LECTURE 5 - Maximal Domain of Existence

Cf ① $\begin{cases} \dot{x}=f(x) \\ x(0)=x_0 \end{cases}$, $f\in C^1(E)$, $E\subset\mathbb{R}^n$ open, $x_0\in\mathbb{R}^n$

We've seen that for some $a>0$

- there exists a solution $x(t)\subset E$ for $t\in[-a,a]$
- this is local existence

Questions: - What is the largest interval J s.t

- ① has a unique solution $x(t)\subset E$
- If J is finite, what happens as its endpoint is approached
- And if it's infinite?

Recall a definition: if E is open, the closure $\bar{E}$ is the set of all limit points of E, note $E\subseteq\bar{E}$, the boundary $\partial E=\bar{E}\setminus E$

Four illustrative examples:

1) $\begin{cases} \dot{x}=ax \\ x(0)=x_0 \end{cases}$ here $E=\mathbb{R}$, $x=x_0e^{at}$ and $J=(-\infty,\infty)$

2) $\begin{cases} \dot{x}=1+x^2 \\ x(0)=0 \end{cases}$ $E=\mathbb{R}$, $x=\tan t$ and $J=(-\frac{\pi}{2},\frac{\pi}{2})$ note $x\to\pm\infty$ as $t\to\pm\frac{\pi}{2}$

3) $\begin{cases} \dot{x}=-\frac{1}{2x} \\ x(0)=1 \end{cases}$ $E=(0,\infty)$ is the largest connected open set on which $f(x)$ is continuous
$x(t)=\sqrt{1-t}$ is an exact sol'n and $x(t)\in E$ for $t\in(\cancel{0},\infty)=J$
Note also as $t\downarrow -1$, $x(t)\to\partial E$

4) $\begin{cases} \dot{x}=\frac{-y}{z} \\ \dot{y}=\frac{x}{z} \\ \dot{z}=1 \end{cases}$ $(x,y,z)\big|_{t=\frac{1}{\pi}}=(0,-1,\frac{1}{\pi})$, $E=\{(x,y,z)\mid z>0\}$

- exact sol'n $(x,y,z)=(\sin\frac{1}{t},\cos\frac{1}{t},t)$, $J=(0,\infty)$
- as $t\to0^+$, the solution oscillates infinitely often and approaches ∂E

Observation: J is an open set in all examples

Lecture 5-2

Theorem 1 Consider IVP ①. Then for each $x_0 \in E$, there exists a maximal interval J s.t ① has a unique solution $x(t) \subset E$.
That is: if ① has a solution $y(t)$ defined on an interval I, then $I \subseteq J$ and $x(t) = y(t)$ for all $t \in I$.
Further: J is an open interval $J = (\alpha, \beta)$

To prove this we first need a lemma:

Lemma 1 let $u_1(t)$ and $u_2(t)$ solve ① on intervals I_1 & I_2 respectively.
Then: 1) $0 \in I_1 \cap I_2$
2) If I an open interval with $0 \in I$ & $I \subset I_1 \cap I_2$, then $u_1(t) = u_2(t)$ $\forall t \in I$

Proof of lemma (1) follows from initial conditions

(ii) Suppose $0 \in I \subset I_1 \cap I_2$. Then by existence/uniqueness thm, $\exists a > 0$ s.t ① has a unique solution on $(-a, a) \subset I$

- let $I^* = \bigcup \{$ open intervals contained in I on which $u_1 = u_2 \}$
- Note that I^* is nonempty since it contains (at least) the interval $(-a, a)$
- Then I^* is the largest, open subinterval of I on which $u_1 = u_2$

(BEGIN PROOF BY CONTRADICTION)
- Suppose $I^* \subsetneq I$, Then one of its endpoints t_0 is in $I_0 \subset I_1 \cap I_2$
then $\lim_{t \to t_0} u_1(t) = \lim_{t \to 0} u_2(t)$ by continuity.

Call this limit u_0 (see fig 1)

- Then, by existence/uniqueness thm $\exists a_0 > 0$ s.t ① has unique sol'n with init'l cond $u(t_0) = u_0$ on $I_0 (t_0 - a_0, t_0 + a_0)$
see fig 2

Thus $u_1(t) = u_2(t)$ on $I_0 \cup I^* \subset I$. Contradicting that I^* is max'l interval of uniqueness.

FIG 1

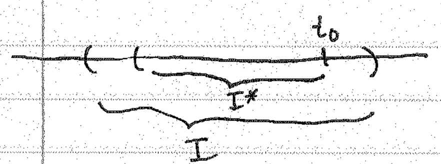
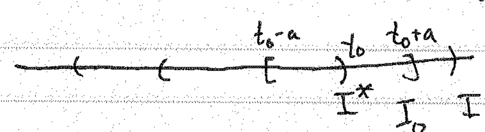


FIG 2



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Lecture 5-3

PROOF OF THM 1: - By existence-uniqueness, (1) has a solution on $(-a, a)$

- let $(\alpha, \beta) = \bigcup\{\text{all open intervals } I \text{ s.t. (1) has a solution on } I\}$
- define $x(t)$ as follows:

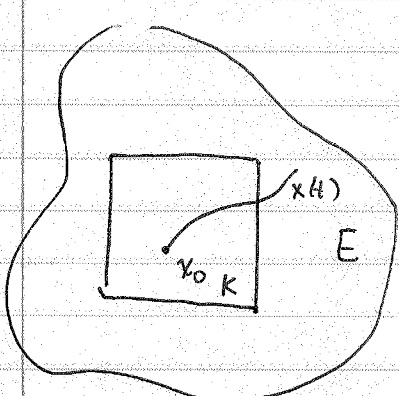
  given $t \in (\alpha, \beta)$, $\exists I \subset (\alpha, \beta)$ s.t. $t \in I$ and (1) has a solution $u(t)$ on I. Let $x(t) = u(t)$ on I
- claim $x(t)$ is a well-defined function on (α, β) because if $t \in I \subset I_1 \cap I_2$
- also $x(t)$ solves (1) since each point $t \in (\alpha, \beta)$ is in some interval I on which (1) has a unique solution $u(t)$ and $x(t) = u(t)$ on I

To show J is open, suppose $J = (\alpha, \beta]$, then $\lim_{t \to \beta^-} x(t) = x(\beta)$, but we can uniquely continue

$x(t)$ to some $(\alpha, \beta + \alpha)$ contradicting that J is the maximal interval $\square$

Definition: (α, β) called the maximal interval

Theorem 2 Let $E \subset \mathbb{R}^n$ open, $x_0 \in E$, $f \in C^1(E)$.
Let (α, β) be the maximal interval of existence for (1).
Assume $\beta < \infty$. Then given any compact $K \subset E$, there exists $t \in (\alpha, \beta)$ s.t. $x(t) \notin K$.



IF $\beta < \infty$, the solution eventually leaves

LECTURE 5-4

PROOF: - Since f continuous on compact set K, $f(x)$ bounded on K

i.e. $\exists M \geq 0$ st $|f(x)| \leq M \quad \forall x \in K$

- Let $x(t)$ solve (1) with maximal interval (α, β),
- Assume $\beta < \infty$ and $x(t) \in K \quad \forall t \in (\alpha, \beta)$

(Beginning of proof by contradiction)

- Then for all $\alpha < t_1 < t_2 < \beta$

$$|x(t_2) - x(t_1)| \leq \int_{t_1}^{t_2} |f(x(s))| ds \leq M(t_2 - t_1)$$

thus as $t_1, t_2 \nearrow \beta$, $|x(t_2) - x(t_1)| \to 0$

which shows by the Cauchy Criterion that $\lim_{t \nearrow \beta} x(t)$ exists

- let $x_1 = \lim_{t \nearrow \beta} x(t)$, then $x_1 \in K \subset E$ since K is closed + bounded

- Define $u(t) = \begin{cases} x(t) & \alpha < t < \beta \\ x_1 & t = \beta \end{cases}$

then $u(t)$ ~~solves (1)~~ differentiable and solves

$$u(t) = x_0 + \int_0^t f(u(s)) ds \quad \text{for all } t \in (\alpha, \beta]$$

this defines a continuation of the solution $x(t)$ to $(\alpha, \beta]$

- Since $x_1 \in E$, the IVP $\begin{cases} \dot{x} = f(x) \\ x(\beta) = x_1 \end{cases}$ has a unique sol'n $x_1(t)$ on $(\beta - a_1, \beta + a_1)$ for some $a_1 > 0$

- by lemma 1, $u(t) = x_1(t)$ on $(\beta - a_1, \beta)$, further $x_1(\beta) = u(\beta)$

define $v(t) = \begin{cases} u(t) & t \in (\alpha, \beta] \\ x_1(t) & t \in (\beta, \beta + a_1) \end{cases}$

Then $v(t)$ solves (1) on $(\alpha, \beta + a_1)$ contradicting the assumption that (α, β) is the maximal interval of existence

$\square$

- Note - A SIMILAR RESULT APPLIES IF $\alpha > -\infty$

Lecture 5-3

Revisit example 2 $\begin{cases} \dot{x} = 1+x^2 \\ x(0)=0 \end{cases}$, note $x = \tan t$, $J = (-\frac{\pi}{2}, \frac{\pi}{2})$

if $M>0$ then $K = [-M, M]$ is a compact subset of $E = \mathbb{R}$

as $t \nearrow \frac{\pi}{2}$, $x(t) \nearrow +\infty$ and in particular it leaves K

(Note since $E = \mathbb{R}$, the only way a solution can have finite interval of existence is to diverge to $\pm\infty$ in finite time)

Revisit example 4 : $E = \{(x,y,z) \mid z>0\}$

any compact subset K must satisfy $z \geq z_0 > 0$ for some z_0

but as $t \searrow 0$, the solution approaches $z=0$, leaving K

Theorem 3 - The same as Thm 2, but on half intervals $[\alpha, 0]$ or $[0, \beta]$

Corollary 1 : Under the hypotheses of Theorem 2, if $\beta < \infty$ and $\lim_{t\to\beta^-} x(t)$ exists, then $\lim_{t\to\beta^-} x(t) \in \partial E$

Proof Assume $\lim_{t\to\beta^-} x(t) = x^-$ exists and let $u = \begin{cases} x(t) & 0 \leq t < \beta \\ x_1 & t = \beta \end{cases}$

let $K = \{x \mid x = u(t)$ for some $t \in [0, \beta]\}$, the image of t under the map $u(t)$

Then K is compact

Assume $x_1 \in E$. Then $K \subset E$ and it follows from theorem 3 that $\exists t \in [0, \beta)$ s.t. $x(t) \notin K$ contradiction

Thus $x_1 \notin E$

But since $x(t) \in E$ $\forall t \in [0, \beta)$, it follows $x_1 = \lim_{t\to\beta^-} x(t) \in \overline{E}$

So $x(t) \in \overline{E} \setminus E = \partial E$

$\square$

Lecture 5-6

Example 3 Revisited $\quad \dot{x} = \frac{-1}{2x}$ $\quad$ so $E = \{x \mid x > 0\}$

$x(0) = 1$ $\quad$ $x(t) = \sqrt{1-t}$, $\quad J = (-\infty, 1)$

as $t \to 1^-$, $x \to 0 \in \partial E$

illustrating the behavior described by Corollary 1

Corollary 2 $\quad$ Let $E \subset \mathbb{R}^n$ be an open set, $x_0 \in E$, $f \in C^1(E)$ and $[0, \beta)$ = right-maximal interval of existence to ①. Assume $\exists$ compact set $K \subset E$ s.t. $x(t) \in K$ $\forall t \in [0, \beta)$. Then $\beta = \infty$.

Proof $\quad$ This is simply the contrapositive of Theorem 3. $\square$

Example 1 Revisited

let $a = -1$: $\quad \dot{x} = -x$ $\quad$ then $\quad x(t) = x_0 e^{-t}$

$x|_0 = x_0 > 0$

The solution remains in $[0, x_0]$ compact $\forall t > 0$ and we see the solution exists on $[0, \infty)$.