Lecture 13-1

Limit Sets and attractors - we'd like to generalize fixed pts to larger sets

$\dot{x} = f(x)$, $f \in C^1(E)$ for some open $E \subset \mathbb{R}^n$

in section 3.1 he shows one can always "reparameterize time" so that for each initial condition $x_0 \in E$, the maximal interval of existence for $x_0 = (-\infty, \infty)$ so $\phi_t x_0 \in E$ $\forall t \in \mathbb{R}$

& such an eqn he calls a "dynamical system"

define the trajectory $\Gamma_{x_0} = \{ x \in E \mid x = \phi_t(x_0), t \in \mathbb{R} \}$

positive half-trajectory $\Gamma^+_{x_0} = \{ x \in E \mid x = \phi_t(x_0), t \geq 0 \}$

negative half-trajectory $\Gamma^-_{x_0} = \{ x \in E \mid x \in \phi_t(x_0), t \leq 0 \}$

define the point $p \in E$ is an ω-limit pt of $\Gamma(x_0)$ if $\exists$ sequence $t_n \to \infty$ s.t. $\lim_{n \to \infty} \phi(t_n, x) = p$

the point $q \in E$ is an α-limit point of $\Gamma(x_0)$ if $\exists$ sequence $t_n \to -\infty$ s.t. $\lim_{n \to \infty} (t_n, x) = q$

$\omega(\Gamma)$ = the set of all ω-limit points is the ω-limit set

$\alpha(\Gamma) = \{ x \mid x$ an α-limit pt of $\Gamma \}$ is the α-limit set, $\alpha(\Gamma) \cup \omega(\Gamma)$ the limit set of the trajectory

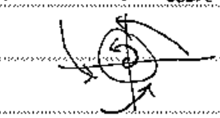
eg $\dot{r} = (1 - r^2) r$ $\qquad x = r\cos\theta$ $\qquad y = r\sin\theta$

$\dot{\theta} = 1$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = (1-r)r\cos\theta - r\sin\theta = -y + (1 - \sqrt{x^2+y^2})\,x$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = x + (1 - \sqrt{x^2+y^2})\,y$$

if $\binom{x^*}{y^*}$ on the unit circle then $\binom{x^*}{y^*}$ is an ω-limit point, thus $\omega(\Gamma)$ = unit circle



& if $|\vec{x}_0| < 1$ $\lim_{t \to -\infty} \phi_t(x_0) = \binom{0}{0}$, so the origin is an α-limit point,

in fact, it is the only α-limit pt so

<u>Theorem</u> the α- and ω-limit sets of Γ are ~~compact~~ closed subsets of E. If Γ is contained in a compact subset of $\mathbb{R}^n$, then $\alpha(\Gamma)$ and $\omega(\Gamma)$ are non-empty, connected, compact subsets of E
(note, this is all limit sets of a particular trajectory

~~proof~~: by definition, any pt of $\omega(\Gamma) \in E \Rightarrow \omega(\Gamma) \subset E$

to show $\omega(\Gamma)$ closed, let p_n be a sequence of points in $\omega(\Gamma)$ s.t $p_n \to p$, need to show $p \in \omega(\Gamma)$

let $p_n \in \omega(\Gamma)$, $x_0 \in \Gamma$ $\Rightarrow \exists$ sequence $t_k^{(n)} \xrightarrow[k\to\infty]{} \infty$ and $\forall$ $\varphi_{t_k^{(n)}}(x_0) \longrightarrow p_n$

assume ~~φ~~ $t_k^{(n+1)} > t_k^{(n)}$ since we can always choose a subsequence with this property

thus we can find a sequence of integers $K(n) > K(n-1)$ s.t. $\forall k \geq K(n)$, $|\varphi(t_k^{(n)}, x_0) - p_n| < \frac{1}{n}$

$$\text{let } t_n = t^{(n)}_{K(n)}$$

$$\text{then } |\varphi(t_n, x_0) - p| \leq |\varphi(t_n, x_0) - p_n| + |p_n - p|$$

$$\leq \frac{1}{n} + |p_n - p| \xrightarrow[n\to\infty]{} 0$$

so $\omega(\Gamma)$ is closed

~~Now assume Γ is~~
now assume $\Gamma \subset K$ a compact subset of E

if $\varphi_{t_k}(x_0) \xrightarrow[k\to\infty]{} p$ then $p \in K$ since $\varphi_t(x_0) \in \Gamma \subset K$ compact
so $\omega(\Gamma) \subset K$
$\omega(\Gamma)$ closed + bounded hence compact

$\omega(\Gamma)$ nonempty since the sequence $\varphi_n(x_0)$ must have a convergent subsequence $\varphi_n(x_0) \to$ a point in $\omega(\Gamma) \subset K$

Finally, suppose $\omega(\Gamma)$ not connected then $\exists$ closed set A, B s.t
$\omega(\Gamma) = A \cup B$,

Since A, B closed, define $\delta = D(A,B) = \min_{x \in A, y \in B} |x-y| > 0$

Since A, B are in ω-limit sets of Γ, exist arb. large t s.t
$d(\varphi_t(x_0), A) < \frac{\delta}{2}$
and t arb large t s.t $d(\varphi_t(x_0), B) < \frac{\delta}{2} \Rightarrow d(\varphi_t(x_0), A) > \frac{\delta}{2}$

since $d(\varphi_t(x_0), A)$ a continuous function of t, $\exists$ ~~arb large~~ sequence $t_n \to \infty$
s.t $d(\varphi_{t_n}(x_0), A) = \delta/2$
since $\varphi_{t_n}(x_0) \subset K$, compact, it has a convergent subsequence
$\varphi_{t_n}(x_0) \to p$, then $p \in \omega(\Gamma)$ but $d(p,A) = \delta/2$, $d(p,B) \geq \frac{\delta}{2}$
$p \notin A$ & $p \notin B$ contradiction

thus $\omega(\Gamma)$ connected

Analogous proofs hold for $\alpha(\Gamma)$

$\square$

Theorem 2 if ~~for a~~ $p \in \omega(\Gamma)$, for a given trajectory Γ then all points of the trajectory $\Gamma_p = \varphi_t(p)$ are in $\omega(\Gamma)$, SIMILARLY if $p \in \alpha(\Gamma)$ then $\Gamma_p \subset \alpha(\Gamma)$

proof: let $p \in \omega(\Gamma)$ where $\Gamma = \varphi_t(x_0)$ for some $x_0 \in E$
let q be a point on $\Gamma_p = \varphi_t(p)$, i.e. $\exists \tilde{t}$ s.t $q = \varphi_{\tilde{t}}(p)$

p an ω-limit pt of $\Gamma \Rightarrow \exists$ sequence $t_n \to \infty$
s.t $\varphi_{t_n}(x_0) \to p$

thus $\varphi_{t_n+\tilde{t}}(x_0) = \varphi_{\tilde{t}}(\varphi_{t_n}(x_0)) \longrightarrow \varphi_{\tilde{t}}(p) = q$

by continuity of solutions w.r.t initial conds
(main theorem of lecture 4)

thus $\tau_n = t_n + \hat{t}$ is a sequence $\tau_n \to \infty$
s.t $\varphi_{\tau_n}(x_0) \to q$ i.e. $q \in \omega(\Gamma)$

a SIMILAR PROOF FOR α limit sets

$\square$

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SAY FIRST

Corollary $\alpha(\Gamma)$ and $\omega(\Gamma)$ are invariant under the flow

back to example 1: clearly the set $r=1$ is invariant under the flow, closed, ~~the sol~~

if $|x_0|<1$ then the trajectory Γ is bounded (by uniqueness the trajectory can't cross $r=1$)

$\Rightarrow$ nonempty α and ω limit set

if $|x_0|>1$, Γ unbounded and ~~we can't find the~~ $\alpha(\Gamma)=\{\}$

also the unit circle is an attractor

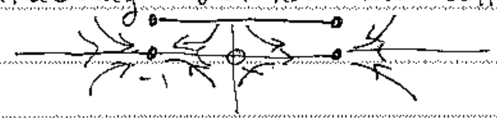
definition a closed, invariant set $A\subset E$ is called an attracting set if $\exists$ nbhd ~~around containing~~ $U\supset A$ s.t. $\forall x\in U$, $\varphi_t x_0 \to A$ as $t\to\infty$

an attractor is an attracting set w/ A dense ~~periodic~~ orbit

i.e. a single trajectory that comes arb close to each pt in A

example
$$\begin{cases}\dot x = x-x^3\\ \dot y=-y\end{cases}$$

note the set $\{|x|\le 1, y=0\}$ is a closed invariant set that is attracting but not an attractor



$\{1\}$, $\{-1\}$ are attractors

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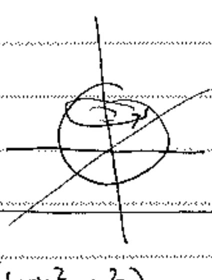
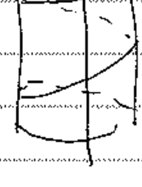
examples
$$\dot x = -y + x(1-x^2-y^2-z^2)$$
$$\dot y = x + y(1-x^2-y^2-z^2)$$
$$\dot z = 0$$

the unit sphere ~~plus~~ $\cup \{(0,0,z)\,|\,|z|\ge 1\}$ is an attracting set

no attractor



$$\dot x = -y + x(1-x^2-y^2)$$
$$\dot y = x + y(1-x^2-y^2)$$
$$\dot z = \alpha$$



both the z-axis, and the cylinder $x^2+y^2=1$ are invariant

but only the cylinder is an attractor

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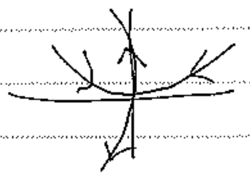
also notice fixed pts are invariant under flow

Asymptotically stable fixed pts are ~~the~~ ω-limit pts for an open set of initial conds, i.e. attracting

Q: are they attractors? A: yes

if x_0 a saddle point



then x_0 the ω-limit set for points on its stable manifold ~~the~~ and the α-limit set for pts on its unstable manifold

READ SECTION 3.2, I've covered through example 3 on p 178

Theorem (LaSalle's Invariance Principle)

Suppose $x=0$ a fixed pt of $\dot{x}=f(x)$, $x\in E\subset \mathbb{R}^n$ and $V(x)$ a Lyapunov function $\left(V(x)\geq 0,\ V(x)=0 \Leftrightarrow x=0,\ \dot{V}(x)\leq 0 \quad \forall x\in \cancel{\text{open set}}\ \text{Nbhd } G\supset 0 \right)$ $\bar{G}$ compact

~~if $x_0\in G$ and, $\Gamma_+(x_0)\subset G$, and~~

if $x_0\in G$, $\Gamma_+(x_0)\subset G$, and $\omega(\Gamma_+(x_0))\subset G$ ~~then~~

and M is the largest invariant ~~set~~ subset of E

then $\phi_t\, x_0 \to M$ as $t\to\infty$

PROOF

since $\Gamma_+(x_0)\subset \bar{G}$, $\omega(\Gamma)\subset \bar{G}$,

since V a Lyapunov function $V(\phi_t(x_0))$ nonincreasing and ≥ 0

hence $\exists\, C\geq 0$ (depending on x_0) s.t $\lim_{t\to\infty} V(\phi_t(x_0))=C$

Now let $y\in\omega(\Gamma_{(x_0)})$ thus, by def'n of limit pt $\exists$ sequence $t_k\to\infty$ st $\phi_{t_k}(x_0)\to y \Rightarrow V(y)=C$

by continuity and monotonicity $V(y)=C \quad \forall\, y\in\omega(\Gamma(x_0))$

$\omega(\Gamma_{x_0})$ invariant, so $y \in \omega(\Gamma) \Rightarrow \varphi_t(y) \in \omega(\Gamma_{x_0})$

$\Rightarrow V(\varphi_t(y)) = c \quad \forall y \in \omega(\Gamma_{x_0}), t \geq 0$

$\Rightarrow \dot{V} = 0 \quad \forall y \in \omega(\Gamma)$

so $\omega(\Gamma(x_0)) \subset M$

then since $\varphi_t x_0 \to \omega(\Gamma(x_0))$

we have $\varphi_t x_0 \to M$ for all $x_0 \in G$ $\square$

~~To use this, we must identify M~~

back to example from last time

$\ddot{x} + c\dot{x} + ax + bx^3 = 0 \qquad a, b, c > 0$

$y = \dot{x}$

letting $V = \frac{1}{2}y^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4$

We had $\dot{V} = -cy^2 \leq 0$

the largest invariant subset of E is $M = \{(0,0)\}$, if we pick any level set trajectories guaranteed to stay inside

thus the origin is an attracting fixed point

DIFFICULT PART USUALLY WANT TO SHOW THAT A WHOLE OPEN SET IS ATTRACTED TO SOME invariant set, then need to find some compact set $\bar{G}$ that trajectories starting in $\bar{G}$ stay in $\bar{G}$

We will return to this subject after (and spend more time on the case when the invariant set more than just a fixed pt) after break)