LECTURE 9

This section contains important definitions + some useful but unexciting theorems — we'll state without proof

<u>The Flow</u> We've looked at fundamental solution matrices $Y(t,t_0)$

and matrix exponentials e^{At}

the map $e^{At} = \varphi_t$ has the following properties $\forall x \in \mathbb{R}^n$

1) $\varphi_0 \vec{x} = \vec{x}$

2) $\varphi_s(\varphi_t \vec{x}) = \varphi_{s+t} \vec{x}$ $\quad \forall s,t \in \mathbb{R}$

3) $\varphi_{-t}(\varphi_t \vec{x}) = \varphi_t(\varphi_{-t})\vec{x} = \vec{x}$ $\quad \forall t \in \mathbb{R}$

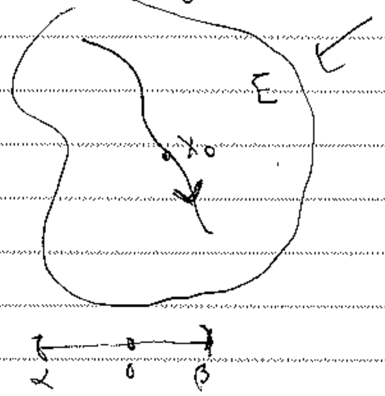
How to generalize to nonlinear eqns $\dot{\vec{x}} = f(\vec{x})$

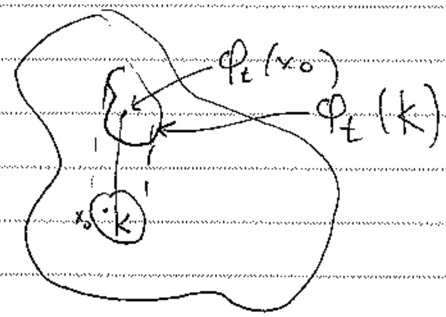
<u>Definition</u> let $(*)$ ~~of~~ $\dot{x} = f(x)$, $f \in C^1(E)$, $x_0 \in E$, $t \in I(x_0)$

define $\varphi_t(x_0) = \varphi(t, x_0)$ = the solution to the initial problem with $x|_{t=0} = x_0$ to $(*)$ at time t

~~this is called the flow of the~~ the set of ~~all~~ $\varphi_t(x_0)$ for all $t \in I(x_0)$ called the flow of $(*)$

Can think of in 2 ways: fix $\vec{x}_0$ vary $t \in I(x_0) = (\alpha, \beta)$ - this gives the trajectory through x_0





We'll skip the proofs of these thms (straightforward corollaries of thms in 2.2–2.4)

<u>Thm 1</u> (only part of Perko's thm 1)

φ is C^1 (follows from continuous dep on init cond)

on $\Omega = \{(t, x_0) \mid x_0 \in E,\ t \in I(x_0)\}$ and Ω is an open set

<u>Theorem 2</u> if $t \in I(x_0)$ and $s \in I(\varphi_t(x_0))$

then $\varphi_s(\varphi_t(x_0)) = \varphi_{s+t}(x_0)$

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Theorem 3 $\varphi_{-t}(\varphi_t(x_0)) = \varphi_t(\varphi_{-t}(x_0)) = x_0$

Definition assume $I(x_0) = \mathbb{R}$ $\forall x_0 \in E$ ~~then~~ and let $S \subset E$
then S is positively (negatively) invariant if
$\varphi_t(S) \subset S$ $\forall t > 0$ $(\forall t < 0)$

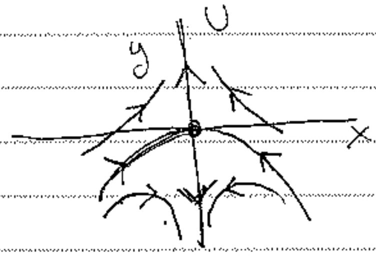
Invariant sets play a very big role in dynamics

example $\dot{x} = -x$
$\dot{y} = y + x^2$

exact solution $\varphi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 e^{-t} \\ y_0 e^{t} + \frac{x_0^2}{3}(e^{t} - e^{-2t}) \end{pmatrix}$

the set $U = \{(x,y) \mid x = 0\}$ is invariant

so is $S = \{(x,y) \mid y = \frac{-x^2}{3}\}$



~~called the~~ U – the unstable manifold
S – the stable manifold

Linearization

(1) $\dot{x} = f(x)$ $\quad f \in C^1$ . . .
useful to start analyzing this near the simplest solutions
fixed points x_0 s.t $f(x_0) = 0$, also a critical pt or a singular pt
~~define~~ of the vector field given
x_0 called a hyperbolic fixed point if $A = Df(x_0)$
~~if~~ all the eigenvalues of A have nonzero real part

the linearization is given by leading order taylor series
$x(t) = x_0 + \tilde{x}(t)$
$\dot{x} = \frac{d}{dt}\tilde{x} = f(x_0) + Df(x_0)\tilde{x} + \cancel{O(|\tilde{x}|^2)}\ D^2 f(x_0)(\tilde{x}, \tilde{x})$

(2) $\frac{d}{dt}\tilde{x} = A\tilde{x}$

Drop ~, then Ax called the linear part of $f(x)$

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Q: Is it valid (i.e. a reasonable approximation) to replace (1) by (2)?
A: We will see that as long as ~~all~~ the fixed pt is hyperbolic, then this is a good approximation

if all eigenvalues λ have real $\lambda < 0$, x_0 called a sink
real $\lambda > 0$ x_0 called a source
if $\exists \lambda_1, \lambda_2$ st $Re\,\lambda_1 > 0 > Re\,\lambda_2$ then x_0 called a saddle pt

example Duffing equation

$\ddot{x} - x + x^3 = 0$

write $\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases}$ FIND & CLASSIFY FIXED PTS

$y = 0$, $x - x^3 = 0$

$(-1,0)$, $(1,0)$, $(0,0)$

~~linearize at $(0,0)$ $\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$ $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$~~ $-\lambda^2 - 1 = 0$ $\lambda = \pm 1$ saddle

$Df = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$ $Df(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\lambda = \pm 1$ saddle

$Df(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ $\lambda = \pm\sqrt{2}\, i$ not hyperbolic

example

$f = \begin{bmatrix} x - xy \\ y - x^2 \end{bmatrix}$ $x(1-y) = 0$ $x = 0$ or $y = 1$
$y = x^2$

$(0,0)$, $(1,1)$, $(-1,1)$

$Df = \begin{bmatrix} 1-y & -x \\ -2x & 1 \end{bmatrix}$ $Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\lambda = 1 \Rightarrow$ source

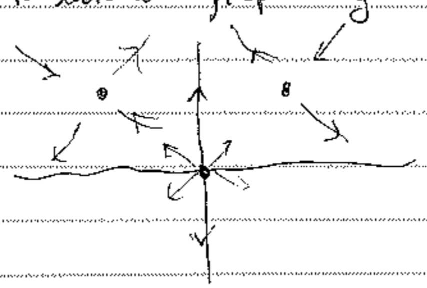
$Df(-1,1) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ $\lambda(\lambda - 1) - 2 = 0$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1)$

$Df(1,1) = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$ $\lambda = 2$, $\lambda = -1$ saddle $\lambda = 2$ or $\lambda = -1$ saddle

Useful to look at graphically: pplane7 for Matlab



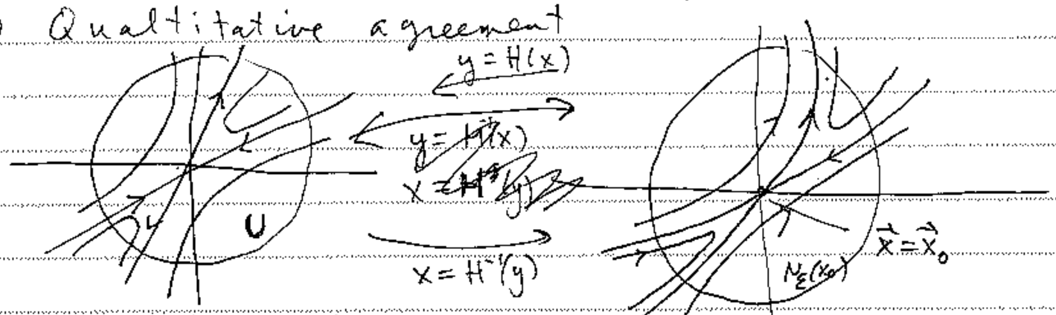
Lecture 9-4

What would ~~it~~ be the best we could hope a linearized flow could tell us?

1) Quantitative agreement $\forall \varepsilon > 0$, $\exists \delta > 0$, T not too small (eg $T = O(\frac{1}{\varepsilon})$)

st $|x_{full}(0) - x_{linear}(0)| < \delta \Rightarrow \max_{0 \le |t| \le T} |x_{full}(t) - x_{linearized}| < \varepsilon$

2) Qualitative agreement



y solves linearized eqn $\dot{y} = Ay$ $\qquad$ x solves $\dot{x} = f(x)$

Topological equivalence: a continuous 1-1 map H from a neighborhood of x_0 to an open set U containing the origin which (i) transforms (1) into (2)

(ii) maps trajectories of (1) in $N_\varepsilon(x_0)$ into trajectories of (2) in U

(iii) preserves orientation of trajectories w.r.t time and time-parameterization

eg consider $\cancel{\dot{x} = -x \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y - x^2 \end{pmatrix}}$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 + x_1^2 \end{pmatrix} = f(x)$$

$$A = Df(0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{a saddle at the origin}$$

let $y = H(x) = \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^2}{3} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\frac{dy}{dt} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 + \frac{2}{3}x_1 \dot{x}_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 + x_1^2 + \frac{2}{3}x_1(-x_1) \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} -x_1 \\ x_2 + x_1^2 - x_1^3 \end{pmatrix}} = \begin{pmatrix} -x_1 \\ x_2 + \frac{x_1^2}{3} \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix}$$

$x_1 = y_1$

$x_2 = y_2 - \frac{x_1^2}{3} = y_2 - \frac{y_1^2}{3}$

Note $x = H^{-1}(y) = \begin{pmatrix} y_1 \\ y_2 - \frac{y_1^2}{3} \end{pmatrix}$

So ~~What~~ Does this tell us

$\cancel{y = H(x)}$

$\frac{dy}{dt} = \cancel{(D_x H)\frac{dx}{dt} = (D_x H(x)) f(x)} = Ay$