Lecture 21 Structural stability + Bifurcations of vector ~~fi~~ fields

(*) $f \in C^1(E)$, $E \subset \mathbb{R}^n$ open

the C^1-norm of f = $\|f\|_{C^1} = \sup_{x\in E} |f(x)| + \sup_{x\in E} \|Df(x)\|$

definition the vector field f given in (*) is <u>structurally stable</u>

if ~~$\exists g$~~ $\exists \varepsilon > 0$ st $\forall g \in C^1(E)$,

$\|f-g\|_{C^1} < \varepsilon \Rightarrow$ the vector fields are equivalent orientation-preserving

ie $\exists$ Homeomorphism $H: E \to E$, which maps trajectories of $\dot{x} = f(x)$ onto trajectories of $\dot{x} = g(x)$

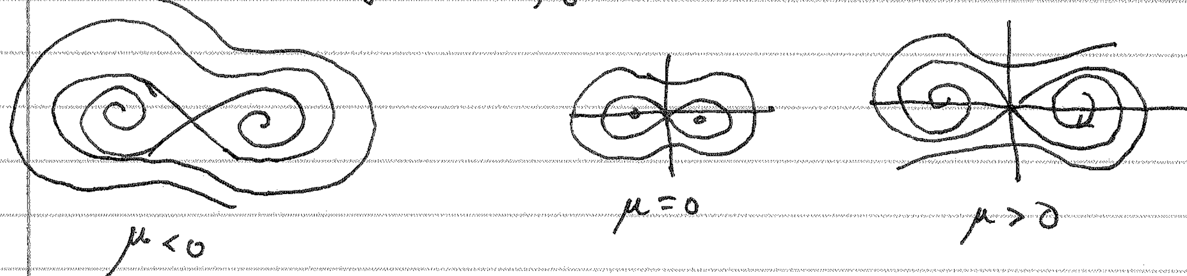
may restrict this definition to say vector field is locally structurally ~~st~~ stable on some compact $K \subset E$

Now assume $f = f(x,\mu)$

saw 2 examples last time

example $\dot{x} = y$

$\dot{y} = x - x^3 + \mu y$



See that the only structurally stable feature of the $\mu = 0$ system is the ~~pt~~ hyperbolic fixed pt

Periodic orbits and homoclinic connections destroyed

~~note on any small compact set $N_\delta(x_0)$ s.t. $x_0 \neq (\pm 1, 0)$~~

Note that $\forall$ pts ~~not~~ $x_0 \neq (\pm 1, 0)$, ~~$N_\delta(x_0)$~~

$\exists \varepsilon > 0$ st the vector field is structurally stable on ~~some nbhds~~

$\overline{N_\delta(x_0)}$ for all ~~δ~~ $0 < \delta < \varepsilon$

LEC 21-2

Thm if $f \in C^1(E)$, has a hyperbolic critical pt x_0 then $\forall \varepsilon > 0$ $\exists \delta > 0$ st $\forall g \in C^1(E)$ s.t $\|f-g\|_{C^1} < \delta \Rightarrow \exists y_0 \in N_\varepsilon(x_0)$ hyperbolic critical pt w/ same # of +ve & -ve eigenvalues

~~further~~ SIMILARLY FOR Hyperbolic periodic orbits of f

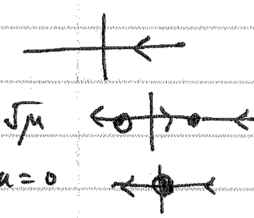
Why? Implicit function thm, $f(x_0,\mu_0)=0$ $D_x f(x_0,\mu_0) \neq 0 \Rightarrow x = X(\mu)$ near x_0, μ_0

Bifurcations - $\dot{x} = f(x,\mu)$ how does structure of solutions depend on μ?

Simplest examples : 1d

$\dot{x} = \mu - x^2$ $\quad$ $\mu < 0$ no fixed pts

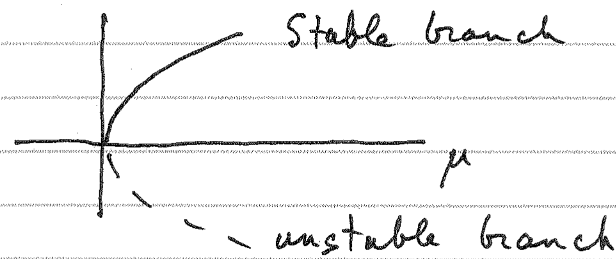
$\mu > 0$ 2 fixed pts $x_\pm = \pm\sqrt{\mu}$



Stability given by sign $f'(x_\pm) = -2x_\pm = \mp\sqrt{\mu}$

$\Rightarrow x_+$ stable, x_- unstable

draw ~~fix~~ <u>bifurcation diagram</u>



this is called a saddle-node bifurcation

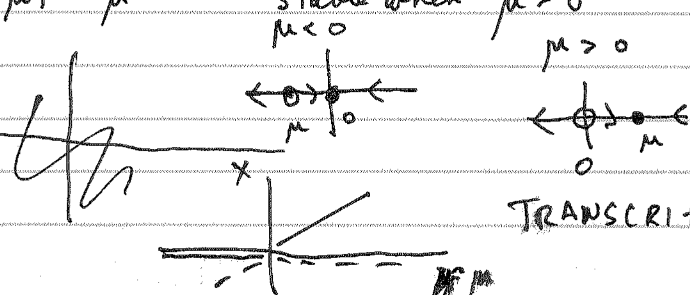
<u>example 2</u>

$\dot{x} = \mu x - x^2 = x(\mu - x)$

$f'(x) = \mu - 2x$

$f'(0) = \mu$ $\quad$ stable when $\mu < 0$

$f'(\mu) = -\mu$ $\quad$ stable when $\mu > 0$



TRANSCRITICAL BIFURCATION

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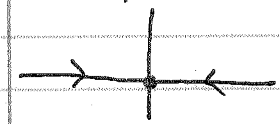
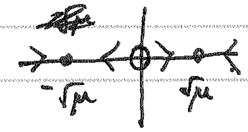
$\dot{x} = \mu x - x^3$ (System with symmetry, no even terms)

$\mu < 0$ $\mu > 0$

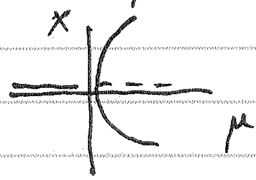




$f' = \mu - 3x^2$

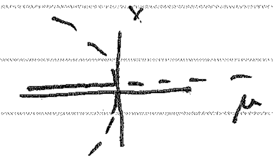
$f'(0) = \mu \Rightarrow$ stable if $\mu < 0$

$f'(\pm\sqrt{\mu}) = -2\mu$ stable when $\mu > 0$ ie when $\pm\sqrt{\mu}$ is a fixed pt



PITCHFORK BIFURCATION

SIMILAR $\dot{x} = \mu x + x^3$ find



SUBCRITICAL PITCHFORK

note that in all 3 examples $\frac{\partial}{\partial x} f(x,\mu)\Big|_{x=0,\mu=0} = 0$, Saddle-node + transcritical $f_{xx}(0,0) \neq 0$

pitchfork $f_{xx}(0,0) = 0$ but $f_{xxx}(0,0) \neq 0$

so there is a center subspace associated to the linearization about fixed pt $x=0$ at the parameter value $\mu = 0$

In general $f(x_0,\mu_0) = Df(x_0,\mu_0) = 0$ then x_0 is a nonhyperbolic critical pt. The type of bifurcation depends on which derivative $\dfrac{\partial^m f(x_0,\mu_0)}{\partial x^{j} \partial \mu^k}$ is nonzero

Note there exist in higher dimension

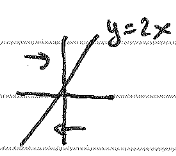
eg trivial $\dot{x} = \mu - x^2$

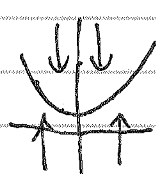
$\dot{y} = -y$

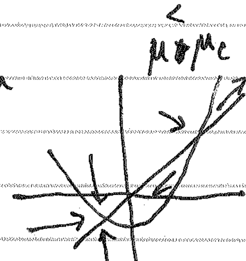
non-trivial

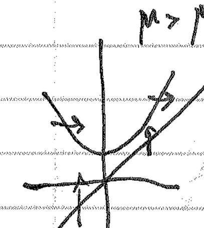
$\dot{x} = y - 2x$

$\dot{y} = \mu + x^2 - y$









$\mu = \mu_c$

$x^2 + \mu = 2x$ and

$2x = 2 \Rightarrow x = 1 \Rightarrow \mu = 1$

Can solve for fixed pts since quadratic

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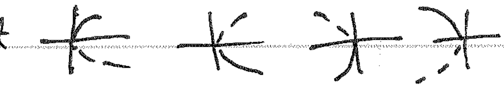
Now there exist theorems that tell us if the Taylor series of $\dot{x} = f(x, \mu)$ ~~takes a certain form, then~~ near a non-hyperbolic fixed pt x_0 at parameter-value μ_0 takes a certain form, then the bifurcation is of a given type

$\dot{x} = \pm\mu \pm x^2$ the <u>normal form</u> for the saddle-node bifurcation
depending on signs, get 

theorems can tell us "which way parabola opens" + "which branch is stable" but usually easier to just work it out by hand

~~<u>Theorem</u> (Simplification of §4.2 Thm 1) to 1 dimension~~
~~$\dot{x} = f(x, \mu)$, $f(\cdot,\mu) \in C^\infty(\mathbb{R})$~~

We want to talk about the fact that the saddle-node bifurcation is "generic" while the transcritical & pitchfork bifurcations are not, i.e. structurally stable. If you change equations slightly get same bifurcation behavior

In general, (in 1D) $\dot{x} = f(x, \mu)$ has a bifurcation at $x = x_0$, $\mu = \mu_0$ if

$$f(x_0, \mu_0) = 0 \quad \text{and} \quad \partial_x f(x_0, \mu_0) = 0$$ because then x_0 a multiple root

~~$\dot{x} = f(x_0, \mu_0) + \partial_x f(x_0, \mu_0)(x - x_0) + \frac{1}{2}\partial_x^2 f(x_0, \mu_0)(x - x_0)^2 +$~~ Implicit function thm inapplicable

~~Saddle-node $\partial_\mu f(x_0, \mu_0) \neq 0$ and $\partial_x^2 f(x_0, \mu_0) \neq 0$~~

Expand in Taylor series

$$\dot{x} = \cancel{f(x_0, \mu_0)}^{\,0} + \cancel{f_x(x_0, \mu_0)}^{\,0}(x - x_0) + f_\mu(x_0, \mu_0)(\mu - \mu_0) + \tfrac{1}{2} f_{xx}(x_0, \mu_0)(x - x_0)^2 + f_{x\mu}(\mu - \mu_0)(x - x_0) + \dots$$

Saddle node: $f_\mu(x_0, \mu_0) \neq 0$, $f_{xx}(x_0, \mu_0) \neq 0$

Transcritical: $f_\mu(x_0, \mu_0) = 0$, $f_{x\mu}(x_0, \mu_0) \neq 0$, $f_{xx}(x_0, \mu_0) \neq 0$

pitchfork: $f_\mu = 0$, $f_{x\mu} \neq 0$, $f_{xx} = 0$, $f_{xxx} \neq 0$

LEC 21-5

Now we'll let μ be a vector of parameters
but we'll keep $x \in \mathbb{R}$
Then we'll go dip into chapter 2 for center manifold theory which
we'll need to extend this to $x \in \mathbb{R}^n$

assume that $f_0(x)$ defines a structurally unstable vector field (locally)
eg $f_0(x) = x^2$
now we embed f_0 in an m-parameter family of vector fields ~~st~~
~~f(x)~~ $f(x,\mu)$, $\mu \in \mathbb{R}^m$, s.t $f_0(x) = f(x, \mu_0)$

this family is called an <u>unfolding</u> of the vector field $f_0(x)$

it is called the <u>universal unfolding</u> of $f_0(x)$ at a nonhyperbolic
fixed point if it is an unfolding and if every other unfolding
of $f_0(x)$ is topologically equivalent to $f(x,\mu)$ in a nbhd of x_0
the codimension of a bifurcation is the minimum # of parameters to describe a universal unfolding
The normal form of the saddle node bifurcation is
$f_0(x) = ax^2$, scaling x and t, we can reset $a=-1$ WLOG

So set $f_0(x) = -x^2$
if we add higher ~~deg~~ degree terms, then behavior near $x=0$
unaffected ie if $f(x,\mu) = -x^2 + \mu_3 x^3 + \dots$
then if $\mu_3 << 1$ this just adds another fixed pt at $x \approx \frac{1}{\mu_3} >> 1$
so this doesn't change behavior near 0
so a universal unfolding is
$f(x,\mu) = \mu_1 + \mu_2 x - x^2$
complete the square $y = x - \frac{\mu_2}{2}$
~~f(y,\mu)~~ $f(y,\mu) = (\mu_1 + \mu_2^2/4) - y^2$ define $\mu = \mu_1 + \mu_2^2/4$

$f(x,\mu) = \mu - x^2$

Thus all possible types of behavior ~~for systems~~ that can occur in an unfolding
of $\dot{x} = -x^2$ are given by the saddle-node normal form
it ~~is~~ is a codimension 1 bifurcation since it is described by $\mu \in \mathbb{R}^1$