0.1 A Fixed Point Theorem

The local existence proof we have given can be viewed as an application of general fixed point theorem in an abstract function space. We shall prove the theorem, and show its connection with what we have done. We begin with some preliminaries.We remind the reader that a linear space (over the real numbers) is a set of objects (vectors) $u, v, \ldots$ for which the operations of addition and multiplication by a real number are defined and which are closed under these operations; i.e. if u and v are elements of the space and λ is a real number then $u+v$ and λu also belong to the space. For the additive operation there exists a zero vector, and a unique inverse image for every vector, and the usual commutative, associative, and distributive rules hold for the two operations.

A *normed* linear space is one on which a non-negative function is defined, denoted by $P(u) = \|u\|$, which satisfies the following properties:

1. $\|\lambda u\| = |\lambda| \, \|u\|$
2. $\|u+v\| \leq \|u\| + \|v\|$
3. $\|u\| \geq 0$ and $\|u\| = 0 \iff u = 0$.

The double bar is sometimes used to avoid confusion between the norm of a vector and the absolute value of a scalar. The norm defines a distance between two elements of the space, and introduces a natural notion of convergence. Thus the sequence $\{u^m\}$ of elements of the space is said to converge to an element u of the space (u is the *limit* of $\{u^m\}$) whenever the sequence of positive numbers $\{u^m - u\}$ converges to zero. Every convergent sequence is a *Cauchy sequence*; i.e. for $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that $\left\|u^m - u^k\right\| < \epsilon$ whenever $m, k \geq N$. The space is called *complete* when the converse is true, i.e. if every Cauchy sequence converges to an element of the space. A normed linear space which is complete is called a *Banach space*.

Let S be a subset of elements of the space. u is a *limit point* of S (u not necessarily in S) if u is the limit of some sequene $\{u^m\}$ of elements of S. S is *closed* if it contains all of its limit points.

Let T be a vector-valued function defined on a subset S, which maps S into itself. That is $Tu \in S$ if $u \in S$. The map T is called *contracting* if $\|Tu - Tv\| \leq \Lambda \|u - v\|$ for vectors $u, v \in S$ and if $\Lambda < 1$.

In terms of this language, we can now state and prove the following fixed point theorem.

Theorem 1 *(Contraction Mapping Theorem) If B is a Banach space, S a closed subset of B, and T is a contracting map which takes S into itself, then the equation $u = Tu$ has a unique solution in S.*

The proof is accomplished by Picard iteration, starting with any $u^0 \in S$. By hypothesis the scheme $u^{n+1} = Tu^n, n \geq 0$, is well-defined, and $\left\|u^{n+1} - u^n\right\| < \Lambda^n \left\|u^1 - u^0\right\|$. Use of the triangle inequality and comparison with the geometric series implies that $\{u^n\}$ is a

Cauchy sequence, which converges to a $u \in B$ since B is complete. Furthermore, $u \in S$ since S is closed. This u is a fixed point of T in S, since

$$\|Tu^n - Tu\| \leq \Lambda \|u^n - u\|,$$

so that $u = Tu$ is a consequence of passing to the limit on both sides of $u^{n+1} = Tu^n$ and the positive definiteness of the norm. Finally, u is the only fixed point in S, for if v were also, then

$$\|u - v\| = \|Tu - Tv\| \leq \Lambda \|u - v\|,$$

and $u \neq v$ would imply $\Lambda \geq 1$.

As promised, we can fit the special situation of our existence theorem for the initial value problem to this general theorem. To reiterate, we want a vector-valued $y = y(x)$ defined and continuous on some I_α which satisfies the first order system

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned}$$

and is such that $|y(x) - y_0| < b$ for $x \in I_\alpha$ (the single bar notation for vectors and vector valued functions has the same meaning as before). We assume that f is continuous on $R = \{(x, y) \big| \, |x - x_0| < a, |y - y_0| < b\}$ and Lipschitz continuous in y

$$|f(x, y) - f(x, z)| \leq L\,|y - z|$$

for $(x, y), (y, z)$ in R.

Set $\alpha = \min{(a, \frac{b}{M})}$. Take B to be the linear space of function $y = y(x)$ continuous on $I_\alpha = \{x \big| \, |x - x_0| < \alpha\}$. For $y \in B$ the function

$$Py = \|y\| = \max_{|x - x_0| < \alpha} e^{-L'|x - x_0|}\,|y(x)| \text{ where } L' > L$$

is a norm on B (convergence in this norm amounts to uniform convergence on I_α) and B is complete under this norm (excercise **??**). Take for S, $S = \{y \in B \big| \, |y(x) - y_0| < b\}$; S is closed under the norm. Take for T the map defined by

$$Ty = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))d\xi$$

for $y \in S$. Our choice of α already implies that T maps S into itself. T is contracting in the sense of our norm, and the existence (and uniqueness) theorem is proved.

Exercise 1 *Show that $Py = \|y\|$ is a norm on B. Show that B is a Banach space with this norm. Show that S is closed with respect to P. Show that T is a contracting map with $\Lambda = \frac{L}{L'}$.*