

Name: Solutions

SID: \_\_\_\_\_

## Midterm 2 Version 1

This exam has 7 pages and 6 problems. Make sure that your exam has all 7 pages and that your name is on every page.

Put your name and student ID on every page.

You must show your work and justify your answers to receive full credit unless otherwise stated.

If you need more space, use the back of the pages; clearly indicate when you have done this.

You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

Name: \_\_\_\_\_

SID: \_\_\_\_\_

1. (6 points) Let  $f(x, y, z) = yx^2 + yz^2$  and  $\mathbf{c}(t) = (\sin(t), 2t, \cos(t))$ ,  $0 \leq t \leq 1$ .

Calculate  $\int_{\mathbf{c}} f ds$ .

$$\int_{\mathbf{c}} f ds = \int_0^1 f(\mathbf{c}(t)) \|\mathbf{c}'(t)\| dt$$

$$\mathbf{c}'(t) = (\cos(t), 2, -\sin(t))$$

$$\|\mathbf{c}'(t)\| = \sqrt{\cos^2(t) + 4 + \sin^2(t)} = \sqrt{5}$$

$$f(\mathbf{c}(t)) = f(\sin(t), 2t, \cos(t)) = 2t \sin^2(t) + 2t \cos^2(t) = 2t$$

$$\int_0^1 f(\mathbf{c}(t)) \|\mathbf{c}'(t)\| dt = \int_0^1 2t \cdot \sqrt{5} dt = 2\sqrt{5} \left( \frac{t^2}{2} \Big|_0^1 \right) = \sqrt{5}$$

Name: \_\_\_\_\_

SID: \_\_\_\_\_

2. (6 points) Let  $F(x, y, z) = (x + y, yz, x^2)$  and  $\mathbf{c}(t) = (t^2, 2t^2, 2)$ ,  $0 \leq t \leq 1$ . Calculate  $\int_{\mathbf{c}} F \cdot d\vec{s}$ .

$$\int_{\mathbf{c}} F \cdot d\vec{s} = \int_0^1 F(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

$$\mathbf{c}'(t) = (2t, 4t, 0)$$

$$F(\mathbf{c}(t)) = F(t^2, 2t^2, 2) = (t^2 + 2t^2, 4t^2, t^4) = (3t^2, 4t^2, t^4)$$

$$F(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = (3t^2, 4t^2, t^4) \cdot (2t, 4t, 0) = 6t^3 + 16t^3 + 0 = 22t^3$$

$$\int_0^1 F(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_0^1 22t^3 dt = \frac{22t^4}{4} \Big|_0^1 = \boxed{\frac{11}{2}}$$

Name: \_\_\_\_\_

SID: \_\_\_\_\_

- 3(a). (2 points) Let  $f(x, y, z)$  be a scalar valued function, and let  $\mathbf{c}(t)$ ,  $0 \leq t \leq 2$  be a parametrization of a curve  $C$ . Let  $\mathbf{p}(t)$ ,  $0 \leq t \leq 1$  be another parametrization of  $C$  that travels  $C$  in the same direction as  $\mathbf{c}(t)$ , but twice as fast. If  $\int_{\mathbf{c}(t)} f(x, y, z) ds = 5$ ,

what is  $\int_{\mathbf{p}(t)} f(x, y, z) ds$ ?

5

path integrals do not depend on parametrization

- 3(b). (2 points) Let  $F(x, y, z)$  be a vector field, and let  $\mathbf{c}(t)$ ,  $0 \leq t \leq 1$ , be a parametrization of a curve  $C$ . Let  $\mathbf{p}(t)$ ,  $0 \leq t \leq 1$ , be another parametrization of  $C$  such that  $\mathbf{p}(1/2) = \mathbf{c}(1/2)$  and  $\mathbf{p}'(1/2) = -\mathbf{c}'(1/2)$ . If  $\int_{\mathbf{c}(t)} F(x, y, z) \cdot d\mathbf{s} = 10$ , what is

$\int_{\mathbf{p}(t)} F(x, y, z) \cdot d\mathbf{s}$ ?

-10

$\mathbf{p}(t)$  travels  $C$  in the opposite direction as  $\mathbf{c}(t)$  because  $\mathbf{p}'(1/2) = -\mathbf{c}'(1/2)$  and

- 3(c). (2 points) Let  $f(x, y, z)$  be a scalar valued function, and let  $\Phi(u, v)$ ,  $(u, v)$  in  $D$ , be a parametrization of a surface  $S$ . Let  $\Psi(s, t)$ ,  $(s, t)$  in  $E$ , be another parametrization of  $S$ . Let  $p_0$  be a smooth point on  $S$  and say that  $\Phi(u_0, v_0) = p_0$  and  $\Psi(s_0, t_0) = p_0$ . Let  $\mathbf{n}_0 = \mathbf{T}_u \times \mathbf{T}_v(u_0, v_0)$  be the normal vector at  $p_0$  determined by  $\Phi$ . If the normal vector at  $p_0$  determined by  $\Psi$  (which is  $\mathbf{T}_s \times \mathbf{T}_t(s_0, t_0)$ ) is  $-\mathbf{n}_0$  (so  $\mathbf{T}_s \times \mathbf{T}_t(s_0, t_0) = -\mathbf{n}_0$ ), and  $\iint_{\Phi(u, v)} f(x, y, z) dS = 8$ , what is  $\iint_{\Psi(s, t)} f(x, y, z) dS$ ?

8

Surface integrals of scalar valued functions do not depend on parametrization.

- 3(d). (2 points) Let  $F(x, y, z)$  be a vector field, and let  $\Phi(u, v)$ ,  $(u, v)$  in  $D$ , be a parametrization of a surface  $S$ . Let  $\Psi(s, t)$ ,  $(s, t)$  in  $E$ , be another parametrization of  $S$ . Let  $p_0$  be a smooth point on  $S$  and say that  $\Phi(u_0, v_0) = p_0$  and  $\Psi(s_0, t_0) = p_0$ . Let  $\mathbf{n}_0 = \mathbf{T}_u \times \mathbf{T}_v(u_0, v_0)$  be the normal vector at  $p_0$  determined by  $\Phi$ . If the normal vector at  $p_0$  determined by  $\Psi$  (which is  $\mathbf{T}_s \times \mathbf{T}_t(s_0, t_0)$ ) is  $2\mathbf{n}_0$  (so  $\mathbf{T}_s \times \mathbf{T}_t(s_0, t_0) = 2\mathbf{n}_0$ ), and  $\iint_{\Phi(u, v)} F(x, y, z) d\mathbf{s} = 11$ , what is  $\iint_{\Psi(s, t)} F(x, y, z) d\mathbf{s}$ ?

11

Since  $2\mathbf{n}_0$  and  $\mathbf{n}_0$  both point in the same direction,  $\Psi$  and  $\Phi$  both determine the same orientation of  $S$ . Therefore the surface integrals of  $F$  over  $\Phi$  and  $\Psi$  are the same.

Name: \_\_\_\_\_

SID: \_\_\_\_\_

4. Consider the parametrization  $\Phi(u, v) = (u^2, u^3, v)$ ,  $(u, v)$  in  $\mathbb{R}^2$ , of the surface  $S$  given by the equation  $x^3 = y^2$ .

- (a) (4 points) Calculate  $\mathbf{T}_u \times \mathbf{T}_v$ , the normal vector of  $S$  at  $\Phi(u, v)$  determined by  $\Phi$ .

$$\mathbf{T}_u = \left( \frac{\partial(u^2)}{\partial u}, \frac{\partial(u^3)}{\partial u}, \frac{\partial(v)}{\partial u} \right) = (2u, 3u^2, 0)$$

$$\mathbf{T}_v = \left( \frac{\partial(u^2)}{\partial v}, \frac{\partial(u^3)}{\partial v}, \frac{\partial(v)}{\partial v} \right) = (0, 0, 1)$$

$$\mathbf{T}_u \times \mathbf{T}_v = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 3u^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3u^2 \mathbf{i} - 2u \mathbf{j} + 0 \mathbf{k} = \boxed{(3u^2, -2u, 0)}$$

- (b) (3 points) What is the equation of the tangent plane to  $S$  at the point  $(1, -1, 5)$ ?

$$(1, -1, 5) = \Phi(u, v) \quad \cancel{\Phi(1, 1) = (1, 1, 1)}$$

$$(1, -1, 5) = (u^2, u^3, v) \quad \Phi(-1, 5) = (1, -1, 5)$$

$$u = -1 \quad \begin{cases} 1 = u^2 \\ -1 = u^3 \\ 5 = v \end{cases}$$

$$\mathbf{T}_u \times \mathbf{T}_v(-1, 5) = (3 \cdot (-1)^2, -2 \cdot (-1), 0) = (3, 2, 0)$$

Tangent Plane equation:

$$3(x-1) + 2(y+1) + 0(z-5) = 0$$

$$\boxed{3(x-1) + 2(y+1) = 0}$$

- (c) (3 points) At which points is  $S$  not smooth? Describe them in terms of  $x, y$ , and  $z$ . (Smoothness as determined by  $\Phi$ .)

Not smooth if  $\mathbf{T}_u \times \mathbf{T}_v = (0, 0, 0)$

$$(3u^2, -2u, 0) = (0, 0, 0)$$

$$3u^2 = 0, \quad -2u = 0$$

$$u = 0$$

$v$  - anything

$$\Phi(0, v) = (0, 0, v)$$

$S$  is not smooth at the points where  $x=0, y=0$  and  $z$  is anything:  $z$ -axis.

Name: \_\_\_\_\_

SID: \_\_\_\_\_

5. (5 points) Parametrize the curve that is the intersection of the surfaces given by equations  $z + y^3 - 1 = 0$  and  $x - yz = 0$ .

$$z + y^3 - 1 = 0 \text{ solve for } z: z = 1 - y^3 \quad \text{set } \cancel{t=y} \\ z = 1 - t^3 \quad t = y.$$

$$x - yz = 0, \quad x = yz = t(1 - t^3)$$

$$\boxed{\cancel{c(t)} \quad c(t) = (t(1 - t^3), t, 1 - t^3), \quad -\infty < t < \infty}$$

Name: \_\_\_\_\_

SID: \_\_\_\_\_

6. (5 points) Parametrize the cylinder in  $\mathbb{R}^3$  that is given by the equation  $y^2 - 6y + z^2 - 4z = -12$ . (Hint: Complete the squares.)

Complete squares  $y^2 - 6y = y^2 - 6y + 9 - 9 = (y-3)^2 - 9$

$$z^2 - 4z = z^2 - 4z + 4 - 4 = (z-2)^2 - 4$$

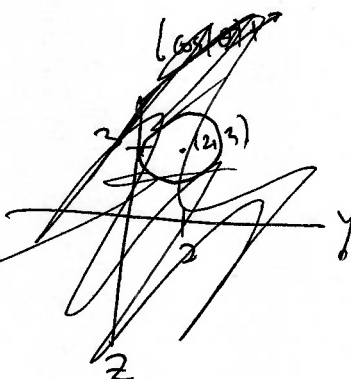
$$y^2 - 6y + z^2 - 4z = -12$$

same as

$$y^2 - 6y + 9 - 9 + z^2 - 4z + 4 - 4 = -12$$

$$(y-3)^2 - 9 + (z-2)^2 - 4 = -12$$

$$\rightarrow (y-3)^2 + (z-2)^2 = 1$$



$$\Phi(u, v) = (v, \cos(u) + 3, \sin(u) + 2), \quad 0 \leq u \leq 2\pi, \quad -\infty < v < \infty$$

