lect 24-5

LECTURE 25 (2 lectures)

Homoclinic orbits + global bifurcations)

$\mu = 149$ — Periodic orbit

147.5 doubled

147 doubled again

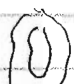
A result for near Hamiltonian systems (Baby Melnikov)

$$\dot{x} = f_1(x,y) + \varepsilon g_1(x,y) \qquad f_1 = \frac{\partial H}{\partial y}$$
$$\dot{y} = f_2(x,y) + \varepsilon g_2(x,y) \qquad f_2 = -\frac{\partial H}{\partial x}$$

but assume the perturbation is not Hamiltonian

and for $\varepsilon = 0$ $\exists$ center surrounded by periodic orbits

nested periodic orbits are not generic for non-Hamiltonian problems



Question: Which ~~periodic~~ closed orbits will survive

suppose Γ_ε a ~~periodic~~ closed orbit of the perturbed system

then $\oint_{\Gamma_\varepsilon} dH = 0$

$$\oint_{\Gamma_\varepsilon} \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy = 0 \qquad \frac{\partial H}{\partial x} = -f_2 = -\dot{y} + \varepsilon g_2$$

$$\frac{\partial H}{\partial y} = f_1 = \dot{x} - \varepsilon g_1$$

$$\int_{\Gamma_\varepsilon} (-\dot{y} + \varepsilon g_2)dx + (\dot{x} - \varepsilon g_1)dy = 0 \qquad \text{but } \int -\dot{y}\,dx + \dot{x}\,dy$$
$$= \int -\dot{y}\dot{x} + \dot{x}\dot{y}\,dt = 0$$

$$= \int \varepsilon g_2\,dx - \varepsilon g_1\,dy = 0$$

So $\int_{\Gamma_\varepsilon} g_2\,dx - g_1\,dy = 0$, Γ_ε is a small perturbation to Γ

~~to leading order~~ $\int_0^T (g_2\dot{x} - g_1\dot{y}) = 0 \; = \int_0^T g_2 f_1 - g_1 f_2\,dt + O(\varepsilon) = 0$

$$\Rightarrow \int_0^T g_2 f_1 - g_1 f_2\,dt = 0$$

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Example The Van-der Pol oscillators

$$\dot{x} = y$$
$$\dot{y} = -x - \varepsilon y(x^2-1) \qquad \varepsilon = 0 \qquad H = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2$$

$x = r\sin t$, $y = r\cos t$

$$\int_0^{2\pi} f_1 g_2 - f_2 g_1 \, dt = \int_0^{2\pi} [y(-y(x^2-1)) - (-x)\cdot 0]\, dt$$
$$= \int_0^{2\pi} -y^2(x^2-1)\, dt$$
$$= \int_0^{2\pi} r^2\cos^2 t\,(r^2\sin^2 t - 1)\, dt$$
$$= r^4\int_0^{2\pi} \sin^2 t\cos^2 t\, dt - r^2\int_0^{2\pi}\cos^2 t\, dt$$
$$= r^4\cdot\frac{1}{4}\int_0^{2\pi}\sin^2 2t\, dt - r^2\int_0^{2\pi}\cos^2 t\, dt$$
$$= \frac{r^4}{4}\cdot\frac{1}{2}\cdot 2\pi - r^2\cdot\frac{1}{2}\cdot 2\pi = 0$$
$$= \cancel{2\pi}\ \pi r^2\left(\frac{r^2}{4} - 1\right) = 0$$
$$\Rightarrow r = 2$$

So when $\varepsilon \ll 1$ the circle of radius 2 survives

Homoclinic bifurcations

$$\ddot{x} + a\dot{x} - x + x^3$$

$a > 0$ $a = 0$ $a < 0$

lecture 25-3 A global bifurcation

$\dot{x} = y$

$\dot{y} = -a + x^2 + \varepsilon(by + xy)$ $\qquad \varepsilon > 0$ small, fixed

$\boxed{\cancel{\varepsilon = 0}}$

$a < 0$ no fixed pts , $a > 0$, fixed pts at $(\pm\sqrt{a}, 0)$

saddle node for ~~b=0~~ $a = 0$ ie the b-axis

let $a = c^2$

$\dot{x} = y$

$\dot{y} = -c^2 + x^2 + \varepsilon(by + xy)$

$$J = \begin{pmatrix} 0 & 1 \\ 2x + \varepsilon y & \varepsilon(b+x) \end{pmatrix}$$

so $J(c,0) = \begin{pmatrix} 0 & 1 \\ 2c & \varepsilon(b+c) \end{pmatrix}$, $\delta = -2c < 0 \Rightarrow (c,0)$ a saddle

$J(-c,0) = \begin{pmatrix} 0 & 1 \\ -2c & \varepsilon(b-c) \end{pmatrix}$ $\qquad \tau = \varepsilon(b-c)$, $\delta = 2c > 0 \Rightarrow$ source, sink or center

$$\lambda = \frac{\varepsilon(b-c) \pm \sqrt{\varepsilon^2(b-c)^2 - 8c}}{2}$$

note if ε small, square root gives imaginary

stability depends on sign $(b-c)$

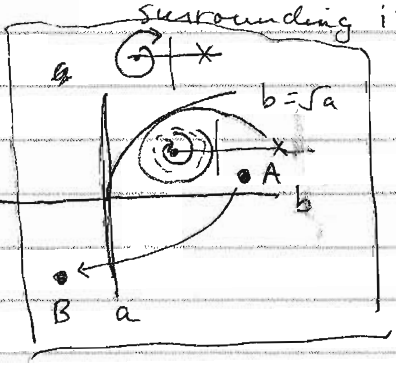
i.e. a Hopf bifurcation at $b = c = \sqrt{a}$ where $\lambda = \pm i\sqrt{2c}$

Fixed pt $(-c,0)$ stable if $b < c = \sqrt{a}$

This Hopf bifurcation turns out to be subcritical so ~~for~~

when the fixed pt is stable, $\exists$ unstable periodic orbit

surrounding it ie when $b < \sqrt{a}$ $\exists$ unstable periodic orbit



if we start at point A, move to point B

lose 2 fixed pts in saddle-node

— by index theory, no fixed pts

$\Rightarrow$ no periodic orbits

but we did not see any bifurcation

that destroyed the periodic orbit

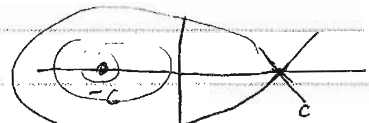
we must have missed it

LEC 25-4

Claim $\exists$ ~~the~~ curve of Homoclinic bifurcations where ~~the~~ periodic orbit is destroyed

~~Recall other~~ Note when $\varepsilon = 0$ this system is Hamiltonian

$$H = \frac{1}{2}y^2 + c^2 x - \frac{1}{3}x^3$$



want to find a curve in (a,b) space where $\exists$ homoclinic orbit

the homoclinic orbit persists if

$$\int_{\Gamma_h} f_1 g_2 - f_2 g_1 = 0 = \int_{\Gamma_h} y \cdot \varepsilon(by + xy)\,dt$$

$$0 = \int_{\Gamma_h} (by^2 + xy^2)\,dt$$

Can we find $x(t)$, $y(t)$ on homoclinic orbit?

on Homoclinic orbit $\quad \frac{1}{2}y^2 + c^2 x - \frac{1}{3}x^3 = \frac{1}{2}\cdot 0^2 + c^2\cdot c - \frac{1}{3}c^3 = \frac{2}{3}c^3$

$$\frac{1}{2}\dot{x}^2 + c^2 x - \frac{1}{3}x^3 = \frac{2}{3}c^3$$

any mistakes?

$$\frac{1}{2}\dot{x}^2 = \frac{1}{3}x^3 - c^2 x + \frac{2}{3}c^3$$

let $x = cz$

$$\frac{1}{2}c^2\dot{z}^2 = \frac{1}{3}c^3 z^3 - c^3 z + \frac{2}{3}c^3$$

$$\frac{1}{2c}\dot{z}^2 = \frac{1}{3}z^3 - z + \frac{2}{3}$$

$$\frac{3}{2c}\dot{z}^2 = z^3 - 3z + 2$$

$$\frac{dz}{(z-1)\sqrt{z+2}} = \sqrt{\frac{2c}{3}}\,dt$$

$$\begin{array}{r} z^2 + z - 2 \\ z-1\,\overline{)\,z^3 - 3z + 2} \\ \underline{z^3 - z^2} \\ z^2 - 3z + 2 \\ \underline{z^2 - z} \\ -2z + 2 \end{array}$$

$(z-1)(z-1)(z+2)$

a bunch more work, find

$$x_h = c - 3c\,\mathrm{sech}^2\sqrt{\tfrac{c}{2}}t$$

$$y_h = 3\sqrt{2c^3}\,\mathrm{sech}^2\sqrt{\tfrac{c}{2}}t\,\tanh\sqrt{\tfrac{c}{2}}t$$

put this into integral find

$$0 = \int_{-\infty}^{\infty} 18bc^3\,\mathrm{sech}^4\left(t\sqrt{\tfrac{c}{2}}\right)\tanh^2\left(t\sqrt{\tfrac{c}{2}}\right)dt + \int_{-\infty}^{\infty} 18c^4\left(1 - 3\,\mathrm{sech}^2\left(\sqrt{\tfrac{c}{2}}t\right)\right)\mathrm{sech}^4\left(\sqrt{\tfrac{c}{2}}t\right)\cdot\tanh^2\left(\sqrt{\tfrac{c}{2}}t\right)dt$$

let $\tau = t\sqrt{\frac{c}{2}}$

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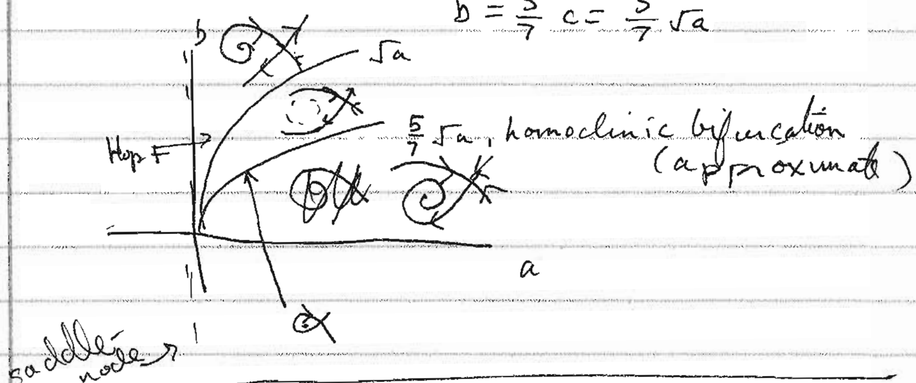
$$b\int_{-\infty}^{\infty} \operatorname{sech}^4\tau \tanh^2\tau \, d\tau + c\int_{-\infty}^{\infty} (1-3\operatorname{sech}^2\tau)\operatorname{sech}^4\tau\tanh^2\tau\, d\tau = 0$$

$$\frac{4}{15}b + c\left(\frac{4}{15} - 3\cdot\frac{16}{105}\right) = 0$$

$$28b + c(28 - 48) = 0$$

$$28b - 20c = 0$$

$$b = \frac{5}{7}c = \frac{5}{7}\sqrt{a}$$



Melnikov Integrals: Persistence of homoclinic orbits in a Hamiltonian System

(*) $\dot{x} = f(x) + \varepsilon g(x,t)$, $x = \binom{u}{v} \in \mathbb{R}^2$, $g(x, t+T) = g(x,t)$

(**) Rewrite as "suspended flow" $\dot{x} = f(x) + \varepsilon g(x,\theta)$ $x \in \mathbb{R}^2$
$\dot{\theta} = 1$ $\theta \in S^1$ (unit circle)

Assumption $f = \binom{f_1}{f_2} = \binom{\partial H/\partial v}{\partial H/\partial u}$ and $g \in C^r$, $r \geq 2$

A1) For $\varepsilon = 0$, (*) has a homoclinic orbit $q^0(t)$ to a hyperbolic fixed point p_0

A2) let $\Gamma_0 = \{p_0\} \cup \{q_0(t)\}$ the interior of Γ^0 is filled with periodic orbits $q^\alpha(t)$, $\alpha \in (-1, t)$

define the poincare map $P_\varepsilon^{t_0} : \Sigma^{t_0} \to \Sigma^{t_0}$, $\Sigma^{t_0} = \{(x,t) \mid t = t_0 \in [0,T)\}$
this is transverse to the suspended flow (**) at $\theta = t_0$

note we will let t_0 vary in the following

~~Assumption A1 implies $P_0^{t_0}$ (ie $P_\varepsilon^{t_0}|_{\varepsilon=0}$) has a saddle pt~~

Assumption A1 $\Rightarrow$ p_0 is a saddle-type fixed pt of $P_0^{t_0}\left(= P_\varepsilon^{t_0}\big|_{\varepsilon=0}\right)$
and that $\Gamma_0 = W^s(q_0) \cap W^u(q_0)$ is made up of nontransverse homoclinic pts to q_0 – this is very non-generic
We expect this structure to break up when $\varepsilon > 0$ and ask the question: Do any homoclinic pts survive after the perturbation?

Lemma 1 Under above assumptions, for ε suff small, ⊛ has a periodic orbit $\gamma_\varepsilon(t) = p_0 + O(\varepsilon)$. Correspondingly $P_\varepsilon^{t_0}$ has a ~~fixed~~ unique hyperbolic saddle point $p_\varepsilon^{t_0} = p_0 + O(\varepsilon)$

Note: this is actually a family of periodic orbits $\gamma_\varepsilon(t, t_0)$
We proved this lemma when we defined Poincaré sections

Lemma 2 the local stable and unstable manifolds $W^s_{loc}(\gamma_\varepsilon)$ and $W^u_{loc}(\gamma_\varepsilon)$ are C^r-close to those of the unperturbed periodic orbit $p_0 \times S^1$. Further, orbits $q_\varepsilon^s(t, t_0)$, $q_\varepsilon^u(t, t_0)$ lying in $W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$ respectively and based on Σ^{t_0} can be expressed as follows, uniformly on intervals indicated

$$q_\varepsilon^s(t, t_0) = q^0(t - t_0) + \varepsilon q_1^s(t, t_0) + O(\varepsilon^2) \qquad t \in [t_0, \infty)$$
$$q_\varepsilon^u(t, t_0) = q^0(t - t_0) + \varepsilon q_1^u(t, t_0) + O(\varepsilon^2) \qquad t \in (-\infty, t_0]$$

We'll skip proof: main ideas 1) existence of stable and unstable manifolds follows since hyperboli fixed pts structurally stable
2) look at $\frac{d}{dt}(\gamma_\varepsilon^s(t) - \gamma_0^s(t))$
3) behavior of q_ε^s governed by linearization
Use a Gronwall-type estimate to show these stay close

Note: Uniform in time for any t_0!

let $q(t) = q^0(t-t_0) + \varepsilon q_1^s(t, t_0)$ for $t \geq t_0$

$$\dot q = \dot q_0(t-t_0) + \varepsilon \dot q_1^s(t, t_0)$$
$$= f(q_0 + \varepsilon q_1^s) + \varepsilon g(q_0, t + \varepsilon q_1^s(*))$$
$$= f(q_0) + \varepsilon Df(q_0) q_1^s + \varepsilon g(q_0, t) + O(\varepsilon^2)$$

First variational equation

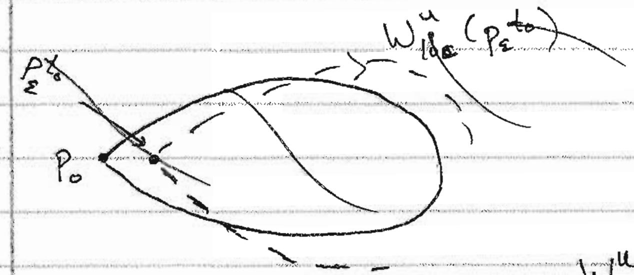
$$\dot q_1^s(t, t_0) = Df(q_0(t-t_0)) q_1^s + g(q_0, t)$$

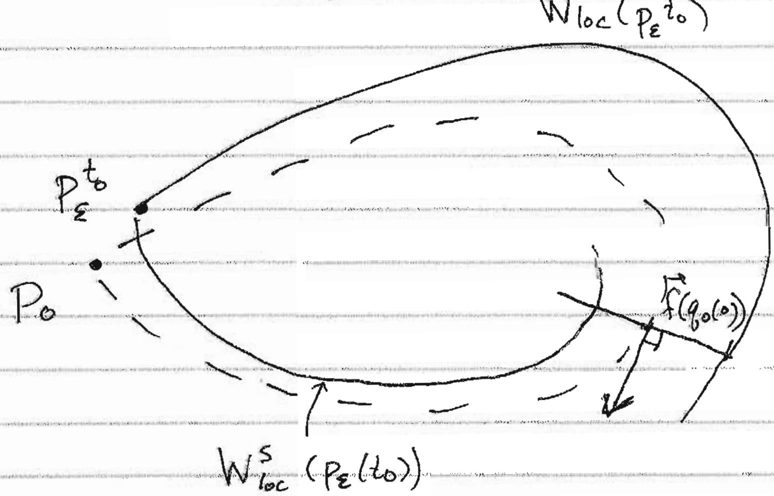
(by lemma 2 this is uniformly valid on $[t_0, \infty)$)

Note: initial time appears explicitly since (*) not invariant under time translation

Looking to see if q_1^s intersects q_1^u. If so, and if transversal, then $\exists$ sequence of pts homoclinic to $p_\varepsilon^{t_0}$

picture





Now we define the separation between $W^u_{loc}(p_\varepsilon^{t_0})$ and $W^s_{loc}(p_\varepsilon^{t_0})$ at $q_0(0)$ by $d(t_0) = |q^u_\varepsilon(t_0) - q^s_\varepsilon(t_0)|$, $\boxed{\text{these are the nearest points to } q_0(0) \text{ on } W_\varepsilon^{u,s}(p_\varepsilon^{t_0}), \text{ lying on normal } f^\perp(q_0(0))}$

$= q_\varepsilon^{u,s}(t_0, t_0)$

$$f^\perp(q_0(0)) = (-f_2(q^0(0)), f_1(q^0(0)))^T$$

Then by lemma 2

$$d(t_0) = \varepsilon \frac{f^\perp \cdot (q_1^u(t_0) - q_1^s(t_0))}{|f^\perp|} + O(\varepsilon^2) = \varepsilon \frac{f \wedge (q_1^u(t_0) - q_1^s(t_0))}{|f(q_0^0(t_0))|}$$

where $a \wedge b = a_1 b_2 - a_2 b_1$

so $f \wedge (q_1^u - q_1^s)$ is the projection of $q_1^u - q_1^s$ onto $f(q_0(0))$

finally define the Melnikov Function

$$M(t_0) = \int_{-\infty}^{\infty} f(q^0(t-t_0)) \wedge g(q^0(t-t_0), t)\, dt$$

<u>theorem</u> if $M(t_0)$ has simple zeros and is independent of ε then for $\varepsilon > 0$ suff small $W^u_{p_\varepsilon^{t_0}}(t_0)$ and $W^s_{p_\varepsilon^{t_0}}$ intersect transversely. If $M(t_0)$ has no zeros, the intersection is empty

(Remark, this is often an important step in showing a diff eqn has chaotic orbits)

<u>Proof</u> let $\Delta(t, t_0) = f(q^0(t-t_0)) \wedge (q_1^u(t, t_0) - q_1^s(t, t_0))$
$= \Delta^u(t,t_0) - \Delta^s(t, t_0)$

note $d(t_0) = \dfrac{\varepsilon \Delta(t_0, t_0)}{|f(q_0(0))|}$

The main idea of the proof. Find a differential eqn satisfied by Δ, then integrate it

$$\frac{d}{dt}\Delta^s(t,t_0) = Df(q^0(t-t_0))\dot{q}(t-t_0)\wedge q_1^s + f(q^0(t-t_0))\wedge \dot{q}_1^s$$
$$= Df(q^0(t-t_0))f(q_0)\wedge q_1^s + f(q^0(t-t_0))\wedge(Df(q^0)q_1^s + g(q_0))$$
$$= \big(Df(q^0(t-t_0))f(q_0)\wedge q_1^s + f(q^0(t-t_0))\wedge(Df(q^0)q_1^s)\big) + f(q^0(t-t_0))\wedge g(q^0(t-t_0),t)$$

lemma
$$\boxed{(Ax)\wedge y + x\wedge(Ay) = (\operatorname{tr} A)(x\wedge y)}$$

$$\frac{d}{dt}\Delta^s(t,t_0) = \underbrace{\big(\operatorname{tr} Df(q^0(t-t_0))\big)}_{=0 \text{ since } f \text{ Hamiltonian } Df\equiv 0}\big(f(q_0)\wedge q_1^s\big) + f(q^0(t-t_0))\wedge g(q^0(t-t_0),t)$$

(~~Abel's form~~

so $\Delta^s(\infty,t_0) - \Delta^s(t_0,t_0) = \int_{t_0}^{\infty} f(q^0(t-t_0))\wedge g(q^0(t-t_0),t)\,dt$

but $\Delta^s(\infty,t_0) = \lim_{t\to\infty} f(q^0(t-t_0))\wedge q_1^s(t,t_0)$
$= f(p^0)\wedge q_1^s(t,t_0)$
$= 0\wedge q_1^s(t,t_0) = 0$

similarly $\Delta^u(t_0,t_0) = \int_{-\infty}^{t_0} f(q^0(t-t_0))\wedge g(q^0(t-t_0),t)\,dt$

so $\Delta = \Delta^u(t_0,t_0) - \Delta^s(t_0,t_0) = \int_{-\infty}^{\infty} f(q^0(t-t_0))\wedge g(q^0(t-t_0),t)\,dt$

so $d(t_0) = \dfrac{\varepsilon M(t_0)}{|f(q^0(0))|} + O(\varepsilon^2)$

Since $|f(q^0(0))| = O(1)$ this is a good measurement

need $M(t_0)$ independent of ε to justify ignoring the $O(\varepsilon^2)$ bit

$\Rightarrow \exists\, \tau\in[0,T]$ s.t. $q_\varepsilon^{(s)}(\tau) = q_\varepsilon^{(u)}(\tau)$
ie $\exists q \in W^s(p_\varepsilon^{t_0})\cap W^u(p_\varepsilon^{t_0})$
but all poincaré maps are equivalent
so $\exists q \in$ intersection $\forall t\in[0,T]$

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if zeros are ~~transversals~~ simple $M'(t)|_{t=t_0} \neq 0$ then intersection $W^u \cap W^s$ is a transversal, so it survives under small perturbation and a range of ε

if $M(t_0) \neq 0 \ \forall t_0 \in [0,T]$ then no crossing

finally if $q \in W^u \cap W^s$ then $P_\varepsilon^{t_0}(q) \in W^u \cap W^s$

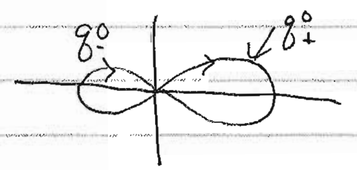
and $(P_\varepsilon^{t_0})^n(q) \in W^u \cap W^s$

Example $\ddot{x} = x - x^3 + \varepsilon(\gamma \cos \omega t - \delta \dot{x})$ (forced, damped Duffing eqn)

write as $\dot{u} = v$

$\dot{v} = u - u^3 + \varepsilon(\gamma \cos \omega t - \delta v)$

$$H(u,v) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4}$$



$$q_+^0 = (\underbrace{\sqrt{2}\,\text{sech}\,t}_{u}, \underbrace{-\sqrt{2}\,\text{sech}\,t \tanh t}_{v})$$

thus $f = \begin{bmatrix} v \\ u - u^3 \end{bmatrix}$, $g = \begin{pmatrix} 0 \\ \gamma \cos \omega t - \delta v \end{pmatrix}$

$$f \wedge g = -\overset{v}{\cancel{(u-u^3)}}(\gamma \cos \omega t - \delta v)$$

$$\text{so } M(t_0) = \int_{-\infty}^{\infty} -\sqrt{2}\,\text{sech}(t-t_0)\tanh(t-t_0)\left[\gamma \cos \omega t - \delta\sqrt{2}\,\text{sech}(t-t_0)\tanh(t-t_0)\right] dt$$

$$= -\frac{4\delta\sqrt{2}}{3} + \overset{2}{\sqrt{2}}\,\gamma\pi\omega\,\text{sech}\frac{\pi\omega}{2}\sin\omega t_0$$

has a transverse zero if

$$\sin \omega t_0 = \frac{2\sqrt{2}\gamma\pi\omega\,\text{sech}\frac{\pi\omega}{2}}{\left(\frac{4\delta\sqrt{2}}{3}\right)}$$

so if $\frac{3\sqrt{2}\gamma\pi\omega\,\text{sech}\frac{\pi\omega}{2}}{4\delta} < 1$, $\exists$ transverse