Lecture 6 - Linear Systems + Fundamental Solutions

Consider The homogeneous linear system

$$(*)\begin{cases} \vec{x}' = A(t)\vec{x} & A(t) \in C^1([a,b] \to \mathbb{R}^{n\times n}),\ t_0 \in [a,b] \\ \vec{x}(t_0) = \vec{x}_0 & x \in \mathbb{R}^n \end{cases}$$

and the ~~h~~ inhomogeneous system $(*')\begin{cases} \vec{x}' = A(t)\vec{x} + f(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$ as above with $f \in C^1[a,b]$

- By Picard Iteration/Contraction mapping both $(*)$ and $(*')$ have unique solutions on all $[a,b]$
- Principle of superposition: if $y^{(1)}$ & $y^{(2)}$ satisfy $(*)$, then so does $a_1 y^{(1)} + a_2 y^{(2)}(t)$
- We define a fundamental solution of $Y(t)$ ~~$\in GL_n(\mathbb{R})$~~ as a matrix whose columns are linearly independent solutions to $(*)$

$$Y = [y^{(1)}(t) \quad y^{(2)}(t) \quad \cdots \quad y^{(n)}(t)]$$

Note then that $\frac{dY}{dt} = A(t)Y$

Question: ~~Can~~ Does there always exist such a fundamental solution matrix $Y(t)$? Clearly, we can find a linearly independent set of vectors at $t=t_0$. The question is whether they remain independent $\forall t \in [a,b]$.

Answer: Such a $Y(t)$ exists.

<u>Proof</u> ~~Sup~~ by contradiction:

Suppose there exist $y^{(1)}(t), \dots, y^{(n)}(t)$ that are linearly independent at $t=t_0$ but dependent at $t=t_1$.

Then $\exists a_1, \dots, a_n$ s.t. $\sum a_i y^{(i)}(t_1) = 0$ but $\sum a_i y^{(i)}(t_0) \neq 0$

Consider the init'l value problem $(*)$ with ~~y~~ initial cond $y(t_1) = 0$.
Then it has two solutions $y = 0$ and $y = \sum a_i y^{(i)}(t)$.
These are different at $t = t_1$, violating uniqueness

Lecture 6-2

An alternate proof is to show that $W(t) = \det Y(t)$,
also called the Wronskian of the solutions
~~of~~ $y^{(1)}(t), \ldots, y^{(n)}(t)$ is never zero

Liouville's Theorem: $\frac{d}{dt} W(t) = (\operatorname{tr} A(t)) W(t)$

(note this implies $W(t) = e^{\int_{t_0}^{t} \operatorname{tr}(A(s))\,ds} W(0) \neq 0$)

Proof: $\cancel{\frac{W(t+h) + W(t)}{h}}$ $W(t+h) = \det(Y(t+h))$

$$= \det\left(Y(t) + \frac{dY}{dt} h\right) + o(h)$$
$$= \det(Y(t) + A(t)Y(t)h) + o(h)$$
$$= \det Y(t) \det(I + A(t)h) + o(h)$$

now it is easy to see that $\det(I + hA) = 1 + h \operatorname{Tr}(A) + O(h^2)$

so $W(t+h) = W(t)(1 + h\operatorname{Tr}(A)) + o(h)$

therefore $\lim_{h \to 0} \frac{W(t+h) - W(t)}{h} = \frac{(1 + h \operatorname{Tr} A(t))W(t) - W(t)}{h} = \operatorname{tr} A(t) \cdot W(t)$

$\square$

Aside on nth order systems

let $x^{(n)} + p_{n-1} x^{(n-1)} + \ldots + p_1 x' + p_0 x = 0$ (**)

then by the usual trick $u_1 = x,\ u_2 = x', \ldots, u_n = x^{(n-1)}$

we can rewrite this as

$$\frac{du}{dt} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} \end{bmatrix}$$

if $y_1, \ldots, y_n$ are solutions to (**), the Wronskian is defined as

$$W(t) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \vdots \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}$$

This is zero $\Leftrightarrow$ $y_1(t) \ldots y_n(t)$ are linearly dependent

Lecture 6-3

Applying Liouville's Theorem to the matrix form of (**), we find

$$\frac{dW}{dt} = -p_{n-1}(t)W(t)$$

$$\Rightarrow W(t) = e^{-\int_{t_0}^{t} p_{n-1}(s)\,ds}\, W(t_0)$$

This result is known as Abel's formula and tells us that $y_1(t), \dots, y_n(t)$ are either always dependent or always independent

EXAMPLE $x'' + 2x' + x = 0$ * has $p_{n-1} = 2$ so should get $W = Ce^{-2t}$

this has exact solutions $y_1 = e^{-t}$ and $y_2 = te^{-t}$

$$W(t) = \det\begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{bmatrix} = e^{-2t}\begin{vmatrix} 1 & t \\ -1 & 1-t \end{vmatrix} = e^{-2t}(1-t+t) = e^{-2t} \checkmark$$

Back to Fundamental Solutions

Suppose $Y(t)$ and $Z(t)$ are both fundamental sol'ns

$$\dot{Y} = AY \quad \text{and} \quad \dot{Z} = AZ$$

let $W(t) = Y(t)Y(t_0)^{-1}Z(t_0)$

then $\dot{W} = \dot{Y}Y(t_0)^{-1}Z(t_0)$

$= AY(t)Y(t_0)^{-1}Z(t_0)$

$= AW(t)$

So W is also a fundamental solution.

Further $W(t_0) = Y(t_0)Y(t_0)^{-1}Z(t_0) = Z(t_0)$

By uniqueness then $W(t) = Z(t)$

Let $C = Y(t_0)^{-1}Z(t_0)$

Conclusion $Z(t) = Y(t)C$. Any two fundamental solutions are identical up to right-multiplication by a constant matrix

Further $Y(t)Y(t_0)^{-1} = Z(t)Z(t_0)^{-1}$, and note both $= I$ at $t = t_0$

Define $Y(t, t_0)$ as the fundamental solution to (*) such that $Y(t_0, t_0) = I$

Then the general solution to (*) is $X(t) = Y(t, t_0)x(t_0)$.

Lemma $Y(t, t_0) = Y(t, s)Y(s, t_0)$

proof: for fixed s, $Y(t, s)$ is a fundamental solution matrix and $Y(s, t_0)$ is an invertible matrix

therefore $Y(t,s)Y(s,t_0)$ is a fundamental solution matrix

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Further: At $t = s$

$LHS|_{t=s} = Y(s, t_0)$, $RHS|_{t=s} = Y(s,s)Y(s,t_0) = Y(s,t_0)$

So the two sides of the eqn are equal at $t = s$ and both solve ⊛.

By uniqueness they are equal $\forall t \in [a,b]$ □

The inhomogeneous problem (*') $x' = A(t)x + f(t)$

$x|_{t=t_0} = x_0$

to solve let $x(t) = Y(t,t_0)y(t)$, then $x(t_0) = Y(t_0,t_0)y(t_0) = y(t_0)$.

further $\frac{dx}{dt} = \frac{dY}{dt}(t,t_0)y + Y(t,t_0)\frac{dy}{dt}$

$= A(t)\frac{dY}{dt}(t,t_0)y + Y(t,t_0)\frac{dy}{dt} =$

$= A(t)x + Y(t,t_0)\frac{dy}{dt} = A(t)x + f(t)$ by (*')

$Y(t,t_0)\frac{dy}{dt} = f(t)$

$\cancel{Y}$ $y' = Y^{-1}(t,t_0)f(t)$

So by quadratures $y = x_0 + \int_{t_0}^{t} Y^{-1}(s,t_0)f(s)ds$

$$Y^{-1}(t,t_0)x(t) = x_0 + \int_{t_0}^{t} Y^{-1}(s,t_0)f(s)ds$$

$$x(t) = Y(t,t_0)\left[x_0 + \int_0^t Y^{-1}(s,t_0)f(s)ds\right]$$

The Variation of Parameters Formula or Duhamel's Formula

All, so far in great generality. When $A(t)$ is constant, you have all seen $Y(t,t_0)$ many times.

It is $Y(t,t_0) = e^{A(t-t_0)}$

(Note that it is tempting but very wrong when A depends on time to write $A = e^{\int_{t_0}^{t} A(s)ds}$)

Let's review the theory of matrix exponentials

For the rest of the lecture A is an $n \times n$ real matrix

(***) $\begin{cases} \frac{dx}{dt} = Ax \\ x|_0 = x_0 \end{cases}$

Lecture 6-5

We look for solutions $\vec{x} = e^{\lambda t} v$ and find that

$$\lambda v = A v$$

$$(A - \lambda I) v = 0$$ has solutions only if $\det(A - \lambda I) = 0$

i.e. if λ are eigenvalue & v its associated eigenvector

If A has n linearly independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$
with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$

Then we may write $\vec{x}_0 = \sum_{k=1}^{n} a_k \vec{v}_k$ and $\vec{x}(t) = \sum a_k \vec{v}_k e^{\lambda_k t}$

Letting $D = [\vec{v}_1 \cdots \vec{v}_n]$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Then $AD = D\Lambda \Rightarrow A = D \Lambda D^{-1}$

$$\frac{dx}{dt} = D \Lambda D^{-1} x$$

Letting $y = D^{-1} x$ then $\frac{dy}{dt} = D^{-1} \frac{dx}{dt} = D^{-1} D \Lambda D^{-1} x$

$$\frac{dy}{dt} = \Lambda y$$

In particular $\frac{dy_i}{dt} = \lambda_i y_i$ so $y = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \vec{y}_0$

So for the diagonalized problem $Y(t, t_0) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$

$$D^{-1} x = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} D^{-1} x_0$$

$$x = D \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} D^{-1} x_0$$

So for a Diagonalizable matrix, the fundamental solution matrix
is $e^{At} = D \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} D^{-1}$

Note that from this ~~defn~~ construction, ~~we~~ it is difficult to show that the Fundamental Solution matrix e^{At} is unique.

Lecture 6 -6

Recall if A has $m<n$ different eigenvalues, and fewer than n linearly independent eigenvectors. Then there exist eigenvalues λ_k with multiplicity $n_k>1$. Corresponding to λ_k is an eigenvector v_k s.t. $(A-\lambda_k I)v_k$ and generalized eigenvectors

$w_1, \dots, w_{n_k-1}$ s.t. $(A-\lambda_k I)w_1 = v_k$

and $(A-\lambda_k I)w_j = w_{j-1}$ $\quad j=1,\dots,n_k-1$

not $(A-\lambda_k I)^{n_k}\cancel{w_j}\, v = 0$ for all $v \in \operatorname{span}\{v_k, w_1, \dots, w_{n_k-1}\}$

the generalized eigenspace of λ_k

In this Case, A can't be diagonalized. A can only be put in Jordan Canonical Form $A = DJD^{-1}$ where J is composed of $n_k \times n_k$ blocks of the Form

$$J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \quad (n_k \text{ by } n_k)$$

Our one tool to solve (***) is Picard iteration

$$x_{n+1} = x_0 + \int_0^t A x_n(s)\,ds \quad \Rightarrow x_1 = x_0 + \int_0^t A x_0\,ds = \cancel{x_0}\,(I+At)x_0$$

$$x_n = \sum_{k=0}^{n} \frac{A^k t^k}{k!} \quad (\text{exactly Taylor series of exponential}$$

So we define the matrix exponential $\quad e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}$

Then $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$ converges uniformly on any compact interval

<u>Claim</u> $\quad e^{At} = Y(t,0)$ $\quad$ (for this we will need to ~~$e^{A+B}=e^Ae^B$~~ that $AB=BA \Rightarrow e^{A+B}=e^Ae^B$. the lack of this identity is why $Y(t,0) \neq e^{\int_0^t A(s)\,ds}$ for the variable coefficient system (*))

Two parts: Clearly $e^{A0} = I$, so they agree at $t=0$.

Also $\cancel{\frac{d}{dh}}$ $\quad e^{A(t+h)} = e^{Ah}e^{At}$ $\quad$ so $\frac{1}{h}\left(e^{A(t+h)} - e^{At}\right) = \frac{1}{h}\left(e^{Ah} - I\right)e^{At}$

$$= \frac{1}{h}\left(Ah + \sum_{k=2}^{\infty} \frac{(Ah)^k}{k!}\right)e^{At}$$

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now ~~use~~ take the limit as $h \to 0$ (~~and note the two side commute~~)

to get $\frac{d}{dt} e^{At} = A \cdot e^{At}$

So e^{At} is a fundamental solution $\square$

if A is diagonalizable, we can reproduce our earlier result

if $A = D \Lambda D^{-1}$

$A^2 = D\Lambda D^{-1} \cdot D \Lambda D^{-1} = D \Lambda^2 D^{-1}$

induction $\Rightarrow A^n = D \Lambda^n D^{-1}$

So $e^{At} = D\left(\sum \frac{(\Lambda t)^k}{k!}\right) D$ ~~which~~ $= D\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix} D^{-1}$ as before

EXPONENTIALS OF JORDAN BLOCKS

example $A = \begin{bmatrix} 8 & 9 \\ -4 & 4 \end{bmatrix}$ has double eigenvalue $\lambda = 2$

with eigenvector $v_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

& generalized eigenvector $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ let $D = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$

then $D^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$

then $A = D \Lambda D^{-1} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$

We can check that for a 2×2 Jordan block

$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{bmatrix}$

$$(e^{Jt})_{11} = (e^{Jt})_{22} = e^{\lambda t} \quad \text{as before}$$

$$(e^{Jt})_{12} = 0 + t + \frac{1}{2!} \cdot 2\lambda t^2 + \frac{1}{3!} 3\lambda^2 t^3 + \dots$$

$$= t\left(1 + \tfrac{1}{1}\lambda t + \tfrac{1}{2}(\lambda t)^2 + \dots\right) = t e^{\lambda t}$$

$$e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Lecture 6-8

General Jordan block

Let Λ be an $n\times n$ Jordan block

$\Lambda = \lambda I + D^{n,1}$ where $(D^{n,m})_{jk} = \delta_{j,k-m}$ has ones in mth diagonal above the main + zeros elsewhere

(This $D^{n,m}$ notation is not-standard)

In this notation $D^{n,0} = I$, $D^{n,m} = 0$ if $m \geq n$

Note that $(D^{n,1})^b = D^{n,b}$ and that I commutes with $D^{n,m}$

then $\displaystyle \Lambda^b = \sum_{k=0}^{\min(b,n-1)} \binom{b}{k} \lambda^{b-k} D^{n-k}$

Then we find $\displaystyle e^{\Lambda t} = e^{\lambda t} \sum_{k=0}^{n-1} \frac{\lambda^k D^{n,k}}{k!} = \begin{bmatrix} 1 & t & t^2/2 & t^3/3 & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & & & \vdots \\ & & 1 & \ddots & & \vdots \\ & & & \ddots & t & \vdots \\ & & & & 1 & t \\ & & & & & 1 \end{bmatrix}$

COMPLEX EIGENVALUES

Note that all this goes through unchanged for complex eigenvalues.
But the power series definition implies e^{At} is real for real A.
Using the eigen-decomposition would require complex arithmetic.
This can be avoided using the

REAL JORDAN FORM: If A is a 2×2 matrix w/ eigenvalue $\alpha \pm i\beta$ then $\exists$ real matrix D s.t.

$$A = D\Lambda D^{-1} \quad \text{and} \quad \Lambda = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

using the power series, one finds $e^{\Lambda t} = e^{\alpha t} \begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix}$

The real Jordan form states that a real $n\times n$ Matrix can be ~~decompose~~ conjugated to a form Λ given by $A = D\Lambda D^{-1}$ (D real) where Λ is composed of blocks of four types:

Lecture 6-9

1) Diagonal, corresponding to real eigenvalues with complete sets of eigenvectors

2) Non-trivial real Jordan blocks (rank deficient real eigenvalues)

3) 2x2 blocks $\begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}$ corresponding to complex eigenvalues w/ complete eigenspaces

4) Nontrivial complex Jordan blocks

$$\begin{bmatrix} B_i & I & & \\ & B_i & I & \\ & & \ddots & I \\ & & & B_\ell \end{bmatrix} \quad \text{where } B_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}$$

Together, these show that the matrix exponential consists of sums of terms like $t^k e^{\alpha t} \cos\beta t$ and $t^k e^{\alpha t} \sin\beta t$