LECTURE 15-1

Let's review where we were before vacation (2 weeks since last lecture!)

We introduced the idea of an ~~+~~ ω-limit set $\omega(\Gamma)$ for a trajectory $\Gamma = \varphi_t x_0$, which is the set of points $\hat{x}$ s.t. $\exists$ sequence $t_n \to \infty$ st $\varphi_{t_n} x_0 \to \hat{x}$

We ~~say~~ saw that this set is closed and that if Γ contained in a compact subset of $\mathbb{R}^n$ then $\omega(\Gamma)$ is non-empty, connected + compact

Then we defined attracting sets - a set S ~~that contain~~ such that $\exists$ open set of inition conditions $X \supset S$ st ~~$\varphi_t x_0 \to$~~ ~~$\omega(\varphi_t x_0)$~~ $d(\varphi_t x_0, S) \xrightarrow[t\to\infty]{} 0$

an attractor is an attracting set with a dense trajectory

Then we were considering fixed points that are attractors, used Lyapunov functions Lasalle invariance principal, geometric reasoning to show regions of attraction to an asymptotically stable fixed pt.

~~There are (at least) 3~~

There are several other types of sets that can be attractors

the SIMPLEST : Periodic orbits

We all now $x(t)$ a periodic orbit if $\exists T > 0$ st $x(t+T) = x(t)$ and $T > 0$ is the minimum value s.t. this is true. (want to exclude fixed pts from definition)

If an equation autonomous, ~~$\Rightarrow$~~ uniqueness

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a periodic orbit Γ called stable if $\forall \varepsilon > 0$ $\exists$ nbhd $U \supset \Gamma$ st

$$x \in U \Rightarrow |\varphi_t x - \Gamma| < \varepsilon \quad \forall t > 0$$

unstable if not stable

asymptotically stable if stable and $\exists$ a nbhd $U \supset \Gamma$ st

$$\forall x \in U \quad d(\varphi_t x, \Gamma) \xrightarrow[t \to \infty]{} 0$$

example ~~(in polar coords)~~

$$\cancel{\dot{x}}\ \dot{r} = r(1 - r^2) \qquad x = r\cos\theta$$

$$\dot{\theta} = 1 \qquad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases}$$

then Γ given by $\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t < 2\pi$

is asymptotically stable and attracts all initial conditions except $(0,0)$

Just like a fixed pt, a periodic orbit (in 2 or more dims) (in 3 or more dims) may have stable & unstable manifolds

example

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$
$$\dot{z} = z$$

~~there~~ Γ given by $(\cos t, \sin t, 0)$

~~How should we~~

if $x = (x, y, 0) \neq (0,0,0)$, $\varphi_t x \xrightarrow[t \to \infty]{} \Gamma$

if $x^2 + y^2 = 1$ but $z \neq 0$, $d(\Gamma, \varphi_t \vec{x}) \xrightarrow[t \to -\infty]{} 0$

We define $W_s(\Gamma) = \{x \mid d(\varphi_t(x), \Gamma) \xrightarrow[t \to +\infty]{} 0\}$

$$W_u(\Gamma) = \{x \mid d(\varphi_t(x), \Gamma) \xrightarrow[t \to -\infty]{} 0\}$$

clearly for the SIMPLEST system

$\dot{x} = -y$
$\dot{y} = x$ the ~~solution~~ periodic orbits are stable but not asymptotically stable

a limit cycle is a periodic orbit Γ which is the ω-limit set (or α-limit set) of some trajectory other than itself

~~a stable limit cycle~~

if a limit cycle is the ω-limit set of every trajectory in a neighborhood containing Γ it is called a stable limit cycle

if it is the α-limit set of all trajectories in a nbhd of Γ, it is an unstable limit cycle

if it is the α-limit set of one trajectory & the ω limit set of another, Γ is a semistable limit cycle

eg previous example or section 3.3 example 2

Perko loves to show crazy examples $\dot{x} = f(x)$, f not analytic

eg $\dot{r} = r^3 \sin \frac{1}{r}$
$\dot{\theta} = 1$ } clearly has countable sequence of limit cycles at $r = \frac{1}{n\pi}$

Nice Theorem if $f \in C^1(\mathbb{R}^2)$ and Γ is a periodic orbit and $\exists$ trajectory exterior to Γ s.t. Γ is its ω-limit set then $\exists$ Nbhd U exterior to Γ with w/ ω-limit set Γ, further every trajectory in U spirals around Γ ∞-often

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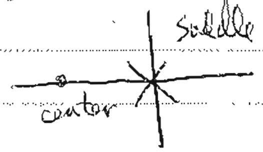
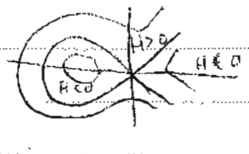
Homoclinic orbits

eg $\ddot{x} = x + x^2$ fixed pts $x=0,\ x=-1$

$\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 = H$



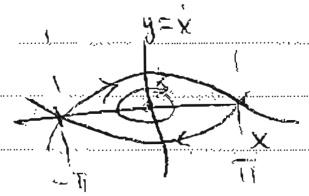


The curves $H=0$ are called separatrices since they separate solutions w/ different asymptotic behavior

Note these are the stable & unstable manifolds of the origin

Heteroclinic orbits

pendulum $\ddot{x} + \sin x = 0$



Note $\frac{1}{2}\dot{x}^2 + 1 - \cos x = E$

also note sometimes it is easier to find heteroclinic orbits than general periodic orbits

$\cos\pi = -1$

so $E=2$ on the heteroclinic orbits

$$\frac{1}{2}\dot{x}^2 + 1 - \cos x = 2$$

$$\frac{1}{2}\dot{x}^2 = 1 + \cos x$$

$$\frac{dx}{\sqrt{1+\cos x}} = \pm\sqrt{2}\,dt$$

use $\cos x = \frac{1}{2} + \frac{1}{2}\cos^2 x$

$\cos 2x$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\cos^2\frac{x}{2} = \frac{1}{2} + \frac{1}{2}\cos x$$

$$2\cos^2\frac{x}{2} = 1 + \cos x$$

$$\frac{dx}{\sqrt{2\cos^2\frac{x}{2}}} = \pm\sqrt{2}\,dt$$

etc

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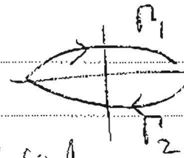
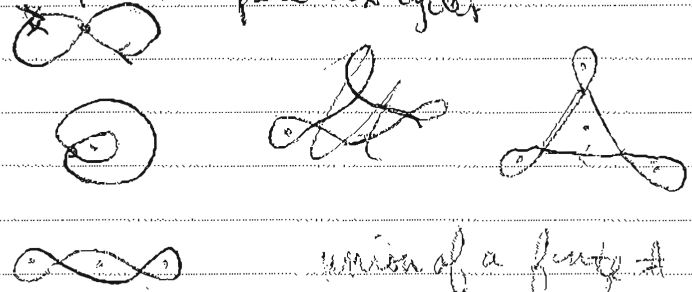
separatrix cycles – 

above $\Gamma_1 \cup \Gamma_2 = \{(\pi,0),(-\pi,0)\}$ is a SIMPLE CLOSED CURVE

& compound separatrix cycles



union of a finite # of separatrices + their limit points

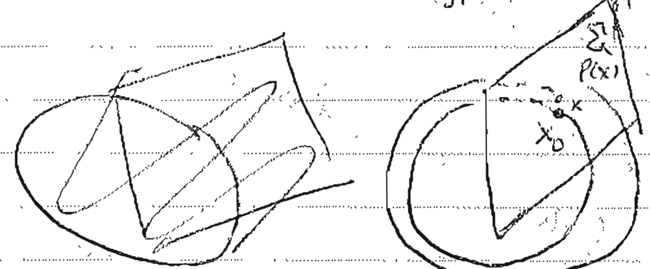
~~The Poincare~~

The goal of the next few lectures: methods for determining existence, non-existence, stability etc of periodic orbits, homoclinic, heteroclinic orbits

Important tool: the Poincaré map, $x \in \mathbb{R}^n$

$\dot{x} = f(x)$, $f \in C^1(E)$, $E \subset \mathbb{R}^n$ open

Suppose $\Gamma =$ ~~...~~ is a periodic orbit to $\dot{x} = f(x)$ through x_0 and Σ_i is a hyperplane perpendicular to Γ at x_0



if x is a point on Σ_i near x_0, then the trajectory will cross the plane Σ_i in another point $P(x)$

In fact, as long as Σ_i is any ~~...~~ $(n-1)$-dimensional hypersurface such that Γ ~~...~~ crosses Σ_i such that Γ is not tangent to Σ_i, (the trajectory is said to intersect Σ transversally) we can make this argument