Lecture 17 - Linearization about Periodic Orbits

Hypothesis List: $\dot{x} = f(x)$ (1), $f \in C^1(E)$, $E \subset \mathbb{R}^n$ open
has a periodic orbit through $x = 0 \in E$
$\Gamma = \{\gamma(t) = \varphi_t(0) \mid 0 \leq t \leq T\}$ w/ minimal period T

let Σ = poincare section through 0 - an n-1 dim'l hyperplane
$= \{\vec{x} \mid \vec{x} \cdot f(\vec{0}) = 0\}$
$P(x) = \varphi_{\tau(x)}\, x$ the Poincaré map s.t $P(0) = 0$

The derivative $DP(0)$ is an $(n-1) \times (n-1)$ matrix

We define the linearization about the periodic orbit by

$$\dot{y} = Df(\gamma(t))\, y = A(t)\, y \qquad (2)$$

this is a linear equation w/ periodic coefficients - we studied this - Floquet theory - for just this reason

Recall a fundamental solution is a (square) matrix-valued solution Φ to (2) with linearly independent columns

then $y(t) = \Phi(t)\,\Phi(0)^{-1}\, y_0$

define $\Omega = \Phi(T)\,\Phi(0)^{-1}$ then the stability of the periodic orbit $\gamma(t)$ related to

Further, $\exists$ T-periodic matrix $Q(t)$, const matrix B (possibly complex) s.t $\Phi(t) = Q(t)e^{Bt}$

if $z = Q^{-1}(y)$ then
$\dot{z} = Bz \Rightarrow$ periodic-coeff problem can be reduced to const coeff, but, sadly, we can rarely find B and $Q(t)$

Recall further: the eigenvalues of e^{BT} are given by

$$\mu_j = e^{\lambda_j T}, \quad \lambda_j \text{ the eigenvalues of } B$$

λ_j = Floquet exponents of $\gamma(t)$, μ_j = Floquet multipliers of $\gamma(t)$

~~define~~ let $H(x, t) = D_x \varphi_t(x)$ is an $n \times n$ matrix-valued function

$$\frac{\partial H}{\partial t} = Df(\varphi_t(x))\, H(x, t)$$

define $\Phi(t) = H(0, t)$

then
$$\frac{d\Phi}{dt} = Df(\varphi_t(0))\,\Phi(t) = Df(\gamma(t))\,\Phi(t)$$

ie $\Phi(t)$ is a fundamental solution matrix

further $H(0,0) = D_x \varphi_0(x)\big|_{x=0} = D_x x\big|_{x=0} = I$

so $\Phi(0) = I$

$\Phi(t) = H(t, 0) = Q(t)e^{Bt}$

$\Phi(0) = H(0,0) = Q(0) = I$

$\Phi(T) = Q(T)e^{BT} = Q(0)e^{BT} = e^{BT}$

<u>Theorem</u> under the above hypotheses with $P(x)$ defined as above for $x \in N_\delta(0) \cap \Sigma$ ~~and DP(x) has eigenvalues~~ if $\lambda_1, \dots, \lambda_n$ are the Floquet exponents, Then

(A) One of the λ_j is zero, WLOG $\lambda_n = 0$ and for the other λ_1 through λ_n ~~are~~, the eigenvalues of $DP(0)$ are $\mu_j = e^{\lambda_j T}$

(B) <u>Further</u> if the basis of $\mathbb{R}^n$ is chosen such that $\vec{f}(0) = (0, 0, \dots, 0, 1)^T = \hat{e}_n$

then the last column of $H(0,T) = D\varphi_T(0)$ is $(0, 0, \dots, 0, 1)^T$ and $DP(0)$ obtained by deleting the last row and column of $H(0,T)$

Proof Since $\gamma'(t) = f(\gamma(t))$

$\gamma''(t) = Df(\gamma(t))\gamma'(t)$

Thus $\gamma'(t)$ is a solution to 2

and $\gamma'(0) = f(\gamma(0)) = f(\varphi_0(0)) = f(0) \Rightarrow \gamma'(T) = f(0)$

so $\gamma'(t) = \Phi(t) f(0) = H(0,t) f(0)$

$\gamma'(T) = H(0,T) f(0)$

but γ' is periodic so $f(0) = H(0,T) f(0)$

$\Rightarrow H(0,T)$ has eigenvalue $\mu = e^{\lambda} = 1$

$\Rightarrow \lambda = 0$

i.e. A proven

* Interpret!

DO NOT DO IN LECTURE

To prove B Choose a basis of $\mathbb{R}^n$ s.t. $\lambda_n = 0$ and $f(0) = \hat{e}_n$ is the eigenvector corresponding to λ_n

$\Rightarrow$ last column of $H(0,T)$ is $\hat{e}_n$

define $h(x) = \varphi(\tau(x), x)$ for $x \in N_\delta(0)$

then P is the restriction of h to the subspace Σ

then $Dh(x) = \frac{\partial \varphi}{\partial t}(\tau(x), x) D\tau(x) + D\varphi_{\tau(x)}(x)$

$= \frac{\partial \varphi}{\partial t}(\tau(x), x) D\tau(x) + H(x, \tau(x))$

$$Dh(0) = f(0) D\tau(0) + H(0,T)$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \tau}{\partial x_1} & \frac{\partial \tau}{\partial x_2} & \cdots & \frac{\partial \tau}{\partial x_n} \end{pmatrix} + \begin{pmatrix} \tilde{H}(0,T) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \frac{\partial \tau}{\partial x_1} & \cdots & & \frac{\partial \tau}{\partial x_n} \end{pmatrix} + \begin{pmatrix} \tilde{H}(0,T) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \cdots & 1 \end{pmatrix}$$

and since $DP(0)$ is the first $(n-1)$ rows + columns

we get $DP(0) = \tilde{H}(0,T)$

so B is proven $\square$

note, although $\Phi(t)$ usually not obtainable in closed form
we can compute it numerically and find
$(DP(0))_{ij} = \left[\frac{\partial \varphi_i}{\partial x_j}(T,0)\right]$, $i, j = 1, \ldots, n-1$

Stability is determined by $\lambda_1, \ldots, \lambda_{n-1}$

Theorem: Stable manifold theorem suppose $\Gamma = \gamma(t) = \varphi_t(0)$
defined as above, suppose k Floquet exponents
have negative real part where $0 \leq k \leq n-1$
and $n-k-1$ of positive real part

1ST EDITION OF PERKO GETS SOME HYPOTHESES WRONG. BORROW A 3RD EDITION!

then $\exists \delta > 0$ st $W^s_{loc}(\Gamma) = S(\Gamma) = \{x \in N_\delta(\Gamma) \mid d(\varphi_t(x), \Gamma) \xrightarrow[t\to\infty]{} 0$ and $\varphi_t(x) \in N_\delta(\Gamma)\ \forall t \geq 0\}$
is a $(k+1)$ dim'l differentiable manifold that is positively
invariant under the flow
and $W^u_{loc}(\Gamma) = U(\Gamma) = \{x \in N_\delta(\Gamma) \mid d(\varphi_t(x), \Gamma) \xrightarrow[t\to-\infty]{} 0$ and $\varphi_t(x) \in N_\delta(\Gamma)\ \forall t \leq 0\}$
is an $(n-k)$-dim'l differentiable manifold negatively
invariant under the flow

These can be extended by invariance to Global stable & unstable
manifolds $W^u(\Gamma)$ and $W^s(\Gamma)$

~~SIMILARLY IF there exists k Floquet exponents w/ $Re(\lambda_j) < 0$~~

If there exist more than one ~~eigen~~ floquet exponent
with $Re\, \lambda_j = 0$ then there exist stable, unstable
and center subspaces and associated invariant
manifolds

Stronger Theorem: Under hypotheses of stable manifold theorem ~~if $x \in W^s_{loc}(\Gamma)$~~
~~$\exists K > 0$, $\alpha > 0$, $t_0 \in [0,T)$ st~~ ~~that~~ s.t $\forall x \in W^s_{loc}(\Gamma)$, $\exists t \in [0,T)$
~~st $|\varphi_t(x) - \gamma(t-t_0)| < Ke^{-\alpha t/T}$~~

If $\operatorname{Re} \lambda_j \neq 0$ for $j = 1, \dots, n-1$ then the periodic orbit is called hyperbolic

examples $\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \\ \dot{z} = z \end{cases}$ Γ $\gamma = (\cos t, \sin t, 0)$ $= \{(x,y,0) \mid x^2+y^2 = 1\}$

$W^s(\Gamma) = \{(x,y,0) \mid x^2+y^2 > 0\}$, $W^u(\Gamma) = \{(x,y,z) \mid x^2+y^2 = 1\}$

$\begin{cases} \dot{x} = -y + x(1-x^2-y^2-z^2) \\ \dot{y} = x + y(1-x^2-y^2-z^2) \\ \dot{z} = 0 \end{cases}$ then

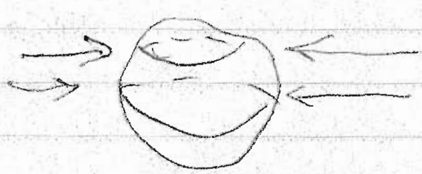
there exist non-isolated periodic orbits

$x^2+y^2 = \cos^2\varphi$, $z = \sin\varphi$ $0 < \varphi < \pi$

$\gamma(x,y,z) = (\cos\varphi \cos t, \cos\varphi \sin t, \sin\varphi)$

$S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$ is a center manifold for $\gamma_0 = \{(\cos t, \sin t, 0)\}$

~~so is the unit cylinder $x^2+y^2=1$~~



Stronger Theorem under hypotheses of stable manifold theorem ~~give~~ for a periodic orbit $\gamma(t)$

$\exists\, \alpha > 0,\ K > 0,$ $\operatorname{Re} \lambda_j < -\alpha$ $j = 1, \dots, k$

s.t. $\forall x \in W^s_{loc}(\Gamma)$, $\exists\, t_0 \in [0,T]$ s.t. $|\varphi_t(x) - \gamma(t-t_0)| < K e^{-\alpha t}$ $\forall t \geq 0$

This says that a trajectory on the stable manifold approaches not just the closed orbit Γ but actually satisfies the dynamics on that orbit as $t \to \infty$

examples $\dot{x} = x - y - x^3 - xy^2 = -y + x(1-x^2-y^2)$

$\dot{y} = x + y - x^2 y - y^3 = x + y(1-x^2-y^2)$

$\dot{z} = \lambda z$

$\Gamma = \gamma(t) = (\cos t, \sin t, 0)$

if λ as above we know the cylinder $\{x^2+y^2=1\} = W^u(\Gamma)$
the plane $\{(x,y,0)\}$ is the stable manifold

$$Df = \begin{bmatrix} 1-3x^2-y^2 & -1-2xy & 0 \\ 1-2xy & 1-x^2-3y^2 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A(t) = Df(\cos t, \sin t, 0) = \begin{bmatrix} -2\cos^2 t & -1-\sin 2t & 0 \\ 1-\sin 2t & -2\sin^2 t & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

can find $\Phi(t)$ explicitly

$$\Phi = \begin{bmatrix} e^{-2t}\cos t & -\sin t & 0 \\ e^{-2t}\sin t & \cos t & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} e^{t\begin{bmatrix} -2 & & \\ & 0 & \\ & & \lambda \end{bmatrix}}$$

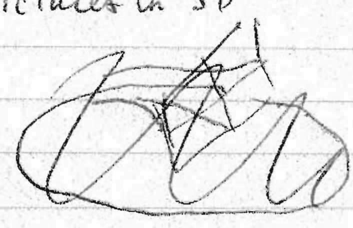
$$= Q(t)e^{Bt}$$

note $Q(0) = I = Q(2\pi)$
so cross out row & column of e^{Bt} with $\operatorname{Re}\lambda = 0$

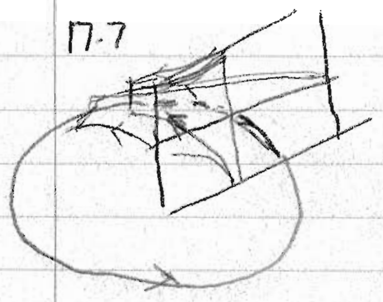
$$P(0) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \quad \text{so } \lambda_1 = -2, \ \lambda_2 = \lambda$$

if $\lambda > 0$, periodic orbit unstable
We could go to cylindrical coords + this would be very easily

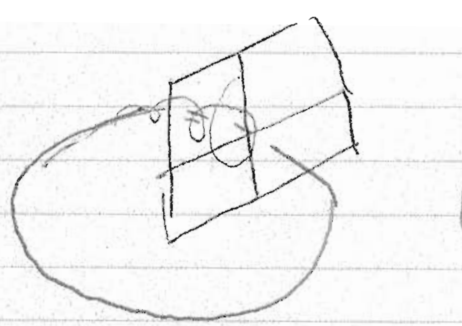
Pictures in 3D



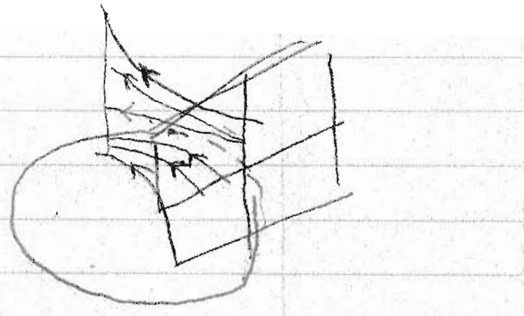
17.7



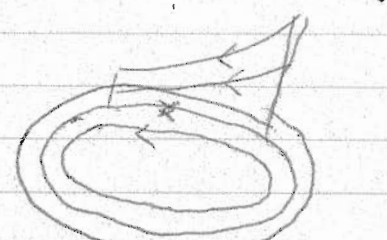
2 real negative eigenvalues



complex conj eigenvalues w/ re $\lambda < 0$



A SADDLE TYPE PERIODIC ORBIT



center manifold (plane)
stable manifold
cylinder

all eigenvalue pure
complex
Stable but not asymptotically

Finally

Theorem under above hypotheses
$\gamma(t)$ not asymptotically stable unless

$$\int_0^T \nabla\cdot f(\gamma(t))\,dt \le 0$$

i.e. $\int_0^T \nabla\cdot f(\gamma(t))dt > 0 \Rightarrow \gamma(t)$ not asymptotically stable

~~cylinder~~ example

$\dot{x} = -y + x(1-x^2-y^2)$
$\dot{y} = x + y(1-x^2-y^2)$
$\dot{z} = \lambda z$

$\nabla\cdot f = (1-3x^2-y^2) + (1-x^2-3y^2) + \lambda$
$= 2-4(x^2+y^2)+\lambda$

$\oint_0^{2\pi} \nabla\cdot f(\gamma(t))\,dt = (2-4\pi) + \lambda \Big|_0^{2\pi} = 2\pi(\lambda-2) > 0$ only if $\lambda > 2$ but unstable if $\lambda > 0$