Name: Solutions
SID: ____________

20C, Ferrara, Nov 14, Version 1

Midterm 2

This exam has 7 problems. Make sure that your exam has all 7 problems and that your name is on every page.
Put your name and student ID on every page.
You must show your work and justify your answers to receive full credit unless otherwise stated.
If you need more space, use the pack of the pages; clearly indicate when you have done this.
You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

Name:
SID:

20C, Ferrara, Nov 14, Version 1

1. (5 points) Consider the following limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4+x^2y^2}{x^3y+2y^4}.$$

If the limit exists, compute it. If the limit does not exist, prove that it does not exist. (In either case, you do not need to use ϵ-δ.)

Limit along line $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^4+x^2y^2}{x^3y+2y^4} = \lim_{(0,y)\to(0,0)} \frac{0+0}{0+2y^4}$$

$$= \lim_{y\to 0} \frac{0}{2y^4} = \lim_{y\to 0} 0 = 0$$

Limit along line $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{x^4+x^2y^2}{x^3y+2y^4} = \lim_{(x,x)\to(0,0)} \frac{x^4+x^4}{x^4+2x^4}$$

$$= \lim_{x\to 0} \frac{2x^4}{3x^4} = \lim_{x\to 0} \frac{2}{3} = \frac{2}{3}$$

The limit does not exist because we got different answers when doing the limit along different directions.

20C, Ferrara, Nov 14, Version 1

Name: ____________

SID: ____________

2. (5 points) Find the linear approximation of

$$f(x,y) = x^7y^5 + 7x^4y^2 + 3x^2 + 2x + y + 4$$

at the point $(0,0)$.

Linear approximation of $f(x,y)$ at (x_0,y_0) is:

$$L(x,y) = f(x_0,y_0) + Df(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$f(0,0) = 4$$

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 7x^6y^5 + 28x^3y^2 + 6x + 2 & 5x^7y^4 + 14x^4y + 1 \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$L(x,y) = 4 + \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 4 + 2x + y$$

$$\boxed{\text{Answer: } L(x,y) = 4 + 2x + y}$$

Name: ____________________

20C, Ferrara, Nov 14, Version 1 SID:____________________

3. Let $f(x, y) = x^2 + 3xy + \frac{1}{2}y^2 + 4x + y$.

(a) (5 points) Compute the six partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}.$$

$\frac{\partial f}{\partial x} = 2x + 3y + 4$ $\quad \frac{\partial f}{\partial y} = 3x + y + 1$

$\frac{\partial^2 f}{\partial x^2} = 2$ $\quad \frac{\partial f}{\partial y \partial x} = 3 = \frac{\partial^2 f}{\partial x \partial y}$ $\quad \frac{\partial^2 f}{\partial y^2} = 1$

(b) (5 points) Find the critical points of f and determine which are local maximums, which are local minimums, and which are saddle points.

$\frac{\partial f}{\partial x} = 0$

$2x+3y+4=0$

$2x+3(-3x-1)+4=0$

$2x-9x-3+4=0$

$-7x+1=0$

$x=\frac{1}{7}$

$\frac{\partial f}{\partial y} = 0$

$3x+y+1=0$

$y=-3x-1$

$y=-3(\frac{1}{7})-1$

$y=\frac{-3}{7}-1$ 7

$y=-\frac{11}{7}$

$\boxed{\text{critical point: } (\frac{1}{7}, -\frac{11}{7})}$

$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$= 2 \cdot 1 - 3^2$

$= -7$

$\boxed{(\frac{1}{7}, -\frac{11}{7}) \text{ is a saddle point}}$

Name: ____________________
SID: ____________________

4. (a) (5 points) Calculate the derivative of the function

$$f(x, y) = (2e^x \cos(y), 4y^2 \sin(x))$$

at the point $(0, \frac{\pi}{4})$.

$f_1(x,y) = 2e^x \cos(y)$
$f_2(x,y) = 4y^2 \sin(x)$

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2e^x \cos(y) & -2e^x \sin(y) \\ 4y^2 \cos(x) & 8y \sin(x) \end{pmatrix}$$

$$Df(0, \tfrac{\pi}{4}) = \begin{pmatrix} 2e^0 \cos(\frac{\pi}{4}) & -2e^0 \sin(\frac{\pi}{4}) \\ 4(\frac{\pi}{4})^2 \cos(0) & 8(\frac{\pi}{4}) \sin(0) \end{pmatrix} = \boxed{\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \frac{\pi^2}{4} & 0 \end{pmatrix}}$$

(b) (5 points) Let $g(x, y, z) = (x + y, \pi z)$. Calculate the derivative of $f \circ g$ at the point $(0, 0, \frac{1}{4})$.

$g_1(x,y,z) = x+y, \quad g_2(x,y,z) = \pi z$

$$D(f \circ g)(0,0,\tfrac{1}{4}) = Df(g(0,0,\tfrac{1}{4})) \cdot Dg(0,0,\tfrac{1}{4}), \quad g(0,0,\tfrac{1}{4}) = (0+0, \tfrac{\pi}{4}) = (0, \tfrac{\pi}{4})$$

$$Df(g(0,0,\tfrac{1}{4})) = Df(0, \tfrac{\pi}{4}) = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \frac{\pi^2}{4} & \frac{-\sqrt{2}}{2\pi} \end{pmatrix}$$

$$Dg = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix}$$

$$Dg(0,0,\tfrac{1}{4}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix},$$

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \frac{\pi^2}{4} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & -\pi\sqrt{2} \\ \frac{\pi^2}{4} & \frac{\pi^2}{4} & 0 \end{pmatrix}$$

$$\boxed{\text{Answer: } \begin{pmatrix} \sqrt{2} & \sqrt{2} & -\pi\sqrt{2} \\ \frac{\pi^2}{4} & \frac{\pi^2}{4} & 0 \end{pmatrix}}$$

Name: ____________

SID: ____________

5. (5 points) Parametrize the solutions to the equationi $y = x^3$ in $\mathbb{R}^2$ by a curve $c(t)$ that has speed 3 at the point $(0, 0)$, and such that the velocity vector of the curve $c(t)$ at the point $(0, 0)$ has negative x-component.

Parametrization of $y = x^3$

$c(t) = (at, a^3t^3)$ where a is a nonzero real number.

$c(0) = (0,0)$ so $t=0$ corresponds to point $(0,0)$

velocity vector $c'(t) = (a, 3a^3t^2)$

at $t=0$ $c'(0) = (a, 0)$

speed $\|c'(0)\| = \sqrt{a^2} = |a|$.

speed 3 $|a| = 3$ $a = 3$ or -3

negative x-component $a = -3$

$\boxed{\text{Answer } c(t) = (-3t, -27t^3)}$

Name: ______________________

SID: ______________________

6. (5 points) Calculate the directional derivative of $f(x, y, z) = xy + xz$ at the point $(1, 1, 1)$ in the direction with positive x-component that is normal to the surface given by $-2x^2 - y^2 + 3z^2 = \cancel{1}\ 0$ at the point $(1, 1, 1)$.

SEE NEXT PAGE FOR SOLUTION

Normal to surface $-2x^2 - y^2 + 3z^2 = 0$ is

$$(-4x, -2y, 6z)$$

any scalar multiple of this is normal to the surface.

at $(1,1,1)$ normal is $(-4, -2, 6)$

make normal have positive x-component: $(4, 2, -6)$

make normal have magnitude 1: $\sqrt{4^2 + 2^2 + (-6)^2} = \sqrt{16+4+36} = \sqrt{56}$

Direction for directional derivative is $\frac{(4,2,-6)}{\sqrt{56}}$

Directional derivative: $\nabla f(\vec{x}_0) \cdot \vec{v}$, $\vec{v} = \frac{(4,2,-6)}{\sqrt{56}}$

$\vec{x}_0 = (1,1,1)$

$\nabla f = (y+z, x, x)$, $\nabla f(1,1,1) = (2,1,1)$

$$\nabla f(1,1,1) \cdot \frac{(1,1,-3)}{\sqrt{11}} = \frac{1}{\sqrt{11}}\left((2,1,1)\cdot(1,1,-3)\right) = \frac{1}{\sqrt{11}}(2+1-3)$$

$$= 0$$

$\boxed{\text{Answer: } 0}$

Solution for #6:

normal to surface $-2x^2 - y^2 + 3z^2 = 0$ is

$(-4x, -2y, 6z)$

at $(1,1,1)$ normal is $(-4, -2, 6)$ ← any scalar multiple of this is normal to the surface.

make normal have positive x-component: $(4, 2, -6)$

make normal have magnitude 1: $\|(4,2,-6)\| = \sqrt{4^2+2^2+(-6)^2}$

$= \sqrt{16+4+36} = \sqrt{56}$

$$\frac{(4,2,-6)}{\sqrt{56}}$$

Directional derivative is $\nabla f(\vec{x}_0)\cdot\vec{v}$, $\vec{x}_0 = (1,1,1)$

$\vec{v} = \dfrac{(4,2,-6)}{\sqrt{56}}$

$\nabla f = (y+z, x, x)$, $\nabla f(1,1,1) = (2,1,1)$

$$\nabla f(1,1,1)\cdot\frac{(4,2,-6)}{\sqrt{56}} = \frac{1}{\sqrt{56}}(2,1,1)\cdot(4,2,-6) = \frac{1}{\sqrt{56}}(8+2-6) = \frac{4}{\sqrt{56}}$$

Answer: $\dfrac{4}{\sqrt{56}}$

Name: ________________

SID: ________________

7. (10 points) Find the point at which the function $f(x, y, z) = 9x + y + 6z$ has a maximum on the surface S given by $\frac{1}{3}x^3 + y + 2z^3 = 1$. Also find the point at which f has a minimum on S.

Lagrange: $\nabla f = \lambda \nabla g$, $g(x,y,z) = \frac{1}{3}x^3 + y + 2z^3$

$\nabla f = (9, 1, 6)$, $\nabla g = (x^2, 1, 6z^2)$

4 equations:

$9 = \lambda x^2$

$1 = \lambda$

$6 = \lambda 6z^2$

$\frac{1}{3}x^3 + y + 2z^3 = 1$

$\lambda = 1$

$6 = 6z^2$ $\quad$ $9 = x^2$

$1 = z^2$ $\quad$ -3 or $3 = x$

-1 or $1 = z$

$y = 1 - \frac{1}{3}x^3 - 2z^3$

$x = 3, \; z = 1$

$y = 1 - \frac{27}{3} - 2 = 1 - 9 - 2 = -10$

$(3, -10, 1)$

$x = 3, \; z = -1$

$y = 1 - \frac{27}{3} + 2 = 1 - 9 + 2 = -6$

$(3, -6, -1)$

$x = -3, \; z = 1$

$y = 1 + \frac{27}{3} - 2 = 1 + 9 - 2 = 8$

$(-3, 8, 1)$

$x = -3, \; z = -1$

$y = 1 + \frac{27}{3} + 2 = 1 + 9 + 2 = 12$

$(-3, 12, -1)$

4 points: $(3, -10, 1)$, $(3, -6, -1)$, $(-3, 8, 1)$, $(-3, 12, -1)$

$f(3, -10, 1) = 27 - 10 + 6 = 23$ ← max

$f(3, -6, -1) = 27 - 6 - 6 = 15$

$f(-3, 8, 1) = -27 + 8 + 6 = -13$

$f(-3, 12, -1) = -27 + 12 - 6 = -21$ ← min.

$\boxed{\begin{array}{l} f \text{ has a max at } (3, -10, 1) \\ f \text{ has a min at } (-3, 12, -1) \end{array}}$