Lecture 10 : The stable manifold Theorem

(*) $\dot{x} = f(x) = Ax + F(x)$ where $f(0)=0$, $x \in \mathbb{R}^n$, $A = Df(0)$

Whose linearization is

(**) $\dot{x} = Ax$ and $F(0)=0$, $DF(0)=0$

Goal: Find out what the solution to (**) tells us about the solution to (*)

Suppose $E^c = \{\}$, $\dim E^s = k$, $\dim E^u = n-k$

define $W^s = \{x \in \mathbb{R}^n \mid \phi_t(x) \to 0$ exponentially as $t \to +\infty\}$

$W^u = \{x \in \mathbb{R}^n \mid \phi_x(t) \to 0$ exponentially as $t \to -\infty\}$

(Most of what we have to say does not require the formal definition of manifolds, especially for local theory. See perko to review the diff. geom.)

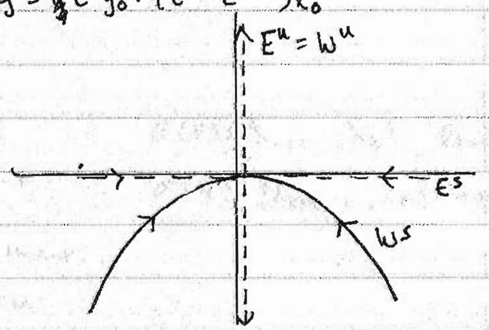
EXAMPLE FROM LAST TIME:

$\dot{x} = -x$ exact solution $x = e^{-t}x_0$

$\dot{y} = y + x^2$ $y = e^t y_0 + (e^t - e^{-2t})x_0^3$

$W^u = \{(x,y) \mid x = 0\}$

$W^s = \{(x,y) \mid y = -\frac{1}{3}x^2\}$



From linearization $E^u = \{(0,y)\}$, $E^s = \{(x,0)\}$

We note that at $(0,0)$ W^u tangent to E^u

& W^s tangent to E^s

Further • W^u is a graph $\{(g(y), y)\}$ where the variable in the stable direction x is a function of the unstable direction y

W^s is a graph $\{(x, f(x))\}$ (unstable direction a function of the stable direction)

Lecture 10-2

In fact. This behavior is generic whenever $E^c = \{\}$

Theorem (Local stable manifold theorem)

Let E be an open set containing 0. Let $f \in C^1$, $f(0)=0$, and $Df(0)$ have $\dim E^s(Df(0)) = k$ and $\dim E^u(Df(0)) = n-k$. Then $\mathbb{R}^n = E^s \oplus E^u$

i.e. $\forall x \in \mathbb{R}^n$, $\exists \xi \in E^s, \eta \in E^u$ st $x = \xi + \eta$.

Then $\exists$ neighborhood $N_\delta \subset E$ s.t. the local stable manifold

$$W^s_{loc} = \{ x \in N_\delta \mid \varphi_t(x) \in N_\delta, \forall t > 0 \text{ and } \lim_{t\to\infty} \varphi_t(x) = 0 \}$$

may be written as a graph $\eta = \psi_s(\xi)$.

Further, W^s_{loc} is tangent to E^s at the origin

Before we prove it, we will make some additional assumptions + definitions. Our proof is based on Perko's but uses the contraction mapping principle to simplify it.

Define $F(x) = f(x) - Ax$. Then $F(0) = 0$ and $DF(0) = 0$.

$$\Rightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } |x| < \delta \ \&\ |y| < \delta \Rightarrow |F(x) - F(y)| < \varepsilon |x-y| \quad (1)$$

We will pick ε & δ as necessary for our proof.

PROOF:

Since A is invertible $\exists C$ s.t

$$C^{-1} A C = B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix} \qquad \begin{matrix} \sigma(P) = \{\lambda_1, \dots, \lambda_k\},\ \text{Real } \sigma(P) < 0 \\ \sigma(Q) = \{\lambda_{k+1}, \dots, \lambda_n\}\ \text{Real } \sigma(Q) > 0 \end{matrix}$$

So without loss of generality, assume

$A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ of this form

then $E^s = \text{span}\{\hat{e}_1, \dots, \hat{e}_k\}$

$E^u = \text{span}\{\hat{e}_{k+1}, \dots, \hat{e}_n\}$

define $U(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix}$ & $V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}$

then $\dot{U} = AU$ and $\dot{V} = AV$

Choose $\alpha > 0$ s.t. $\text{Re}\,\lambda_1, \dots, \text{Re}\,\lambda_k < -\alpha$

Then $\exists\ \sigma > 0$ st $\|U(t)\| \le K e^{-(\alpha+\sigma)t}$ $t \geq 0$

$\|V(t)\| \le K e^{\sigma t}$ $t \le 0$

Lecture 10-3

We will show $\exists$ functions $\psi_j(x_1, \dots, x_k)$ $\quad j = k+1, \dots, n$

such that $x_j = \psi_j$ define an invariant set, i.e.

$\{x_1, \dots, x_k, \psi_{k+1}, \dots, \psi_n\}$ invariant and $\lim_{t \to +\infty} \{x_1, \dots, x_k, \psi_{k+1}, \dots, \psi_n\} = 0$

Define the weighted norm $\|x(t)\|_\beta = \sup_{t \geq 0} |e^{\beta t} x(t)|$, $\beta > 0$

this defines a vector space $C^0_\beta([0,\infty))$ of functions

$x(t)$ st $\exists K \geq 0$ with $|e^{\beta t} x(t)| < K$

$|x(t)| < K e^{-\beta t}$

if $\beta > 0$ this is a space of continuous functions with sufficienty fast exponential decay

~~Notation~~

In our proof we use the Contraction mapping theorem on a closed bounded subset of the Banach space C^0_α

$S = \{x(t) \in C^0_\alpha \text{ st } \|x\|_\alpha < \delta\}$ for some δ yet to be determined

Define a map $T_a u = U(t)a + \int_0^t U(t-s)F(u(s))ds - \int_t^\infty V(t-s)F(u(s))ds$

$a \in E^s$ and a "suitably small"

Then Claim: a continuous solution to the integral equation

(2) $T_a u = u$ solves (*) (Exercise)

To apply Contraction Mapping Principle, we need

(a) $u \in S \Rightarrow T_a u \in S$

(b) $\exists \Lambda < 1$ st $\|T_a u - T_a v\|_\alpha \leq \Lambda \|u - v\|_\alpha$

proof of a $\quad$ assume $u \in S$, $t > 0$

$$|T_a u| \leq \|U(t)\| \cdot |a| + \int_0^t \|U(t-s)\| |F(u(s))| ds + \int_t^\infty \|V(t-s)\| \cdot |F(u(s))| ds$$

$$\leq K e^{-(\alpha+\sigma)t} |a| + \int_0^t K e^{-(\alpha+\sigma)(t-s)} \cdot \varepsilon |u(s)| ds + \int_t^\infty K e^{\sigma(t-s)} \cdot \varepsilon |u(s)| ds$$

where we have used (2a) and, since $|u| < \delta$ can apply (1) with

Lecture 10-4

Next we use that $u \in S \Rightarrow |u(s)| e^{\alpha s} < \delta$

$$|T_a u| \le K e^{-(\alpha+\sigma)t} |a| + K\varepsilon \int_0^t e^{-(\alpha+\sigma)(t-s)} \delta e^{-\alpha s} ds + K\varepsilon \int_t^\infty e^{\sigma(t-s)} \delta e^{-\alpha s} ds$$

multiply by $e^{\alpha t}$

$$|e^{\alpha t} T_a u| \le K e^{-\sigma t} |a| + K\varepsilon\delta \int_0^t e^{-\sigma(t-s)} ds + K\varepsilon\delta \int_t^\infty e^{\sigma(t-s)} \cdot \underbrace{e^{\alpha(t-s)}}_{<1} ds$$

$$\|T_a u\|_\alpha \le K|a| + \frac{K\varepsilon\delta}{\sigma} e^{-\sigma t}(e^{\sigma t}-1) + \frac{K\varepsilon\delta}{\sigma} e^{\sigma t} e^{-\sigma t}$$

$$\le K|a| + \frac{K\varepsilon\delta}{\sigma} + \frac{K\varepsilon\delta}{\sigma}$$

$$\le K|a| + \frac{2\varepsilon K}{\sigma}\delta$$

Choose $\varepsilon < \frac{\sigma}{4K}$, then in (1) this fixes δ

then choose $|a| < \frac{\delta}{2K}$

then $\|T_a u| < \delta$

proving claim (a). A SIMILAR ARGUMENT, using the same values of ε and α proves claim (b), that this is a contraction

Thus $T_a u = u$ has a unique solution $u(t,a)$ in S' which implies that $|u(t,a)| \le \delta e^{-\alpha t}$ for $t \ge 0$ and $\vec{a} \in E^s$ with $|a| < \frac{\delta}{2K}$

Note that $u_a(0) = a - \int_0^\infty V(-s) F(u_a(s)) ds$

If we introduce $P_{s,u}$ = orthogonal projection onto $E^{s,u}$

$$P_s = \begin{bmatrix} I_k & \\ & 0_{n-k} \end{bmatrix}, \quad P_u = \begin{bmatrix} 0_k & \\ & I_{n-k} \end{bmatrix}$$

then $P_u^2 = P_u$, $P_s^2 = P_s$. $P_u P_s = P_s P_u = 0$

Then $$u_a(0) = \underbrace{P_s a}_{\in E^s} - \underbrace{\int_0^\infty P_u e^{-As} F(u(s)) ds}_{\in E^u}$$

Lecture 10-5

Thus at $t=0$

$$u_j(0) = a_j \qquad j=1,\dots,k$$
$$u_j(0) = -\left[\int_0^\infty V(-s)\, F(u_{a_1,\dots,a_k}(s))\, ds\right]_j \overset{def}{=} \psi_j(a_1,\dots,a_k) \qquad k=K+1,\dots,n$$

These equations define the differentiable manifold W^s_{loc} for $\sqrt{x_1^2+\dots+x_k^2} \le \frac{\delta}{2K}$

We can also show if $x_0 \notin W^s_{loc}$ then $|\phi_t x_0| \underset{t\to\infty}{\not\to} 0$. This is done, for example, in Coddington + Levinson

They also show that $\frac{\partial\psi_j}{\partial x_i}(0) = 0$ for $j=k+1,\dots,n$ and $i=1,\dots,k$

By taking $t \to -t$ and interchanging E^u and E^s, we can find the unstable manifold in the same way.

Remark: If $f \in C^r$, one can show W^s_{loc} & W^u_{loc} are also C^r

Note that we can use Picard Iteration $u_{j+1} = Tu_j$ to approximate W^s_{loc}

new term from previous example

Example $\begin{cases} \dot{x} = -x - y^2 \\ \dot{y} = y + x^2 \end{cases}$

Then $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} -y^2 \\ x^2 \end{bmatrix}$, $U = \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 0 \\ 0 & e^t \end{bmatrix}$

$\vec{u}_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}$, $\vec{a} = \begin{bmatrix} a \\ 0 \end{bmatrix}$

The iteration is

$$\vec{u}_{j+1} = \begin{bmatrix} e^{-t}a \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} -e^{-(t-s)}\, y_j^2(s) \\ 0 \end{bmatrix} ds - \int_t^\infty \begin{bmatrix} 0 \\ e^{t-s}\, x_j^2(s) \end{bmatrix} ds$$

$$\vec{u}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_1 = \begin{bmatrix} e^{-t}a \\ 0 \end{bmatrix}$$

Lecture 10-6

$$\vec{u}_2 = \begin{bmatrix} e^{-t}a \\ 0 \end{bmatrix} - \hat{e}_2\, e^{t}\int_t^\infty e^{-s}(e^{-s}a)^2\,ds = \begin{bmatrix} (e^{-t})a \\ -\frac{1}{3}e^{-2t}a^2 \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} e^{-t}a \\ 0 \end{bmatrix} + \hat{e}_1 \int_0^t - e^{-(t-s)}\left(\frac{e^{-4s}}{9}a^4\right)ds - \hat{e}_2\int_t^\infty e^{t-s}(e^{-s}a)^2\,ds$$

$$= \begin{bmatrix} e^{-t}a \\ 0 \end{bmatrix} - \hat{e}_1 e^{-t}\cdot\frac{a^4}{9}\int_0^t e^{-3s}\,ds - a^2e^{t}\int_t^\infty e^{-3s}\,ds$$

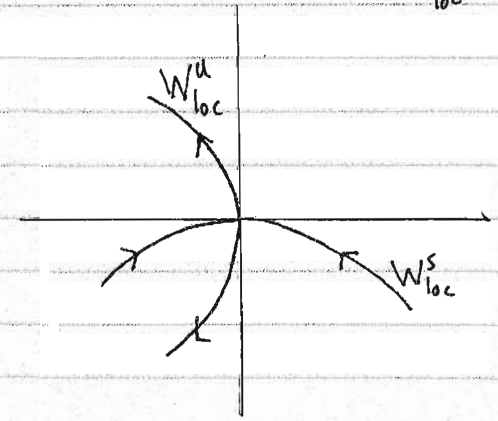
$$= \begin{bmatrix} e^{-t}a + \frac{1}{27}(e^{-4t} - e^{-t})a^4 \\ -\frac{1}{3}e^{-2t}a^2 \end{bmatrix}$$

Notice that on the next iterate, we will only pick up a term in the 2nd component and it will be $o(a^5)$

So $u_4(0) = \begin{bmatrix} a \\ -\frac{1}{3}a^2 + o(a^5) \end{bmatrix} \Rightarrow \psi_2(a) = -\frac{1}{3}a^2$

and the unstable manifold is $y = -\frac{1}{3}x^2 + o(x^5)$

a SIMILAR calculation yields W^u_{loc} given by $x = -\frac{1}{3}y^2 + o(y^5)$ as $y \to 0$



Now, we'd like to show that W^s_{loc} as constructed describes all initial conditions in N_δ that stay near zero. We will show if $x_0 \in N_\delta(0)\setminus W^s_{loc}$ then $\varphi_t(x_0)$ leaves N_δ

<u>Theorem</u> if $x_0 \in N_\delta(0)\setminus W^s_{loc}$ then $\exists t>0$ st $\varphi_t(x_0) \notin N_\delta(0)$

Proof: let $x_0 = a + b$ where $a \in E^s$, $b \in E^u$

and assume $|x(t)| < \delta \quad \forall t > 0$ (beginning of proof by contradiction)

Lecture 10-7

by our standard techniques (integrating factor method

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} F(x(s))\,ds$$

$$= U(t)\,a + V(t)\,b + \int_0^t U(t-s)\,F(x(s))\,ds + \int_0^t V(t-s)\,F(x(s))\,ds$$

$$= U(t)a + V(t)b + \int_0^t U(t-s)F(x(s))ds + \int_0^\infty V(t-s)F(x(s))ds - \int_t^\infty V(t-s)F(x(s))ds$$

These last two integrals converge since $\|V(-s)\| \leq Ke^{-\sigma s}$ for $s>0$ and $|x(s)| \leq \delta$ for all $s>0$ by our assumption

$$x(t) = U(t)a + V(t)\left[b + \int_0^\infty V(-s)F(x(s))ds\right] + \int_0^t U(t-s)F(x(s))ds - \int_t^\infty V(t-s)F(x(s))ds$$

Note $\int_t^\infty V(t-s)|F(x(s))|ds$ is finite

but if $c = b + \int_0^\infty V(-s)F(x(s))ds \neq 0$

then $V(t)c \to \infty$ since $V(t)$ is unbounded.

Therefore, for $x(t)$ to be bounded must have $b = -\int_0^\infty V(-s)F(x(s))ds$

otherwise we obtain a contradiction.

This is equivalent to setting $x_j = \psi_j(x_1, \ldots, x_k)$, $j = k+1, \ldots, n$

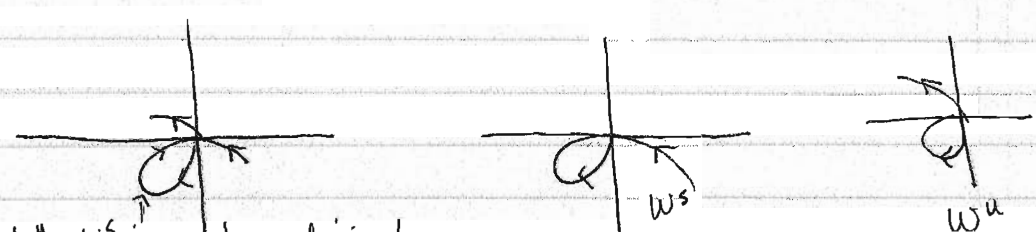
Global Unstable Manifold

To extend $W^{u,s}$ outside $N_\delta(0)$,

define $W^{u,s} = \bigcup_t \varphi_t(W^{u,s}_{loc})$

one can show these are unique and, clearly, these are invariant under the flow

CONTINUING THE PREVIOUS EXAMPLE NUMERICALLY



Lecture 10-8

Corollary: If $E^s = \mathbb{R}^n$, ie if all eigenvalues have negative real part, then $\exists$ Neighborhood $N_\delta(0)$ s.t $\phi_t(x) \xrightarrow[t\to\infty]{} 0 \quad \forall x \in N_\delta$

Proof: In the iteration scheme, the term $-\int_t^\infty V(t-s)F(x(s))$ drops out, and the proof goes through as before but for any initial condition in $N_\delta(0)$.

Center Manifold Theorem If $f \in C^r(E)$, $E \subset \mathbb{R}^n$ open, $f(0)=0$ and $A = Df(0)$ has $\text{Dim } E^s(A) = j$

$\text{Dim } E^u(A) = k$

$\text{Dim } E^c(A) = m = n-k-j > 0$

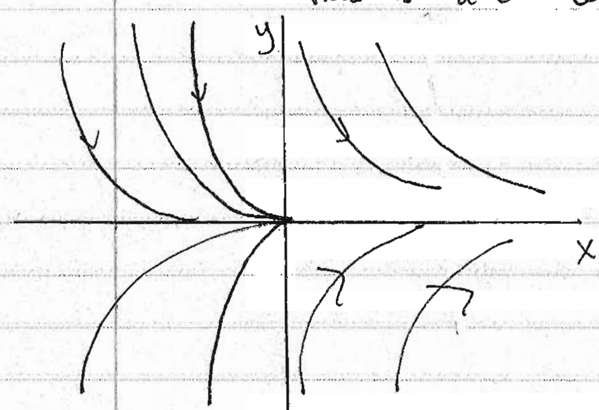
Then there exist unique stable and unstable manifolds as before and there exists a (NOT NECESSARILY UNIQUE) C^r, m-dim'l center manifold $W^c(0)$ tangent to E^c at 0 and invariant under the flow.

- Note that the theorem makes no claims about behavior of solutions on E^c
- Proof is significantly more involved.

EXAMPLE: $\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases}$ Any solution with $x<0$ approaches the origin tangent to E^c, the x-axis

Construct E^c as the union of $\{(x,0) \mid x \geq 0\}$ with any solution whose trajectory lies in the left half-plane.

This is a C^∞ center manifold



Note there does exist a unique analytic center manifold, namely the x-axis

10-9 Persistence of Hyperbolic Stationary Points

What happens to a hyperbolic fixed point when the equation is changed

$\dot{x} = f(x) + \varepsilon v(x)$ s.t. $f(0)=0$, $A = Df(0)$ nonsingular since it is hyperbolic

near $x=0$, $\dot{x} = \cancel{f(0)}^{0} + \overset{A}{Df(0)}x + \varepsilon v(0) + \varepsilon Dv(0)x + O(\varepsilon, x)$

fixed pts satisfies

$$-(A + \varepsilon Dv(0))x = \varepsilon v(0)$$

$$\text{so} \quad \tilde{x} = -(A+\varepsilon Dv(0))^{-1}\varepsilon v(0) + O(\varepsilon^2)$$

If ε sufficiently small $A + \varepsilon Dv(0)$ is invertible

A SIMILAR Perturbation calculation shows the eigenvalues of $(A + \varepsilon Dv(0))$ are within $O(\varepsilon)$ of the eigenvalues of A, so for ε suff small, the real parts of the eigenvalues will not cross zero

Alternatively, when $Df(0)$ nonsingular, the implicit function theorem guarantees

$\exists$ nbhd $N_{\varepsilon_0}(0)$ s.t $\forall \varepsilon < \varepsilon_0$, $\exists x^*(\varepsilon)$ s.t $f(x^*(\varepsilon)) = 0$