Lecture 4: Continuous Dependence on Init Cond + Parameters

$$(*)\begin{cases} \dot{x} = f(x) \\ x(0) = y \end{cases}$$

think of the initial condition as a parameter

a family of solutions to $*$ is given by $x(t) = u(t;y)$

use semicolon to emphasize y a parameter

if we assume f is C^1 we can define the linearization

let $\Phi = D_y u = \frac{\partial u}{\partial y}$ (a matrix

note since $u(0,y) = y$ we have $\Phi(0,y) = \frac{\partial y}{\partial y} = I$

What ODE does Φ solve?

$$\frac{d\Phi}{dt} = \frac{d}{dt}\frac{\partial}{\partial y}(u(t,y)) = \frac{\partial}{\partial y}\dot{u}(t,y) = \frac{\partial}{\partial y} f(u(t,y))$$

$$= Df(u(t,y))\frac{\partial y}{\partial y} = Df(u(t,y))\Phi$$

$$\begin{cases} \frac{d\Phi}{dt} = Df(u(t,y))\Phi \\ \Phi(0) = I \end{cases}$$ the linearized eqn

first we need to show it makes sense to write $x(t) = u(t,y)$

Lemma 1 Neighborhood existence

suppose for some $x_0 \in \mathbb{R}^n$, $\exists b > 0$ s.t. $f: B_b(x_0) \to \mathbb{R}^n$ is Lipschitz w/ const K

and let $M = \max_{x \in B_b(x_0)} |f(x)|$

Then a family of solutions $u(t,y)$ exist $\forall y \in B_{b/2}(x_0)$

and is unique $\forall t \in [a,a] = J$ where $a < \min\left[\frac{1}{K}, \frac{b}{2M}\right]$

Proof define

the closed set of functions $V = C^0(J, B_b(x_0))$

label the operator T by its initial condition

$$T_y(u) = y + \int_0^t f(u(\tau))d\tau \quad \text{for all } y \in B_{b/2}(x_0)$$

$$|T_y(u) - x_0| \le |y - x_0| + \int_0^t f(u(\tau))d\tau$$

$$\le \frac{b}{2} + Ma \le \frac{b}{2} + \frac{b}{2} \quad \text{provided } a \le \frac{b}{2M}$$

Lecture 4-2

Further $T_y u$ is a contraction if $a < \frac{1}{K}$ (can relax w/ weighted norm)

Contraction mapping thm $\Rightarrow \exists!$ soln $u(t,y)$ $\forall t \in J$ $\square$

note we had to vary initial conds over ball half the size, could take a larger ball, but would need to take a smaller

eg $B_{\frac{3b}{4}}$, $a < \frac{1}{4}\frac{b}{M}$

Recall from last time

$$\boxed{\begin{array}{l}\text{GRONWALL LEMMA} \\ \text{if } g(t) \leq C + \int_0^t k(s)g(s)ds, \qquad g(t), k(t), C \geq 0 \\ \qquad\qquad \text{then } g \leq Ce^{\int_0^t k(s)ds}\end{array}}$$

~~Lemma~~

Theorem 1 Lipschitz dependence on Initial Conditions

let $x_0 \in \mathbb{R}^n$, suppose $\exists b > 0$ st $f: B_b(x_0) \to \mathbb{R}^n$ Lipschitz w/ const. K and $J = [-a, a]$ is the common interval of existence for solutions $u(t,y)$ ~~of~~ ⊛ : $J \times B_{b/2}(x_0) \longrightarrow B_b(x_0)$

then $u(t,y)$ is uniformly Lipschitz in y w/ Lipschitz const e^{Ka}

uniform w.r.t. t

Pf $|u(t,y) - u(t,z)| \leq |y-z| + \int_0^t |f(u(\tau,y)) - f(u(\tau,z))| d\tau$

$$\leq |y-z| + K\int_0^t |u(\tau,y) - u(\tau,z)| d\tau$$

by gronwall w/ $k(t) = K$, $C = |y-z|$

$$|u(t,y) - u(t,z)| \leq |y-z| e^{Kt} \leq |y-z| e^{Ka}$$

Lecture 4-4

Suppose $|h| < b/2$ and define

$$g(t) = |u(t, y+h) - u(t,y) - \Phi(t,y)h|$$

$$= \cancel{|u(t), y+h}$$

$$= \Big| y+h + \int_0^t f(u(\tau, y+h))\,d\tau - \Big(y + \int_0^t f(u(\tau,y))\,d\tau\Big)$$

$$- \Big(\Phi(t,0)h + \int_0^t Df(u(\tau,y))\,\Phi(\tau,y)h\,d\tau\Big)\Big|$$

$$= \Big|\int_0^t \Big[\cancel{\ldots}\, f(u(\tau,y+h)) - f(u(\tau,y)) - Df(u(\tau,y))\Phi(\tau,y)\,h\Big]\,d\tau\Big|$$

we need to show $\lim_{\cancel{h\to 0}} g(t) \to 0$ faster than $|h|$ as $h \to 0$

which would imply $\Phi = \frac{\partial u}{\partial y}$

Since f is C^1

we can write $f(w) = f(u) + Df(u)(w-u) + R(u,w)$

where $R(u,w)$ is small in the sense that

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon, u) > 0 \text{ s.t. } |w-u| < \delta \Rightarrow \boxed{|R(u,w)| < \varepsilon|w-u|} \quad (3)$$

thus, using $w = u(\tau, y+h)$, $u = u(\tau, y)$, get

$$g(t) \leq \int_0^t \big| Df(u(\tau,y))(u(\tau,y+h) - u(\tau,y)) + R(u,w) - Df(u(\tau,y))\Phi(\tau,y)h\big|\,dt$$

$$\leq \int_0^t \big| Df(u(\tau,y))(u(\tau,y+h) - u(\tau,y) - \Phi(\tau,y)h\big|\,d\tau + \underbrace{\int_0^t |R(u,w)|\,d\tau}_{\int_0^t |R(u(\tau,y+h),\, u(\tau,y))|\,d\tau}$$

$$\leq \int_0^t \cancel{\|Df\|}\, g(\tau)$$

~~so if $|h| < \ldots$, but using (3) and we have~~

$\int_0^t$ ~~...~~

define $r = \int_0^t |R(u(\tau, y+h)), u(\tau,y)|\,d\tau$

Lecture 4-3

Review: what is a derivative in $\mathbb{R}^n$?

def $f: \mathbb{R}^n \to \mathbb{R}^n$ is differentiable at x_0 if there is a linear transformation (a matrix) denoted $Df(x_0)$ satisfying

$$\lim_{|h| \to 0} \frac{|f(x_0+h) - f(x_0) - Df(x_0)h|}{|h|} = 0$$

Theorem f is differentiable at $x_0 \Leftrightarrow$ all the mixed partials $\frac{\partial f_i}{\partial x_j}$ are well-defined

in which case $(Df(x_0))_{ij} = \frac{\partial f_i}{\partial x_j}$

Theorem 2 (Smooth dependence on initial conditions)
Suppose $f: E \to \mathbb{R}^n$ is C^1 on an open set E
Then $\exists a > 0$ s.t. $u(t,y)$ solving $\circledast$ is a C^1 function of y for $t \in [-a,a]$

Proof - Since f is C^1 then f locally Lipschitz
For each $x_0 \in E$, $\exists b > 0$ w/ $B_b(x_0) \subset E$ and f Lipschitz on $B_b(x_0)$ w/ const $K(x_0, b)$
- by lemma 1, $\exists$ unique sol'n $u(t,y)$ $\forall y \in B_{b/2}(x_0)$ on a common interval J
- since $f \in C^1$, can define Φ as above by

$$(**) \begin{cases} \frac{d}{dt}\Phi = Df(u(t,y))\Phi \\ \Phi(0,y) = I \end{cases}$$

let $A(t) = Df(u(t,y))$
Since $u \in C^1$ and Df continuous, A is a continuous function of t
thus for ~~each~~ column i of $**$ solves

$$\begin{cases} \dot{x} = A(t)x \\ x(0) = \hat{e}_i \end{cases}$$ has a unique solution on J

these solutions define the columns of J

Lecture 4-5

$g(t) \leq r + \int_0^t K g(s)\, ds$

$$\Rightarrow g(t) \leq r e^{Kt} \leq r e^{Ka}$$

need to show $\frac{r}{h} \to 0$ as $h \to 0$

~~RF~~ R_F by ~~(2)~~ $\cancel{|R(u(\tau, y+h)), u(\tau,y))| \leq}$

by (3) $|R(u(\tau, y+h), u(\tau,y)| \leq \varepsilon\, |u(\tau,y+h) - u(\tau, y+h)|$

if $|u(\tau, y+h) - u(\tau,y)| \leq |h|\, e^{Ka} < \delta(\varepsilon, b)$

i.e. if $|h| < \delta e^{-Ka}$

in which case $\int |R(u(\tau, y+h), u(\tau,y)|\, d\tau \leq \varepsilon \int_0^t |u(\tau,y+h), u(\tau,y)|\, d\tau$

$r \leq \varepsilon \int_0^t |h| e^{Ka}\, d\tau$

$r \leq \varepsilon a |h| e^{Ka}$

So we end up with $g(t) \leq r e^{Ka} \leq \varepsilon a |h| e^{Ka} e^{Ka} = \varepsilon a |h| e^{2Ka}$

since this is true for all $\varepsilon > 0$

we have $\frac{g(t)}{|h|} \to 0$ as $h \to 0 \Rightarrow u \in C^1$

Theorem Suppose (☺) $\begin{cases} \dot{x} = f(x, \mu) \\ x(0) = x_0 \end{cases}$, $f : B_b(x_0) \times B_r(\mu_0) \to \mathbb{R}^n$

has uniformly Lipschitz dependence on $x \in B_b(x_0)$

(~~Lipschitz~~ K independent of μ)

and f is uniformly continuous function of parameters

then (☺) has unique solution $u(t; y, \mu)$ for $y \in B_{b/2}(x_0)$

that is a uniformly continuous function of μ on some ~~w~~ time interval J

Lecture 4-6

Theorem 3 if $\frac{dx}{dt} = f(x, \mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ $x|_{t=0} = y$

then the solution

$x(t) = u(t; y, \mu)$ depends continuously ~~of~~ on both y & μ

proof let $Z = \begin{bmatrix} x \\ \mu \end{bmatrix} \in \mathbb{R}^{n+m}$

$$g(Z) = \begin{bmatrix} f(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \}n \text{ coords} \\ \}m \text{ coords} \end{matrix}$$

apply previous proof to $\dot{Z} = g(Z)$
$Z|_{t=0} = \binom{y}{\mu}$

Final note : if $f(t)$ is C^k then dependence on initial conds/parameters is also C^k