Lecture 11 - Topological Conjugacy + Linearization

(*) $\dot{x} = f(x) = Ax + F(x)$ where $f(0)=0$, and $F(x) = f(x) - Ax$ so that

$x \in E \subset \mathbb{R}^n$ $A = Df(0)$ $F(0)=0$ and $DF(0)=0$

<u>Additional assumption</u>: $F(x)$ is real analytic on $\mathbb{R}^n$, i.e.

$$F(x) = \sum_{m \geq 2} a_m x^m$$ where $a_m x^m$ is a sum of terms of the form $a_{m_1 m_2 \dots m_n} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$

$m_i \in \mathbb{Z}^{+,0}$ & $\sum_{k=1}^{n} m_k = m$

As we saw last time, we will try to make a change of variables $y = H(x)$ so that $y(t)$ solves the linear equation

(**) $\dot{y} = Ay = AH(x)$

Recall $y = H(x)$

$\dot{y} = DH(x)\dot{x}$

$Ay = DH(x)(Ax + F(x))$

$$\boxed{AH(x) = DH(x)(Ax + F(x))}$$

We accomplish H by a sequence of near identity changes of variables

$y = x + by^2 = g_2(x)$

such that $\dot{y} = Ay + F_2(y)$ where $F_2 = \sum_{m \geq 3} a'_m y^m$

iterate, let $z = y + cz^3 = g_3(y) = g_3(g_2(x))$

$\dot{z} = Az + F_3(z)$ where $F_3 = \sum_{m \geq 4} a''_m z^m$

- At each step, we "push" the nonlinearity off to a higher order, which is smaller near the origin, so that the equation is "more linear"
- If this works, then we iteratively generate a power series for H
- If this power series has a positive radius of convergence, then we have shown there exists an analytic change of variables that linearizes (*) ~~the problem~~

SIMPLE example

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y + x^2 \end{cases}$$

let $u = x + ax^2 + bxy + cy^2$

$v = y + dx^2 + exy + fy^2 \quad \cancel{\Rightarrow y = v}$

Clearly $a = b = c = 0$, otherwise we'd just introduce new nonlinearities

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\dot{v} = \dot{y} + 2dx\dot{x} + ex\dot{y} + e\dot{x}y + 2fy\dot{y}$$

$$\dot{v} = y + x^2 + 2dx(-x) + ex(y+x^2) + e(-x)y + 2fy(y+x^2)$$

$$= y + x^2 - 2dx^2 + \cancel{exy} + ex^3 - \cancel{exy} + 2fy^2 + 2fyx^2$$

$$= y + (1-2d)x^2 + 2fy^2 + O(3)$$

$$\cancel{y} + dx^2 + exy + fy^2 = \cancel{y} + (1-2d)x^2 + 2fy^2 + O(3)$$

x^2: $\quad d = 1 - 2d \quad \Rightarrow d = \frac{1}{3}$

xy: $\quad e = 0$

y^2: $\quad f = 2f \Rightarrow f = 0$

$v = y + \frac{1}{3}x^2$ as we got before

<u>example 2</u>

$\dot{x} = x$

$\dot{y} = 2y + x^2$

again $u = x$

$v = y + dx^2 + exy + fy^2$

$$\cancel{\dot{v} = 2v = 2(y + dx^2 + exy + fy^2) =}$$

$$2v = \dot{v} = \dot{y} + 2dx\dot{x} + ex\dot{y} + e\dot{x}y + 2fy\dot{y}$$

$$2(y + dx^2 + exy + fy^2) = (2y + x^2) + 2dx \cdot x + ex(2y + x^2) + exy + 2fy(2y + x^2)$$

$$= 2y + x^2 + 2dx^2 + 2exy + ex^3 + exy + 4fy^2 + 2fx^2y$$

$$2y + 2dx^2 + 2exy + 2fy^2 = 2y + (2d+1)x^2 + 3exy + 4fy^2 + O(|x|^3)$$

x^2 term $\quad 2dx^2 = (2d+1)x^2$

$\Rightarrow 0 = x^2$ $\quad$ No condition ~~of~~ on d will allow us to eliminate x^2 terms!

This is called a resonance

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General Theory for Resonances

Suppose $A = Df(0)$ has eigenvalues $\lambda_1, \dots, \lambda_n$ then A is resonant if $\exists m_1, \dots, m_n \in \mathbb{Z}^{+,0}$ s.t $\sum_{k=1}^{n} m_k \geq 2$ and $(m, \lambda) = \sum m_k \lambda_k = \lambda_s$ for some $s \in 1, \dots, n$

The quantity $m = \sum |m_k|$ is called the order of the resonance

let g_k be defined as above and let G_k be defined as

$$G_2(x) = g_2(x)$$
$$G_k(x) = g_k(G_{k-1}(x)) \quad \text{by mathematical induction}$$

- If we can construct G_k, it is a polynomial of degree k, so if $\lim_{k \to \infty} G_k(x) \to G(x)$ in some neighborhood of the origin, $G(x)$ is an analytic function defined by a power series.
- To prove convergence of this power series is difficult. We concern ourselves with the formal theory

Assume A is (for now) diagonalizable with distinct eigenvalues and that A is in Jordan form

So, componentwise

$$\dot{x}_j = \lambda_j x_j + \text{higher order terms}$$

as a vector

$$\dot{x} = Ax + v_r(x) + v_{r+1}(x) + \dots \qquad \text{where } v_j \text{ contains terms only of order } j \text{ and } r \geq 2$$

let $M_r = \{ m \in \mathbb{Z}^n \mid m_j \geq 0 \text{ and } |m| = \sum m_j = r \}$

eg if $x \in \mathbb{R}^2$, $r=2$, $M_r = \{ \binom{2}{0}, \binom{1}{1}, \binom{0}{2} \}$

$x \in \mathbb{R}^2$, $M_3 = \{ \binom{3}{0}, \binom{2}{1}, \binom{1}{2}, \binom{0}{3} \}$

Then $v_r = \sum_{m \in M_r} a_m x^m$ where $x^m = \prod_{j=1}^{n} x_j^{m_j}$

So if $x \in \mathbb{R}^2$, $v_2 = a_{(2,0)} x_1^2 + a_{(1,1)} x_1 x_2 + a_{(2,0)} x_2^2$

Try to construct a near-identity change-of-variables

$$y = x + \dots$$
$$\text{s.t} \quad \dot{y} = Ay + V_{r+1}(y) + \dots$$

~~Suppose~~ Now $\dot{x}_i = \lambda_i x_i + \sum_{m \in M_r} a_{m,i} x^m + V_{r+1,i}(x) + \dots$

let $y = x_i + \sum_{m \in M_r} b_{m,i} x^m$ and try to pick $b_{m,i}$ s.t

$$\dot{y}_i = \lambda_i y_i + V_{r+1,i}(y) \quad \text{(B)}$$

Note $\frac{d}{dt} x^m = \sum_k \frac{\dot{x}_k}{x_k} m_k x^m = \left(\sum_{k=1}^n \lambda_k m_k\right) x^m + o(|x|^{r+1})$

$$= (m,\lambda) x^m + o(|x|^{r+1}) \quad \text{(C)}$$

so

$$y_i = x_i + \sum_{m \in M_r} b_{m,i} x^m$$
$$\dot{y}_i = \dot{x}_i + \sum_{m \in M_r} b_{m,i} (m,\lambda) x^m + o(|x|^{r+1}) \qquad \text{from (C)}$$

$$\lambda_i y_i + \underbrace{V_{r+1,i}(y)}_{=O(|x|^{r+1})} = \left(\lambda_i x_i + \sum_{m \in M_r} a_{m,i} x^m + V_{r+1,i}(x)\right) + \sum_{m \in M_r} b_{m,i} (m,\lambda) x^m + o(|x|^{r+1})$$

$$\lambda_i \left(\cancel{x_i} + \sum_{m \in M_r} b_{m,i} x^m\right) = \cancel{\lambda_i x_i} + \sum_{m \in M_r} a_{m,i} x^m + \sum_{m \in M_r} b_{m,i} (m,\lambda) x^m$$

$$0 = \sum_{m \in M_r} \left(a_{m,i} + b_{m,i}\left[(m,\lambda) - \lambda_i\right]\right) x_m + o(|x|^{r+1})$$

if $(m,\lambda) - \lambda_i \neq 0$, set $\boxed{b_{m,i} = \frac{a_{m,i}}{\lambda_i - (m,\lambda)}}$

Repeat inductively on r as long as there are no resonant terms

Theorem If A is nonresonant & diagonal, there exits a formal change of variables that linearizes (*) (Note, this does not say that the radius of convergence is ~~non~~ positive)

Problem: near-resonances: if $|\lambda_i - (m,\lambda)|$ is nonzero but small then $b_{m,i}$ can be large, but of course we need $|b_m| \to 0$ sufficiently rapidly for the series to converge. This is called a small divisor problem. Exact resonance is called a zero divisor.

- Poincaré proved (in his Ph.D. thesis!) proved the power series converges if the eigenvectors are nonresonant and $Re\,\lambda_i > 0\ \forall i$ or $Re\,\lambda_i < 0\ \forall i$
- Siegel proved that if $\exists C > 0$ and $\nu > 0$ s.t. $|\lambda_i - (m,\lambda)| > \frac{C}{|m|^\nu}$ for all $m \in \mathbb{Z}^n_{+,0}$, the so-called Siegel condition
- If A not diagonalizable but only Jordan-izable, the argument goes through
- Note if $\lambda_1 = 0$ then $\lambda_i = \lambda_1 + \lambda_i$ for $i = 2, \ldots, n$, so this is automatically resonant

Poincaré's Linearization Theorem

If the eigenvalues are non-resonant, $Re\,\lambda_i > 0\ \forall i$ or $Re\,\lambda_i < 0\ \forall i$, or λ_i satisfy the Siegel condition, then the formal power series converges in some open set containing the origin.

Back to example 2

$\dot{x} = x$ has $\lambda = (1,2)$ and $\lambda_2 = 2\lambda_1$.

$\dot{y} = 2y + x^2$ Setting $m = (2,0)$, then $(m,\lambda) - \lambda_2 = 0$, resonance.

looking at the linearized problem, then we can use the phase equation to find the trajectories

$$\begin{cases} \dot{x} = x \\ \dot{y} = 2y \end{cases} \qquad \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2y}{x}$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\ln|y| = 2\ln|x| + c$$

$$y = Kx^2$$

Applying the same method to the full equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2y + x^2}{x} = \frac{2y}{x} + x$$

$$y' - \frac{2}{x}y = x \qquad \text{use the integrating factor } \rho = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$$

$$x^{-2}y' - 2x^{-3}y = x^{-2} \cdot x = x^{-1}$$

$$\frac{d}{dx}(x^{-2}y) = x^{-1}$$

$$x^{-2}y = \ln|x| + K$$

$$y = (K + \ln|x|)x^2$$

Topologically indistinguishable from parabolas, but the change of variables is

When $f(x)$ is merely C^1 and not analytic, then there is no way ~~to~~ find an ~~exact~~ ~~the~~ analytic change of variables to the linearization

When the linearization has a resonance, again there is no analytic change of variables to the linearization

In these cases the Hartman-Grobman Theorem applies:

if $x=0$ is a hyperbolic fixed pt, $f \in C^1$, then $\exists$ continuous invertible change of variables $y = h(x)$ (a homeomorphism) which takes orbits of the nonlinear flow φ_t onto those of the linear flow $e^{Df(0)t}$ and the map can be chosen to preserve time-parameterization of orbits.

- This is much weaker than Poincaré (continuous vs analytic)
- ~~No~~ Not even differentiable