean	0.7494	0.7683	2.2060	0.7859	0.8084	2.3211	1.4303	1.4995	4.4730
	Standard deviation	1.2143	1.1889	3.6508	1.2367	1.2133	3.7345	2.2295	2.2588	6.9943
10^{-3}	Mean	1.2636	1.2642	3.7371	1.2744	1.2789	3.7804	3.1115	3.5568	11.1608
	Standard deviation	1.9035	1.9468	5.9640	1.9249	1.9669	6.0256	3.2244	3.4703	10.8910
					NDX					
10^{-6}	Mean	0.6222	0.6833	1.9211	1.0449	1.0982	3.1821	1.9342	1.9776	5.8650
	Standard deviation	0.7968	0.8129	2.4495	1.5753	1.5951	4.8827	3.1340	3.1926	9.7768
10^{-5}	Mean	1.0538	1.0808	3.1362	1.2625	1.3023	3.7948	2.1339	2.1646	6.4374
	Standard deviation	1.4725	1.4858	4.4605	1.8680	1.8768	5.6847	3.3612	3.4246	10.4463
10^{-4}	Mean	1.5887	1.5865	4.6747	1.6471	1.6537	4.8735	2.7380	2.7687	8.2967
	Standard deviation	2.3181	2.3420	7.0119	2.4270	2.4506	7.3597	3.9347	3.9343	11.9107
10^{-3}	Mean	2.4024	2.3680	7.0371	2.4223	2.3927	7.1111	5.8675	6.5957	20.8493
	Standard deviation	3.5684	3.5462	10.5408	3.5969	3.5706	10.6150	7.4308	7.3282	22.7155
					MID					
10^{-6}	Mean	0.4520	0.5105	1.4177	0.7708	0.7996	2.3238	1.5054	1.5233	4.5534
	Standard deviation	0.5555	0.5517	1.6456	0.8161	0.8064	2.4738	1.6891	1.6791	5.1004
10^{-5}	Mean	0.7412	0.7819	2.2658	0.9174	0.9464	2.7838	1.6482	1.6942	5.0633
	Standard deviation	0.9359	0.9290	2.8262	1.0295	1.0322	3.1520	1.8669	1.8723	5.7261
10^{-4}	Mean	1.2253	1.2223	3.6347	1.2965	1.2970	3.8609	2.5156	2.6480	8.0552
	Standard deviation	1.3860	1.4192	4.2927	1.4119	1.4448	4.3810	2.6038	2.7070	8.3896
10^{-3}	Mean	2.2765	2.2557	6.7799	2.2934	2.2761	6.8430	7.1193	9.6291	33.0852
	Standard deviation	2.4441	2.4877	7.5572	2.4615	2.5059	7.6145	5.9431	8.2481	28.6164

Notes. The mean and standard deviation are calculated from the 132 daily values from September 17, 2020, to March 30, 2021. The three strategies are applied on the minutely data of the underlying index over these 132 days.

Table 1 demonstrates the advantage of the periodic-DN strategy over the periodic-D and one-day strategies using simulated underlying sample paths generated from the geometric Brownian motion model (1). However, does this advantage still exist in a real-world test? To this end, we repeat the slippage test in panel B of Table 1 on the real intraday values of the four indices SPX, INDU, NDX, and MID. Specifically, for each index, we obtain the minutely values from September 17, 2020, to March 30, 2021. As a result, there are 132 trading days (excluding two days on which the market closed earlier than usual), and 391 data points per day. On every trading day, we apply the three strategies on the intraday data and rebalance on a minutely basis. Note that because the closing index value on one day can differ from the opening value on the next day, the overnight price jump still exists, therefore we still expect that the periodic-DN strategy outperforms the periodic-D strategy. As shown in Table 2, the testing results show that the pattern in panel B of Table 1 still holds. That is, for all four indices, the periodic-DN strategy generally leads to the lowest slippage, whereas the one-day strategy leads to the highest slippage. This suggests that the benefit of aiming in front of target is still relevant in a real-world setting.[20]

6.2. Impact of Market Closure

Although the periodic-DN strategy provides the optimal rebalancing strategy to better prepare for the market closure and overnight jump risk, the market closure can still be a contributing factor to the slippage under this strategy. To find out such cost, we compare

Table 3. Cost of Market Closure

		Long market closure			Normal market closure			No market closure		
Λ	Statistics	(2x)	(−1x)	(−2x)	(2x)	(−1x)	(−2x)	(2x)	(−1x)	(−2x)
10^{-6}	Mean	0.6178	0.8623	1.7074	0.5590	0.8332	1.5702	0.4361	0.7963	1.2853
	Standard deviation	0.5414	0.6648	1.7117	0.2505	0.5818	1.6114	0.1929	0.3271	0.4671
10^{-5}	Mean	0.8863	1.0366	2.4351	0.7628	0.9568	2.1332	0.5492	0.8352	1.5714
	Standard deviation	1.0452	1.0209	2.6054	0.7171	0.6776	1.6809	0.4302	0.4636	1.4761
10^{-4}	Mean	1.4290	1.5069	4.0712	1.1308	1.2068	3.0859	0.7972	0.9674	2.1760
	Standard deviation	1.6663	1.9474	4.5514	1.1404	1.3398	3.7767	0.5489	0.8965	2.9398
10^{-3}	Mean	2.6988	2.7629	8.0162	1.9925	2.0507	5.8098	1.3686	1.4333	3.8467
	Standard deviation	2.5767	4.0329	28.8919	3.5890	3.6416	11.4236	1.9423	1.6857	4.0751

Notes. The statistics for simulation results are calculated over the daily slippage for 2,520 × 20,000 day (20,000 sample paths, 2,520 days per sample path), and the estimator is constructed as the 1% trimmed mean of the daily slippages. No market closure means 24-hour trading, normal market closure means 6.5-hour trading, and 17.5-hour market closure, and long market closure means 2.17-hour trading and 21.83-hour market closure. The daily average $\sigma = 0.2$ in all three cases.

the slippage presented in panel B of Table 1 with a 17.5 hours market closure by following the periodic-DN strategy, against the slippage in an otherwise identical market that opens for 24 hours a day (with the volatility $\sigma = 0.2$) by following the periodic-D strategy (which is the same as the optimal periodic-DN strategy in this market as there is no market closure).

The result of this comparison is shown in Table 3. As expected, without market closure, the average slippage becomes lower across all Λ and β, suggesting that the presence of market closure is indeed costly to the fund and increases the slippage. Such reduction is most significant when Λ is small. For instance, with $\Lambda = 10^{-6}$, for the (2x) fund, the mean slippage reduces by 22% from 0.5590 bps to 0.4361 bps. However, for larger Λ, the reduction becomes less significant. This is because the larger frictions limit the fund's ability to rebalance during daytime, which makes the inability to rebalance during nighttime less constraining in comparison.

7. Empirical Findings and Implications

In this section, we discuss empirical findings and implications using our rebalancing model, including the weekend effect, intraday trading volume pattern, and compounding effect.

7.1. Weekend Effect

The long market closure before Monday, as it includes the weekend, can have a particular impact on the

Table 4. Statistics for Historical Slippage Grouped by the Days of the Week

Index	Multiple	Statistics	Monday	Tuesday	Wednesday	Thursday	Friday
SPX	(2x)	Mean	1.8883	1.4806	1.3794	1.5509	1.5385
		Standard deviation	2.0741	1.6002	1.2943	1.5441	1.6803
	(−1x)	Mean	2.0368	1.5724	1.6107	1.6464	1.6175
		Standard deviation	2.3450	1.8130	1.4712	1.3370	1.2954
	(−2x)	Mean	3.2053	2.4861	2.5589	2.5508	2.6483
		Standard deviation	4.0658	2.9198	2.3393	2.2615	2.7407
INDU	(2x)	Mean	2.0597	1.5171	1.5227	1.6463	1.4989
		Standard deviation	2.2424	2.0524	1.6306	1.8397	1.6954
	(−1x)	Mean	2.1050	1.6379	1.6238	1.7639	1.6051
		Standard deviation	2.5846	1.7282	1.5834	1.6663	1.2247
	(−2x)	Mean	3.3248	2.6025	2.6729	2.8084	2.6684
		Standard deviation	4.0078	3.3140	2.5565	2.8942	2.4565
NDX	(2x)	Mean	1.8772	1.4737	1.4868	1.5926	1.5151
		Standard deviation	1.9163	1.9057	1.6145	1.7266	1.8549
	(−1x)	Mean	2.0534	1.5774	1.5997	1.6557	1.6631
		Standard deviation	2.3910	1.6385	1.4576	1.4291	1.3093
	(−2x)	Mean	3.2119	2.4287	2.4812	2.6331	2.3857
		Standard deviation	3.8823	2.7625	2.4076	2.4484	2.2843
MID	(2x)	Mean	2.1454	1.6006	1.7329	1.6869	1.7567
		Standard deviation	2.5895	1.7788	2.6767	2.0334	2.3592
	(−1x)	Mean	2.4490	1.9455	1.9124	1.9372	1.9351
		Standard deviation	2.4577	1.8259	1.6758	1.5145	1.7314
	(−2x)	Mean	3.3692	2.5688	2.3804	2.6562	2.6123
		Standard deviation	4.3976	3.4666	2.7779	2.6760	3.3758

rebalancing.[21] Intuitively, our model suggests that the slippage will be largest on Monday, as the long market closure during the weekend increases the overnight jump risk. We call this the weekend effect.

We first perform a simulation test to estimate the model slippage with a long market closure, for which we assume that a daily trading hour of 2.17 hours and market closure hours of 21.83 hours. This ratio matches the scenario from Friday market opening to Monday market opening, where there is a 6.5-hour trading period followed by a (17.5 + 48)-hour market closure.[22] By comparing the slippages in the long market closure columns and the normal market closure columns in Table 3, our model predicts that the slippage should also be more prominent on Monday, as the fund will not be able to rebalance during the large proportion of market closure.

Next, we test the weekend effect empirically. To this end, we redo the slippage calculation in panel A of Table 1 for 2006–2020 by grouping them on the day of the week. The results are shown in Table 4. For all 12 LETFs, the mean slippage on Monday is always the highest among all days of the week. Furthermore, the slippage on Monday is 23%–33% higher than the average of remaining days for all LETFs. Therefore, the empirical results support the implication of the weekend effect from our model.

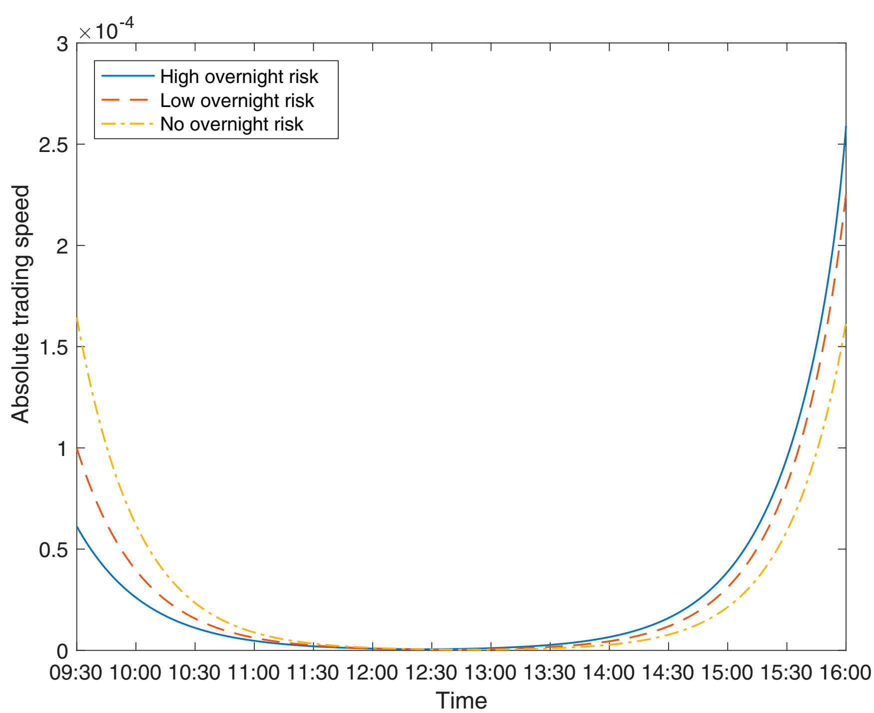
7.2. Intraday Trading Volume

To investigate the intraday trading pattern, we compare strategies with three overnight risk profiles: *high overnight risk* (the benchmark case with $\delta = 17.5/6.5$, $k = 3$), *low overnight risk* (with $\delta = 17.5/6.5, k = 5$) and *no overnight risk* ($\delta = 0$). By taking a larger value of k, the underlying nighttime volatility is smaller compared with daytime, which leads to smaller overnight jumps on average.

Figure 5 shows under all three profiles, the intraday absolute trading speed exhibits the well-known U-shaped curve (see, for instance, Admati and Pfleiderer 1988, Dai et al. 2015). When there is no overnight risk, the curve is symmetric, that is, the absolute trading speed after the market opening is about the same as that before market closing. In the benchmark case with high overnight risk, the trading speed is much higher before market closing, and this bias is less significant for the low overnight risk case. In other words, a larger portion of rebalancing occurs right before market closing when the overnight risk is greater. This is consistent with the observation based on one sample path in Section 4, reflecting the effort of accounting for overnight jump risk.

This implies that the fund's intraday trading activity is at a high level before market closing, a medium level after market opening, and a low level in the

Figure 5. (Color online) Intraday Absolute Trading Speed Distribution



Notes. The distribution is estimated via Monte Carlo simulation. We simulate 20,000 sample paths based on the default parameters, each containing a 10-year period (2,520 days including daytime and nighttime). We calculate the absolute trading speed over daytime of each day, and then calculate the average across all days on all sample paths.

Table 5. Intraday Trading Volume

From To	9:30 10:00	10:00 10:30	10:30 11:00	11:00 11:30	11:30 12:00	12:00 12:30	12:30 13:00	13:00 13:30	13:30 14:00	14:00 14:30	14:30 15:00	15:00 15:30	15:30 16:00	15:55 16:00	Whole day	Market frictions
Periodic-DN strategy																
$\Lambda = 10^{-6}$	0.0032	0.0014	0.0006	0.0002	0.0001	0.0001	0.0001	0.0001	0.0003	0.0008	0.0020	0.0049	0.0122	0.0031	0.0260	0.0005
$\Lambda = 10^{-5}$	0.0024	0.0018	0.0014	0.0011	0.0009	0.0008	0.0009	0.0011	0.0014	0.0018	0.0025	0.0034	0.0046	0.0009	0.0242	0.0014
$\Lambda = 10^{-4}$	0.0014	0.0013	0.0013	0.0012	0.0012	0.0012	0.0012	0.0012	0.0013	0.0014	0.0015	0.0016	0.0017	0.0003	0.0175	0.0060
$\Lambda = 10^{-3}$	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0001	0.0108	0.0205
Periodic-D strategy																
$\Lambda = 10^{-6}$	0.0080	0.0030	0.0011	0.0004	0.0002	0.0001	0.0000	0.0001	0.0001	0.0004	0.0011	0.0029	0.0078	0.0019	0.0252	0.0004
$\Lambda = 10^{-5}$	0.0036	0.0027	0.0020	0.0015	0.0011	0.0009	0.0008	0.0008	0.0010	0.0013	0.0019	0.0026	0.0035	0.0007	0.0236	0.0014
$\Lambda = 10^{-4}$	0.0016	0.0015	0.0014	0.0013	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0013	0.0014	0.0016	0.0003	0.0172	0.0059
$\Lambda = 10^{-3}$	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0008	0.0001	0.0107	0.0190
One-day strategy																
$\Lambda = 10^{-6}$	0.0142	0.0060	0.0026	0.0011	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0248	0.0007
$\Lambda = 10^{-5}$	0.0058	0.0045	0.0034	0.0026	0.0020	0.0015	0.0011	0.0008	0.0006	0.0004	0.0003	0.0002	0.0001	0.0000	0.0233	0.0019
$\Lambda = 10^{-4}$	0.0020	0.0018	0.0016	0.0014	0.0012	0.0011	0.0009	0.0008	0.0006	0.0005	0.0003	0.0002	0.0001	0.0000	0.0124	0.0036
$\Lambda = 10^{-3}$	0.0006	0.0006	0.0005	0.0005	0.0004	0.0004	0.0003	0.0003	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0042	0.0044
One-shot strategy																
	0	0	0	0	0	0	0	0	0	0	0	0	0.0246	0.0246	0.0246	$+\infty$

Notes. The unit of the trading volume is the number of shares. The market friction is daily average, with unit 10^{-4} squared dollars. The statistics are calculated using the trading volume during each of the 13 half-hour periods for 2,520 × 20,000 days (20,000 sample paths, 2,520 days per sample path). Initial values: $s = x = 1, z = 2$.

middle of daytime. This conjecture may be tested in future empirical studies using high-frequency intraday data on LETF rebalancing activity, although the data set does not seem to be available now.

Next, we compare the intraday trading volume of the periodic-DN, periodic-D, and one-day strategies. We split the daytime on each trading day from market opening time, 9:30, to market closing time, 16:00, into 13 subintervals, each with 30 minutes. Then we calculate the total expected trading volume for each subinterval, the trading volume during the last five minutes, as well as total trading volume via simulation, as reported in Table 5. As a comparison, we also include the one-shot strategy that only performs a single lump-sum trade at the market closing each day.

For each Λ, the intraday trading volume exhibits the same U-shaped characteristics as in Figure 5. For each strategy, a larger Λ results in a smaller daily total trading volume, because larger market friction discourages rebalancing. On the other hand, a larger Λ also results in a flatter U-shape; in particular, for $\Lambda = 10^{-3}$ the intraday volume is almost constant, ranging from 0.0009 to 0.0010. Indeed, given a daily total trading volume, spreading the rebalancing more evenly across daytime leads to a smaller (quadratic) friction cost.

Compared with the one-shot trading strategy (the strategy ignoring market frictions and assuming $r = 0$ given in Theorem 1), the periodic-DN strategy has a significantly smaller trading volume near market closing. For instance, the periodic-DN strategy for $\Lambda = 10^{-6}$ has a trading volume 0.0031 during the last five minutes (as opposed to 0.0246 for the one-shot strategy), implying a much smaller market friction cost right before market closing. On the other hand, the periodic-DN strategy has a higher volume right before market closing compared with the periodic-D and one-day strategies, due to accounting for overnight risk and aiming in front of target as discussed earlier.

On a daily level, the total trading volume and the market friction costs from the periodic-DN and periodic-D strategies are on the same level; however, both are greater than those from the one-day strategy, especially when Λ is large. This suggests that the one-day strategy does not rebalance sufficiently. Indeed, following the myopic aim, the one-day strategy barely does any rebalancing in the second half of daytime. Therefore, at the beginning of the next day, the position is way off target, and the fund cannot do enough rebalancing in a limited amount of time due to the market frictions. As a result, under the myopic one-day strategy, the fund ends up with insufficient rebalancing and large slippage, as discussed in Section 6.1. This is especially the case for large Λ, because the amount of rebalancing after the market opening needs to be cut down even further to keep a reasonable level of cost.

8. Conclusion

We study how to rebalance leveraged ETFs in a comprehensive setting, including overnight market closure and market frictions, and obtain analytical solutions in terms of a system of periodic ordinary differential equations. Interestingly, the optimal rebalancing (hedging) strategy confirms the principle of aiming in front of target in Gârleanu and Pedersen (2013), introduced in the setting of asset allocation, although the focus here is on hedging. Our optimal strategy yields a lower slippage and smoother trading pattern compared with existing strategies. The framework in this paper may be extended, in principle, to study more general hedging problems with market frictions and market closure. The challenge is to get analytical solutions for specific hedging problems, although we expect that numerical procedures suggested in the current framework may still work.

We present empirical findings and implications for the weekend effect and return deviations during multiday periods. The discussion about the intraday trading volume leads to the following conjecture from the optimal strategy of our model: LETFs' daily rebalancing activity will be at a high level before market closing, at a medium level after market opening, and at a low level in the middle of daytime. This conjecture may be tested in future empirical studies using high-frequency intraday data on LETF rebalancing activity, although the data set does not seem available now.

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M. Dai is on leave from the National University of Singapore, where part of this work was done.

Endnotes

[1] A fund may continue rebalancing via trading derivatives such as futures after the normal trading hours of the stock market. However, during this period, the trading volume of futures is typically extremely low, making any large rebalancing impractical due to the potentially high price impact. Therefore, it is still optimal for the fund to finish the major, if not all, rebalancing before stock market closing time. Indeed, many researchers, such as Cheng and Madhavan (2009), Bai et al. (2011), and Tuzun (2014), pointed out that the LETFs carry out their rebalancing during the final hour before market closing.

[2] In a system of standard ordinary differential equations, the terminal or initial conditions are specified exogenously. However, in a system of periodic ordinary differential equations, the terminal or initial conditions are determined endogenously using periodicity requirements.

[3] The LETF rebalancing problem in this paper can be linked to the optimal execution problem, which aims to achieve a prespecified position level within a fixed amount of time (e.g., from market opening to market closing) under frictions (see Bertsimas and Lo 1998, Almgren and Chriss 2001, Frei and Westray 2015). However, whereas the LETF rebalancing problem also aims at achieving a position level at the market closure subject to market friction costs, there are some key differences: (i) For our LETF rebalancing problem, the position target at market closure is unknown right after market opening and only fully revealed at market closure, as it depends on the realized fund value X and underlying price S at market closure. As a result, one has to learn the future target as time goes on and watch for the impact of the past strategy from the market opening on the fund value X. In contrast, in the optimal execution literature, the target position at the end of the period is deterministic and known right at the beginning of trade. (ii) The measurement of the deviation from the benchmark in our problem is different from that in the optimal execution literature with performance benchmarks (e.g., the volume-weighted average price in Frei and Westray 2015), which leads to significant technical differences. In optimal execution, the deviation is measured as the aggregated instantaneous distance between stock position (number of shares) and an exogenous target position over a finite horizon. However, in our problem, the deviation is measured as the aggregated distance between simple returns at an infinite set of discrete time points (i.e., daily market closure). This, together with the continuous-time rebalancing, creates a periodic structure in our problem, which is not covered by the optimal execution literature. Such a formulation combining continuous control and discrete monitoring is also used in Keppo et al. (2021) in the different context of optimal dividend policy.

[4] As the fund rebalancing is assumed to cause a temporary price impact, the fundamental price S here is not changed by the rebalancing.

[5] In practice, the LETFs also use (total return) swaps and futures to achieve their daily target exposure. For example, as of April 1, 2021, Ultra S&P 500 from ProShares (a bull LETF on S&P 500 with a multiple of + 2) has a total exposure (notional plus profit and loss) to swaps and futures of \$4.01 billion, in addition to the \$2.93 billion exposure to stocks. By using the total return swaps, the tracking problem is effectively transferred to the swap counterparty. However, the impact on the index is still present, albeit caused by the tracking effort of the swap counterparty. Also, the slippage cost to the fund manager is still present, because the counterparty can pass on the hedging cost to the fund (e.g., in forms of the LIBOR spread). By using futures, the impact is passed on similarly to the underlying asset via arbitrageurs; see Avellaneda and Dobi (2012), Wagalath (2014), and Cheng and Madhavan (2009) for more details. Such slippage cost can be passed on to the fund or to both the fund and the counterparty, depending on variables such as the fund's bargaining power compared with the counterparty. For example, when the fund has a very high bargaining power overall, only a tiny fraction of the costs will be passed on to the fund.

[6] This agrees with the definition of LETFs' daily return in practice. For instance, ProShare Trust (2022) states in its prospectus (p. 306) that "[a] single day is measured from the time the Fund calculates its NAV to the time of the Fund's next NAV calculation," and (p. 671) "the NAV of each Fund . . . is generally determined each business day as of the close of regular trading on the Exchange on which it is listed."

[7] As stated in the prospectus of ProShares Trust (2022) fund (p. 306), "The fund seeks daily investment results, before fees and expenses, that correspond to two times (2x) the daily performance of the index. The fund does not seek to achieve its stated investment objective over a period of time greater than a single day." Here, two times refers to the multiple β and is changed accordingly for other funds with different multiples.

[8] There is a strand of empirical literature on the impact of LETF rebalancing on the underlying index. Tuzun (2014) finds that daily LETF rebalancing leaves an imprint on various U.S. equity categories, triggering price reactions and increased volatility, especially on volatile days. Bai et al. (2011) show similar impacts for LETFs on real estate indices. More recently, Charupat and Miu (2016) give partial support that LETF rebalancing contributes to the index movement before market closing, especially for indices with less liquidity, such as Russell 2000. One may refer to Acharya and Pedersen (2005) and Brunnermeier and Pedersen (2005) for the impact of liquidity risk on asset pricing or trading, and also Cont et al. (2014) and Cartea et al. (2015) about the micro-foundations behind it at the limit order book level.

[9] Note that this cost is not actually paid by the fund in practice; instead, one can regard this as a cost from reputational damage. Large return deviations are costly to LETFs in the sense that they make the fund less attractive and drive away investors.

[10] An alternative formation is to pay the cost out of X. However, this alternative formulation has one major drawback: if X achieved a return higher than the target return right before market closing, the fund manager can reduce the deviation by deliberately incurring cost from market frictions and pulling down X. This strategy seems unethical, because it throws away fund's value via triggering market friction cost intentionally. Also, this alternative formulation brings difficulty to the modeling because the optimal strategy may not be well defined.

[11] We keep the terms $\bar{x}^2$ and $X^2_{t_{2i-1}}$ due to tractability and comparison with the general cases with market friction.

[12] Specifically, $R_i^X = \theta^*_{t_{2i-1}} \cdot \frac{S_{t_{2i+1}} - S_{t_{2i-1}}}{X_{t_{2i-1}}} + \frac{X_{t_{2i-1}} - \theta^*_{t_{2i-1}} S_{t_{2i-1}}}{X_{t_{2i-1}}}(e^{rT}-1) = \frac{\beta X_{t_{2i-1}}}{S_{t_{2i-1}}} \frac{S_{t_{2i+1}} - S_{t_{2i-1}}}{X_{t_{2i-1}}} + (1-\beta)(e^{rT}-1) = \beta \cdot R_i^S + (1-\beta)(e^{rT}-1)$.

[13] More precisely, we define $\mathcal{A} = \left\{\varphi : \varphi_u = 0 \text{ for } u \in \cup_{i=0}^{\infty}(t_{2i+1}, t_{2i+2}) \text{ and } E\left[\sum_{i=0}^{\infty} e^{-\rho t_{2i}} \int_{t_{2i}}^{t_{2i+1}} \varphi_u^2 S_u^2 du\right] < \infty\right\}$.

[14] For $\rho = +\infty$, the value function (10) simplifies to

$$V(t,s,x,z,\bar{s},\bar{x}) = \inf_{\theta \in \mathcal{A}} E\left[\frac{1}{2}\bar{x}^2\left(\beta\left(\frac{S_{t_1}}{\bar{s}}-1\right)-\left(\frac{X_{t_1-}}{\bar{x}}-1\right)\right)^2 + \int_t^{t_1}\frac{1}{2}\Lambda S_u^2 \varphi_u^2 du\right].$$

The corresponding coefficients can be calculated using (A.9)–(A.18) and plugging $\rho = +\infty$ into (A.19) (see the online supplement).

[15] The parameter ρ is a subjective discount rate that represents the relative importance between short-term slippage (e.g., today and the next few days) and long-term slippage. Note that ρ is not the rate for discounting cash flow. If $\rho = 0.6$, then the daily slippage one year later only accounts for about half the importance of the same slippage today. The value for ρ should not be very small (i.e., not on the level of interest rate), because if the current slippage is large, then it can already cause damage to the fund's reputation. In terms of aiming in front of target, a larger ρ means the aim is less in front of target, so as to achieve a better performance today while sacrificing future performance. In the extreme case where $\rho = +\infty$, the fund only cares about today's slippage and does not care about the future slippage at all.

[16] Specifically, by normalizing $T = 1$ in the following calculations, the corresponding effective values are $\mu_d = \mu_n = \frac{0.1}{252(1+\delta)}$, $\sigma = 0.2\sqrt{252 \times (1+\delta)}$, and $\rho = \frac{0.6}{252 \times (1+\delta)}$, $r = \frac{0.01}{252 \times (1+\delta)}$ per trading session (6.5 hours), and $\gamma = \frac{0.01}{252}$ per day.

[17] Specifically, Robert et al. (2012) reported a cost level C_T that is 0.1009% of the dollar transaction amount for all the stock transactions in their sample. In our case, denote $\Delta\theta$ as the daily transaction amount in number of shares, $\frac{\tilde{\Lambda}}{2}S^2(\Delta\theta)^2 = C_T^2$, meaning $\tilde{\Lambda} = 2 \times \left(\frac{C_T}{S\Delta\theta}\right)^2 = 2 \times (0.1009\%)^2 = 2 \times 10^{-6}$.

[18] Note that the paths of $\bar{\theta}^O$ and $\bar{\theta}^{PD}$ illustrated in Figure 3 are based on the same sample path of $\tilde{S}$, as well as the path of $\tilde{X}$ under the periodic-DN strategy, which are different from $\bar{\theta}^O$ and $\bar{\theta}^{PD}$ calculated based on the path of X by following the one-day and periodic-D strategies, respectively. The purpose of doing so is to get a fair comparison: under the periodic-DN strategy, we want to examine what the aim would be at any time, if we were standing in one-day and periodic-D strategies' shoes. Note that $\bar{\theta}^O$ and $\bar{\theta}^{PD}$ are completely determined given the current underlying and fund value, as well as the reference values.

[19] Tang and Xu (2013) reported daily slippage using daily return calculated from LETFs' market price, whereas here we do so using daily return from NAV. Indeed, the fund only has direct control over its NAV rather than its market price, and minimizing the slippage in terms of NAV is the goal of LETFs. However, as pointed out by Tang and Xu (2013), LETFs' total return deviation can be mainly attributed to NAV return deviation.

[20] It should be noted that although Table 2 uses real index data instead of model-based simulated data (in panel B of Table 1), it is still based on the market friction model in Section 2. As a result, the results may still be biased toward our model.

[21] In a different context, Adrian et al. (2020) found that the market makers tend to speed up their inventory liquidation before the end of Friday.

[22] In our model framework, the fund performance is monitored on a daily basis, and the slippage is measured via the daily fund NAV and the underlying returns from one market closing time to the next. Because the slippage during a weekend is measured via the fund NAV and the underlying return from the Friday market closing time to the Monday market closing time, it is natural to rescale the weekend into a day to be consistent with our model. Ideally the weekend should be treated differently from the remaining days without scaling; however, this ideal treatment would bring extra complexity to the model framework.

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