LECTURE 22 Higher-codimension bifurcations + center manifold reductions

PITCHFORK BIFURCATION RECALL $\dot{x} = \mu x - x^3$ is the supercritical pitchfork normal form

at $\mu = 0$ this is the structurally unstable vector field $\dot{x} = -x^3$

The naive form of the critical unfolding is

$\dot{x} = \mu_1 + \mu_2 x + \mu_3 x^2 - x^3$ but letting $x \leftarrow x - \frac{\mu_3}{2}$ as in previous + redefining μ_1 & μ_2 we get

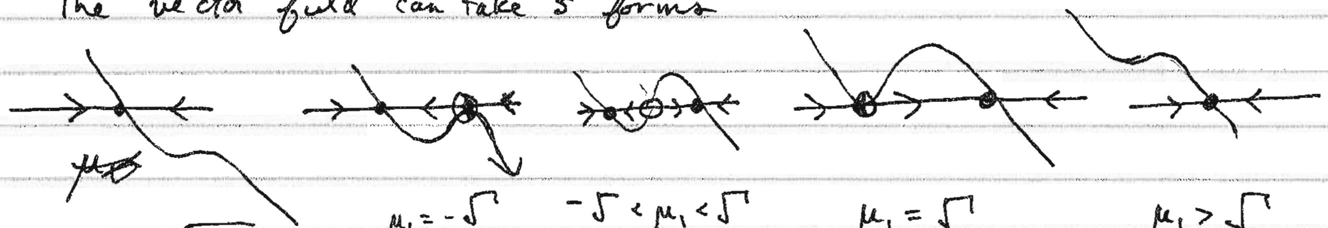
$$\dot{x} = \mu_1 + \mu_2 x - x^3$$

let's assume $\mu_2 > 0$, vary μ_1

fixed pts $x^3 - \mu_2 x - \mu_1 = 0$

this has 3 roots $\iff \mu_1^2 < \frac{4\mu_2^3}{27}$

The vector field can take 5 forms



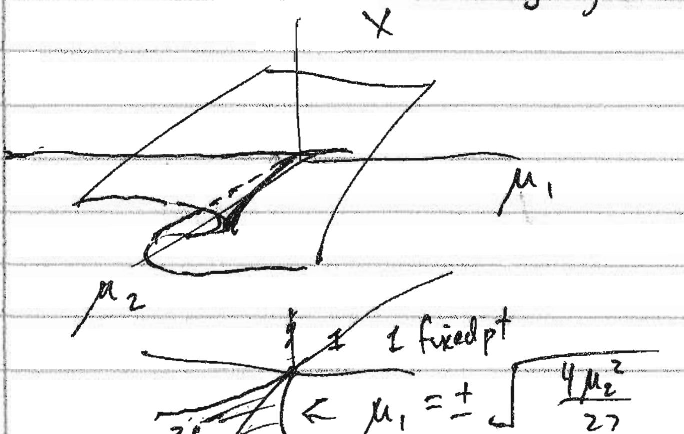
$\mu_1 < -\sqrt{\frac{4\mu_2^3}{27}}$ $\quad\mu_1 = -\sqrt{\ }$ $\quad -\sqrt{\ } < \mu_1 < \sqrt{\ }$ $\quad \mu_1 = \sqrt{\ }$ $\quad \mu_1 > \sqrt{\ }$

the same 5 qualitative behaviors are possible for $\mu_2 < 0$

Note $\dot{x} = \mu x - x^3$ (the pitchfork normalform) is an unfolding of $\dot{x} = -x^3$, just not a universal unfolding because it does not contain all possible phase portraits

So the generic bifurcation is codimension 2, see that at $\mu_1 = \pm\sqrt{\frac{4\mu_2^3}{27}}$, $\mu_2 \neq 0$ the system undergoes a saddle-node bifurcation

only by moving along $\mu_2 = 0$ and μ_1 increasing through zero do we see pitchfork



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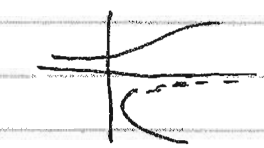

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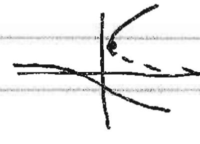

Now back to center manifold theory (Perko 2.12)

suppose $x \in \mathbb{R}^n$, $\dot{x} = f(x)$ $f \in C^1(\mathbb{R}^n)$, $f(0)=0$

suppose $\tilde{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $x \in \mathbb{R}^c$, $y \in \mathbb{R}^s$, $z \in \mathbb{R}^u$

$$\dot{x} = Cx + F(x,y,z) \qquad \text{Where } F(0) = G(0) = H(0) = 0$$
$$\dot{y} = Py + G(x,y,z) \qquad DF(0) = DG(0) = DH(0)$$
$$\dot{z} = Qz + H(x,y,z) \qquad \text{Re } \lambda(C) = 0$$
$$\text{Re } \lambda(P) < 0, \quad \text{Re } \lambda(Q) > 0$$

SIMPLEST CASE, assume $u=0$, no unstable direction
then stability of zero solution depends on the flow on the center manifold

Local center manifold thm
$f \in C^r(E)$, $r \geq 1$
$$\dot{x} = Cx + F(x,y)$$
$$\dot{y} = Py + G(x,y) \quad \text{as above}$$
then $\exists\, h(x) \in C^r(N_\delta(0))$, $\delta > 0$
$y = h(x)$ ~~satisfies~~ defines the local stable manifold
and substituting $y = h(x)$ into $\dot{y} = Py + G(x,y)$ we find
$$Dh(x)\dot{x} = Ph(x) + G(x, h(x))$$
$$Dh(x)(Cx + F(x, h(x))) = Ph(x) + G(x, h(x))$$
This is in general a partial differential eqn for x and an ugly one, but it gives us a recipe for constructing $h(x)$ via Taylor series
(h not unique but if f analytic, $\exists$ unique analytic h)

eg
$$\dot{x} = xy \qquad \text{stable manifold } x=0$$
$$\dot{y} = -y - x^2$$

$$y = ax^2 + bx^3 + \ldots$$
$$\dot{y} = 2ax\dot{x} + 3bx^2\dot{x} = \cancel{-y} \cancel{-(bx^2+b}$$
$$= (2ax + 3bx^2)xy = (2ax + 3bx^2)x(ax^2 + bx^3) + \ldots = -(ax^2 + bx^3) - x^2$$

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note on LHS $\quad 2a^2x^4+\dots = (-a-1)x^2 - bx^3 + \dots$

$\Rightarrow a=-1, \quad b=0$

so we find the center manifold is $y=-x^2+O(x^4)$

so on $y=-x^2+\dots$ we find

$\dot{x} = x(-x^2)+\dots$

$\dot{x} \approx -x^3 \Rightarrow$ $(0,0)$ asymptotically stable

Change problem slightly

$\dot{x} = x^2y$

$\dot{y} = -y - x^2$

$y = ax^2+bx^3+cx^4$

as before only we get, from same ansatz

$\dot{y} = (2ax+3bx^2)x^2y = (2ax+3bx^2)x^2(ax^2+bx^3+\dots) = -(ax^2+bx^3+\dots)-x^2$

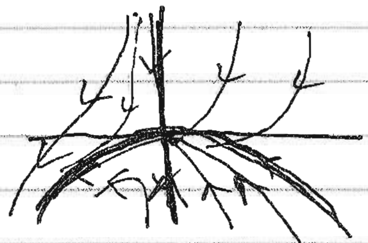
again, find $a=-1$, $b=0 \Rightarrow y=-x^2$

in fact $c=0$ also

$\dot{x} = x^2(-x^2)$

$= -x^4$

origin only semistable on center manifold



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note ~~this~~ it is hard to determine what constitute "low-order" or "high order" terms

eg $\quad \dot{x} = x^2y + \alpha x^4$

$\dot{y} = -y - x^2$

x^4 term looks higher, but when we plug in $y=-x^2$ we get $\dot{x}=0 \Rightarrow$ need include αx^5 term in our expansion

What happens when $\alpha = 1$?

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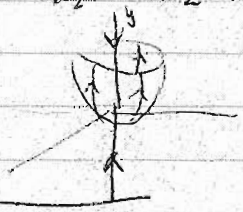
Example

$$\left.\begin{aligned}\dot{x}_1 &= x_1 y \cancel{x} - x_1 x_2^2 \\ \dot{x}_2 &= x_2 y - x_2 x_1^2\end{aligned}\right\} E^c = \text{span}\{\hat{e}_1, \hat{e}_2\}$$

$$\dot{y} = -y + x_1^2 + x_2^2 \quad \} \; E^s = \text{span}\{\hat{e}_3\}$$

Pretty clear $\quad y = h(x_1, x_2) = x_1^2 + x_2^2 + O(|x|^3)$

then on W^c
$$\dot{x}_1 = x_1(x_1^2 + x_2^2) - x_1 x_2^2 = x_1^3$$
$$\dot{x}_2 = x_2(x_1^2 + x_2^2) - x_2 x_1^2 = x_2^3$$

So 

note this is not what we get by crossing out y in $\dot{x}_1, \dot{x}_2$ eqns

EXAMPLE

$$\left.\begin{aligned}\dot{x}_1 &= x_2 + y \\ \dot{x}_2 &= y + x_1^2\end{aligned}\right\} C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{Jordan Block}$$
$$\dot{y} = -y + x_2^2 + x_1 y$$

Try $\quad y = a x_1^2 + b x_1 x_2 + c x_2^2 + O(|x|^3)$

$$2a x_1 \dot{x}_1 + b x_1 \dot{x}_2 + \cancel{1} b \dot{x}_1 x_2 + 2c x_2 \dot{x}_2 = -(a x_1^2 + b x_1 x_2 + c x_2^2) + x_2^2 + x_1(a x_1^2 + b x_1 x_2 + c x_2^2)$$

$$(x_2 + a x_1^2 + b x_1 x_2 + c x_2^2)(2a x_1 + b x_2)$$
$$+ (x_1^2 + a x_1^2 + b x_1 x_2 + c x_2^2)(b x_1 + 2c x_2) = \text{RHS}$$

note there is no x_1^2 term on LHS
$\Rightarrow a = 0$

So $(x_2 + b x_1 x_2 + c x_2) b x_2 + (x_1^2 + b x_1 x_2 + c x_2^2)(b x_1 + 2c x_2) = (x_1 - 1)(b x_1 x_2 + c x_2^2) + x_2^2$

now there are no $x_1 x_2$ terms on left $\Rightarrow b = 0$

$$(x_1^2 + c x_2^2) 2c x_2 = (x_1 - 1) c x_2^2 + x_2^2 + \ldots O(|x|^3)$$
$$\Rightarrow c = 1$$

So $y = x_2^2$

So
$$\dot{x}_1 = x_2 + \text{hot}$$
$$\dot{x}_2 = x_1^2 + x_2^2$$