Lecture 3 The existence + uniqueness Theorem

$$\circledast \begin{cases} \dot{x} = f(x) & \cancel{x \in \mathbb{R}^n} \quad x \in E \subset \mathbb{R}^n \\ x\big|_{t=t_0} = x_0 \end{cases}$$

The question is whether we can find an interval $J \ni t_0$, $J = [t_0 - a, t_0 + a]$

WLOG assume $t_0 = 0$

we can formally integrate $\circledast$ to obtain

$$(\ast\ast) \quad x(t) = x_0 + \int_0^t f(x(\tau))\,d\tau$$

not a sol'n since x on RHS

we will show ~~∄~~ when $\exists$ solution to $(\ast\ast)$ which will imply a solution to $\circledast$

lemma 1 if $f \in C^k(E, \mathbb{R}^n)$ $k \geq 0$ and $x \in C^0(J, E)$ a sol'n to $(\ast\ast)$ then $x \in C^{k+1}(J, E)$

pf first note $x(0) = x_0$

2nd, by RHS is integral of ~~cont~~ C^0 fcn, thus is C^1

by FTC its derivative is $f(x(t))$

so $\dot{x} = f(x(t))$ solves $\circledast$

finally if $x \in C^j$ $0 \leq j < k$ then ~~x~~ $f(x) \in C^j$ and the rhs of $(\ast\ast)$ is $C^{j+1} \Rightarrow x \in C^{j+1} \Rightarrow x \in C^{k+1}$

$\square$

in $(\ast\ast)$ we have really defined an operator

(3) ~~Tu~~ $T(u) = x_0 + \int_0^t f(u(\tau))\,d\tau$

need to define a function space on which T is a contraction

for (3) to be well-defined, need ~~∀~~ $u \in X$ some appropriate space of functions

by the lemma ⊛ has a solution $\iff$ (3) has a fixed pt
$$x^* = T(x^*)$$
Strategy – Picard iteration
start w/ $u_0(t)$ define $u_{j+1} = T(u_j)$
if this has a limit $u_j \to x^* \in C^0$ then by lemma 1
it solves ⊛

example $\begin{cases} \dot{x} = rx \\ x(0) = x_0 \end{cases}$
$$Tu = x_0 + \int_0^t r\,u(s)\,ds$$

let $u_0 = x_0$
$u_1 = x_0 + r\int_0^t x_0\,ds = x_0(1+rt)$
$u_2 = x_0 + r\int x_0(1+rs)\,ds = x_0(1 + rt + \frac{r^2}{2}t^2)$
$$\longrightarrow x_0 e^{rt}$$

exercise try a bad guess, eg $x_0 = t$
show it still converges

To use Picard iteration, need to restrict ~~to~~ ~~u~~ the space of
u s.t. $|f(u)|$ bounded so
– In order to use contraction mapping theorem, need this subset
to be complete
fortunately
~~lemma a closed subset of a complete set is complete~~
~~pf suppose $C \subset X$ C closed, X complete~~
~~let $\langle x_n \rangle \subset C$ be a Cauchy sequence~~
~~then $x_m \to x^* \in X$~~
~~but C closed $\Rightarrow x^* \in C \Rightarrow C$ complete~~

<u>Theorem</u> (Picard 1890, Lindelof 1894) Suppose for $X_0 \in \mathbb{R}^n$ $\exists b>0$ s.t. $f: B_b(x_0) \to \mathbb{R}^n$ is Lipschitz w/ constant K

let $B_\varepsilon(x_0)$ be the "closed ball"

then ⊛ IVP has a unique sol'n for $t \in J =$ ~~$[-a, a]$~~, where $[t_0-a, t_0+a]$

$$a = \frac{b}{M}, \quad M = \max_{x \in B_b(x_0)} |f(x)|$$

<u>PROOF 1</u> (almost works)

<u>First</u> Define $V = C^0(J, B_b(x_0))$ (continuous fcns s.t. $\|x(t) - x_0\| \le b$ in sup norm)

this is closed, C^0 complete

$\Rightarrow V$ complete

$\Rightarrow$ can apply contraction mapping thm if ⊛

1) $x \in V \Rightarrow T(x) \in V$

2) $\exists C < 1$ st $\|T(x) - T(y)\| \le C\|x-y\|$ $\forall x, y \in V$

1) $$|T(x(t)) - x_0| \le \int_{t_0}^{t} |f(x(\tau))| d\tau \le M(t-t_0) \le Ma$$

so if $a < \frac{b}{M}$ then $\|T(x(t)) - x_0\| \le b$

2) Contraction if ~~$x, y \in V$, then~~

$$|T(x(t)) - T(y(t))| \le \int_{t_0}^{t} |f(x(\tau)) - f(y(\tau))| d\tau$$ (use that $x, y \in B_b(x_0)$ $\Rightarrow$ Lipschitz!)

$$\le K \int_0^t |x(\tau) - y(\tau)| \le Ka\|x-y\|$$

if $Ka = C \le 1$ then this is a contraction

a) ~~We use that $|x(t) - x_0| \le b$ in order that the~~

<u>Note</u> b) that we cheated a little bit

we showed if $a \le \frac{b}{M}$ and $a < \frac{1}{K}$

then the proof gives slightly less than we claimed

<u>PROOF 2</u> PICK any $L > K$

define the weighted (Bielecki) norm

$$\|x\|_L = \max_{|t-t_0| \le a} e^{-L|t-t_0|} |x(t)|$$ = space of functions that grow slower than $e^{L|t|}$

in fact (exercise) $\|x\|_L$ and the sup-norm are equivalent norms for $|a|$ finite and $C^0[t_0-a, t_0+a]$) complete under need to show T a contraction in the $\|\cdot\|_L$ norm

$$\|Tx - Ty\| = \|x_0 + \int_0^t f(x(s))ds - (x_0 + \int_0^t f(y(s))ds)\|$$

$$= \|\int_0^t [f(x(s)) - f(y(s))]\, ds\,\|$$

assume $t>0$ $\quad = \max_{0\le t\le t_0} e^{-Lt} \left| \int_0^t [f(x(s)) - f(y(s))] ds \right|$

$$\le \max e^{-Lt} \int_0^t K|x(s) - y(s)|\, ds \qquad (\text{if } x^{(t)}, y^{(t)} \in B_b(x_0))$$

$$\le \max e^{-Lt} \int_0^t e^{Ls} K \underbrace{e^{-Ls}|x-y|}_{\text{take max}} ds$$

$$\le \max e^{-Lt} \int_0^t K e^{Ls} \|x-y\|_L\, ds$$

$$\le \max e^{-Lt}\, K\|x-y\|_L \left(\frac{e^{Ls}}{L}\right)\Big|_0^t$$

$$\le \max e^{-Lt} \frac{K}{L}\|x-y\| (e^{Lt} - 1)$$

$$\le \max \frac{K}{L}\|x-y\| (1 - e^{-Lt}) < \cancel{\max} \frac{K}{L}\|x-y\|$$

$C = \frac{K}{L}$ contraction, SIMILAR FOR $t<0$

□

~~Notice further, if f(x) globally Lipschitz, then~~
Notice ~~this applies for all x, y, we can take b = ∞~~

Corollary if $f(x)$ globally Lipschitz, sol'n exists for all t

Notice that if $f(x)$ globally Lipschitz we can skip step 1 (showing $T(x) \in B_b(x_0)$)

and Proof 2 works $\forall x, y$ and any $t>0$

example $\dot{x} = x^2$ not globally Lipschitz

$x(0) = 1$

$x(t) = \frac{1}{1-t}$, solution blows up as $t \to 1^-$

example In fact, to prove existence, it is sufficient that $f(x)$ is C^0 (e.g. for $\dot{x} = \sqrt{|x|}$)

Coddington - Levinson 1955 Thm 1.1.2

solution is not unique, as we've seen

Note elementary books often give the condition $f \in C^1$ as necessary for uniqueness. Our proof shows the slightly weaker condition — locally Lipschitz continuity — is enough

We know that $C^1 \Rightarrow$ locally Lipschitz

example maximal domain

$\dot{x} = x^2$

$x(0) = x_0 > 0$

on the ball $B_b(x_0)$, $|f(x)| \le (x_0+b)^2 = M$

$$|t| \le a = \frac{b}{M} = \frac{b}{(x_0+b)^2}$$

what b gives the largest t_{max}

$$t_{max} = \frac{b}{(x_0+b)^2}$$

$$\frac{dt_{max}}{db} = \frac{(x_0+b)^2 \cdot 1 - b \cdot 2(x_0+b)}{(x_0+b)^4}$$

$$= \frac{x_0+b-2b}{(x_0+b)^3} = \frac{x_0-b}{(x_0+b)^3}$$

maximized when $b = x_0$

$$t_{max} = \frac{x_0}{(2x_0)^2} = \frac{1}{4x_0}$$

the exact solution is $x = \dfrac{x_0}{1-tx_0}$

the maximal interval is $\left(-\infty, \frac{1}{x_0}\right) \supset \left[-\frac{1}{4x_0}, \frac{1}{4x_0}\right]$

$\square$

This example shows that the proof does not actually find the maximal domain

Lecture 4-0 GRONWALL'S INEQUALITY

Lemma

Suppose, $g, k : [0, a] \longrightarrow \mathbb{R}$ are continuous, $a > 0$

$k(t) \geq 0$ and

$$g(t) \leq c + \int_0^t k(s) g(s)\, ds$$

then $\forall t \in [0, a]$

$$g(t) \leq c\, e^{\int_0^t k(s)\, ds}$$

proof let $G(t) = c + \int_0^t k(s) g(s)\, ds$

then $G(0) = c$ and

$$\dot{G} = k(t) g(t) \leq k G$$

so $\dot{G} - k(t) G \leq 0$

multiply by $e^{-\int_0^t k(s)\, ds} > 0$

$$e^{-\int_0^t k(s)\, ds} (\dot{G} - k(t) G) = \frac{d}{dt}\left(e^{-\int_0^t k(s)} G(t)\right) \leq 0$$

$$\Rightarrow \quad e^{-\int_0^t k(s)\, ds} G(t) \leq c$$

$$G(t) \leq c\, e^{\int_0^t k(s)\, ds}$$

$\square$

often choose $k(t) = k_0 > 0$ const