20C, Ferrara, Nov 14, Version 2

Name: ____________________
SID: ____________________

Midterm 2

This exam has BLANK pages and BLANK problems. Make sure that your exam has all BLANK pages and that your name is on every page.
Put your name and student ID on every page.
You must show your work and justify your answers to receive full credit unless otherwise stated.
If you need more space, use the pack of the pages; clearly indicate when you have done this.
You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

20C, Ferrara, Nov 14, Version 2

Name:

SID:

1. (5 points) Consider the following limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4+2x^2y^2}{x^3y+y^4}.$$

If the limit exists, compute it. If the limit does not exist, prove that it does not exist. (In either case, you do not need to use ϵ-δ.)

Limit along line $X=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^4+2x^2y^2}{x^3y+y^4} = \lim_{(0,y)\to(0,0)} \frac{0+0}{0+y^4}$$

$$= \lim_{y\to 0} \frac{0}{y^4} = \lim_{y\to 0} 0 = 0$$

Limit along line $y=X$

$$\lim_{(x,y)\to(0,0)} \frac{X^4+2x^2y^2}{x^3y+y^4} = \lim_{(x,x)\to(0,0)} \frac{x^4+2x^4}{x^4+x^4}$$

$$= \lim_{x\to 0} \frac{3x^4}{2x^4} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}$$

The limit does not exist because we get different answers when doing the limit along different directions.

20C, Ferrara, Nov 14, Version 2

Name: ____________

SID: ____________

2. (5 points) Find the linear approximation of

$$f(x, y) = x^8y^4 + 6x^3y^2 + 2x^2 + 3x + 2y + 1$$

at the point $(0, 0)$.

Linear approximation of $f(x,y)$ at (x_0, y_0) is.

$$L(x,y) = f(x_0,y_0) + Df(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$f(0,0) = 1$

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 8x^7y^4 + 18x^2y^2 + 4x + 3 & 4x^8y^3 + 12x^3y + 2 \end{bmatrix}$$

$Df(0,0) = \begin{bmatrix} 3 & 2 \end{bmatrix}$

$$L(x,y) = 1 + \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 + 3x + 2y$$

Answer: $\boxed{L(x,y) = 1 + 3x + 2y}$

20C, Ferrara, Nov 14, Version 2

Name: ____________________

SID: ____________________

3. Let $f(x, y) = \frac{1}{2}x^2 + 5xy + 2y^2 - 3x + 6y$.

(a) (5 points) Compute the six partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}.$$

$\frac{\partial f}{\partial x} = x + 5y - 3$ $\quad \frac{\partial f}{\partial y} = 5x + 4y + 6$

$\frac{\partial^2 f}{\partial x^2} = 1$ $\quad \frac{\partial^2 f}{\partial y \partial x} = 5 = \frac{\partial^2 f}{\partial x \partial y}$ $\quad \frac{\partial^2 f}{\partial y^2} = 4$

(b) (5 points) Find the critical points of f and determine which are local maximums, which are local minimums, and which are saddle points.

$\frac{\partial f}{\partial x} = 0$

$x + 5y - 3 = 0$

$x = 3 - 5y$

$x = 3 - 5 = -2$

$\frac{\partial f}{\partial y} = 0$

$5x + 4y + 6 = 0$

$5(3 - 5y) + 4y + 6 = 0$

$15 - 25y + 4y + 6 = 0$

$-21y + 21 = 0$

$y = 1$

critical point: $(-2, 1)$

$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$= 4 \cdot 1 - 5^2 = -21$

$(-2,1)$ is a saddle point

20C, Ferrara, Nov 14, Version 2

Name: ____________
SID: ____________

4. (a) (5 points) Calculate the derivative of the function

$$f(x,y) = (4y^2\sin(x), 2e^x\cos(y))$$

at the point $(0, \frac{\pi}{4})$.

$f_1(x,y) = 4y^2\sin(x)$
$f_2(x,y) = 2e^x\cos(y)$

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 4y^2\cos(x) & 8y\sin(x) \\ 2e^x\cos(y) & -2e^x\sin(y) \end{pmatrix}$$

$$Df\left(0, \tfrac{\pi}{4}\right) = \begin{pmatrix} 4(\frac{\pi}{4})^2\cos(0) & 8(\frac{\pi}{4})\sin(0) \\ 2e^0\cos(\frac{\pi}{4}) & -2e^0\sin(\frac{\pi}{4}) \end{pmatrix} = \boxed{\begin{pmatrix} \frac{\pi^2}{4} & 0 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}}$$

(b) (5 points) Let $g(x,y,z) = (y, \pi z + \pi x)$. Calculate the derivative of $f \circ g$ at the point $(\frac{1}{8}, 0, \frac{1}{8})$.

$g_1(x,y,z) = y$, $g_2(x,y,z) = \pi z + \pi x$

$$D(f\circ g)\left(\tfrac{1}{8}, 0, \tfrac{1}{8}\right) = Df\left(g\left(\tfrac{1}{8}, 0, \tfrac{1}{8}\right)\right) \cdot Dg\left(\tfrac{1}{8}, 0, \tfrac{1}{8}\right), \quad g\left(\tfrac{1}{8}, 0, \tfrac{1}{8}\right) = \left(0, \tfrac{\pi}{8} + \tfrac{\pi}{8}\right) = \left(0, \tfrac{\pi}{4}\right)$$

$$Df\left(g\left(\tfrac{1}{8}, 0, \tfrac{1}{8}\right)\right) = Df\left(0, \tfrac{\pi}{4}\right) = \begin{pmatrix} \frac{\pi^2}{4} & 0 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

$$Dg = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \pi & 0 & \pi \end{pmatrix} = Dg\left(\tfrac{1}{8}, 0, \tfrac{1}{8}\right)$$

$$\begin{pmatrix} \frac{\pi^2}{4} & 0 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \pi & 0 & \pi \end{pmatrix} = \begin{pmatrix} 0 & \frac{\pi^2}{4} & 0 \\ -\pi\sqrt{2} & \sqrt{2} & -\pi\sqrt{2} \end{pmatrix}$$

$$\boxed{\text{Answer: } \begin{pmatrix} 0 & \frac{\pi^2}{4} & 0 \\ -\pi\sqrt{2} & \sqrt{2} & -\pi\sqrt{2} \end{pmatrix}}$$

5. (5 points) Parametrize the solutions to the equationi $y = x^4$ in $\mathbb{R}^2$ by a curve $c(t)$ that has speed 2 at the point $(0, 0)$, and such that the velocity vector of the curve $c(t)$ at the point $(0, 0)$ has negative x-component.

Parametrization of $y = x^4$

$c(t) = (at, a^4 t^4)$ where a is a nonzero real number.

$c(0) = (0,0)$ $t=0$ corresponds to point $(0,0)$

velocity vector $c'(t) = (a, 4a^4t^3)$

at $t = 0$ $c'(0) = (a, 0)$

speed $\|c'(0)\| = \sqrt{a^2} = |a|$

speed 2 $|a| = 2$ so $a = 2$ or -2

negative x-component $a = -2$

$$\boxed{\text{Answer } c(t) = (-2t, 8t^4)}$$

Name: ______________

SID: ______________

6. (5 points) Calculate the directional derivative of $f(x, y, z) = yz + xz$ at the point $(1, 1, 1)$ in the direction with positive x-component that is normal to the surface given by $-2x^2 - z^2 + 3y^2 = \cancel{1}$ 0 at the point $(1, 1, 1)$.

normal to surface $-2x^2 - z^2 + 3y^2 = 0$ is

$$(-4x, 6y, -2z)$$

at $(1,1,1)$ normal is $(-4, 6, -2)$

any scalar multiple of this is normal to the surface.

Make normal have positive x-component: $(4, -6, 2)$

Make normal have magnitude 1: $\|(4,-6,2)\| = \sqrt{4^2 + (-6)^2 + 2^2}$

$= \sqrt{16+36+4}$

$= \sqrt{56}$

$\frac{(4,-6,2)}{\sqrt{56}}$

Directional derivative is

$\nabla f(\vec{x}_0)\cdot\vec{v}$, $\vec{x}_0 = (1,1,1)$

$\vec{v} = \frac{(4,-6,2)}{\sqrt{56}}$

$\nabla f = (z, z, x+y)$, $\nabla f(1,1,1) = (1,1,2)$

$\nabla f(1,1,1)\cdot\frac{(4,-6,2)}{\sqrt{56}} = \frac{1}{\sqrt{56}}(1,1,2)\cdot(4,-6,2) = \frac{1}{\sqrt{56}}(4-6+4) = \frac{2}{\sqrt{56}}$

$\boxed{\text{Answer: } \frac{2}{\sqrt{56}}}$

Name: ______________
SID: ______________

7. (10 points) Find the point at which the function $f(x, y, z) = 9x + y + 6z$ has a maximum on the surface S given by $\frac{1}{3}x^3 + y + 2z^3 = 1$. Also find the point at which f has a minimum on S.

Lagrange: $\nabla f = \lambda \nabla g$, $g(x,y,z) = \frac{1}{3}x^3 + y + 2z^3$

$\nabla f = (9, 1, 6)$, $\nabla g = (x^2, 1, 6z^2)$

4 equations:

$9 = \lambda x^2$

$1 = \lambda$

$6 = \lambda 6 z^2$

$\frac{1}{3}x^3 + y + 2z^3 = 1$

$\lambda = 1$

$6 = 6z^2$

$1 = z^2$

-1 or $1 = z$

$9 = x^2$

-3 or $3 = x$

$y = 1 - \frac{1}{3}x^3 - 2z^3$

$x = 3$, $z = 1$

$y = 1 - \frac{27}{3} - 2 = 1 - 9 - 2 = -10$

$(3, -10, 1)$

$x = 3$, $z = -1$

$y = 1 - \frac{27}{3} + 2 = 1 - 9 + 2 = -6$

$(3, -6, -1)$

$x = -3$, $z = 1$

$y = 1 + \frac{27}{3} - 2 = 1 + 9 - 2 = 8$

$(-3, 8, 1)$

$x = -3$, $z = -1$

$y = 1 + \frac{27}{3} + 2 = 1 + 9 + 2 = 12$

$(-3, 12, -1)$

4 points: $(3, -10, 1)$, $(3, -6, -1)$, $(-3, 8, 1)$, $(-3, 12, -1)$

$f(3, -10, 1) = 27 - 10 + 6 = 23$ ← max

$f(3, -6, -1) = 27 - 6 - 6 = 15$

$f(-3, 8, 1) = -27 + 8 + 6 = -13$

$f(-3, 12, -1) = -27 + 12 - 6 = -21$ ← min.

f has ~~a~~ max at $(3, -10, 1)$

f has ~~a~~ min at $(-3, 12, -1)$