LECTURE 8 Linear Systems w/ Periodic Coefficients

$(*)\begin{cases} x'(t) = A(t)x \\ x \in \mathbb{R}^n \end{cases}$ $A(t+p) = A(t)$ is a smooth, periodic matrix-valued function (w/ minimal period p)

VERY IMPORTANT CASES: Hill's Eqn: $x'' + g(t)x = 0$

Mathieu's Eqn: $x'' + (a + b\cos\Omega t)x = 0$

these come up in studying stability of planetary orbits (& in our class general stability of periodic orbits), vibration of elliptical membranes

Lemma let $Y(t)$ be a fundamental matrix, then so is $Y(t+p)$ therefore $\exists$ nonsingular matrix st $Y(t+p) = Y(t)\Omega$

proof: $\frac{d}{dt}Y(t+p) = A(t+p)Y(t+p) = A(t)Y(t+p)$ $\square$

Note - $Y(t+np) = Y(t)\Omega^n$

- So if we know $Y(t)$ on $[0,p]$ and Ω, then $Y(t)$ known everywhere

- $Y(t+p) = Y(t)\Omega$

$Y(p) = Y(0)\Omega$

$Y(0)^{-1}Y(p) = \Omega$

If a different fundamental solution matrix $Z_1(t)$ then $Z_1(t) = Y(t)C$

& $Z_1(0)^{-1}Z_1(p) = (Y(0)C)^{-1}Y(p)C = C^{-1}Y(0)^{-1}Y(p)C = C^{-1}\Omega C$

A SIMILAR MATRIX

Definition - if $Y(0) = I$ (ie $Y(t) = Y(t,0)$) then $\Omega = Y(p)$ called the monodromy matrix

Clearly the behavior of solutions as $t \to \pm\infty$ determined by the eigenvalues of Ω

Let ω_i be the eigenvalues of Ω

Thm 1) if all $|\omega_i| < 1$ then $\lim_{t\to\infty}|y(t)| = 0$, ~~$\lim_{t\to-\infty}|y(t)| = \infty$ for all nonzero $y(0)$~~

Thm 2) if $\exists |\omega_i| > 1$ then $\exists$ solution $y(t)$ st $|y(t)| \xrightarrow[t\to\infty]{} \infty$

Thm 3) if $|\omega_i| = 1$ $\forall i$ and Ω has a complete set of eigenvectors then all solutions are bounded as $t \to \pm\infty$

LECTURE 8-2

EXAMPLE: HILL'S EQUATION $x'' + g(t)x = 0$, $x \in \mathbb{R}$, $g(t+p) = g(t)$

- Rewrite as $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -g(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- For SIMPLICITY, CHOOSE $Y(0) = I$, then the wronskian $W(t) = \det Y(t)$

satisfies $\frac{dW}{dt} = \text{trace} \begin{pmatrix} 0 & 1 \\ -g(t) & 0 \end{pmatrix} = 0$

$W(0) = 1$

$\Rightarrow W(t) = 1 \Rightarrow W(P) = 1$

but $W(P) = \det \Omega = \omega_1 \omega_2 \Rightarrow \omega_2 = \frac{1}{\omega_1}$

3 possibilities

(i) $\omega_1 \neq \omega_2$ and both real then $\omega_2 = \frac{1}{\omega_1}$ $\Rightarrow$ 1 of them has $|\omega_i| > 1$

$\Rightarrow \exists$ unbounded solutions

(ii) $\omega_1 \neq \omega_2$ & both have nonzero imaginary part

the $\omega_2 = \bar{\omega}_1$, $\exists$ complete set of eigenvectors & $|\omega_1| = |\omega_2| = 1$

$\Rightarrow$ All solutions are bounded & periodic or quasiperiodic

(iii) $\omega_1 = \omega_2 = \pm 1$ TWO CASES

(a) Ω diagonalizable $\Rightarrow \Omega = \pm I$ (only 2x2 diagonalizable matrix with $\lambda_1 = \lambda_2 = \lambda$ is λI)

$\Rightarrow$ all solutions bounded

(b) Ω not diagonalizable $\Rightarrow$ exist unbounded solutions

A useful criterion (often used numerically) for stability in Hill's equation

Let $Y(0) = I$, $Y(p) = \Omega = \begin{bmatrix} x_1(p) & x_2(p) \\ x_1'(p) & x_2'(p) \end{bmatrix}$, we know $\det \Omega = 1$

also know $\sum_i \omega_i = \text{tr}\, \Omega$

So $x_1(p) + x_2'(p) = \omega_1 + \omega_2$

look at 3 cases above

case (i) $\Leftrightarrow |\text{tr}\, \Omega| > 2$

(ii) $\Leftrightarrow |\text{tr}\, \Omega| < 2$

(iii) $\Leftrightarrow |\text{tr}\, \Omega| = 2$

So stability condition: $|\text{trace of monodromy matrix}| < 2$

LECTURE 8-3

Thm 4 <u>Floquet's Thm</u> let $Y(t)$ be a fundamental matrix for (*)
then $\exists$ constant matrix R, periodic matrix w/ $P(t+p) = P(t)$
such that $Y(t) = P(t)e^{tR}$

To prove this, we first need a lemma:

<u>Lemma</u> for any invertible matrix C, there exists a matrix B s.t. $C = e^B$
(i.e. there exists a logarithm)

<u>proof of lemma</u> • let $J = DCD^{-1}$ be the Jordan form of C

• then it suffices to find $K = \log J$

• then $C = D^{-1} J D = D^{-1} e^K D = e^{-D^{-1}KD}$ so we let $B = D^{-1}KD$

• In fact, it's sufficient to just find the logarithm of a <u>single</u> Jordan Block

• Assume $C = \lambda I + D^1$ where $D^b_{jk} = \delta_{j,k-b}$, Note $(D^1)^b = D^b$

• pick $\mu \in \mathbb{C}$ st $e^\mu = \lambda$ (this specifies a branch of the logarithm)
and define $\log(\lambda I) = \mu I$

• define $\log(I + \lambda^{-1} D^1)$ by Power series (formally)

then $B = \log C = \log(\lambda I(I + \lambda^{-1} D^1))$ (since any matrix commutes with λI)

$= \log \lambda I + \log(I + \lambda^{-1} D^1)$

$= \mu I + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\lambda^{-k} D^k}{k}$ but $D^k = 0$ for $k \geq n$

$= \mu I + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}\lambda^{-k} D^k}{k}$ so we don't need to worry about radius of convergence

This B is well-defined & you may check by direct substitution that $e^B = C$ □

<u>Note</u> we can replace μ by $\mu + 2\pi i k$ for any k (choosing a branch)
& we have one such choice for <u>each</u> Jordan block, so the logarithm is far from unique

<u>proof of theorem</u> let $\Omega = e^{pR}$ by the lemma, and $P(t) = Y(t)e^{-tR}$

then $Y(t) = P(t)e^{tR}$

and $P(t+p) = Y(t+p)e^{-(t+p)R} = Y(t)\Omega e^{-(t+p)R}$

$= Y(t) e^{pR} e^{-(t+p)R}$

$= Y(t) e^{-tR} = P(t)$ □

LECTURE 8-4

Comments - The μ are clearly not unique since $e^{\mu + 2\pi i k} = e^{\mu}$
but the eigenvalues of Ω are unique and are called the Floquet multipliers, the eigenvalues of R are the Floquet exponents

- $P(t)$ and R may be complex even though $Y(t)$ is real although one may show that ~~t~~ one can always find real R, $P(t)$ such that $P(t+2p) = P(t)$

- let $y(t) := P^{-1}(t)x(t)$

$$\frac{d}{dt}P(t)y(t) = \frac{dx}{dt} = A(t)x(t) = A(t)P(t)y(t)$$
$$= A(t)Y(t)e^{-tR}y(t)$$

ALSO

$$\frac{d}{dt}P(t)y(t) = P'(t)y(t) + P(t)y'(t)$$
$$= \frac{d}{dt}(Y(t)e^{-tR})y(t) + Y(t)e^{-tR}y'(t)$$
$$= \left(A(t)Y(t)e^{-tR} - Y(t)e^{-tR}R\right)y(t) + Y(t)e^{-tR}y'(t)$$

setting these equal

$$\cancel{A(t)Y(t)e^{-tR}y(t)} = \left(\cancel{A(t)Y(t)e^{-tR}} - Y(t)e^{-tR}R\right)y(t) + Y(t)e^{-tR}y'(t)$$

$$Y(t)e^{-tR}y'(t) = Y(t)e^{-tR}Ry(t)$$
$$y'(t) = Ry(t) \quad \text{since } Y(t)e^{-tR} \text{ is invertible}$$

So this allows us to replace a periodic coeff problem with a constant coeff one (if we could find p)

Finally, from Liouville's Theorem, we know

$$\det Y(t) = e^{\int_{t_0}^{t} \operatorname{tr} A(s)\,ds} \det Y(t_0)$$

$$\det \Omega = e^{\int_0^P \operatorname{tr} A(s)\,ds} \det I$$

$$\prod_{k=1}^{n} \omega_i \quad \cancel{\omega_1 \omega_2 \cdots} \quad = e^{\int_0^P \operatorname{tr} A(s)\,ds}$$