LECTURE 23-1 The extended center manifold

Suppose $\dot{x} = f(x, \varepsilon)$ $x \in \mathbb{R}^n$, etc. has a fixed point x^* at $\varepsilon = 0$

ie $f(x^*, 0) = 0$

and further this fixed point is degenerate

~~$\partial_x f(x^*, 0) = 0$~~ $D_x f(x, \varepsilon)|_{(x^*, 0)}$ has an eigenvalue w/ zero real part

let's suppose $x = \binom{u}{v}$ in such a way that

$u \in \mathbb{R}^k$

$v \in \mathbb{R}^{n-k}$

and we can write $\dot{u} = Au + O(x^2) + O(\varepsilon)$

$\dot{v} = Bu + O(x^2) + O(\varepsilon)$

s.t. A has eigenvalues w/ zero real part

and B has eigenvalues w/ nonzero real part

(A and B constant matrices independent of ε)

then $\dim E^c = k$, $\dim E^u \oplus E^s = n-k$

and we know there exists a k-dimensional center manifold W^c defined by $v = \varphi(u)$ in a neighborhood of the origin st $\varphi(u) = O(u^2)$.

Now we'd like to extend this to the case $0 < \varepsilon \ll 1$ in order to determine the normal form of the bifurcation. We do this by introducing the trivial modification of appending the equation $\dot{\varepsilon} = 0$

then we can define the extended center manifold $v = \varphi(u, \varepsilon)$ which is $(k+1)$ dimensional

This is best illustrated with an example.
We have already seen by carefully drawing the nullclines that

$\dot{x} = y - 2x$

$\dot{y} = \mu + x^2 - y$

has a saddlenode bifurcation at $\mu = 1$ ~~at the~~

At $\mu = 1$, $(x, y) = (1, 2)$ is a non-hyperbolic fixed point

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let $u = x - 1 \quad \Rightarrow \quad x = u + 1$

$v = y - 2 \qquad\qquad y = v + 2$

$\varepsilon = \mu - 1$

then $\dot{u} = \dot{x} = (v+2) - 2(u+1) = v - 2u$

$\dot{v} = 1 + \varepsilon + (u+1)^2 - (v+2) = 2u - v + \varepsilon + u^2$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon + u^2 \end{pmatrix}$$

note this is linear in ε

Append the trivial ε dynamics

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 \\ 0 \end{pmatrix}$$

$A = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ has $\lambda_1 = -3, \ \lambda_2 = \lambda_3 = 0$

~~with correspondin~~

If we try to linearizes, we find that the double eigenvalue $\lambda = 0$ has an irreducible Jordan block

We get eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

and generalized eigenvector $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

which satisfies $A\vec{v}_3 = \vec{v}_2$

(of course the generalized eigenvector $\vec{v}_3$ is highly non-unique!

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now we let $\Lambda = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then $\quad A\Lambda = \Lambda V$

$$V = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad A = \Lambda V \Lambda^{-1}$$

$$V^{-1} = \frac{1}{9}\begin{bmatrix} 6 & -3 & 1 \\ 3 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

so $\frac{d}{dt}\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix} = A\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 \\ 0 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix} = \Lambda V \Lambda^{-1}\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 \\ 0 \end{pmatrix}$$

left-multiply by Λ^{-1}, get

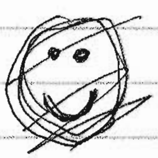
$$\frac{d}{dt}\left(\Lambda^{-1}\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix}\right) + V\left(\Lambda^{-1}\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix}\right) + \Lambda^{-1}\begin{pmatrix} 0 \\ u^2 \\ 0 \end{pmatrix}$$

let $\begin{pmatrix} z \\ w \\ \delta \end{pmatrix} = \Lambda^{-1}\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix}$

then $\frac{d}{dt}\begin{pmatrix} z \\ w \\ \delta \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} z \\ w \\ \delta \end{pmatrix} + \Lambda^{-1}\begin{pmatrix} 0 \\ u^2 \\ 0 \end{pmatrix}$

now $\begin{pmatrix} u \\ v \\ \varepsilon \end{pmatrix} = V\begin{pmatrix} z \\ w \\ \delta \end{pmatrix}$ so $u = z + w$

also and $\varepsilon = 3\delta \Rightarrow \delta = \frac{\varepsilon}{3}$



So we end up with
$$\boxed{\begin{aligned} \dot z &= -3z + \tfrac{1}{3}(z+w)^2 \\ \dot w &= \delta + \tfrac{1}{3}(z+w)^2 \\ \dot\delta &= 0 \end{aligned}}$$

(~~note in the earlier calculation at ☹ there was an ε in the both the $\dot z$ and w terms!~~)

Now in the extended system, $E^s = \{(z,0,0)\}$
$E^c = \{(0,w,\delta)\}$

So on the extended center manifold $z = \underbrace{aw^2 + bw\delta + c\delta^2}_{\text{Quadratic terms in } \binom{w}{\delta}} + O(\cdot^3)$

$$\dot z = 2aw\dot w + b\delta\dot w = (2aw + b\delta)\left[\delta + \tfrac{1}{3}(w + \dots)^2\right]$$
$$= 2aw\delta + b\delta^2 + \text{cubic terms}$$

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also, $\dot{z} = -3z + \frac{1}{3}(z+w)^2$
$= -3(aw^2 + bw\delta + c\delta^2) + \frac{1}{3}w^2 +$ cubic terms

matching the 2 ~~formats~~ formulas for $\dot{z}$

$$2aw\delta + b\delta^2 = (-3a + \tfrac{1}{3})w^2 - 3bw\delta - 3c\delta^2$$

matching terms
w^2: $0 = -3a + \frac{1}{3} \Rightarrow a = \frac{1}{9}$
$w\delta$: $2a = -3b \Rightarrow b = -\frac{2}{3}a = \frac{-2}{27}$
δ^2: $b = -3c \Rightarrow c = -\frac{b}{3} = \frac{2}{81}$

so the extended center manifold is
$$z = \frac{1}{9}w^2 - \frac{2}{27}w\delta + \frac{2}{81}\delta^2$$

finally, the flow on the extended center manifold is
$$\dot{\delta} = 0$$
$$\dot{w} = \delta + \frac{1}{3}(z+w)^2 = \delta + \frac{1}{3}w^2 + \text{cubic terms}$$

since $\delta = \varepsilon/3$ this reduces to

$$\boxed{\dot{w} = \frac{1}{3}(\varepsilon + w^2) + \ldots}$$ a saddle node bifurcation

lecture 23-25 Non-degeneracy condition

transcritical
~~Saddle-node~~ bifurcation $\quad \dot{x} = \mu x - x^2$

but the general form : suppose $\dot{x} = f(x, \mu)$
has a fixed pt at $x = 0$, ~~st~~ $f(0,0) = f_x(0,0) = f_\mu(0,0)$
$f_{\mu x}(0,0) \neq 0$, $f_{xx}(0,0) \neq 0$

so how do we know $f_{\mu\mu}$ not important (& higher terms)?
be more careful

$$\dot{x} = \frac{1}{2}\left(f_{xx} x^2 + 2 f_{\mu x} \mu x + f_{\mu\mu} \mu^2\right) + O(\cdot^3)$$

Solve for $\quad x = \dfrac{-f_{\mu x}\mu \pm \sqrt{f_{\mu x}^2 \mu^2 - f_{xx} f_{\mu\mu} \mu^2}}{f_{xx}}$

so we see that if $\Delta^2 = f_{\mu x}^2 - f_{xx} f_{\mu\mu} > 0$ then

$$x = \frac{-f_{\mu x}\mu \pm \Delta \mu}{f_{xx}}$$

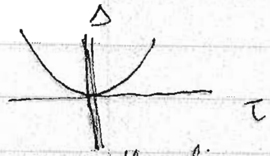
$f_{xx} \neq 0$, $f_{\mu x} \neq 0$ by the normal form assumptions
to see a saddle-node bifurcation, also need
$f_{\mu x}^2 - f_{xx} f_{\mu\mu} > 0$ (non-degeneracy condition

~~There~~ Many types of bifurcation have these add'l conditions

Hamiltonian bifurcations

if $\quad \dot{x} = \dfrac{\partial H}{\partial y}$, $\dot{y} = -\dfrac{\partial H}{\partial x}$ $\quad$ we've seen that this is
area preserving $\Rightarrow \operatorname{tr} J = 0$

how does this effect bifurcations?



Example

$\ddot{x} = \mu - x^2$ $\quad$ saddle node

fixed points $\quad \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \pm\sqrt{\mu} \\ 0 \end{pmatrix}$ $\quad \mu > 0$

all on line $\tau = 0$
$\Rightarrow$ saddle or center

$\dot{x} = y$
$\dot{y} = \mu - x^2$, $\quad J = \begin{bmatrix} 0 & 1 \\ -2x & 0 \end{bmatrix}$, ~~$J(\sqrt{\mu}, 0)$~~

$\delta = 2x$, $\quad \delta(\sqrt{\mu}, 0) = 2\sqrt{\mu}$ center
$\delta(-\sqrt{\mu}, 0) < 0$ saddle

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Hamiltonian Pitchfork

$$\ddot{x} - \mu x + x^3 = 0 \qquad x = 0,\ x = \pm\sqrt{\mu} \text{ when } \mu > 0$$

$$\dot{x} = y$$
$$\dot{y} = \mu x - x^3$$
$$J = \begin{pmatrix} 0 & 1 \\ \mu - 3x^2 & 0 \end{pmatrix}$$

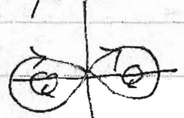
$\delta = 3x^2 - \mu$

$\delta(0) = -\mu \Rightarrow$ center, $\mu < 0$; saddle $\mu > 0$

$\delta(\pm\sqrt{\mu}) = 2\mu \Rightarrow$ ~~saddle~~ center $\mu > 0$

$\mu < 0$

$\mu > 0$



Both cases, eigenvalues do this  not generic for Non Hamiltonian

Some other bifurcations can only occur in higher dimensions

let $\dot{x} = f(x)$ be a Hamiltonian system in $\mathbb{R}^{2n}$

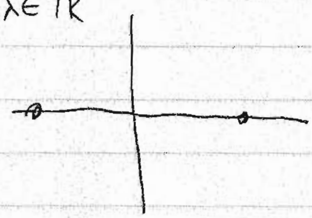
what's important in higher dimensions

area preservation $\Rightarrow$ if λ an eigenvalue then $-\lambda$ an eigenvalue

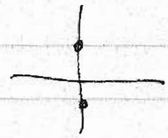
also, since real λ an eigenvalue $\Rightarrow$ λ^* an eigenvalue

$\Rightarrow -\lambda^*$ an eigenvalue

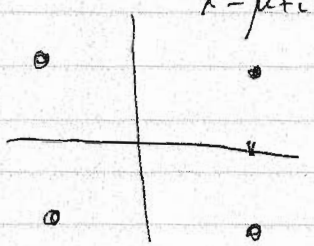
$\lambda \in \mathbb{R}$



λ pure imaginary



$\lambda = \mu + i\nu$



quartets, $\lambda = \pm\mu \pm i\nu$