Lecture 8 The Poincaré-Bendixson Theorem

The topology of $\mathbb{R}^2$ ~~st~~ severely restricts the dynamics of

(1) $\dot{x} = f(x)$, $f(x) \in C^1(E)$, $E \subset \mathbb{R}^2$ open

In particular, in $\mathbb{R}^2$ the Jordan curve theorem says if J a simple closed curve in $\mathbb{R}^2$, then J splits $\mathbb{R}^2$ an "inside" & an "outside", ie $\mathbb{R}^2 \setminus J = \text{int } J \cup \text{ext } J$ where $\text{int } J \cap \text{ext } J = \emptyset$

Notation: $x_0 \in E$

Γ_{x_0} ~~$\Gamma(x_0)$~~ $= \{\varphi_t(x_0) \mid -\infty \le t \le \infty\}$

$\Gamma^+_{x_0} = \{\varphi_t(x_0) \mid t \ge 0\}$

$\Gamma^-_{x_0} = \{\varphi_t(x_0) \mid t \le 0\}$

$\omega(x_0)$, $\omega(\Gamma)$ the omega-limit sets of the orbits Γ_{x_0} or Γ

~~$\alpha(x_0)$, $\alpha(\Gamma)$ " alpha-limit sets " " " " "~~

~~we will use $\sigma = \pm$~~

definition: A line segment l is a *transversal* if

(i) It has no critical pts (i.e. $f(x_0) \neq 0 \quad \forall x_0 \in l$)

(ii) $f(x)$ is never tangent to l

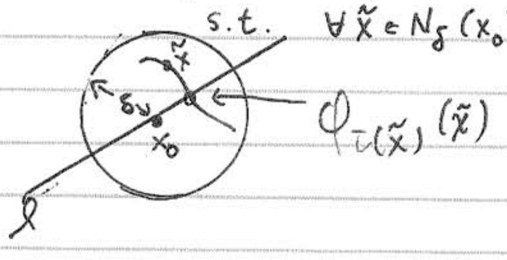
(note that l could be a segment of a curve rather than a line

def x_0 is a *regular pt* of $f(x)$ if it is not a critical point

Lemma 1 (a) Every regular pt x_0 is on the interior of some transversal l

(b) a trajectory through $x_0 \in l$ crosses l

(c) if $x_0 \in l$ is an interior pt of l then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall \tilde{x} \in N_\delta(x_0)$, $\exists t = \tau(\tilde{x})$ st $|t| < \varepsilon$ and $\varphi_t \tilde{x} \in l$



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proof (a) let $\vec{v} = f^{\perp}(x_0) = (-f_2(x_0), f_1(x_0))$

and $l_\varepsilon = \{x = x_0 + \delta v \mid |\delta| \le \varepsilon\}$

Define $F(\delta) = \begin{cases} \dfrac{(x-x_0)\times f(x)}{\delta} & \delta \ne 0 \\ |f(x_0)|^2 & \delta = 0 \end{cases}$

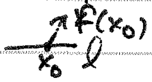
note as $\delta \to 0$ $\dfrac{(x-x_0)\times f(x)}{\delta} \approx \dfrac{\delta v \times f(x_0)}{\delta} = f(x_0)\cdot f(x_0) > 0$

then $\exists\, \varepsilon$ st $F(\delta) > 0 \quad \forall |\delta| < \varepsilon$

but $F(\delta) = 0 \Leftrightarrow l_\varepsilon$ not a transversal at $x = x_0 + \delta v$

so l_ε a transversal

(b) for t suff small $x(t) \approx x_0 + f(x_0)t$



(c) exactly like our proof that the poincaré map exists

Define $L(x,t) = (\varphi_t(x) - x_0)\cdot f(x_0)$

$L(x_0, 0) = (x_0 - x_0)\cdot f(x_0) = 0$

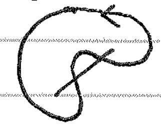
$\left.\dfrac{\partial L}{\partial t}\right|_{(x_0,0)} = f(x_0)\cdot f(x_0) > 0$

$\Rightarrow \exists\, \tau(x)$ st $L(x, \tau(x)) = 0 \ \forall x$ in a nbhd of x_0

Lemma 2 (i) IF a finite closed (i.e. includes end pts) arc of a trajectory Γ crosses a transversal l, it does so in a finite # of pts

MONOTONE CROSSING LEMMA → (ii) if t_n an increasing sequence of times and $x_n = \varphi_{t_n}(x_0)$ are intersections of Γ and l, then the x_n cross l monotonically

(iii) If Γ periodic, it can cross l in only 1 point



the line segment l cannot be a transversal proof of (i) let $\Gamma = \{x \in E \mid x = x(t) \quad -\infty \le t \le \infty\}$

$A = \{x \in \Gamma,\ x = x(t) \quad a \le t \le b\}$

and suppose A meets l in ∞-many pts $x_n = x(t_n)$

since $[a,b]$ closed, bounded $\{t_n\}$ must have a convergent subsequence $\{t_{n_k}\} \to t^*$ ~~thus~~ thus $\{x_{t_{n_k}}\} \to x^* \in l$

but $\dfrac{x(t_{n_k}) - x(t^*)}{t_{n_k} - t^*} \to \dot{x}(t^*) = f(x(t^*))$

but $x(t^*_{n_k})$ and $x(t^*) \in l \Rightarrow v_n = \dfrac{x_{n_k} - x^*}{t_{n_k} - t^*}$

tangent to l contradicting that l a transversal

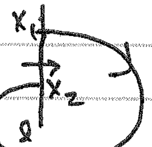
proof of ii

let $t_1 < t_2$ and $x_1 = x(t_1)$, $x_2 = x(t_2)$ be 2 consecutive crossings of l by Γ

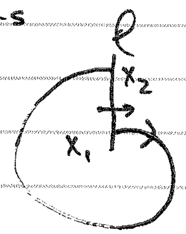
let $A_{12} = \{x(t) \mid t_1 \le t \le t_2\}$, $J = A_{12} \cup \overline{x_1 x_2}$ is a Jordan curve

Separating E into 2 regions

Case a



Case b



Vector field through l points into interior

Vector field points outward (vector field pts in along l and A_{12} is a trajectory)

Case a for $t > t_2$, $\Gamma^+_{x_2}$ must lie inside $J \Rightarrow x_2$ between x_3 + x_1 ie pts ordered

Now use mathematical induction

Case b SIMILAR

<u>proof of iii</u> if Γ a periodic orbit that crosses l in 2 points x_1 & x_2, then part ii shows ~~a 3rd crossing~~ subsequent crossings are on wrong side of x_2 and can never return to x_1

$\square$

<u>Lemma 3</u> if Γ and $\omega(\Gamma)$ have a point in common then Γ is a fixed pt or a periodic orbit

<u>proof</u> let $x_1 = x(t_1) \in \Gamma \cap \omega(\Gamma)$

then x_1 a critical point <u>or</u>

- if x_1 a regular pt, then it is ~~the~~ an interior pt to a transversal l (Lemma 1a)
- since $x_1 \in \omega(\Gamma)$ def'n of ω-limit set $\Rightarrow$ any circle C centered at x_1 must contain in its interior of pt $x^* = x(t^*)$, $t^* > t_1 + 2$
- if C the circle $N_\delta(x_1)$ defined by $\varepsilon = 1$ in lemma 1c then $\exists\, t_2$ s.t $|t_2 - t^*| < 1$ and ~~$x_2 \in l$~~ $x_2 = x(t_2) \in l$ (note that $t_2 - t_1 > 1$)

if $x_1 = x_2$ then Γ a periodic orbit. Assume not then the arc $A_{12} \subset \Gamma$ intersects l in a finite # of pts (lemma 2a)

and successive iterations form a monotone sequence (lemma 2b) tending away from x_1

$\Rightarrow x_1 \notin \omega(\Gamma)$ contradiction

$\square$

Remark - this same argument can be used to show $\omega(\Gamma)$ intersects l in at most 1 point

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<u>Lemma 4</u> If $\omega(\Gamma)$ contains no periodic orbits and there exist a periodic orbit $\Gamma_0 \subset \omega(\Gamma)$ then $\Gamma_0 = \omega(\Gamma)$

I don't understand Perko's proof!

<u>Proof</u> from Teschl notes

We know $\omega(\Gamma)$ connected (lecture 13)

Assume $\omega(\Gamma)\setminus\Gamma_0$ nonempty

by connectedness, $\exists y_0 \in \Gamma_0$, $z \in \omega(\Gamma)\setminus\Gamma_0$ arbitrarily close together

Pick a transversal l through y_0 (lemma 1a)

then $\exists t$ s.t $y_1 = \phi_t z \in l$, and $y_1 \notin \Gamma_0$ since $\phi_t z \in \omega(\Gamma)\setminus\Gamma_0$

but since $z \in \omega(\Gamma)$, $\phi_t z \in \omega(\Gamma)$

so ~~$\phi_t z \in l \cap \omega(\Gamma)$~~ so $y_1 \in l \cap \omega(\Gamma)$

but $y_0 \in l \cap \Gamma_0 \subset l \cap \omega(\Gamma)$

and our Remark says $l \cap \omega(\Gamma)$ is at most one point

$\Rightarrow y_1 = y_0$ contradiction

$\Rightarrow \omega(\Gamma)\setminus\Gamma_0$ empty

$\square$

The Poincaré-Bendixson Theorem

~~4~~ Suppose (1) has a trajectory $\Gamma(x_0)$ s.t $\Gamma^+(x_0) \subset F$, a compact set, then if $\omega(\Gamma)$ contains no fixed points, $\omega(\Gamma)$ is a periodic orbit

<u>Proof</u>

- if Γ a periodic orbit, then $\Gamma \subset \omega(\Gamma)$ and lemma 4 $\Rightarrow$ $\Gamma = \omega(\Gamma)$
- if not, then $\omega(\Gamma)$ nonempty and consists only of regular pts then $\exists$ limit orbit $\Gamma_0 \subset \omega(\Gamma)$
- since ~~Γ^+ contained in~~ $\Gamma^+ \subset F$ then $\Gamma_0 \subset F$
- Thus Γ_0 has an ω-limit point $y_0 \in \omega(\Gamma)$ since $\omega(\Gamma)$ closed
- let l be a transversal through y_0 (lemma 1a)

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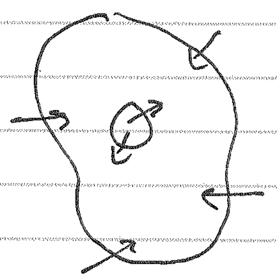
- then since Γ_0 and y_0 both in $\omega(\Gamma)$, $l \cap \overset{\omega(\Gamma)}{\cancel{\Sigma}} = \{y_0\}$ (by Remark)

- Since y_0 a limit point of Γ_0, l must intersect Γ_0 in some pt
- by lemma 2, this $l \cap \Gamma_0 = y_0$

why? thus Γ_0 and $\omega(\Gamma_0)$ have the pt y_0 in common
- lemma 3 $\Rightarrow$ Γ_0 periodic
- lemma 4 $\Rightarrow$ $\Gamma_0 = \omega(\Gamma)$

$\square$

HOW IS THIS USED? CONSTRUCT A TRAPPING REGION F W/ NO FIXED PTS



PICK A POINT $x_0 \in \partial F$, then $\Gamma^+(x_0) \subset F$ and ~~$\Gamma^+(x_0)$ contains~~ $\omega(\Gamma)$ can contain no fixed pts since any limit pt of Γ is in the closed set F which has no fixed pts

We've looked at the example

$\dot{r} = r(1-r^2)$

$\dot{\theta} = 1$

then take $0 < r_1 < 1$

$r_2 > 1$

then the annulus $r_1 \le r \le r_2$ is a trapping region

harder

$\dot{r} = r(1 - r^2 + \mu\cos\theta)$ $\quad \mu > 0$

$\dot{\theta} = 1$

~~along~~ Find an annulus $r_1 \le r \le r_2$

Want $1 - r_1^2 + \mu\cos\theta > 0$ $\qquad$ $1 - r_2^2 + \mu\cos\theta < 0$

$1 - r_1^2 - \mu > 0$ $\qquad$ $1 - r_2^2 + \mu < 0$

$r_1^2 < 1 - \mu$ $\qquad$ $r^2 > 1 + \mu$

can do if $\mu < 1$