LECTURE 7-1 LINEAR SYSTEMS IN 2D

Generic matrix in $\mathbb{R}^2$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\dot{\vec{x}} = A\vec{x}$$

Defines a vector field in $\mathbb{R}^2$, note $\vec{x} = \binom{0}{0}$ an exact solution

Note the vector field is singular (no well-defined direction) at $\binom{0}{0}$

Let's look at solutions in detail

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + ad - bc$$
$$= \lambda^2 - \tau\lambda + \delta$$

$\delta = ad - bc$ the determinant

$\tau = a + d$ the trace

eigenvalues $\lambda, \mu = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$, note $\lambda\mu = \delta$ & $\lambda + \mu = \tau$

$$\vec{x} = c_1 e^{\lambda t}\vec{v}_1 + c_2 \vec{v}_2 e^{\mu t}$$

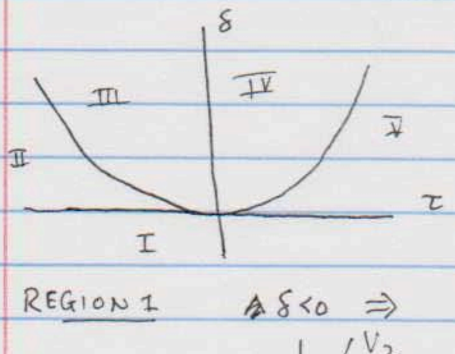
What can we learn about solutions from just τ & δ?

(Warning this is specific to 2D!)

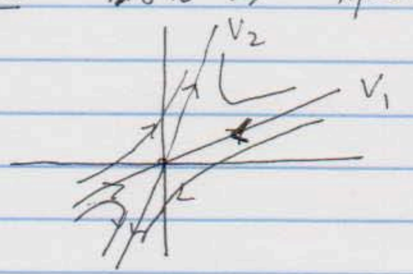
if $\tau^2 - 4\delta < 0$ then $\lambda, \mu = \alpha \pm i\beta$

$\tau^2 - 4\delta > 0$ then λ, μ real (Note $\delta < 0$ puts us in this case)

$\tau^2 - 4\delta = 0$ then $\lambda = \mu$, may be diagonal or Jordan-Block



REGION 1 $\quad \delta < 0 \Rightarrow \lambda\mu < 1$ so WLOG $\lambda < 0 < \mu$



$$\vec{x} = c_1\vec{v}_1 e^{\lambda t} + c_2 v_2 e^{\mu t}$$

Note only way $\|x\| \xrightarrow[t\to\infty]{} 0$ is if $c_2 = 0$

only way $\|x\| \xrightarrow[t\to-\infty]{} 0$ is if $c_1 = 0$

$\binom{0}{0}$ A SADDLE POINT

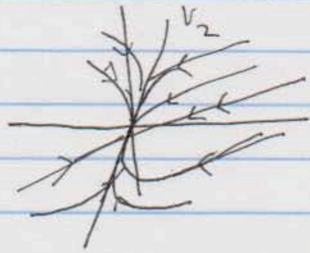
LECTURE 7-2

REGION II, III, BOTH EIGENVALUES HAVE NEGATIVE REAL PART

REGION II WLOG $\lambda < \mu < 0$

then as $t \to +\infty$ the $\vec{v}_1$ COMPONENT DECAYS FASTER

SO SOLUTIONS APPROACH ORIGIN THE $\vec{v}_2$ DIRECTION UNLESS $C_2 = 0$

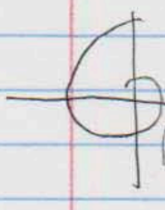


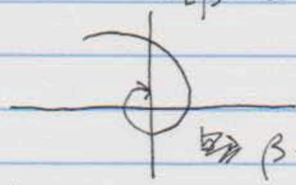
STABLE NODE

REGION 3 $\quad A = D\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} D^{-1} \quad a < 0$

~~Dues~~ STABLE FOCUS

STABLE SPIRAL





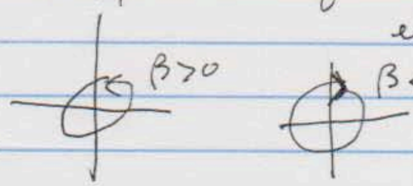
BETWEEN 3 & 4 $\quad \tau = 0, \Delta$ positive

$$\lambda = \pm i\beta$$

$A = D\begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} D^{-1}$, $\quad \beta > 0$ counterclockwise

$\beta < 0$ clockwise

D responsible for ~~eccentricity of~~ closed orbits ellipticity & angle



Regions IV & V much like II & III with solutions moving away from origin

Boundary of region 1 $\quad \lambda = 0$ is an eigenvalue $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 v_1 + c_2 e^{vt}$

SO if $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \vec{v}_1$ then $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

line of fixed points

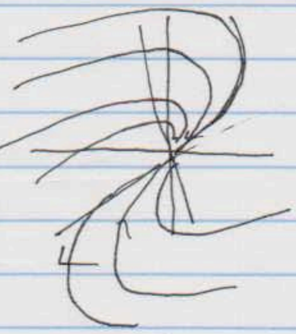
LECTURE 7-3

~~LECTU~~ BOUNDARY II-III

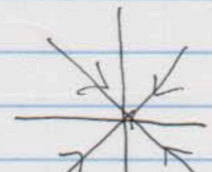
$\lambda = \mu = \frac{\tau}{2}$

if Nontrivial Jordan Form

$\vec{x} = c_1 \underbrace{\vec{v}_1}_{\text{eigenvector}} e^{\lambda t} + c_2 \underbrace{v_2}_{\text{generalized eigenvector}} t e^{\lambda t}$



if Diagonalizable $\quad A = D^{-1}\begin{bmatrix}\lambda & \\ & \lambda\end{bmatrix}D = D^{-1}\lambda D = \lambda D^{-1} D = \lambda I$



all solutions approach origin at same rate $\vec{x}(t) = \vec{x}(0)e^{\lambda t}$

BOUNDARY IV-V SIMILAR W/ARROWS REVERSED

IN GENERAL if $A \in GL_n(\mathbb{R})$

then there exists a complete set of ~~eigenvals~~ eigenvectors + generalized eigenvectors

let $w_j = u_j + i v_j$ be a generalized eigenvector with eigenvalue $\lambda_j = a_j + i b_j$

Suppose we have k real eigenvalues up to and m pairs of complex conjugate pairs

then $B = \{u_1, \dots, u_k, u_{k+1}, v_{k+2}, \dots, u_{n-1}, v_n\}$

then B are linearly independent

define $E^s = \text{span}\{u_i, v_i \mid a_i < 0\}$ the stable subspace

$E^u = \text{span}\{u_i, v_i \mid a_i > 0\}$ unstable subspace

$E^c = \text{span}\{u_i, v_i \mid a_i = 0\}$ unstable subspace

EXAMPLE

$$A = \begin{bmatrix} 3 & -1 & & & & \\ 1 & 3 & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{bmatrix}$$

$E^s = \text{span}\{\hat{e}_3\}$

$E^u = \text{span}\{\hat{e}_1, \hat{e}_2, \hat{e}_4\}$

$E^c = \text{span}\{\hat{e}_5, \hat{e}_6\}$

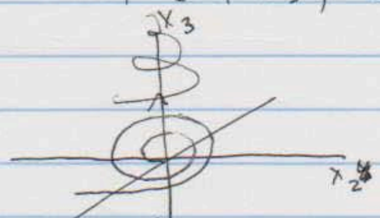
LECTURE 7-4

Want to think geometrically

example $A = \begin{bmatrix} -2 & -1 & \\ 1 & -2 & \\ & & 3 \end{bmatrix}$ $\quad \lambda_1 = -2+i$, $v_1 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{e}_2 + i\hat{e}_1$

$\lambda_2 = 3$, $v_3 = \hat{e}_3$

So $E^s = \text{span}\{\hat{e}_1, \hat{e}_2\}$, $E^u = \text{span}\{\hat{e}_3\}$, $E^c = \{\}$



A spiral orbit in x_1-x_2 plan

more examples in text

~~Definition~~ We know the general solution is $\vec{x}(t) = e^{At}\vec{x}_0$
this is a map from the initial cond $\vec{x}_0 \in \mathbb{R}^n$ to another point $\vec{x} \in \mathbb{R}^n$ call this mapping
the flow of the diff eqn

A flow is hyperbolic if all the eigenvalues have non-zero real part

A subspace $E \subset \mathbb{R}^n$ is invariant under the flow if $e^{At}E \subset E \quad \forall t \in \mathbb{R}$

The usefulness of the subspaces E^s, E^u, E^c are useful because they are invariant under the flow

To show this we first show this

Lemma Let E be the generalized eigenspace corresponding to eigenvalue λ of a matrix A, then $AE \subset E$

Pf let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a basis for the generalized eigenspace E

Given $\vec{v} \in E$, $\vec{v} = \sum_{j=1}^{k} c_j v_j$

so $A\vec{v} = \sum c_j A v_j$

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since each $v_j \in E$, $\exists$ minimal $k_j \leq k$ s.t $(A-\lambda I)^{k_j} v_j = 0$

so $(A-\lambda I)^{k_j - 1} v_j = V_j$ for some
$V_j \in Ker(A-\lambda I)^{k_j - 1} \subset E$

So $A v_j = \lambda v_j + V_j \in E$

~~by~~ Since E a subspace $\sum c_j A v_j \in E$

$\square$

Theorem 1.9.1

$\mathbb{R}^n = E^s \oplus E^u \oplus E^c$

The subspaces E^s, E^u, & E^c are each invariant w.r.t flow of e^{At}

if $x_0 \in E^s$ then $x_0 = \sum_{j=1}^{n_s} c_j V_j$ where $V_j = u_j$ or v_j

then $e^{At} x_0 = \sum_{j=1}^{n_s} c_j e^{At} V_j$

$$e^{At} V_j = \lim_{k \to \infty} \left[I + At + \dots + \frac{A^k t^k}{k!} \right] V_s \in E^S$$

since each partial sum is in E^S
by the above lemma
and E^S is complete

Proof for E^u, E^c identical

Def If all eigenvalues of A have negative real part, the origin is a sink, positive $\Rightarrow$ ~~sink~~ source

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Theorem The following three statements are equivalent

(a) All the eigenvalues of A have negative real part

(b) For all $x_0 \in \mathbb{R}^n$ $\lim_{t \to \infty} e^{At} x_0 = 0$ and $\lim_{t \to -\infty} \| e^{At} x_0 \| = \infty$

(c) $\exists$ positive constants M, m, C, c st $\forall t$
$m e^{-ct} < \| e^{At} \vec{x}_0 \| \leq M e^{-Ct}$

proof: Nothing interesting, see perko §1.9