Name: Solutions SID:

Midterm 2 Version 2

This exam has 7 pages and 6 problems. Make sure that your exam has all 7 pages and that your name is on every page.

Put your name and student ID on every page.

You must show your work and justify your answers to receive full credit unless otherwise stated.

If you need more space, use the pack of the pages; clearly indicate when you have done this.

You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

Name: ______________________ SID:______________________

1. (6 points) Let $f(x, y, z) = xy^2 + xz^2$ and $\mathbf{c}(t) = (t, \cos(t), \sin(t))$, $1 \le t \le 3$. What is $\int_{\mathbf{c}} f ds$?

$$\int_c f ds = \int_1^3 f(c(t)) \|c'(t)\| dt$$

$$c'(t) = (1, -\sin(t), \cos(t)).$$

$$\|c'(t)\| = \sqrt{1 + \sin^2(t) + \cos^2(t)} = \sqrt{2}$$

$$f(c(t)) = f(t, \cos(t), \sin(t)) = t\cos^2(t) + t\sin^2(t) = t$$

$$\int_1^3 f(c(t))\|c'(t)\| dt = \int_1^3 \sqrt{2}t \, dt = \sqrt{2}\left(\frac{t^2}{2}\Big|_1^3\right)$$

$$= \sqrt{2}\left(\frac{9}{2} - \frac{1}{2}\right) =$$

$$= \sqrt{2}\left(\frac{8}{2}\right) = \boxed{4\sqrt{2}}$$

Name: ______________________ SID: ______________________

2. (6 points) Let $F(x, y, z) = (x + y, yz, x^2)$ and $\mathbf{c}(t) = (2t^2, t^2, 1)$, $0 \leq t \leq 1$. What is $\int_{\mathbf{c}} F \cdot d\vec{s}$?

$$\int_C F \cdot d\vec{s} = \int_0^1 F(c(t)) \cdot c'(t)\, dt$$

$$c'(t) = (4t, 2t, 0)$$

$$F(c(t)) = (2t^2 + t^2, t^2, 4t^4) = (3t^2, t^2, 4t^4)$$

$$F(c(t)) \cdot c'(t) = (3t^2, t^2, 4t^4) \cdot (4t, 2t, 0)$$

$$= 12t^3 + 2t^3 = 14t^3$$

$$\int_0^1 F(c(t)) \cdot c'(t)\, dt = \int_0^1 14t^3\, dt = 14\left(\frac{t^4}{4}\Big|_0^1\right) = \frac{14}{4} = \boxed{\frac{7}{2}}$$

Name: ____________________ SID:____________________

3(a). (2 points) Let $f(x,y,z)$ be a scalar valued function, and let $\mathbf{c}(t)$, $0 \le t \le 1$, be a parametrization of a curve $\mathcal{C}$. Let $\mathbf{p}(t)$, $0 \le t \le 1$, be another parametrization of $\mathcal{C}$ such that $\mathbf{p}(1/2) = \mathbf{c}(1/2)$ and $\mathbf{p}'(1/2) = -\mathbf{c}'(1/2)$. If $\int_{\mathbf{c}(t)} f(x,y,z)ds = 5$, what is $\int_{\mathbf{p}(t)} f(x,y,z)ds$? $\boxed{5}$ path integrals do not depend on parametrization.

3(b). (2 points) Let $F(x,y,z)$ be a vector field, and let $\mathbf{c}(t)$, $0 \le t \le 2$, be a parametrization of a curve $\mathcal{C}$. Let $\mathbf{p}(t)$, $0 \le t \le 1$ be another parametrization of $\mathcal{C}$ that travels $\mathcal{C}$ in the same direction as $\mathbf{c}(t)$, but twice as fast. If $\int_{\mathbf{c}(t)} F(x,y,z)\cdot d\vec{s} = 10$, what is $\int_{\mathbf{p}(t)} F(x,y,z)\cdot d\vec{s}$? $\boxed{10}$ $p(t)$ and $c(t)$ both trace $\mathcal{C}$ in the same direction

3(c). (2 points) Let $f(x,y,z)$ be a scalar valued function, and let $\Phi(u,v)$, (u,v) in D, be a parametrization of a surface S. Let $\Psi(s,t)$, (s,t) in E, be another parametrization of S. Let p_0 be a smooth point on S and say that $\Phi(u_0,v_0) = p_0$ and $\Psi(s_0,t_0) = p_0$. Let $\vec{\mathbf{n}}_0 = \mathbf{T}_u \times \mathbf{T}_v(u_0,v_0)$ be the normal vector at p_0 determined by Φ. If the normal vector at p_0 determined by Ψ (which is $\mathbf{T}_s \times \mathbf{T}_t(s_0,t_0)$) is $2\vec{\mathbf{n}}_0$ (so $\mathbf{T}_s \times \mathbf{T}_t(s_0,t_0) = 2\vec{\mathbf{n}}_0$) and $\iint_{\Phi(u,v)} f(x,y,z)dS = 12$, what is $\iint_{\Psi(t,s)} f(x,y,z)dS$? $\boxed{12}$ surface integrals of scalar functions do not depend on parametrization.

3(d). (2 points) Let $F(x,y,z)$ be a vector field, and let $\Phi(u,v)$, (u,v) in D, be a parametrization of a surface S. Let $\Psi(s,t)$, (s,t) in E, be another parametrization of S. Let p_0 be a smooth point on S and say that $\Phi(u_0,v_0) = p_0$ and $\Psi(s_0,t_0) = p_0$. Let $\vec{\mathbf{n}}_0 = \mathbf{T}_u \times \mathbf{T}_v(u_0,v_0)$ be the normal vector at p_0 determined by Φ. If the normal vector at p_0 determined by Ψ (which is $\mathbf{T}_s \times \mathbf{T}_t(s_0,t_0)$) is $-\vec{\mathbf{n}}_0$ (so $\mathbf{T}_s \times \mathbf{T}_t(s_0,t_0) = -\vec{\mathbf{n}}_0$), and $\iint_{\Phi(u,v)} F(x,y,z)d\vec{\mathbf{S}} = 7$, what is $\iint_{\Psi(t,s)} F(x,y,z)d\vec{\mathbf{S}}$? $\boxed{-7}$

Since $\vec{n}_0$ and $-\vec{n}_0$ point in opposite directions, Φ and Ψ give S opposite orientations. Therefore

$$\iint_{\Psi} F\cdot d\vec{S} = -\iint_{\Phi} F\cdot d\vec{S}$$

Name: ____________________ SID:____________________

4. Consider the parametrization $\Phi(u,v) = (v, u^2, u^3)$, (u,v) in $\mathbb{R}^2$, of the surfact S given by the equation $y^3 = z^2$.

(a) (4 points) Calculate $\mathbf{T}_u \times \mathbf{T}_v$, the normal vector of S at $\Phi(u,v)$, determined by Φ.

$T_u = \left(\frac{\partial(v)}{\partial u}, \frac{\partial(u^2)}{\partial u}, \frac{\partial(u^3)}{\partial u}\right) = (0, 2u, 3u^2)$ $T_v = \left(\frac{\partial(v)}{\partial v}, \frac{\partial(u^2)}{\partial v}, \frac{\partial(u^3)}{\partial v}\right) = (1,0,0)$

$T_u \times T_v = \det\begin{pmatrix} i & j & k \\ 0 & 2u & 3u^2 \\ 1 & 0 & 0 \end{pmatrix} = 0\cdot i + 3u^2 j - 2uk = \boxed{\cancel{(0, +3u^2, -2u)}}$

$\boxed{(0, 3u^2, -2u)}$

(b) (3 points) What is the equation of the tangent plane to S at the point $(7,4,8)$?

$\Phi(u,v) = (7,4,8)$ $\Phi(2,7) = (7,4,8)$

$(v, u^2, u^3) = (7,4,8)$ $T_u \times T_v(2,7) = (0, +3\cdot 4, -2\cdot 2)$

$v = 7$ $= (0, 12, -4)$

$u^2 = 4$, $u^3 = 8$ } $u = 2$

$\boxed{12(y-4) - 4(z-8) = 0}$

Tangent Plane: $0\cdot(x-7) + 12\cdot(y-4) - 4(z-8) = 0$

(c) (3 points) At which points is S not smooth? Describe them in terms of x, y and z. (Smoothness as determined by Φ.)

not smooth if $T_u \times T_v = (0,0,0)$.

$(0, 3u^2, -2u) = (0,0,0)$

$3u^2 = 0$, $-2u = 0$

$u = 0$, v-anything.

$\Phi(0,v) = (v, 0, 0)$

All x,y,z where $y=0$ and $z=0$ and x is anything: x-axis.

Name: ____________________ SID:____________________

5. (5 points) Parametrize the curve that is the intersection of the surfaces given by the equations $y + z^2 - 2 = 0$ and $x - yz^2 = 0$.

$y + z^2 - 2 = 0$, solve for y: $y = 2 - z^2$

set $z = t$, then.

$y = 2 - t^2$

$x - yz^2 = 0$, solve for x: $x = yz^2$

$z = t$, $y = 2 - t^2$ so

$x = (2-t^2)t^2$

$$\boxed{c(t) = \left((2-t^2)t^2,\ 2-t^2,\ t \right), \quad -\infty < t < \infty}$$

Name: ____________________ SID:____________________

6. (5 points) Parametrize the cylinder in $\mathbb{R}^3$ given by the equation $y^2-4y+z^2-6z=-12$. (Hint: Complete the squares.)

Complete squares: $y^2-4y = y^2-4y+4-4 = (y-2)^2-4$

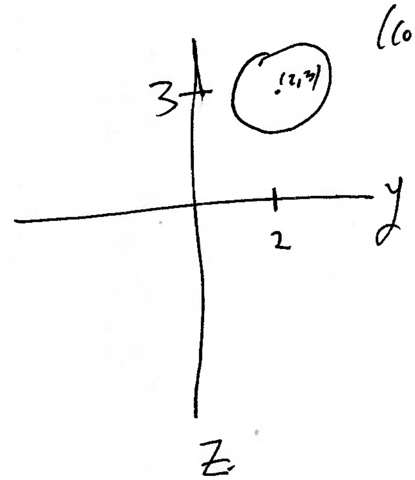
$z^2-6z = z^2-6z+9-9 = (z-3)^2-9$

$y^2-4y+z^2-6z=-12$

$(y-2)^2-4+(z-3)^2-9=-12$

$(y-2)^2+(z-3)^2=1.$

x- free.



$(\cos(\theta)+2, \sin(\theta)+3)$

$0 \le \theta \le 2\pi$

$$\boxed{\Phi(u,v) = (v, \cos(u)+2, \sin(u)+3), \quad 0\le u\le 2\pi, \quad -\infty < v < \infty}$$