Name: Solutions

20E, Ferrara, Jan 28, Version 2

SID:

Midterm 1

This exam has 7 pages and 6 problems. Make sure that your exam has all 7 pages and that your name is on every page.

Put your name and student ID on every page.

You must show your work and justify your answers to receive full credit unless otherwise stated.

If you need more space, use the pack of the pages; clearly indicate when you have done this.

You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

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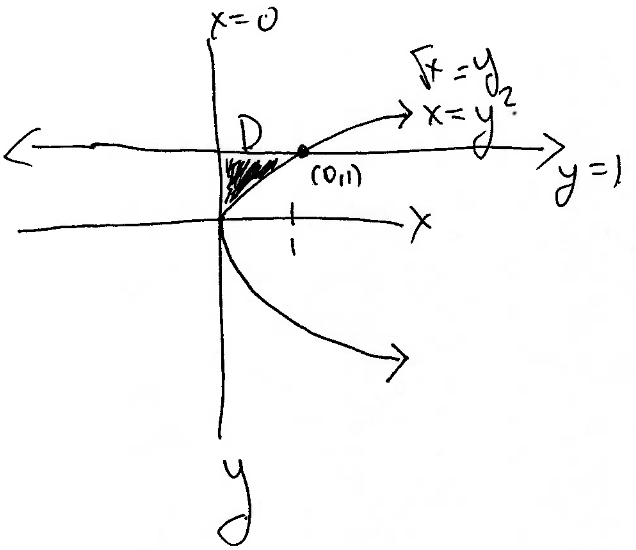
1. (10 points) Calcuate $\int\int\int_D \frac{x}{y^5+1} dA$ where D is the region bounded by $x = 0$, $x = y^2$, and $y = 1$.

TYPO, should be $\iint_D \frac{x}{y^5+1} dA$



D x-simple

$0 \le y \le 1$

$0 \le x \le y^2$

D y-simple

$0 \le x \le 1$

$\sqrt{x} \le y \le 1$

$\int_0^1 \int_{\sqrt{x}}^1 \frac{x}{y^5+1} dy\, dx$ HARD so use x-simple

$$\int_0^1 \int_0^{y^2} \frac{x}{y^5+1} dx\, dy = \int_0^1 \frac{x^2}{2(y^5+1)} \Big|_0^{y^2} dy = \frac{1}{2}\int_0^1 \frac{y^4}{y^5+1} dy \qquad u = y^5+1 \quad du = 5y^4 dy$$

$$\frac{1}{10}\int_1^2 \frac{du}{u} = \frac{1}{10}\left(\log(u)\Big|_1^2\right) = \frac{1}{10}\left(\log(2) - \log(1)\right) = \boxed{\frac{\log(2)}{10}}$$

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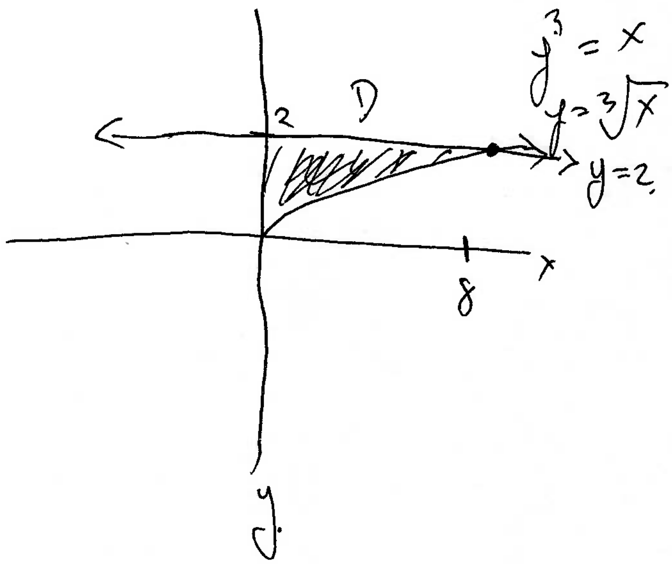
2. (10 points) Calculate $\int_0^8 \int_{\sqrt[3]{x}}^2 e^{y^4}\,dydx$ by changing the order of integration.



y-simple

D: $\sqrt[3]{x} \le y \le 2$

$0 \le x \le 8$

x-simple

$0 \le y \le 2$

$0 \le x \le y^3$

$$\int_0^8 \int_{\sqrt[3]{x}}^2 e^{y^4}\,dydx = \int_0^2 \int_0^{y^3} e^{y^4}\,dxdy = \int_0^2 xe^{y^4}\Big|_0^{y^3} dy$$

$$\int_0^2 y^3 e^{y^4}\,dy \overset{\substack{u=y^4 \\ du=4y^3dy}}{=} \frac{1}{4}\int_0^{16} e^u\,du = \frac{1}{4}\left(e^u\Big|_0^{16}\right) = \boxed{\frac{1}{4}\left(e^{16}-1\right)}$$

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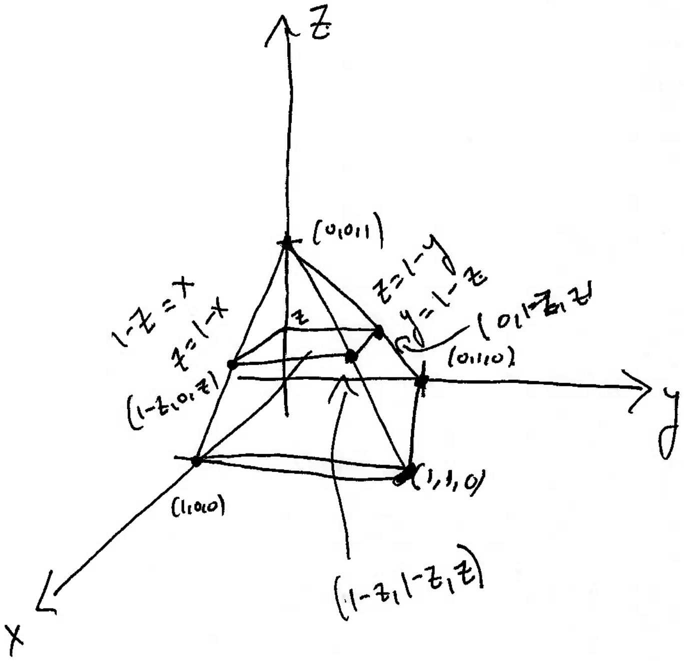
3. (10 points) Calculate $\int\int\int_W z\,dA$ where W is the pyramid with top vertex at $(0,0,1)$ and base vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,0)$.



$$W:\ 0 \le z \le 1$$
$$0 \le y \le 1-z$$
$$0 \le x \le 1-z$$

$$\iiint_W z\,dA = \int_0^1 \int_0^{1-z} \int_0^{1-z} z\,dx\,dy\,dz = \int_0^1 \int_0^{1-z} xz\Big|_0^{1-z} dy\,dz$$

$$\int_0^1 \int_0^{1-z} (1-z)z\,dy\,dz = \int_0^1 y(1-z)z\Big|_0^{1-z} dz = \int_0^1 (1-z)^2 z\,dz$$

$$\int_0^1 z - 2z^2 + z^3\,dz = \frac{z^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4}\Big|_0^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{6}{12} - \frac{8}{12} + \frac{3}{12} = \boxed{\frac{1}{12}}$$

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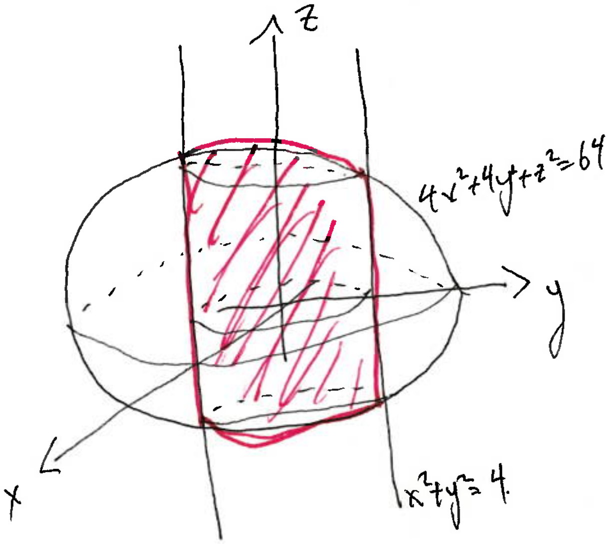
4. (10 points) Use an integral to calculate the volume of the solid that is inside the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$.



Triple Integral: W: $x^2+y^2 \le 4$

$-\sqrt{64-4x^2-4y^2} \le z \le \sqrt{64-4x^2-4y^2}$

$\iiint_W 1\,dx\,dy\,dz$

Use Cylindrical.

$= \int_0^{2\pi}\int_0^2\int_{-\sqrt{64-4r^2}}^{\sqrt{64-4r^2}} r\,dz\,dr\,d\theta$

W^*: $0 \le r \le 2$

$0 \le \theta \le 2\pi$

$-\sqrt{64-4r^2} \le z \le \sqrt{64-4r^2}$

Double Integral: $D: x^2+y^2 \le 4$

Use Polar! $D: 0 \le r \le 2$

$0 \le \theta \le 2\pi$

$\iint_D \sqrt{64-4x^2-4y^2} - -\sqrt{64-4x^2-4y^2}\,dx\,dy = \iint_D 2\sqrt{64-4x^2-4y^2}\,dx\,dy$

$2\int_0^{2\pi}\int_0^2 \sqrt{64-4r^2}\,r\,dr\,d\theta = 4\pi\int_0^2 \sqrt{64-4r^2}\,r\,dr$

$u = 64-4r^2$ $\quad r=0 \quad u=64$

$du = -8r\,dr$ $\quad r=2 \quad u=64-16=48$

$-\frac{4\pi}{8}\int_{64}^{48}\sqrt{u}\,du = -\frac{\pi}{2}\left(\frac{u^{3/2}}{3/2}\Big|_{64}^{48}\right) = -\frac{\pi}{3}\left(48^{3/2}-64^{3/2}\right) = \boxed{\frac{\pi}{3}\left(64^{3/2}-48^{3/2}\right)}$

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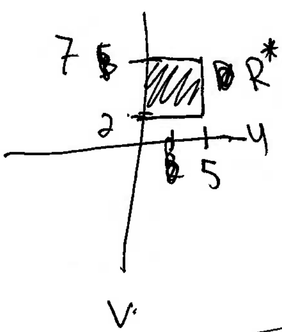
5. (10 points) Calculate $\displaystyle\iint_R \frac{x-3y}{2x-y}\,dA$ where R is the parallelogram enclosed by the lines $x-3y=0$, $x-3y=5$, $2x-y=2$, and $2x-y=7$.

linear change of variables $u = x-3y$, $v = 2x-y$.

$x-3y=0$, $x-3y=5$, $2x-y=2$, $2x-y=7$

$u=0$ $\quad u=5$ $\quad v=2$ $\quad v=7$



$R^*: 0 \le u \le 5$
$2 \le v \le 7$

$$\iint_R \frac{x-3y}{2x-y}\,dA = \int_2^7 \int_0^5 \frac{u}{v}\cdot\frac{1}{5}\,du\,dv.$$

Jacobian: $u = x-3y$ $\quad v-2u = -y - -6y = 5y$ $\qquad y = \frac{v}{5} - \frac{2u}{5}$

$v = 2x-y$ $\quad 3v-u = 6x-x = 5x$ $\qquad x = \frac{3v}{5} - \frac{u}{5}$

$\frac{\partial(x,y)}{\partial(u,v)}$ ~~$= \det\begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5}\end{pmatrix} = -\frac{6}{25} - -\frac{1}{25} = -\frac{5}{25} = -\frac{1}{5}$~~

$$= \det\begin{pmatrix} -\frac{1}{5} & \frac{3}{5} \\ -\frac{2}{5} & \frac{1}{5}\end{pmatrix} = -\frac{1}{25} - -\frac{6}{25} = \frac{5}{25} = \frac{1}{5}$$

$$\frac{1}{5}\int_2^7 \frac{u^2}{2v}\Big|_0^5 dv = \frac{1}{5}\int_2^7 \frac{25}{2v}\,dv = \frac{5}{2}\log(v)\Big|_2^7 = \boxed{\frac{5}{2}\left(\log(7) - \log(2)\right)}$$

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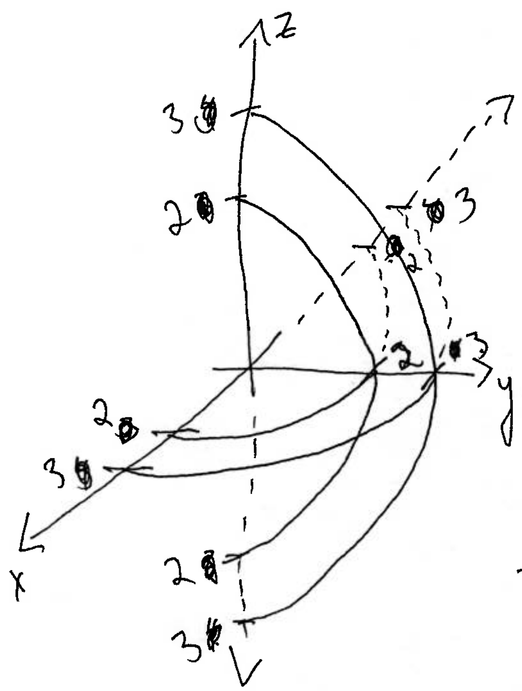
6. (10 points) Calculate $\iiint_W \left(\dfrac{x^2+y^2+z^2}{x^2+y^2}\right)^{1/2} dV$ where W is the region bounded by the xz-plane and the hemispheres $y=\sqrt{4-x^2-z^2}$ and $y=\sqrt{9-x^2-z^2}$.



Spherical: W^*: $2 \le \rho \le 3$

$0 \le \theta \le \pi$

$0 \le \varphi \le \pi$

$$\iiint_W \left(\frac{x^2+y^2+z^2}{x^2+y^2}\right)^{1/2} dV$$

$$\| $$

$$\int_0^{\pi}\int_0^{\pi}\int_2^3 \left(\frac{\rho^2}{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}\right)^{1/2} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

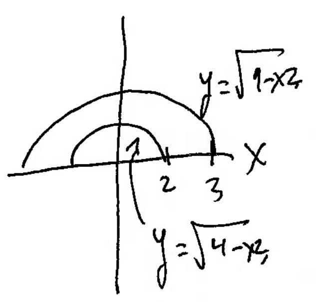
$$\| $$

$$\int_0^{\pi}\int_0^{\pi}\int_2^3 \left(\frac{1}{\sin^2\varphi}\right)^{1/2} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\| $$

$$\int_0^{\pi}\int_0^{\pi}\int_2^3 \rho^2 \, d\rho\, d\varphi\, d\theta = \pi\cdot\pi\left(\frac{\rho^3}{3}\Big|_2^3\right)$$

$$= \pi^2\left(\frac{3^3}{3} - \frac{2^3}{3}\right) = \pi^2\left(\frac{27-8}{3}\right)$$



y $0 \le \theta \le \pi$

$$\boxed{\frac{19}{3}\pi^2}$$