LECTURE 20 - Eliminating the possibility of periodic orbits

- We've spent the last 2 lectures on Poincaré-Bendixson + Lienard systems in order ~~to~~ to show the existence of periodic orbits
- Opposite case: show no periodic orbits in a certain region of phase space

Why? Lasalle's invariance principal said if for all $x \in E$ $\frac{d}{dt}V(x(t)) \leq 0$ and E contains no other invariant sets then $V(x(t))$ approaches a minimum of $V(x)$, so in 2D need to ~~show~~ show no periodic orbits

Several different methods: A survey

1) Gradient systems suppose $\frac{d\vec{x}}{dt} = \vec{f}(x)$

and $f_i = -\frac{\partial V}{\partial x_i}$ for some potential V

Then $\frac{d}{dt} V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial t} = -\sum_i \left(\frac{\partial V}{\partial x_i}\right)^2_{(x(t))} < 0$

unless x is a local minimum of V

No periodic orbits because V can't decrease monotonically & strictly while trajectory periodic

ONLY PROBLEM: Gradient systems are pretty rare in practice

EXAMPLE
$\dot{x} = -x$
$\dot{y} = y$

$V = \frac{1}{2}x^2 - \frac{1}{2}y^2$

so as t since $\dot{V}$ decreasing, but no minima, $|y| \to \infty$
$|x| \to 0$

LEC 20-2

$\dot{x} = f(x), \quad f \in C^1(E), \quad E \subset \mathbb{R}^n$, open ⊛

<u>Theorem</u> Bendixson's Criterion (works only in $\mathbb{R}^2$)

* Consider ⊛ where E is a simply connected subset of $\mathbb{R}^2$. If $\nabla \cdot f$ not identically zero and does not change sign in E, then ⊛ has no periodic orbits entirely inside E

<u>Proof</u> Suppose $\Gamma = \{x(t) \mid 0 \le t \le T\}$ is a periodic orbit lying entirely inside E, and $\underset{\text{has one sign}}{\nabla \cdot f}$ $\forall x \in S = \text{int}\,\Gamma$

~~then $\iint_S \nabla \cdot f \, dA > 0$~~ let $\vec{f} = \begin{pmatrix} f \\ g \end{pmatrix}$

but $$\iint_S \nabla \cdot f \, dA = \oint_\Gamma f\,dy - g\,dx$$
$$= \int_0^T \left(f \frac{dy}{dt} - g \frac{dx}{dt}\right) dt$$
$$= \int_0^T (fg - gf)\,dt = 0$$

~~contradiction~~.

but by assumption $\iint \nabla \cdot f \, dA > 0$ or $\iint \nabla \cdot f \, dA < 0$
contradiction

□

Slight extension:

<u>Theorem</u> Dulac's Criteria

if $\exists$ $\underset{}{\overset{\text{scalar}}{B}} \in C^1(E)$ s.t. $\nabla \cdot Bf$ not identically zero and $\nabla \cdot Bf$ has only one sign then there are no periodic orbits entirely inside E

(note Bendixson's Criteria is to set $B = 1$

<u>EXAMPLE</u> $\dot{x} = x(A_1 - a_1 x + b_1 y)$ a Lotka-Volterra model

$\dot{y} = y(A_2 - a_2 y + b_2 x)$ $\quad A_1, A_2, a_1, a_2 > 0$, $\begin{matrix} x > 0 \\ y > 0 \end{matrix}$

Show there are no periodic orbits w/ $x > 0$ & $y > 0$

try $B = \frac{1}{xy}$ $\quad Bf_1 = \frac{1}{y}(A_1 - a_1 x + b_1 y) \Rightarrow \partial_x Bf_1 = \frac{-a_1}{y}$

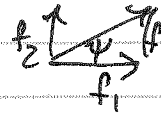
$Bf_2 = \frac{1}{x}(A_2 - a_2 y + b_2 x) \Rightarrow \partial_x Bf_2 = -\frac{a_2}{x}$

$\nabla \cdot B\vec{f} = -\left(\frac{a_1}{y} + \frac{a_2}{x}\right) < 0$ in quadrant I

Lec 20-3 Index Theory

Poincaré Index $\quad \frac{dx}{dt} = f(x), \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C^1(\mathbb{R}^2) \quad (*)$

Then at each point x, the vector field $\vec{f}$ defines a direction
$$\psi = \tan^{-1} \frac{f_2}{f_1}$$



let C be any simple closed curve, not necessarily a trajectory

define the Poincaré index of C w.r.t f as
$$I_C(f) = \frac{1}{2\pi} \int_C d\psi = \frac{1}{2\pi} \int_C d\left(\tan^{-1} \frac{f_2}{f_1}\right)$$

this is the total change in the angle ψ seen as the ~~trajectory~~ curve C is traced through 1 circuit

$$d\left(\tan^{-1} \frac{f_2}{f_1}\right) = \frac{1}{1 + f_2^2/f_1^2} \, \frac{f_1\, df_2 - f_2\, df_1}{f_1^2} = \frac{f_1\, df_2 - f_2\, df_1}{f_1^2 + f_2^2}$$ note undefined at fixed points

EXAMPLES: let $f = (x_1, x_2)$, $g = (-x_1, -x_2)$, $h = (x_2, -x_1)$, $k_1 = (x_1, -x_2)$

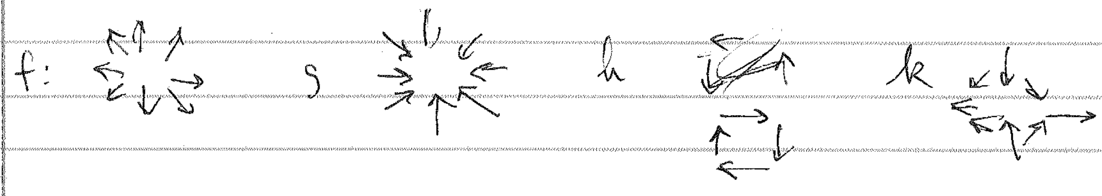
C = unit circle $= \{(\cos s, \sin s) \mid 0 \le s \le 2\pi\}$

$$I_C(f) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos\theta\, d(\sin\theta) - \sin\theta\, d(\cos\theta)}{\cos^2\theta + \sin^2\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

$$I_C(g) = \frac{1}{2\pi} \int_0^{2\pi} \frac{-\cos\theta\, d(-\sin\theta) - (-\sin\theta)\, d(-\cos\theta)}{\cos^2\theta + \sin^2\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

$$I_C(h) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin\theta\, d(-\cos\theta) - (-\cos\theta)\, d(\sin\theta)}{\sin^2\theta + \cos^2\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

$$I_C(k) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos\theta\, d(-\sin\theta) \cancel{+} - \sin\theta\, d(\cos\theta)}{\sin^2\theta + \cos^2\theta} = \frac{1}{2\pi}\int_0^{2\pi} -d\theta = -1$$



Lec 20-4

Theorem if C' is a closed periodic orbit obtained from C by a smooth transformation that doesn't cross any fixed pts of f then $I_C(f) = I_{C'}(f)$

proof let $\mathcal{C}(\tau)$ be a continuous curve-valued function s.t. $\mathcal{C}(0) = C$, $\mathcal{C}(1) = C'$
then $I_{\mathcal{C}(\tau)}(f)$ continuous, but takes only integer values
$\Rightarrow I_{\mathcal{C}(\tau)}$ constant

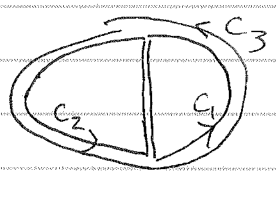
definition if x^* a critical pt of ⊛
then $I_{x^*}(f)$ is the index of any closed curve f that encloses x^* and no other fixed pts (defined because of 1st theorem

Thm if $\Gamma = \gamma(t)$ a closed orbit, then $I_{\Gamma}(f) = 1$ simple
pf the vectorfield is tangent to Γ and Γ a closed curve

Thm if x^* a center, sink, source $I_{x^*}(f) = 1$

Thm if x^* a saddle, $I_{x^*}(f) = -1$

Thm the index is additive



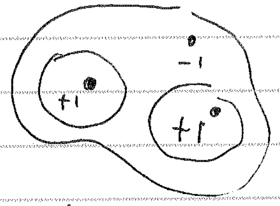
$$I_{C_3} = I_{C_2} + I_{C_1}$$

pf standard techniques about integrals
Thm The index of a curve is the sum of the indices of the fixed pts in its interior
Thm The index of a curve with no stationary pts on its interior is 0 (Homotopically equivalent to a pt)

Theorem ~~and (-f)~~ $I_C(-f) = I_C(f)$

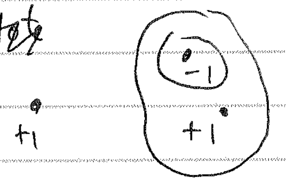
Major Theorem If Γ a periodic orbit, it contains at least one fixed point. If ~~it contains n saddle pts~~
If the fixed pts are hyperbolic, there are $2n+1$ of them, n of them saddles

LEC 20-5



these curves are potential periodic orbits

~~Note~~



these are not potential periodic orbits

Note this is exactly then same as in complex variables
if γ a closed orbit, oriented positively, then

$$\oint_\gamma \frac{f'}{f}\,dz = \Delta_\gamma \log z = 2\pi i\, \Delta\psi = \#\text{ zeros inside } \gamma - \#\text{ poles inside } \gamma$$

interesting exercise - show ~~that~~ ~~p~~ the extent of this correspondence

remark - the index of a non-hyperbolic fixed point may take a value other than ± 1

Lasalle's Invariance principle revisited

$\dot{x} = f(x)$, $f(x) \in C^1(E)$ $\quad E \subset \mathbb{R}^n$

$\exists V$ st $V(0)=0$, $V(x)>0\ \forall x \neq 0$ and $\dot{V} \leq 0$
then if E contains no other invariant sets
$\lim_{t\to\infty} x(t) = 0$

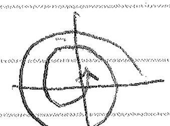
So if we want a method that works in $\mathbb{R}^n$, $n>2$
Just need to look at $\{x \mid \frac{d}{dt}V(x(t))\big|_{t=0} = 0\}$
is that set invariant?

LEC 20-6

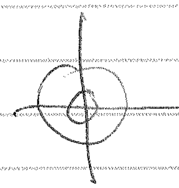
Nonhyperbolic Fixed Pts / Structural stability

$$\dot{x} = -y + \mu x$$
$$\dot{y} = x + \mu y$$

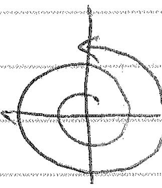
$\mu < 0$



$\mu = 0$



$\mu > 0$

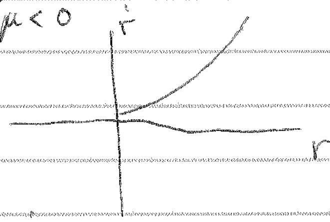


the vector field $\dot{x} = f(x)$ is structurally stable if $\forall g(x) \in C^1(E)$, $\exists\, \varepsilon > 0$ st $\forall |\delta| < \varepsilon$

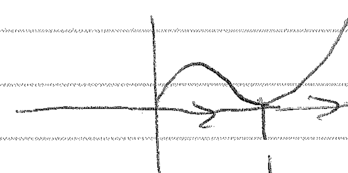
$\dot{x} = f(x) + \delta g(x)$ has the same structure as $\dot{x} = f(x)$

$$\dot{x} = -y + x\left[(x^2+y^2-1)^2 - \mu\right]$$
$$\dot{y} = x + y\left[(x^2+y^2-1)^2 - \mu\right]$$
$$\downarrow$$
$$\dot{r} = r\left[(r^2-1)^2 - \mu\right]$$
$$\dot{\theta} = 1$$

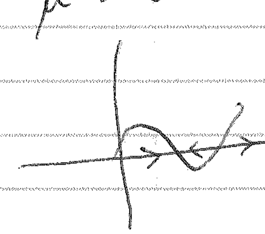
$\mu < 0$



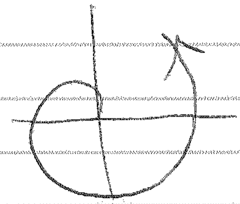
$\mu = 0$

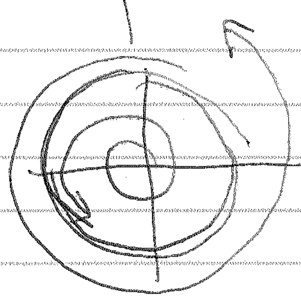


$\mu > 0$



$r > 0$





semistable