A FIXED-POINT THEOREM
(WILL BE USED TO SIMPLIFY EXISTENCE-UNIQUENESS

LECTURE 3-1

Review: What is a vector space?

Closed under addition and multiplication by scalars
ie if S is our space and $x \in S$, $y \in S$, $a, b \in \mathbb{R}$
then $ax + by \in S$ (Note this implies that S has a zero element)

What is a norm? a norm $\|x\|$ is a map from S to $\mathbb{R}^+$ satisfying
(i) $\|ax\| = |a| \cdot \|x\|$
(ii) $\|x+y\| \leq \|x\| + \|y\|$ TRIANGLE INEQUALITY
(iii) $\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$

What is a Banach Space?
A complete normed linear vector space
note the norm defined a distance between x & y
$d(x,y) = \|x-y\|$
& The sequence $\langle u_m \rangle \to u \Leftrightarrow \|u_m - u\| \to 0$

A sequence $\langle u_m \rangle$ is Cauchy if $\forall \varepsilon > 0$ $\exists N > 0$ st $m, n > N$
$\Rightarrow \|u_m - u_n\| < \varepsilon$

A normed linear space $\mathcal{B}$ is a Banach space if it is <u>complete</u> ie if each Cauchy sequence $\langle u_m \rangle \to u$, $u \in S$

Norms play the role of absolute values
let $S \subset B$, S is <u>closed</u> if it contains all its limit points
USING BANACH SPACE IDEAS, WE'LL RE-PROVE EXISTENCE-UNIQUENESS

SUPPOSE $f(t,x)$ COMPONENTWISE CONTINUOUS IN $\mathbb{R}^{n+1}$
in an $(n+1)$ dim'l rectange $R_{a,b} = \{(t,x) \mid |t - t_0| < a, |x - x_0| < b\}$

and $|f(t,x) - f(t,y)| \leq K|x-y|$ for

Lect 2/14 2-2

We talked about $\mathbb{R}^n$ as a vector space

& as a normed vector space – all major norms are equivalent $\Rightarrow$ they have the same Cauchy sequences

Q For any interval I, $C^0(I)$ is a vector space.

~~If we want to~~

Here the norm we use is important

(note, a normed space C^0 is all continuous functions w/ finite norm)

eg the L^∞ norm $\|f\|_\infty = \sup_x |f(x)|$

L^2 norm $\|f\|_2 = \left[\int |f(x)|^2 dx\right]^{1/2}$

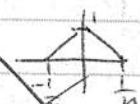
L^p $\|f\|_p = \left[\int |f(x)|^p\right]^{1/p}$, $p \geq 1$

Note different functions are in $C^0(\mathbb{R}, \|\,\|)$ depending on $\|\,\|$

these are not equivalent

see example on next page

for example $f_n(x) = \begin{cases} \frac{x}{n} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$



then $\|f_n(x)\|_\infty = 1$

$\|f_n(x)\| = 2 \cdot \frac{1}{2} \cdot \frac{1}{n} \cdot 1 = \frac{1}{n} \to 0$

So $f_n \to 0$ as an element of L^1

but not as an element of L^∞

Important Theorem $C^0(I)$ is complete under the L^∞-norm

(note convergence in L^∞ is equivalent to uniform convergence, and it is a basic theorem that a uniform limit of continuous functions is continuous)

2-2a

example let $f_n = \begin{cases} 1 & -1 < x \le 0 \\ \frac{1}{1+nx} & 0 < x < 1 \end{cases}$

then $f_n \in C^0[-1,1]$

then the sequence is not Cauchy in L^∞

$$\|f_n - f_m\|_\infty = \max_{0\le x\le 1} \left| \frac{1}{1+nx} - \frac{1}{1+mx} \right| \quad \text{has its maximum at } x^* = \frac{1}{\sqrt{mn}}$$

$$|f_n(x^*) - f_m(x^*)| = \left| \frac{(m-n)\frac{1}{\sqrt{mn}}}{\left(1+\frac{n}{\sqrt{mn}}\right)\left(1+\frac{m}{\sqrt{mn}}\right)} \right| = \left| \frac{\sqrt{\frac{m}{n}} - \sqrt{\frac{n}{m}}}{\left(1+\sqrt{\frac{n}{m}}\right)\left(1+\sqrt{\frac{m}{n}}\right)} \right|$$

if we take $m = 4n$, then $|f_n(x^*) - f_{4n}(x^*)| = \left| \frac{2-\frac{1}{2}}{(1+\frac{1}{2})(1+2)} \right| = \frac{1}{3}$

So for any N, pick $m = 4n$, $n > N$, then $\|f_n - f_m\|_\infty = \frac{1}{3} \not\to 0$

but in L^2

$$\|f_n - f_m\|_2^2 = \int_0^1 \left(\frac{1}{1+mx} - \frac{1}{1+nx} \right)^2 dx$$

$$\le \int_0^1 \left(\frac{1}{(1+mx)^2} + \frac{1}{(1+nx)^2} \right) dx \quad \text{by triangle inequality}$$

$$= \frac{1}{1+m} + \frac{1}{1+n} \le \frac{2}{N} \quad \text{in } m,n > N$$

therefore the sequence is Cauchy in L^2

In fact $f_n \to f = \begin{cases} 1 & -1 \le x \le 0 \\ 0 & 0 < x \le 1 \end{cases}$, $f \notin C^0[-1,1]$

$$\|f_n - f\|_2^2 = \int_0^1 \frac{1}{1+nx}\, dx = \frac{1}{1+n} \xrightarrow[n\to\infty]{} 0$$

this example shows that $C^0[-1,1]$ is not complete w.r.t the L^2-norm lec 2-3

other important norms for C^0

Weighted-norm, given some continuous weight function $W(x) > 0$ on I

$$\|f\|_W = \max |f(x) W(x)|$$

eg $W = e^{|x|}$

then $\|f\|_W \le \infty \Rightarrow e^{|x|}|f(x)| \le C$ for some C

$\Rightarrow |f(x)| \le Ce^{-|x|}$ so these are functions that decay faster than $e^{-|x|}$

whereas $W = e^{-|x|}$ give functions that don't grow too fast $|f(x)| \le Ce^{|x|}$

note if $I = [-a, a]$ then the weighted norms are equivalent to the non-weighted norms

(what are m and M?)

LECTURE ~~2-2~~

let T be a vector valued map defined on a subset S^* $S \subset B$, which maps S^* onto itself ie

$$u \in S^* \Rightarrow Tu \in S^*$$

T called contracting if $\exists \Lambda < 1$ st

$$\|Tu - Tv\| < \Lambda \|u - v\| \text{ for } u, v \in S^*,$$

<u>CONTRACTION MAPPING THM</u>

if B a Banach space and T a contracting map from $S \subset B$, S closed, to itself then $u = Tu$ has a unique solution in S

<u>PROOF</u>: PICARD ITERATION

let $u^{n+1} = Tu^n$

<u>EXISTENCE</u> then $\|u^{n+1} - u^n\| = \|Tu^n - Tu^{n-1}\| \leq \Lambda \|u^n - u^{n-1}\|$

by induction $\|u^{n+1} - u^n\| < \Lambda^n \|u^1 - u^0\|$

Note this implies $\langle u^n \rangle$ cauchy

if $n > m > N$

$$\|u^n - u^m\| \leq \sum_{k=m}^{n-1} \|u^{k-1} - u^k\| \leq \sum_{k=N}^{\infty} \|u^{k-1} - u^k\|$$

$$\leq \frac{1 - \Lambda^N}{1 - \Lambda} \|u^1 - u^0\| \Lambda^N$$

$$\xrightarrow[N \to \infty]{} 0 \qquad \leq \frac{1}{1 - \Lambda} \|u^1 - u^0\| \Lambda^N$$

So $u^n \to u$, $u \in S^*$ since S closed

Thus u a fixed point in S

by passing to limit of $u^{n+1} = Tu^n$

<u>UNIQUENESS</u> SUPPOSE u & v fixed points

$$\|u - v\| = \|Tu - Tv\| \leq \Lambda \|u - v\|$$

$$u \neq v \Rightarrow \Lambda > 1 \quad \text{CONTRADICTION}$$

2-5

(i) pick N large enough s.t. $\rho(f_n, f^*) < \varepsilon \ \forall n > N$

then $\rho(T(f^*), f^*) \le \rho(T(f^*), f_{n+1}) + \rho(f_{n+1}, f^*)$

$\cancel{\le c\rho(f^*, f_n}$

$= \rho(T(f^*), T(f_n)) + \rho(f_{n+1}, f^*)$

$\le c\rho(f^*, f_n)) + \rho(f_{n+1}, f^*)$

$\le c\varepsilon + \varepsilon$

(ii) suppose f^* and g are 2 fixed pts

$\rho(T(f), T(g)) = \rho(f, g)$

but

$\rho(T(f), T(g)) \le c\,\rho(f,g) \cancel{< \rho(f,g)}$

~~contradi~~ ~~contradiction in~~

~~only~~ possible if $\rho(f,g) = 0$

$\square$

example let $X = C^0(S^1)$ where S is the circle of circumference 1

i.e. continuous functions of period 1

let $T(f(x)) = \frac{1}{2}f(2x)$ clearly a contraction w/ $c = 1/2$

What is its fixed point?

For any f_0, the sequence $f_{n+1} = T(f_n) \to f^*$

take $f_0 = \sin 2\pi x$

$f_1 = \frac{1}{2}\sin 4\pi x$

$f_n = \frac{1}{2^n}\sin(2^{n+1}\pi x)$

$\|f_n\| = \frac{1}{2^n} \Rightarrow f_n \to 0$ in sup-norm

example $X = C^0(S^1)$

$T(f(x)) = \cos 2\pi x + \frac{1}{2}f(2x)$ (could also use Weierstrass M-test

again a contraction w/ $c = \frac{1}{2}$

start w/ $f_0 = \sin 2\pi x$

$f_1 = \cos 2\pi x + \frac{1}{2}\sin 4\pi x$

$f_2 = \cos 2\pi x + \frac{1}{2}\cos 4\pi x + \frac{1}{4}\sin 8\pi x$

$f_j = \sum_{n=0}^{j-1} \frac{1}{2^n}\cos(2^{n+1}\pi x) + \underbrace{\frac{1}{2^j}\sin(2^{j+1}\pi x)}_{\to 0 \text{ in sup norm}}$

By contraction mapping this converges $\square$

2-16

Lipschitz functions

let $E \subset \mathbb{R}^n$ be open, $f: E \to \mathbb{R}^n$ is Lipschitz if ~~$\forall x, y \in E$~~, $\exists K$ s.t ~~$f(x,y)$~~ $\forall x, y \in E$

$$|f(x) - f(y)| \leq K|x-y|$$

Lipschitz is stronger than continuity but weaker than differentiability

eg $f(x) = \sqrt{x}$ $E = \mathbb{R}^+$ is continuous, but not Lipschitz or differentiable

$f(x) = \begin{cases} x & x \leq 0 \\ 2x & x > 0 \end{cases}$ is continuous + lipschitz, not differentiable

<u>Lemma 1</u> any lipschitz function is uniformly continuous (review definition).

pf choose any x, then for any ε choose $\delta = \frac{\varepsilon}{K}$

then $|x-y| < \delta = \frac{\varepsilon}{K} \Rightarrow |f(x) - f(y)| < K \cdot \frac{\varepsilon}{K} = \varepsilon$ $\square$

~~Lemma any differentiable function ~~is lipschitz~~ on a compact convex set A, then f is Lipschitz w/ $K = \max_{x \in A} \|Df(x)\|$~~

~~Proof since A convex, the pts on a line between 2 pts $x + y \in A$~~

<u>definition</u> f is locally Lipschitz on E if for any pt $x \in E$ $\exists$ neighborhood $N \ni x$ open set $N \subset E$ f is Lipschitz on N

Now we'll show every differentiable function is locally lipschitz

<u>lemma 2</u> let $f \in C'(A)$ where A compact + convex then f lipschitz w/ const $K = \max_{x \in A} \|Df(x)\|$

<u>proof</u> since A convex, if $x, y \in S$ and ξ on the line connecting them $\xi = sy + (1-s)x \in A$, $0 \leq s \leq 1$

2-~~#~~7

So $f(y)-f(x) = \int_0^1 \frac{d}{ds} f(\xi(s))\,ds \qquad \left(\frac{d\xi}{ds}\right) = y-x$

$= \int_0^1 Df(\xi(s))(y-x)\,ds$

$\|f(y)-f(x)\| \le |y-x| \int_0^1 \|Df(\xi(s))\|\,ds$

$\le K|y-x|$

<u>Corollary 3</u> if f C^1 on E open, it is locally Lipschitz

<u>Pf</u> for any $x \in E$, $\exists\, \varepsilon > 0$ st $B_\varepsilon(x) \subset E$

$B_\varepsilon(x)$ is compact + convex

$\Rightarrow$ Locally Lipschitz

~~<u>corollary 4</u> let $E \subset \mathbb{R}^m$ and $A \subseteq E$ w/ A compact~~

~~If f locally Lipschitz on E then f Lipschitz on A~~

~~pf if A compact then A can be covered by~~

<u>Diagram</u>

<u>example</u> $f(x) = |x|$ ~~[illegible]~~ is Lipschitz but not C^1

$K = 1$

$g(x) = \sqrt{x}$ in locally Lipschitz but not lipschitz

eg if $y = 4\varepsilon > 0$ $\qquad |f(x)-f(y)| = \sqrt{4\varepsilon} - \sqrt{\varepsilon} = \varepsilon$

$x = \varepsilon$ $\qquad |x-y| = 3\sqrt{\varepsilon}$

so $|f(x)-f(y)| = \frac{1}{3\sqrt{\varepsilon}}|x-y|$

so need $K \le \frac{1}{3\sqrt{\varepsilon}}$ not bounded for small ε

$h(x) = x^2$ $\qquad |x^2-y^2| = |x+y||x-y|$

$x \in \mathbb{R}$ not Lipschitz $\qquad$ if $\le (|x|+|y|)|x-y|$, yes Locally Lipschitz