Lecture 1-1

General System of differential eqns, $\vec{x} \in \mathbb{R}^n$

$$\cancel{F(t, \vec{x}, \tfrac{d\vec{x}}{dt}) = 0}$$

Suppose $x(t)$ a real scalar function of 1 variable $x: \mathbb{R} \to \mathbb{R}$

an nth order ODE FOR x is an equation

of the for $\cancel{F(t, x, \tfrac{dx}{dt}, \ldots}$

$$F(t, x, x^{(1)}, \ldots, x^{(n)}) = 0 \qquad x^{(k)} = \frac{d^k x}{dt^k}$$

Put into standard form $x^{(n)} = f(t, x, x^{(1)}, \ldots, x^{(n-1)})$

Special cases: Linear $x^{(n)} = \cancel{a_0(t) + a_1(t) x} \sum_{k=0}^{n-1} a_k^{(t)} x^{(k)} + b(t)$

or in Matrix for $\dot{\vec{x}} = A(t)\vec{x} + B(t)$

Autonomous : $x^{(n)} = f(x, x^{(1)}, \ldots, x^{(n-1)})$

Systems: let $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\frac{du_i}{dt} = f_i(t, u_1, \ldots, u_n)$

or in vector form

$$\frac{d\vec{u}}{dt} = \vec{f}(t, \vec{u})$$

Nonautonomous system $x \in \mathbb{R}^n$ equiv autonomous system in $\mathbb{R}^{n+1}$

nth order linear eqn can be ~~put~~ rewritten as a system of size n

$$\text{let} \quad u_1 = x, \ u_2 = x^{(1)}, \ \ldots, \ u_n = x^{(n-1)}$$

i.e. $u_i = x^{(i-1)}$

then
$$\frac{du_1}{dt} = u_2$$
$$\frac{du_2}{dt} = u_3$$
$$\vdots$$
$$\frac{du_n}{dt} = f(t, u_1, \ldots, u_n)$$

I've been writing $\frac{d\vec{x}}{dt} = f(\vec{x}, t)$ could write

think of t as time

$$\frac{d\vec{y}}{dx} = f(\vec{y}, x)$$

more geometric

Lecture 1-2

What is our goal here? Understand how solutions to DE behave.

Undergraduate (Intro approach) Only look at systems for which one can find $x(t)$ in closed form

Why is this inadequate? (1) Usually can't be done

(2) Even if you could, sometimes doesn't give insight

$$\frac{dx}{dt} = \sin x \qquad \text{SEPARAPLE}$$

$$\frac{dx}{\sin x} = dt$$

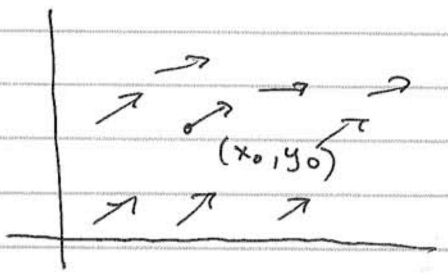
$$-\ln|\csc x + \cot x| = t + c$$

Solve for x??

Algebraic solution gives little insight into what solutions look like

Geometric approach - Qualitative Theory

$\frac{dy}{dx} = f(x,y)$, $\left.\frac{dy}{dx}\right|_{(x_0,y_0)}$ the slope of a curve $= f(x_0,y_0)$



So a solution to the DE is a curve $y(x)$ tangent the the direction field $\frac{dy}{dx} = f(x,y)$

need 1- add'l piece of information $y(x_0) = y_0$

Q: ~~CAN~~

Add vector field in 2D

lecture 1-3

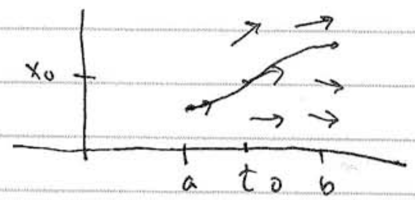
SO WE DEFINE AN INITIAL-VALUE PROBLEM

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

Go BACK TO ~~x-y~~ t-x notation

$$\begin{cases} \frac{dx}{dt} = f(t,x) & (1a) \\ x(t_0) = x_0 & (1b) \end{cases}$$

What would solving this mean?



For some time interval $[a,b]$, $t_0 \in [a,b]$

- There exists a curve $x(t)$ satisfying (1a) & (1b)
- Ideally, this curve should be unique
  i.e. is there only one such curve?
  We'd hope so, since this ODE probably came from a model + we'd want a single ~~an~~ answer
- Slightly harder question: what's the largest interval $[a,b]$ on which we can find ~~an~~ a solution? Can we go to $(-\infty, \infty)$

All of these questions would be boring if the answer were always "yes"? Some examples show this is not the case

~~A problem~~

An example with a unique sol'n for all time (Sol'n by quadratures

$$\frac{dx}{dt} = a(t)\,x + b(t) \quad , \; a, b \text{ defined and continuous, } t \in \mathbb{R}$$

$$x(t_0) = x_0$$

$$\frac{dx}{dt} - a(t)x = b(t)$$

$$\frac{d}{dt}\left[e^{-\int_{t_0}^{t} a(s)ds} x(t)\right] = e^{-\int_{t_0}^{t} a(s)ds}\left(\frac{dx}{dt} - a(t)x\right) = e^{-\int_{t_0}^{t} a(s)ds}\, b(t)$$

$$e^{-\int_{t_0}^{t} a(s)ds}\, x = \int_{t_0}^{t}\left[e^{-\int_{t_0}^{s} a(r)dr}\, b(s)\right] ds + x_0$$

$$x = e^{\int_{t_0}^{t} a(s)ds}\left[x_0 + \int_{t_0}^{t} e^{-\int_{t_0}^{s} a(r)dr}\, b(s)\, ds\right]$$

EXAMPLE WITH NO SOLUTION $\dot{x}^2 + x^2 = -1$ (a little silly)

$\boxed{1-4}$

EXAMPLE Nonunique solution

$$\begin{cases} \dfrac{dx}{dt} = \sqrt{x} \\ x\big|_{t=0} = 0 \end{cases}$$

$$\left.\begin{array}{ll} \text{(i)} & x(t) = 0 \\ \text{(ii)} & \dfrac{dx}{\sqrt{x}} = dt \\ & \displaystyle\int \dfrac{dx}{\sqrt{x}} = t + c \\ & 2x^{1/2} = t + c \quad * \end{array}\right\} \text{2 SOLUTIONS}$$

$$x^{1/2} = \tfrac{1}{2}(t+c)$$

$$x = \tfrac{1}{4}(t+c)^2$$

$$x(0) = \tfrac{c^2}{4} = 0 \Rightarrow x = \left(\tfrac{t}{2}\right)^2$$

(iii) Further note $\dfrac{dx}{dt} = \sqrt{x}$

$x(0) = 1$ has a unique solution

$x(0) = \frac{c^2}{4} = 1 \Rightarrow c = \pm 2$ but from * $c > 0$

so $x = \frac{1}{4}(t+2)^2$

this is only a solution $x > -2$, $[a, b) = [-2, \infty)$

(Note $\frac{dx}{dt} \geq 0$ this gives $\frac{dx}{dt} < 0$ for $t < -2$)

EXAMPLE FINITE DOMAIN OF EXISTENCE

$$\begin{cases} \dfrac{dx}{dt} = 1 + x^2 \\ x(0) = 0 \\ x = \tan t \quad \text{only defined for } -\frac{\pi}{2} < t < \frac{\pi}{2} \end{cases}$$

So questions: – Under what conditions on $f(t, x)$ does

$$\begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$ (1) have a continuous solution? & (2) when is it unique?

local existence + uniqueness

– What is the maximum interval of existence of this sol'n?

Further? – How does the solution change if we change $x_0 \to \tilde{x}_0 = x_0 + \varepsilon$?

– Suppose $\dot{x} = f(x, t, \alpha)$ α a parameter $\dot{x} = 2x$ $\alpha = 2$, $\dot{x} = 2.1x$ $\alpha = 2.1$

How does solution change

Continuous dependence on IC's or parameters

Lecture 1-5

Questions we'll ask later

- What is the behavior of solutions as $t \to \infty$?
- How does qualitative behavior of solutions change as we change parameters? $\ddot{x} + b\dot{x} + x = 0$
- When does eqn have time-periodic solutions?