~~15~~ Lecture 16-1

Theorem let $E \subset \mathbb{R}^n$ open, $f \in C^1(E)$ and $\varphi_t(x_0)$ a periodic solution of $\dot{x} = f(x)$ with period T,
& $\Gamma = \{\varphi_t(x_0) \mid 0 \le t \le T\} \subset E$ $\longrightarrow$ also written $\Gamma = \{\gamma(t) \mid 0 \le t \le T\}$
Σ a hyperplane through x_0 $\perp$ Γ ie

$$\Sigma = \{x \in \mathbb{R}^n \mid (x - x_0) \cdot f(x_0) = 0\}$$

Then $\exists\, \delta > 0$ and a unique function $\tau(x) \in C^1(N_\delta(x_0))$ ~~defined and C^1~~ s.t $\tau(x_0) = T$ and
$\varphi_{\tau(x)}(x) \in \Sigma$

~~Proof~~ Note: Then we can define $P(x) = \varphi_{\tau(x)} x$, the poincaré map for Γ in a nbhd of ~~x_0~~ $N_\delta(x_0)$

Proof application of implicit function theorem

for $x_0 \in \Gamma$, define $F(t, x) = [\varphi_t(x) - x_0] \cdot f(x_0)$
note if $\varphi_t(x) \in \Sigma$ then $F(t, x) = 0$

by periodicity of $\varphi_t(x_0)$, $F(T, x_0) = 0$

$$\left.\frac{\partial F}{\partial t}\right|_{(T, x_0)} = \left.\frac{\partial}{\partial t}\varphi_t(x)\right|_{(T, x_0)} \cdot f(x_0) = f(x_0) \cdot f(x_0) = |f(x_0)|^2 > 0$$

So by IFT, $\exists\, \delta > 0$ and a function $\tau(x)$ st
$F(\tau(x), x) = 0$ which is only true if $\varphi_{\tau(x)}(x) \in \Sigma$

$\square$

from the theorem $P(x) \in C^1(U)$, in fact, since we can run time backwards, it is a diffeomorphism

note that $P(x)$ is essentially a function from $\mathbb{R}^{n-1}$ to itself
WLOG assume $x_0 = 0$

$DP(0)$ is an $(n-1) \times (n-1)$ matrix

~~15~~ → 16-2

<u>Example</u> $\dot{x} = -y + x(1-x^2-y^2)$
$\dot{y} = x + y(1-x^2-y^2)$

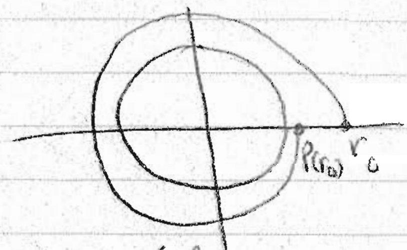
polar coords $\dot{r} = r(1-r^2)$
$\dot{\theta} = 1$

so ~~the~~ $\Sigma_{\theta} = \{\theta = \cancel{\text{so}}\,\theta_0\}$ take map from $\theta_0 = 0$

$$r(t, r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-2t}\right]^{-1/2}$$

so $P(r_0) = r(2\pi, r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right]^{-1/2}$

$P(1) = 1$ is a fixed point



Stability, we will see, is related to $DP(x_0)$

~~P~~ here 1d $DP(x_0) = P'(x_0)$

$$P'(r_0) = e^{-4\pi} r_0^{-3}\left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right]^{-3/2}$$

$$P'(1) = e^{-4\pi} < 1$$

Let's continue with 1d

Intuition about stability. Suppose $P(s_0) = s_0$

if $s_1 = s_0 + \tilde{s}_1$ ~~$P(s)=$~~
$s_2 = s_0 + \tilde{s}_2$

$s_2 = P(s_1)$

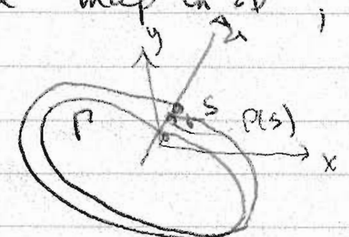
$\cancel{s_0} + \tilde{s}_2 = P(s_0 + \tilde{s}_1) = P(\cancel{s_0}) + P'(s_0)\tilde{s}_1 + O(\tilde{s}_1^2)$

$\tilde{s}_2 \approx P'(s_0)\tilde{s}_1$

Stable if $|P'(s_0)| < 1$

Lecture 16-B

Poincaré map in 2D, assume WLOG OUR FIXED PT is $X=0$



the point 0 divides Σ_1 into 2 pieces

Σ_{1+} and Σ_{1-}

Σ_{1+} entirely outside Γ

~~if $s \in \Sigma_{1+}$ then $p(s) \in \Sigma_{1+}$~~

let s = signed distance along Σ_1, $s>0$ on Σ_{1+}

clearly $s>0 \Rightarrow P(s)>0$ by uniqueness & Jordan curve theorem

Define $d(s) = P(s) - s$

$$d(0)=0, \quad d'(s) = P'(s) - 1$$

by mean value theorem for $|s|<\delta$, $\exists\, \sigma$ betw 0 and s

st $d(s) = d'(\sigma)s$

for δ suff small $\text{sign } d'(\sigma) = \text{sign } d'(0)$

if $d'(0)<0$ then $s>0 \Rightarrow d(s)<0 \Rightarrow 0<P(s)<s$

$s<0 \Rightarrow d(s)>0 \Rightarrow 0>p(s)>s$

either way $|p(s)|<|s| \Rightarrow$ the limit cycle is stable

or an ω-limit set

$$d'(0) = P'(0) - 1 < 0$$

$$\Rightarrow P'(0) < 1$$

SIMILARLY IF $P'(0)>1$ the limit cycle is unstable

Theorem let $\Gamma = \{\gamma(t) \mid 0 \le t \le T\}$ be a periodic orbit to $\dot{x} = f(x) \in \mathbb{R}^2$ & $\mathbb{R}^2$ essential (s.t $\gamma(0)=0$)

then $P'(0) = e^{\int_0^T \nabla\cdot f(\gamma(t))\,dt}$

corollary the limit cycle is stable / unstable if

$$\int_0^T \nabla\cdot f(\gamma(t)) < 1 \;/\; > 1$$

16-~~4~~ example

already

$$\begin{cases} \dot{x} = -y + (1-x^2-y^2)x \\ \dot{y} = x + (1-x^2-y^2)y \end{cases}$$

(know by polar coords that $x = \cos t \quad y = \sin t$ stable)

$\nabla \cdot f = (1-3x^2-y^2) + (1-x^2-3y^2)$

$= 2 - 4x^2 - 4y^2$

$\nabla \cdot f(\gamma(t)) = 2 - 4\cos^2 t - 4\sin^2 t = -2$

$\int_0^{2\pi} -2\, dt = -4\pi$

$P'(0) = e^{-4\pi}$ as we found before

Higher multiplicity periodic orbits

the periodic orbit Γ has multiplicity k if

$d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0, \quad d^{(k)} \neq 0$

$k = 1$, the periodic orbit called SIMPLE

~~FORMAL~~

We can also define $P(s)$ if the origin is a ~~st~~

center or spiral

$\dot{x} = ax - by + \dots$

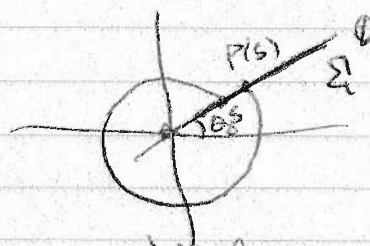
$\dot{y} = bx + ay$

in polars get $\dot{\theta} = b + \dots, \ \dot{r} = a + \dots$

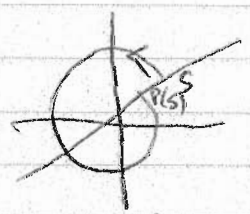
if $a \neq 0$ stability immediate, if not, let's assume $b > 0$

then in a nbhd of origin

we can define $P(r_0) = r(t(\theta_0 + 2\pi, r_0)$



unstable focus



stable focus

look at $d(s) = s - p(s)$

multiplicity: first k st $d^{(k)}(0) \neq 0$

NOTE TO SELF,

Lecture 16-5 ONLY ~~DO~~ MENTION THE STUFF IN BOXES

2nd example

$\dot{x} + x = \cos \cancel{t}\, \omega t$

here instead of a section Σ_i, take fix t_0,

let $P(x_0) = \phi_{\frac{2\pi}{\omega}} x_0$ ie the state of the system after 1 period of the forcing

$$\boxed{x = c e^{-t} + a\cos\omega t + b\sin\omega t}$$

$$\dot{x} = -ce^{-t} - \omega a \sin\omega t + \omega b\cos\omega t$$

$$\dot{x} + x = (a+\omega b)\cos\omega t + (b - \omega a)\sin\omega t$$

$$a + \omega b = 1 \qquad b - \omega a = 0$$

$$a(1+\omega^2) = 1 \qquad b = \omega a$$

$$a = \frac{1}{1+\omega^2}, \; b = \frac{\omega}{1+\omega^2}$$

$$x = ce^{-t} + \frac{1}{1+\omega^2}\cos\omega t + \frac{\omega}{1+\omega^2}\sin\omega t$$

$$x_0 = c + \frac{1}{1+\omega^2}$$

$$c = x_0 - \frac{1}{1+\omega^2}$$

So finally $x(t) = \left(x_0 - \frac{1}{1+\omega^2}\right)e^{-t} + \frac{1}{1+\omega^2}\cos\omega t + \frac{\omega}{1+\omega^2}\sin\omega t$

$$\boxed{P(x_0) = x\left(\frac{2\pi}{\omega}\right) = \left(x_0 - \frac{1}{1+\omega^2}\right)e^{-2\pi/\omega} + \frac{1}{1+\omega^2}}$$

fixed point

$$x_0 = \left(x_0 - \frac{1}{1+\omega^2}\right)e^{-2\pi/\omega} + \frac{1}{1+\omega^2}$$

$$\cancel{x_0(1 - e^{-2\pi/\omega}) = \frac{1}{1+\omega^2}(1-}$$

FIXED PT $\boxed{x_0 = \frac{1}{1+\omega^2}}$

$$P' = e^{-2\pi/\omega} < 1 \quad \Rightarrow \text{fixed pt stable as it must be}$$

How does this generalize from $\mathbb{R}^2$ to $\mathbb{R}^n$

expect that we should get an orientation-preserving map $P: \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$

actually $\Sigma_i \to \Sigma_i$ where Σ_i an $(n-1)$ dim'l hypersurface

so we lose a dimension (that's good!)

16-6 Motivation, non-rigorous

Suppose we have a map $P: \mathbb{R}^n \to \mathbb{R}^n$

s.t $P(x^*) = x^*$ a fixed point

then $P(x^* + \tilde{x}) = P(x^*) + DP(x^*)\tilde{x} + O(|\tilde{x}|^2)$

so if $x_n = x^* + \tilde{x}_n$ we get

$$\tilde{x}_{n+1} \approx DP(x^*)\tilde{x}_n$$

so stability depends on the eigenvalues of $DP(x^*)$

if any eigenvalue satisfies $|\lambda_k| > 1$ then

then can have $|\tilde{x}_{n+1}| > |\tilde{x}_n|$: instability

of course the case $|\lambda_k| = 1$ always more difficult