Lecture #12 Stability

Suppose $\dot{x} = f(x)$ has fixed point x_0 and flow φ_t

<u>definitions</u> x_0 called <u>stable</u> if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |\varphi_t x - x_0| < \varepsilon \quad \forall t > 0$

x_0 called ~~attra~~ unstable if it is not stable

x_0 called <u>asymptotically stable</u> if it is stable and if $\exists \delta > 0$ s.t.
$|x - x_0| < \delta \Rightarrow |\varphi_t x - x_0| \xrightarrow[t \to +\infty]{} 0$

if x_0 is stable but not asymptotically stable, it is <u>neutrally stable</u>

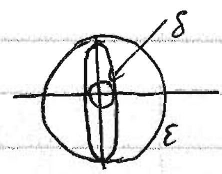
In linear systems,
- centers are neutrally stable
- sinks + stable spirals are asymptotically stable
- sources, unstable spirals, + saddles are unstable

From a corollary to the stable manifold theorem, if $A = Df(0)$ has eigenvalues λ_i with $\operatorname{Re} \lambda_i < 0 \ \forall i$, then x_0 is attracting, and $\forall \varepsilon > 0, \exists \delta > 0, \alpha > 0$ s.t.
$|x - x_0| < \delta \Rightarrow |\varphi_t x - x_0| < \varepsilon e^{-\alpha t}$ (Perko's Thm 1)

Why do we need separate ε & δ?

Neutrally stable example $\dot{x} = y$ $\quad 10x^2 + y^2 = r^2$
$\dot{y} = -10x$

Exact solutions lies on ellipses



To ensure $x^2 + y^2 < \varepsilon$
need $x^2(0) + y^2(0) < \frac{\varepsilon}{\sqrt{10}}$

Asymptotically stable example

$$\dot{x} = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix} x \quad \Rightarrow x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -10 \\ 1 \end{pmatrix}$$

set $c_1 = 10\delta, \quad c_2 = \delta$

at $t = 0$ get $x = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$

after a short time $e^{-2t} \ll e^{-t}$ so $\vec{x}(t)$ can have a large horizontal component left

Bound $|x(0)|^2 \leq |c_1 - 10c_2|^2 + |c_2|^2 \leq \delta^2 \Rightarrow |c_2| \leq \delta$ & $|c_1| \leq 11\delta$

$$|x(t)| \leq |c_1 e^{-t} - 10c_2 e^{-2t}| + |c_2 e^{-2t}|$$
$$\leq |c_1| e^{-t} + 11|c_2| e^{-2t}$$
$$\leq 11\delta + 11\delta = 22\delta$$

to guarantee $|x| < \varepsilon$ need pick $\delta < \frac{\varepsilon}{22}$ (very crude estimate

<u>Theorem 2</u> if x_0 is stable, no eigenvector λ_i of $Df(0)$ satisfies $\operatorname{Re} \lambda_i > 0$

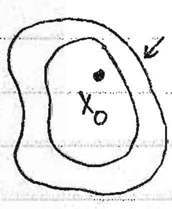
When a fixed point is nonhyperbolic, more subtle approach is needed.
We will apply the Lyapunov methods & Lasalle method
Another approach is the direct classification of nonhyperbolic fixed points as is done in Perko §2.11

<u>Definition</u> given $\dot{x} = f(x)$, $f \in C^1(E)$ with fixed point $x_0 \in E$

$V(x)$ is a <u>Lyapunov function</u> if $V(x) \in C^1(E)$, $V(x_0) = 0$, $V(x) \geq 0$ and

1) $\frac{dV}{dt} = \frac{d}{dt}(V(\varphi_t x))|_{t=0} = DV(x)f(x) \leq 0 \quad \forall x \in E$
2) $\dot{V}(x) = 0 \Leftrightarrow x = x_0$

$V(x)$ a <u>strong Lyapunov function</u> if $\frac{dV}{dt} < 0 \quad \forall x \neq x_0$



level sets of V. Since $\dot{V} < 0$ (in the case of Strong Lyapunov functions), any trajectory crosses a level curve from its exterior to its interior and with nonvanishing speed, so solutions must move toward x_0

<u>Theorem</u> if f and V as in the above definition w/ $V(x_0) = 0$, $V(x) > 0 \quad \forall x \in E \setminus \{x_0\}$

a) if $\frac{d}{dt} V(x(t)) \leq 0 \quad \forall x \in E$, then x_0 is stable
b) if $\frac{d}{dt} V(x) < 0 \quad \forall x \in E \setminus \{x_0\}$ then x_0 is asymptotically stable.
c) if $\frac{d}{dt} V > 0 \quad \forall x \in E \setminus \{x_0\}$ then x_0 is unstable

Note - applying condition (a) is known as Lyapunov's first method; applying (b) is Lyapunov's second method.

Proof: WLOG let $x_0 = 0$ ~~and~~, $V(0) = 0$, and $V(x) > 0 \ \forall x \in E \setminus \{x_0\}$

(a) - Choose ε s.t $\bar{N}_\varepsilon(0) \subset E$

- Let $S_\varepsilon = \{x \mid |x| = \varepsilon\}$ and $m_\varepsilon = \min_{x \in S_\varepsilon} V(x)$
- Since V is continuous, $\exists \delta > 0$ s.t. $V(x) < m_\varepsilon \ \forall x \in N_\delta(0)$
- $\dot{V}(x) \le 0 \Rightarrow V(\varphi_t(x))$ non-increasing, so $\forall \tilde{x} \in N_\delta(0)$

$$V(\varphi_t(\tilde{x})) \le V(\tilde{x}) < m_\varepsilon$$

Now assume $\exists \tilde{x} \in N_\delta(0)$ and $t > 0$ st $|\varphi_t(\tilde{x})| = \varepsilon$, then $V(\varphi_t(\tilde{x})) \ge m_\varepsilon$

CONTRADICTION!

$\Rightarrow \forall \tilde{x} \in N_\delta(0), |\varphi_t \tilde{x}| < \varepsilon$, i.e. 0 is stable

(b) Suppose $\dot{V}(x) < 0 \ \forall x \in E \setminus \{x_0\}$. Define $N_\varepsilon(0), m_\varepsilon, N_\delta$ as above

thus $\frac{d}{dt} V(\varphi_t(x)) = DV(x) f(x) < 0$ along trajectories when $f \ne 0$ (ie when $x \ne 0$).

By part (a), if $\tilde{x} \in N_\delta(0)$, $\varphi_t(x) \subset N_\varepsilon(0) \subset \overline{N_\varepsilon(0)} \Rightarrow$ the forward trajectory of $\tilde{x}$ is compact.

This implies $\exists$ sequence t_n s.t $x_n = \varphi_{t_n}(\tilde{x}) \longrightarrow y_0 \in \overline{N_\varepsilon(0)}$.

It remains to show $y_0 = 0$.

Since $V(x)$ strictly increasing, it follows that $V(\varphi_t(\tilde{x})) > V(y_0) \ \forall t > 0$.

Indirect proof assumption → But if $y_0 \ne 0$, it follows that $\exists s > 0$ with $V(\varphi_s(y_0)) < V(y_0)$.

and for all y sufficiently close to y_0 $V(\varphi_s(y)) < V(y_0)$

Then choose n large enough s.t. $x_n = \varphi_{t_n}(\tilde{x})$ is in this neighborhood.

Then $V(\varphi_{s+t_n}(\tilde{x})) < V(y_0)$. CONTRADICTION

$\Longrightarrow y_0 = 0 \qquad \varphi_t(\tilde{x}) \longrightarrow 0$ for any $\tilde{x} \in N_\delta(0)$.

(c) let $M = \max_{x \in \bar{N}_\varepsilon(0)} V(x)$. Since $\dot{V} > 0$, $V(\varphi_t x)$ increasing on trajectories:

$$V(\varphi_t(x)) > V(x) \text{ and in fact } \inf \dot{V}(\varphi_t(x)) = m > 0$$

so $V(\varphi_t(\tilde{x})) - V(\varphi_t(\tilde{x})) \ge mt$

then for t suff large $mt > M \Rightarrow \varphi_t(\tilde{x}) \notin \overline{N_\varepsilon(0)} \Rightarrow |\varphi_t(\tilde{x})| > \varepsilon$ $\square$

EXAMPLE 1 $\dot{x} = -y^3$, $\dot{y} = x^3$. This looks a lot like $\dot{x} = -y$, $\dot{y} = x$ which has conserved quantity $V = x^2 + y^2$. It is obvious to try $V = x^4 + y^4$

$$\frac{dV}{dt} = 4x^3\dot{x} + 4y^3\dot{y} = -4x^3y^3 + 4x^3y^3 = 0$$

So that the origin is neutrally stable

EXAMPLE 2
$$\begin{cases} \dot{x} = -2y + yz \\ \dot{y} = x - xz \\ \dot{z} = xy \end{cases}$$
Note $Df(0) = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with eigenvalues 0 and $\pm\sqrt{2}\,i$

SIMPLEST POSSIBLE GUESS: $V = c_1 x^2 + c_2 y^2 + c_3 z^2$ (slight generalization of r^2)

TRY TO PICK c_1, c_2, c_3 s.t $\dot{V} < 0$

$$\dot{V} = 2c_1 x\dot{x} + 2c_2 y\dot{y} + 2c_3 z\dot{z}$$
$$= 2c_1(-2xy + xyz) + 2c_2(xy - xyz) + 2c_3(zxy)$$
$$\tfrac{1}{2}\dot{V} = (c_1 - c_2 + c_3)xyz + (c_2 - 2c_1)xy$$

PICK $c_2 = 2c_1$, $c_1 = c_3$ then $\dot{V} = 0$, meaning that flow confined to ellipsoids
$$x^2 + 2y^2 + z^2 = C^2$$

EXAMPLE 2a
$$\begin{cases} \dot{x} = -2y + yz - x^3 \\ \dot{y} = x - xz - y^3 \\ \dot{z} = xy - z^3 \end{cases}$$

Take $V = x^2 + 2y^2 + z^2$ as before. Now $\dot{V} = 2x\dot{x} + 4y\dot{y} + 2\dot{z} = -2(x^4 + 2y^4 + z^4) < 0$

So the origin is asymptotically stable

EXAMPLE $\ddot{x} + c\dot{x} + ax + bx^3 = 0$

If $c = 0$, $V = \frac{1}{2}\dot{x}^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4$ is a conserved energy

When $c \neq 0$, then linear theory tells us that $x = \dot{x} = 0$ is stable since $\mathrm{Re}\,\lambda_i < 0$, but it does not tell us the basin of attraction: How large can x & $y = \dot{x}$ be for $\phi_t\binom{x_0}{y_0} \longrightarrow 0$

letting $\dot{x} = y$
$\dot{y} = -ax - cy - bx^3$
and $V(x,y) = \frac{1}{2}y^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4$

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gives $\dot{V} = y(\dot{y} + ax + bx^3)$

$= y(-ax - bx^3 - cy + ax + bx^3)$

$= -cy^2 \le 0$ but not < 0

This only enough to tell us that $(0,0)$ is stable, not asymptotically stable.

Can we modify $V(x)$ to produce a strong Lyapunov exponent?

If we can find one s.t. $\dot{V}(x) < 0 \ \forall x$ then this tells us the origin is a global attractor

A more general quadratic Lyapunov function would be

$V(x) = x^T M x$ where M positive definite, WLOG can choose M symmetric

$M = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ positive definite $\Rightarrow \mathrm{Tr}(M) = \alpha + \gamma > 0$ and $\mathrm{Det}\, M = \alpha\gamma - \beta^2 > 0$

Try $V = E + \delta xy$, E as before $\quad M = \begin{pmatrix} a/2 & \delta/2 \\ \delta/2 & 1/2 \end{pmatrix}$ $\quad \det M = \frac{a}{4} - \frac{\delta^2}{4} > 0$

$\Rightarrow \boxed{\delta < \sqrt{a}}$

$\dot{V} = \dot{E} + \delta x\dot{y} + \delta\dot{x}y$

$= -cy^2 + \delta x(-ax - cy - bx^3) + \delta y^2$

$= \cancel{-ax^2 - cxy} - (\delta a x^2 + \delta c xy + (c-\delta)y^2) - \delta b x^4$

Clearly if $\boxed{\delta > 0}$ then $-\delta b x^4 \le 0$. Need to make the rest negative definite

$(c-\delta)y^2 + \delta c xy + \delta a x^2 > 0$ need $\boxed{\delta < c}$ Complete square

$y^2 + \frac{\delta c}{c-\delta} xy + \frac{\delta a}{c-\delta} x^2 > 0$

$y^2 + \frac{\delta c}{c-\delta} xy + \frac{1}{4}\left(\frac{\delta c}{c-\delta}\right)^2 x^2 + \left(\frac{\delta a}{c-\delta} - \frac{1}{4}\frac{(\delta c)^2}{(c-\delta)^2}\right) x^2 > 0$

$\underbrace{\left(y + \frac{\delta c}{2(c-\delta)} x\right)^2}_{\ge 0} + \frac{\delta}{c-\delta}\left(a - \frac{1}{4}\frac{\delta c^2}{(c-\delta)}\right) x^2 > 0$

Need $\quad a - \frac{1}{4}\frac{\delta c^2}{(c-\delta)} > 0$

$4a(c-\delta) - \delta c^2 > 0$

$4ac - (4a + c^2)\delta > 0$

$\boxed{\delta < \frac{4ac}{4a + c^2}}$

If we chose $0 < \delta < \min\left\{\sqrt{a}, c, \frac{4ac}{4a+c^2}\right\}$ then $\dot{V} < 0$ which makes the origin

Notes

- If we did not include the $\frac{b}{4}x^4$ term in V, then we still could have used V locally, but by adding this term, $\dot{V}<0$ on all $\mathbb{R}^2$ so all orbits approach origin
- This was a lot of work. Wouldn't it be nice if $\dot{V}=-cy^2\leq 0$ were enough? This is essentially what LaSalle's invariance theorem tells us.