LECTURE 24 Hopf Bifurcations + Bifurcations of limit cycles

$\dot{x} = f(x, \mu)$ suppose at $\mu = 0$, $x = 0$ is a fixed pt and $Df(0,0)$ has 2 imaginary eigenvalues ~~Then for μ non~~ (complex conjugates $\lambda = \pm i\omega$ then when $\mu \neq 0$, these should move off imaginary axis. This is called a Hopf bifurcation

Canonical example

$$\dot{x} = -y + x(\mu - x^2 - y^2)$$
$$\dot{y} = x + y(\mu - x^2 - y^2)$$

$Df(0,\mu) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$, $\lambda = \mu \pm i$

so $\mu > 0 \Rightarrow$ origin unstable

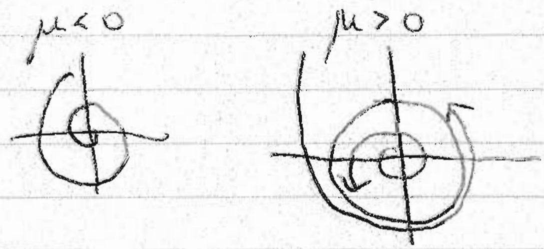
of course, put in polar coordinates

$$\dot{r} = r(\mu - r^2)$$
$$\dot{\theta} = 1$$

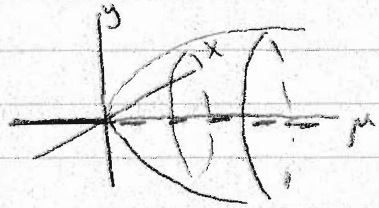
(note similarity to pitchfork)

$\mu < 0$ origin stable, $\mu = 0$ origin nonlinear stable spiral

$\mu > 0$, origin unstable and $r = \sqrt{\mu}$, $\Gamma = \sqrt{\mu}\begin{pmatrix}\cos t \\ \sin t\end{pmatrix}$ stable limit cycle



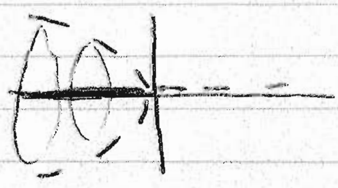
HOPF BIFURCATION



This is a supercritical Hopf bifurcation - stable limit cycles beyond pt where fixed point loses stability

~~STABLE~~ other case

$$\dot{x} = -y + x(\mu + x^2 + y^2)$$
$$\dot{y} = x + y(\mu + x^2 + y^2$$



LECTURE 24-2 Universal unfolding of the Hopf bifurcation (Theorem 2)

$$\dot x = \mu x - y + a x (x^2+y^2) - b y (x^2+y^2)$$
$$\dot y = x + \mu y + b x (x^2+y^2) + a y (x^2+y^2)$$

$$r\dot r = x\dot x + y \dot y = \mu x^2 + \mu y^2 + a(x^2+y^2)^2 = \mu r^2 + a r^4$$
$$\dot r = r(\mu + a r^2)$$

$$\dot\theta = \frac{x\dot y - y \dot x}{x^2+y^2} = \frac{x^2(1+br^2)+y^2(1+br^2)}{r^2} = 1+br^2$$

supercriticality depends on sign $\frac{a}{\mu}$

Text has (theorem 1) analytic test for determining whether supercritical or subcritical

$$\dot x = \mu x - y + (a_{20}x^2 + a_{11}xy + a_{02}y^2) + (a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) + \dots$$
$$\dot y = x + \mu y + (b_{20}x^2 + b_{11}xy + b_{02}y^2) + (b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3) + \dots$$

criticality depends on the sign of
$$\sigma = \frac{3\pi}{2}\left[3(a_{30}+b_{03}) + (a_{12}+b_{21}) - 2(a_{20}b_{20} - a_{02}b_{02}) + a_{11}(a_{02}+b_{20}) - b_{11}(b_{02}+b_{20})\right]$$
$\sigma < 0 \Rightarrow$ supercritical

I do not want you to learn this. If you ever need it, look it up

example

$$\dot x = \mu x - y + x^2$$
$$\dot y = x + \mu y + x^2$$
$a_{20} = b_{20} = 1$ all others zero

So $\sigma = \frac{3\pi}{2}(-2\cdot 1 \cdot 1) = -3\pi < 0$
$\Rightarrow$ supercritical
$\Rightarrow \exists$ stable limit cycle for small $\mu > 0$

lacking this, one can try to use poincare-bendixson theorem or other methods for showing periodic orbits do or do not exist for $\mu > 0$ or $\mu < 0$.

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Bifurcations of periodic orbits

some silly examples: take your favorite bifurcation

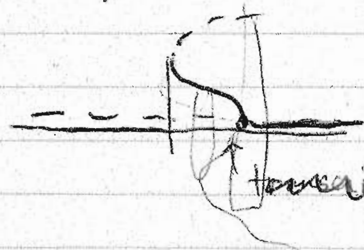
suppose that $\dot{\theta}=1$

~~and $\dot{r} = r(\mu - (r-1)^2$~~

and write $\dot{r}$ in such a way that it has that bifurcation at $r=1$, $\mu=0$

saddlenode $\dot{r} = \cancel{\mu} r(\mu-(r^2-1)^2)$ no ~~fix~~ periodic orbits $\mu<0$

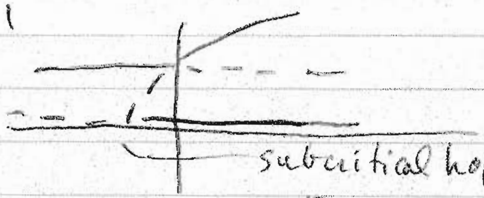
~~fixed~~ periodic orbits at $r=\sqrt{1\pm\sqrt{\mu}}$



~~transcritical~~ Hopf bifurcation at $\mu=1$

transcritical $\dot{r} = -r(1-r^2)(1+\mu-r^2)$

$\dot{\theta}=1$



subcritical hopf

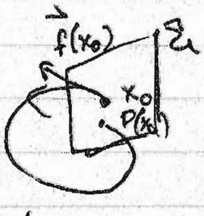
$r=0$, $r=1$, $r=\sqrt{1+\mu}$

PITCHFORK $\dot{r} = r(1-r^2)(\mu-(r^2-1)^2)$

$r=0$, $r=1$, $r=\sqrt{1\pm\sqrt{\mu}}$

so what? These examples are a bit contrived

Recall The main tool for studying periodic orbits is the Poincaré map



if $P(x_0)=x_0$ then there is a periodic orbit through x_0

if for some value of μ $f'(x^*,\mu)=1$ we are in one of the above situations

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$$x_{n+1} = f(x_n, \mu) \quad \text{st} \quad x^*(\mu) = f(x^*(\mu), \mu)$$

Suppose for $\mu < 0$, ~~$-1 < f'(x^*(\mu))$~~ $-1 < D_x f(x^*(\mu), \mu) < 1$

and $\mu > 0$ $\quad D_x f(x^*(\mu), \mu) < -1$

- ~~the above bifurcations take place when $f'(x^*(\mu)) = 1$~~
- then the system has a <u>flip bifurcation</u> or a period-<u>doubling</u> bifurcation

Notice: if $f(0)=0$ and $f'(0)=-1$ were purely linear, then

$x_{n+1} = f(x_n) = -x_n \Rightarrow x = 0$ the only fixed pt

all other initial conditions satisfy $x_{n+2} = x_n$

these are period-2 orbits of this simple map

In fact, near a flip bifurcation nearby ~~stable~~ periodic orbits always appear

Assume $x_{n+1} = f(x_n, \mu)$, $f(0,0)=0$, $f_x(0,0) = -1$

$$= A(\mu) + (-1 + B(\mu))x + C(\mu)x^2 + D(\mu)x^3$$

$A = f_\mu \mu + \frac{f_{\mu\mu}}{2}\mu^2 + O(\mu^3)$ $\quad$ (note $A(0) = 0$)

$B = f_{\mu x}\mu + O(\mu^2)$

$C(\mu) = \frac{1}{2} f_{xx} + O(\mu)$

$D(\mu) = \frac{1}{6} f_{xxx} + O(\mu)$

So fixed pts solve $\quad x^* \approx A(\mu) - x^*$

$$x^* \approx \frac{A(\mu)}{2} = \frac{1}{2} f_\mu \mu + O(\mu^2)$$

Stability Depends on $Df(x^*)$

$$Df = (-1 + B(\mu)) + 2C(\mu)x + O(x^2)$$

$$= -1 + f_{\mu x}(\mu) + 2 \cdot \tfrac{1}{2} f_{xx} \cdot \tfrac{1}{2} f_\mu \mu$$

$$= -1 + (f_{\mu x} + \tfrac{1}{2} f_{xx} f_\mu)\mu$$

stability ~~depends on~~ sign $(f_{\mu x} + \frac{1}{2} f_{xx} f_\mu)$

We need this nonzero to get a non-degenerate bifurcation (Recall our calculation for transcritical

Th 3/24 - 5

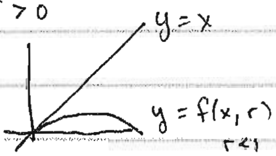
HOW WOULD you SHOW THIS $\quad f(x) = x$

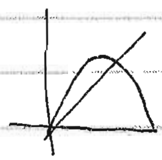
LOOK AT $f^2(x) = f(f(x))$, then if $f'(x) = -1$

$$\frac{d}{dx} f(f(x))\Big|_{x=x^*} = f'(f(x^*))f'(x^*) = (f'(x^*))^2 = 1$$

in fact you carefully show that $f^2(x)$ the 2nd iterate has a pitchfork bifurcation, can be supercritical or subcritical

Example: logistic map $\quad x > 0,\ r > 0$





$$x_{n+1} = r x_n (1 - x_n)$$

f(x) fixed pts $\quad x = rx(1-x)$

$x = 0$ or $1 = r(1-x) \Rightarrow x = 1 - \frac{1}{r}$

$f = r(x - x^2)$

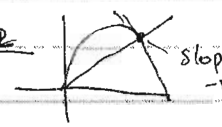
$f' = r(1-2x)$, $\quad f'(0) = r \quad$ stable for $0 < r < 1$

$f'(1-\frac{1}{r}) = r(1 - 2(1 - \frac{1}{r}))$

$= r(-1 + \frac{2}{r}) = 1 - r$

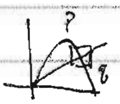
stable if $-1 < 1 - r < 1 \Rightarrow 1 < r < 3$

at $r = \frac{2}{3}$



period 2

$r > 2$



$f(p) = q$

$f(q) = p$

$f^2(q) = q$

$f(f(q)) = r^2 q(1-q)(1 - rq + rq^2)$

know $q = 0$, $q = 1 - \frac{1}{r}$ are solutions, so we can factor out $q(1 - 1 + \frac{1}{r})$

after a bunch of algebra

find $\left(q - \frac{r+1+\sqrt{(r-3)(r+1)}}{2r}\right)\left(q - \frac{r+1-\sqrt{(r-3)(r+1)}}{2r}\right) = 0$

$q = \frac{1}{2} + \frac{1+\sqrt{(r-3)(r+1)}}{2r}$, $\quad p = \frac{1}{2} + \frac{1-\sqrt{(r-3)(r+1)}}{2r}$

LEC 2-6

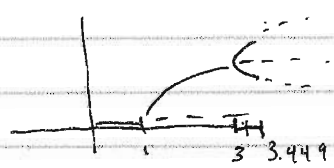
$$\frac{d}{dq} f(f(q)) = f'(f(q))\, f'(q)$$
$$= f'(p) f'(q)$$
$$= r(1-2p)r(1-2q)$$
$$= r^2(1 - 2(p+q) + 4pq)$$
$$= r^2\left(1 - \frac{2(r+1)}{r} + \frac{4(r+1)}{r^2}\right)$$

$$pq = \frac{1}{4r^2}\left((r+1)^2 - (r+1)(r-3)\right)$$
$$= \frac{1}{4r^2}(r+1)(r+1-r+3)$$
$$= \frac{1}{4r^2}(r+1)(4) = \frac{r+1}{r^2}$$

$$= r^2 - 2(r+1)r + 4(r+1)$$
$$= r^2 - 2r^2 - 2r + 4r + 4$$
$$= 4 + 2r - r^2$$

Stable ~~at~~ if $|4+2r-r^2| < 1$

stable for $3 < r < 1+\sqrt{6} = 3.449\ldots$



then $\frac{d}{dq} f^2(q) = -1$ another flip

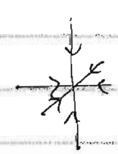
this is the start of a period-doubling cascade

show logistic

~~p plane~~ bifurcation

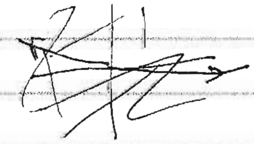
lorenz attractor

$$\dot{x} = 10(y-x)$$
$$\dot{y} = \mu x - y - xz$$
$$\dot{z} = xy - \frac{8}{3}z$$

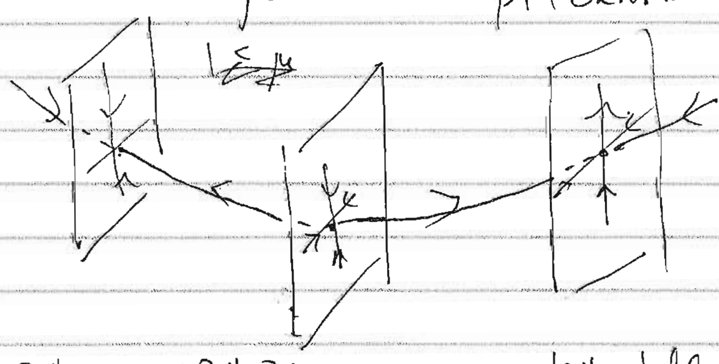


$0 < \mu < 1$ origin stable

$\mu = 1$ pitchfork to



$1 < \mu$



$1.34 < \mu < 24.74$ still stable but oscillatory