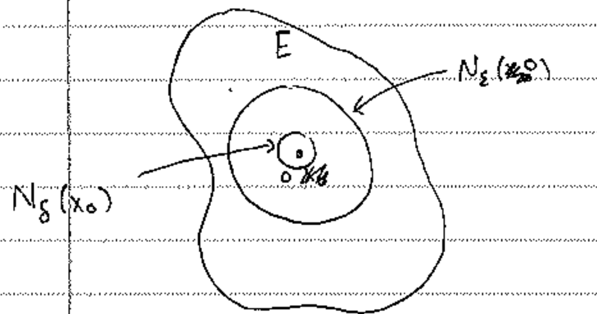
Lecture 14-1 – Applying LaSalle Invariance + Lyapunov's Method

To apply LaSalle's Invariance theorem we need to find a region ~~of~~ G s.t. $x_0 \in G \Rightarrow \varphi_t(x_0) \in G \quad \forall t \geq 0$

If we go back to our proof of Lyapunov's Theorem



s.t. $m_\varepsilon = \min_{|x|=\varepsilon} V(x)$

δ suff small that $V(x) < m_\varepsilon \ \forall x \in N_\delta$

then if $x_0 \in N_\delta$ we have $\varphi_t(x_0) \in N_\varepsilon \ \forall t > 0$

if ~~we rep~~ the same reasoning gives us the following lemma

<u>Bounding lemma</u> G an open bounded region in $\mathbb{R}^n$, Lyapunov fun V defined on $\bar{G}$, if $\exists x_0 \in G$ st $V(x) > V(x_0) \ \forall x \in \partial G$

then the set $S_{x_0} = \{x \mid V(x) \leq V(x_0)\}$ is a bounded subset of G and $\varphi_t x_0 \in S_{x_0} \ \forall t \geq 0$

* SO WE LOOK FOR INVARIANT SETS OF FORM $V_k = \{x \mid V(x) < k\}$

<u>EXAMPLE</u> $\ddot{z} + a(1-z^2)\dot{z} + z = 0 \quad a > 0$ (if $a<0$ van der Pol oscillator)

Eg if z small, see $\ddot{z} + a\dot{z} + z \approx 0$ so origin should be stable

Could convert to 1st order by letting $y = \dot{z}$

instead we write in Liénard form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \qquad f = a(1-x^2),\ g = x \qquad \text{(note } f \text{ even, } g \text{ odd)}$$

let $F(x) = \int^x f(\xi)\,d\xi$

$$\frac{dF(x(t))}{dt} = \frac{dF}{dx}\frac{dx}{dt} = f(x)\dot{x} \qquad \text{here}$$

now let $y = F(x) + \dot{x}$

$$\begin{array}{l|l} \dot{x} = y - F(x) & \dot{x} = y + a\left(\tfrac{1}{3}x^3 - x\right) \\ \dot{y} = -g(x) & \dot{y} = -x \end{array}$$

Now let $V = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$$\dot{V} = x\dot{x} + y\dot{y}$$
$$= ax^2\left(\tfrac{1}{3}x^2 - 1\right)$$

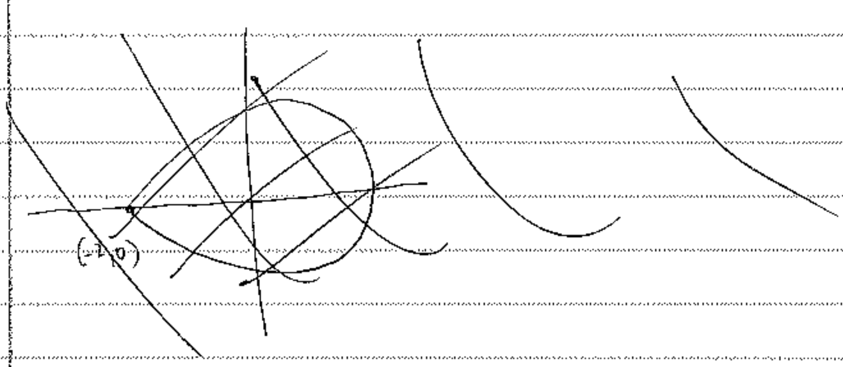
$\dot{V} \leq 0$ if $x^2 < 3$, $-\sqrt{3} < x < \sqrt{3}$

Lecture 14-2

~~Can we find a~~ ~~set~~ the set $x^2+y^2<3$ satisfies $V(x)<\frac{3}{2}$

So $V_{3/2}$ is a trapping region

now $\dot{V}=0$ on $x=0$, but $\dot{x}=y$ on this line
(see how we've used $\dot{V}=0$ to identify a set that contains the invariant set)
trajectories stay on this line only if $y=0$
So the largest invariant set contains only the origin
$\Rightarrow$ the origin is asymptotically stable ~~$a=1, b=\frac{1}{4}$~~



FURTHER, WE WILL LATER SHOW that this system has a periodic orbit enclosing the origin
this shows it must be outside the circle $x^2+y^2=3$

EXAMPLE $\ddot{x}+a\dot{x}+bx+x^3=0$

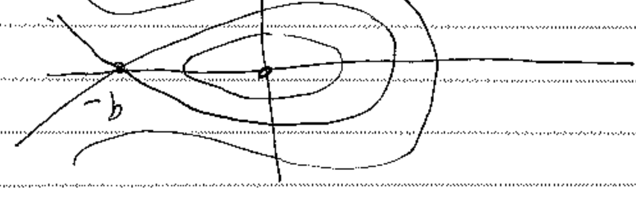
Rewrite as $\begin{cases}\dot{x}=y\\ \dot{y}=-ay-bx-x^2\end{cases}$

$V=\frac{1}{2}y^2+\frac{1}{2}bx^2+\frac{x^3}{3}$ is not positive definite on $\mathbb{R}^2$ but it is on smaller regions containing origin

$\dot{V}=-ay^2$

fixed pt $(-b,0)$ saddle, $(0,0)$ stable spiral
How big is basin of attraction?

level sets of V



lecture 14-3

define 3 quantities V as above

$$W_1 = x + \beta$$

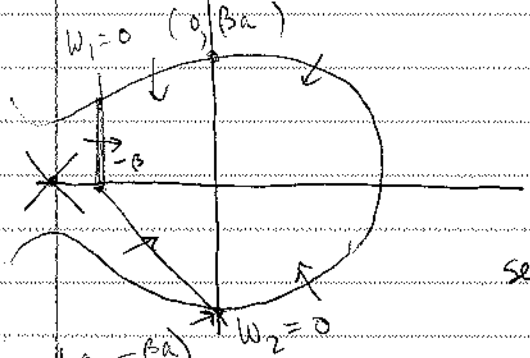
$$W_2 = y + a(x+\beta) \quad \text{some } \beta > 0 \text{ to be determined}$$

define a region $R_\beta = \{(x,y) \mid V \le \frac{1}{2}a^2\beta^2,\ W_1 \geq 0,\ y + a(x+\beta)\}$

$\dot{V} < 0$ on curved boundary!

$W_1 = 0$ a vertical line $x = -\beta$, $\dot{W}_1 = \dot{x} = y$ so our region only bounded by W_1 for $y > 0$

note $W_1 = 0$ intersects $W_2 = 0$ at $(-\beta, 0)$



$$\dot{W}_2 = \dot{y} + a\dot{x}$$
$$= -ay - bx - x^2 + ay$$
$$= -x(b+x)$$

Want $\dot{W}_2 \geq 0$, true for $-b < x < 0$

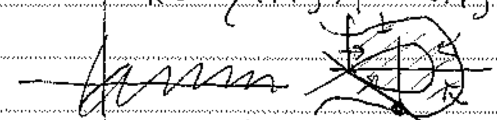
see the segment defined by $W_2 = 0$ satisfies

$$-\beta \le x \le 0$$

so choose $0 < \beta < b$

this is true $\forall \beta < b$ so

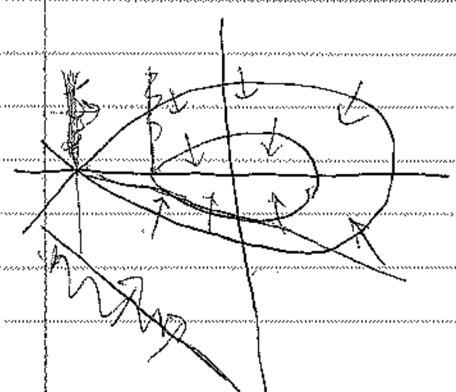
$R = \{(x,y) \mid V(x,y) < \frac{1}{2}a^2b^2,\ x > \beta,\ y + a(x+b) > 0\}$ is a trapping region



note however, on the homoclinic loop $V = V(-b, 0) = \frac{b^3}{6}$

if $\frac{1}{2}a^2b^2 < \frac{b^3}{6}$

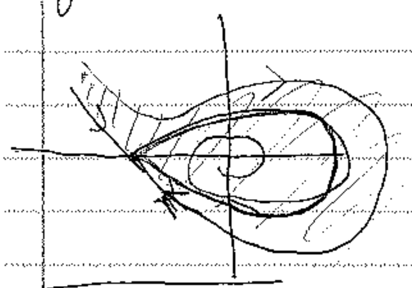
$a^2 < \frac{b}{3}$ we have this picture



so the interior of the homoclinic lobe is a larger trapping region

lecture 14-4

for ~~all~~ $0 < a < \sqrt{\frac{b}{3}}$, look at stable + unstable manifolds of fixed pt $(-b, 0)$



A little bit about Hamiltonian systems

let E be an open set in $\mathbb{R}^{2n}$

$H \in C^2(E)$, $x, y \in \mathbb{R}^n$, scalar

$$\dot{\vec{x}} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x} \qquad \frac{\partial H}{\partial x} = \left(\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n}\right)^T$$

$\vec{x}$ (often written $\vec{q}$) = the "position" variable

$\vec{y}$ (often $\vec{p}$) = the "momentum" variable

Eg $\cancel{H = \frac{1}{2}(x_1^2 + x_2^2 + m y_1^2 + y_2^2)}$

$$\cancel{\dot{x}_1 = y_1} \quad \cancel{\dot{y}_1 = -x_1} \quad \cancel{\ddot{x}_1 + x_1 = 0}$$
$$\cancel{\dot{x}_2 = y_2} \quad \cancel{\dot{y}_2 = -x_2} \quad \cancel{\ddot{x}_2 + x_2 = 0}$$

$$H = \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2)$$

$$\dot{x}_1 = \frac{\partial H}{\partial y_1} = y_1 \qquad \dot{y}_1 = -\frac{\partial H}{\partial x_1} = -x_1 \qquad \Rightarrow \begin{cases} \ddot{x}_1 + x_1 = 0 \\ \ddot{x}_2 + x_2 = 0 \end{cases}$$
$$\dot{x}_2 = y_2 \qquad \dot{y}_2 = -\frac{\partial H}{\partial x_2} = -x_2$$

Slightly generalize $H = \frac{1}{2}\left(k_1 x_1^2 + k_2 x_2^2 + \frac{y_1^2}{m} + \frac{y_2^2}{m}\right)$

$$\dot{x}_i = \frac{\partial H}{\partial y_i} = y_i / m_i \quad \Rightarrow \quad y_i = m\dot{x}_i \text{ the momentum}$$
$$\dot{y}_i = -\frac{\partial H}{\partial x_i} = -k_i x_i \qquad \ddot{x}_i = \frac{\dot{y}_i}{m_i} = -\frac{k_i}{m_i} x_i$$

$$\ddot{x}_i + \frac{k_i}{m_i} x_i = 0$$

Lecture 14-5

Theorem: Conservation of Energy

$\frac{dH}{dt} = 0$ proof $\frac{dH}{dt} = \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial y} \cdot \dot{y} = \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} - \frac{\partial H}{\partial y} \cdot \frac{\partial H}{\partial x} = 0$

$\square$

Theorem Conservation of phase space, if R_0 is a set of initial conditions then $V(\phi_t(R_0)) = V(R_0)$

~~Ys~~ from ~~HW 7~~, in section 2.3 we know

proof

Volume is preserved $\Leftrightarrow \nabla \cdot f = 0$

$$\nabla \cdot f = \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0$$

$\square$

Corollary A hamiltonian has no attracting, (or repelling) fixed pts

Pf suppose 0 is an attracting fixed pt

~~let~~ δ be a ~~st~~ such that ~~neighborhood~~ st $\forall x_0 \in N_\delta(0)$,

$\lim_{t \to \infty} \phi_t(x_0) = 0$

at fixed t, let $\rho(t) = \max_{N_\delta} |\phi_t(x_0)|$, and $\rho(t) \xrightarrow[t \to \infty]{} 0$

thus $V(\phi_t(R_0)) \le C \rho(t)^{2n} \longrightarrow 0$

contradiction

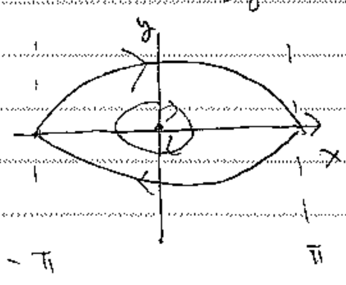
so no attracting fixed points

$\square$

$\Rightarrow$ all fixed pts are (nonlinear) centers + saddles

example you know $\ddot{x} + \sin x = 0$, general

$H = \frac{1}{2} y^2 + (1 - \cos x)$



level sets are trajectories

14-6

$\ddot{x} + U'(x) = 0$

$H = \frac{1}{2}\dot{x}^2 + U(x)$, local minima are centers, local maxima are saddles