LECTURE 19

AN EXAMPLE

$\dot{x} = a - x - \frac{4xy}{1+x^2}$ $\qquad a, b > 0$

$x, y > 0$ Chemical concentrations

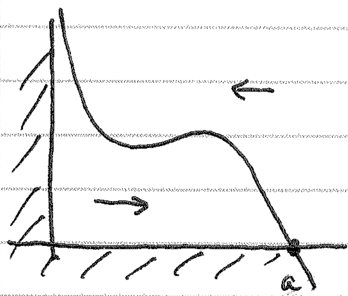
$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$

Where is $\dot{x} = 0$? The <u>null-clines</u> are the solutions to

$$a - x - \frac{4xy}{1+x^2} = 0$$

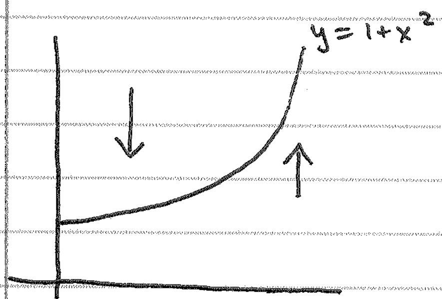
$$\frac{4xy}{1+x^2} = a - x$$

$$y = \frac{(1+x^2)(a-x)}{4x}$$



$\dot{y} = 0$ the null-clines satisfy $bx\left(1 - \frac{y}{1+x^2}\right) = 0$

$x = 0$ or $y = 1 + x^2$



FIXED POINTS $\quad 1 + x^2 = \frac{(1+x^2)(a-x)}{4x}$

$$4x = a - x$$

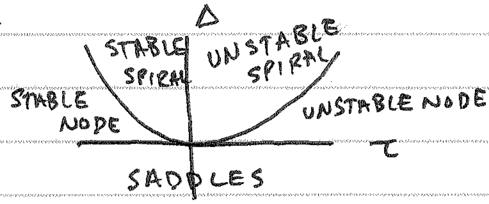
$$x^* = \frac{a}{5}, \quad y^* = 1 + \frac{a^2}{25}$$

FIND (AFTER A LOT OF WORK, AND USING $y^* = 1 + x^{*2}$

$$J(x^*, y^*) = \frac{1}{1+x^2}\begin{bmatrix} 3x^2 - 5 & -4x \\ 2bx^2 & -bx \end{bmatrix}$$

19-2

Recall



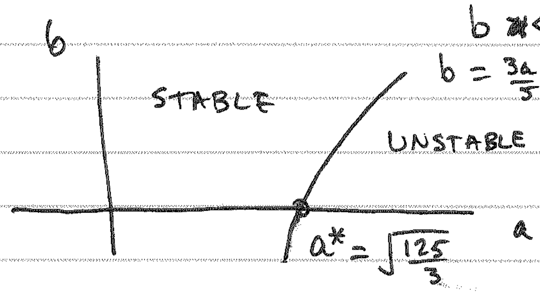
$\Delta = \frac{5bx}{1+x^2} > 0$

$\Rightarrow$ stability entirely due to τ

$\tau = \frac{1}{1+x^2}(3x^2 - 5 - bx)\Big|_{x=x^*=a/5}$

UNSTABLE IF $\frac{3a^2}{25} - 5 - \frac{ba}{5} > 0$

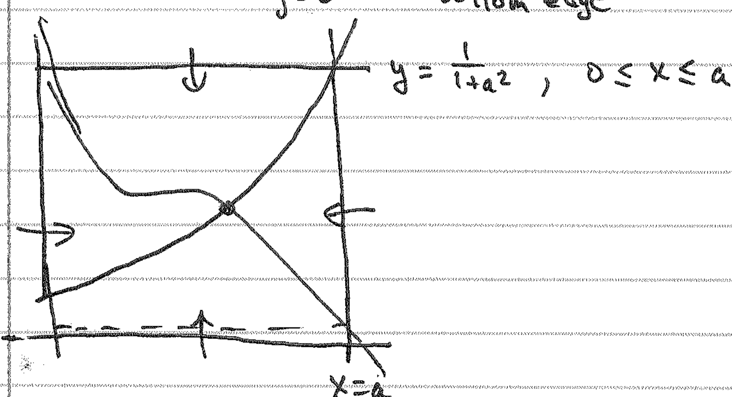
$b < \frac{3a}{5} - \frac{25}{a}$



TRAPPING REGION

CLEARLY $x=0$ left edge

$y=0$ bottom edge



if $b < b_c(a)$ then fixed point unstable $\Rightarrow \exists$ circle around 0 s.t. flow outward

if $b > b_c(a)$, solutions in trapping region approach (x^*, y^*)

Van der Pol oscillator

$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$

only fixed pt $x = \dot{x} = 0$

if $(x^2-1) < 0$ negative damping

$(x^2 > 1) > 0$ positive damping

Q: Is there a balance at moderate amplitude leading to periodic orbit?

A: Yes, but standard $\dot{x} = y$ won't give trapping region

$\dot{y} = -\varepsilon(x^2-1)y - x$

19–3

Lienard equations a generalization of van der Pol

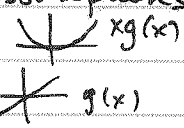
① $\ddot{x} + f(x)\dot{x} + g(x) = 0$

We will prove the existence of a trapping region ∄

$$\text{let } F(x) = \int_0^x f(\xi)\,d\xi$$

$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$ stability depends on $J = \begin{bmatrix} -f(x) & 1 \\ -g'(x) & 0 \end{bmatrix}$ origin unstable if $f(0) < 0$, $g'(0) > 0$ and not a saddle

Theorem Consider ① with assumptions



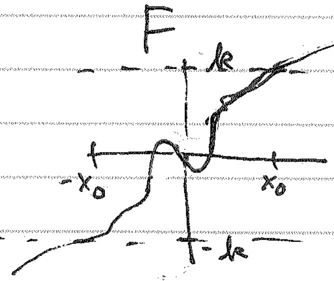
i) $xg(x) > 0$ for $x \neq 0$

$g(0) = 0$, $g'(0) > 0$

$f(0) < 0$

(ii) $\text{sign}\,x \cdot F(x) > k > 0$ for $|x|$ suff large

(iii) $G(x) = \int_0^x g(\xi)\,d\xi \xrightarrow[|x|\to\infty]{} \infty$



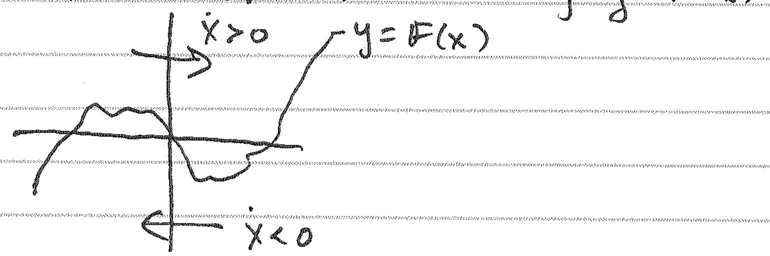
note $F(0) = 0$

$F'(0) < 0$

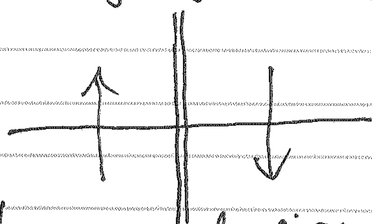
Then ① has at least one periodic orbit

PROOF ~~(i) ⇒ (0,0) the only fixed point and we can take a small circle containing (0,0) as the inner boundary~~

Intuition: Nullclines $\dot{x} = 0$ along $y = F(x)$ } FROM (i) & ii



$\dot{y} = 0$ along $g(x) = 0 \Rightarrow x(0) = 0$



So large scale circulation is finite

19-4

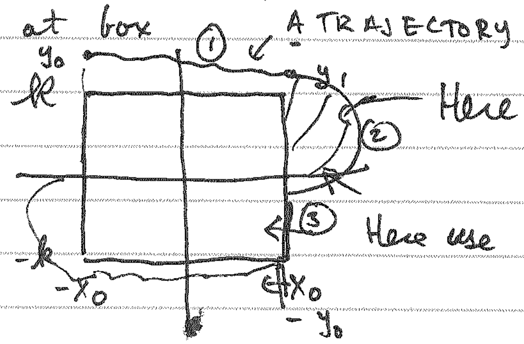
also note general form of $F \Rightarrow$ negative damping for small x
regular damping for large x

proof $(0,0)$ only fixed pt and (i) $\Rightarrow$ it is unstable + not a saddle

Want to use Poincaré-Bendixson. Construct a trapping region
3 Steps

Assume for $|x| > x_0$, $|F(x)| > k$

look at box



Here use a Lyapunov function argument

Here use $y < F(x) \Rightarrow \dot{x} < 0$

MAKE PERIODIC

① Note on any trajectory $\frac{dy}{dx} = \frac{-g(x)}{y - F(x)}$

looking for a solution s.t. for $-x_0 < x < x_0$,

Since $[-x_0, x_0]$ compact, $g(x)$, $F(x)$ continuous
$\exists$ $C > 0$ s.t. $|g(x)| < c$, $|F(x)| < C$ on $[-x_0, x_0]$

pick $y_0 > 2k$ s.t

$$\left|\frac{dy}{dx}\right| = \left|\frac{-g(x)}{y - F(x)}\right| < \frac{k}{2x_0} \quad \forall y \in [y_0 - k, y_0 + k]$$

for all $y \in [y_0 - k, y_0 + k]$

Can I do this? $\left|\frac{g(x)}{y - F(x)}\right| \le \frac{C}{y_0 - k - C} \le \frac{k}{2x_0}$

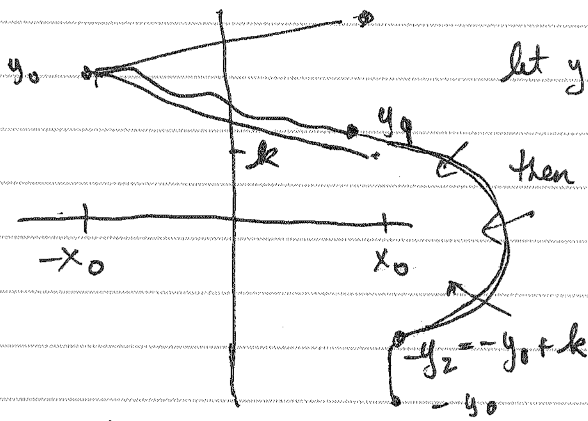
if $\boxed{y_0 > k + C + \frac{2x_0 C}{k}}$

19-5

Thus for ~~$-x_0$~~ $\frac{dy}{dx} < \frac{k}{2x_0}$ for the trajectory through $(-x_0, y_0)$

So $\quad y_0 - \frac{k(x+x_0)}{2x_0} < y(x) < y + \frac{k(x+x_0)}{2x_0}$

$k = 2k - k < y_0 - k < y(x_0) < y_0 + k$



let $y_1 = y(x_0)$

then $y_1 > k$

STEP 2: TO SHOW THE NEXT PART OF THE TRAJECTORY hits $x = x_0$ again at $(x_0, -y_2)$, $y_2 > 0$

We introduce a modified Lyapunov function

$$V(x,y) = \tfrac{1}{2}(y-k)^2 + G(x)$$

This has contour lines that are symmetric about line $y = k$

Since $G(x)$ bounded, a contour line can't approach $y = \infty$ w/ bounded x, nor can it approach ∞ w/ bounded y since $G(x) \to \infty$ as $x \to \infty$

So a ~~count~~ contour through (x_0, y) must cross the line $y = k$ by symmetry, it crosses $x = x_0$ at $-y + k$

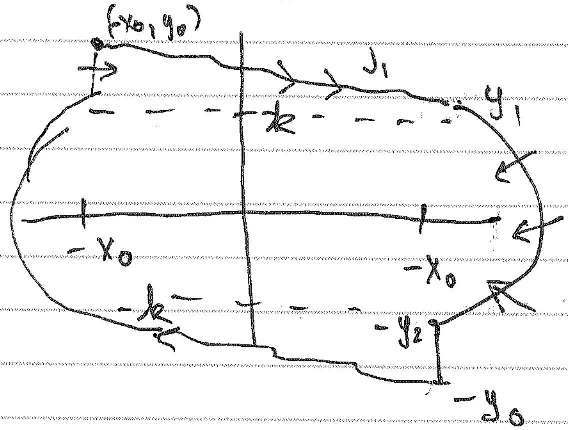
The contour through (x_0, y_0) crosses again at $(x_0, -y_0 + k)$
but $-y_0 + k < -k$ define $y_2 = y_0 - k$

Now ~~also~~ along the contour $V(x,y) = V(x_0, y_0)$

$$\dot{V} = (y-k)\dot{y} + g(x)\dot{x}$$
$$= (y-k)(-g(x)) + g(x)(y - F(x))$$
$$= g(x)(k - F(x)) \quad \text{if } x > x_0 \quad F(x) < k$$
$$\Rightarrow \dot{V} < 0$$

STEP 3 along $x = x_0$, $-y_0 < x < -y_2 < -k$

$$\dot{x} = -g(x) < 0$$



the boundary of our trapping region is ~~Ray~~

$$J = J_1 \cup J_2 \cup J_3 \cup J_4 \cup J_5 \cup J_6$$

J_1 = trajectory from $(-x_0, y_0)$ to (x_0, y_1)

$J_2 = \{(x,y) \mid x \geq x_0 \text{ and } V(x,y) = V(x_0, y_1)\}$

$J_3 = \{(x_0, y) \mid -y_0 \leq k \leq -y_2\}$

$J_4 - J_6$ constructed SIMILARLY

Note Van der Pol

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

$f = x^2 - 1$

$F = \frac{x^3}{3} - x$

$g = x$ satisfy the Hypotheses $\Rightarrow \exists$ periodic orbit

Perko shows if in addition $F(x), g(x)$ odd

$F(x)$ has exactly one root for $x > 0$

then solution is unique

He does not use Poincaré-Bendixson