Rotating Rectangles Task

Joe Xu

November 20, 2016

Abstract

This paper is created to answer the questions given on the "Rotating Rectangles" task sheet. The questions are focused on Trigonometry and Geometry.

This paper is structured in accordance to the task sheet. That is, the first part of the paper focuses on calculating coordinates of an image of a rectangle rotated by given degrees. The second part of this paper focuses on the more general nature of the trigonometric functions sine and cosine.

The approach to calculate the coordinates of the image of a rectangle rotated by given degrees (20° used for illustration of the question) is to use a combination of trigonometry (sine, cosine and tangent) properties and identities, as well as graphical methods such as calculating distance between two points on a 2D plane.

GeoGebra[1] is used for the diagrams, while TeXstudio[2] is used to produce this formatted paper. LaTeX-Tutorials[3] was used for reference for TeX formatting.

[1] International GeoGebra Institute. *GeoGebra*. Software. Version 5.0.297.0-3D. URL: `http://www.geogebra.org/`.

[2] Benito van der Zander, Jan Sundermeyer, Daniel Braun, Tim Hoffmann. *TeXstudio*. Software. Version 2.11.2 (hg 6192:72f68414a729). URL: `http://texstudio.sourceforge.net/`.

[3] *LaTeX Tutorial*. URL: `https://www.latex-tutorial.com/`.

Contents

1 Prerequisites

1.1 Trigonometric Properties and Identities

Trigonometric properties such as the Pythagorean theorem and trigonometric identities such as the Pythagorean identities and compound angle identities are utilized throughout this paper for proofs and for calculations.

Trigonometric Properties These are basic properties involving right-angled triangles as well as the unit circle.

Labels on a Right-Angled Triangle These labels will be used by the following properties and identities.

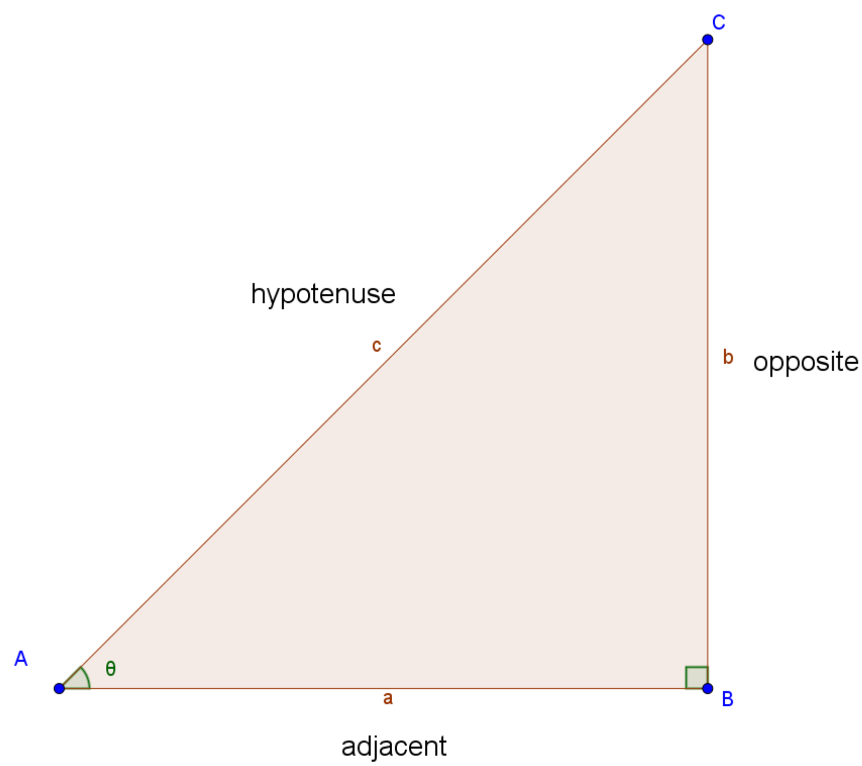

Figure 1: Labeled Right-Angled Triangle

Figure 1 shows a labeled triangle, with vertices A, B and C, and edges a, b, and c. Edge a is the adjacent leg of the right-angle triangle; it forms the angle θ and a right-angle with edges b and c respectively. Edge a is the opposite leg of the triangle; it is the edge opposite to the angle θ. Edge c is the hypotenuse of the triangle; it is the longest edge of the triangle.

Pythagorean Theorem The Pythagorean theorem states that for a right-angled Triangle (see Figure 1), the square of the adjacent leg plus the opposite leg equals to the square of the hypotenuse.

$$adjacent^2 + opposite^2 = hypotenuse^2$$
$$a^2 + b^2 = c^2$$

Basic Trigonometric Functions Sine, Cosine and Tangent, as well as Co-secant(csc) and Secant(sec). The angle θ is used here for the right-angled Triangle, but these functions can take bigger or smaller angles as well (with exceptions for tangent).

$$\cos\theta = \frac{adjacent}{hypotenuse}\left(= \frac{a}{c}\right)$$
$$\sin\theta = \frac{opposite}{hypotenuse}\left(= \frac{b}{c}\right)$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opposite}{adjacent}\left(= \frac{b}{a}\right)$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$

The Unit Circle The unit circle can help to give the exact values of special angles, 30°, 45°, 65°, etc. The radius of the circle is one ($r = 1$). The radius of the circle is the hypotenuse, which can form a right-angled triangle with the x-axis, by the vertical extension from the intersection of the radius with the circle's circumference to the x-axis.

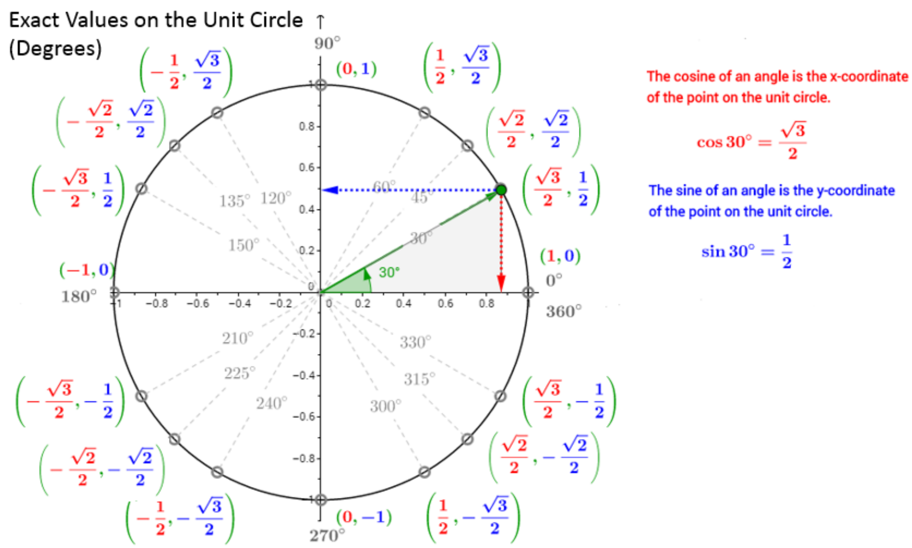

Figure 2: The Unit Circle

This diagram is from OnlineMathLearning.[4]

Trigonometric Identities These include the Pythagorean identities and the compound angle identities.

Pythagorean Identities

$$\cos\theta^2 + \sin\theta^2 \equiv 1 \tag{1}$$

$$1 + \tan\theta^2 \equiv \sec\theta \tag{2}$$

$$\cot\theta + 1 \equiv \csc\theta \tag{3}$$

Compound Angle Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{4}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{5}$$

[4] *Exact Values on the Unit Circle*. 2005. URL: `http://www.onlinemathlearning.com/math-trick-unit-circle.html`.

1.2 Graphical Properties and Calculations

This includes the distance formula.

Calculating Distance Between Two Points This formula calculates the distance (length) between two points, A and B (or the length of segment, AB). Let point A and B be (X_1, Y_1) and (X_2, Y_2) respectively, where X and Y denotes the coordinates of A and B.

$$AB_{\text{DISTANCE}} = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \tag{6}$$

2 Approaching The Problem

As per the questions from the task sheet, I am to calculate the coordinates of the image of a rectangle rotated by given degrees. There exists multiple methods to calculate the coordinates, but to get results as accurate and as exact as possible, I have decided to use trigonometrical functions in conjunction with the unit circle (modifying the radius) to calculate the coordinates. To do so, the investigation is based on a case study of the image of a rectangle $OABC$ rotated 20° from the x-axis, anti-clockwise, about the origin, O; the length and width of the rectangle is 4 units and 3 units respectively.

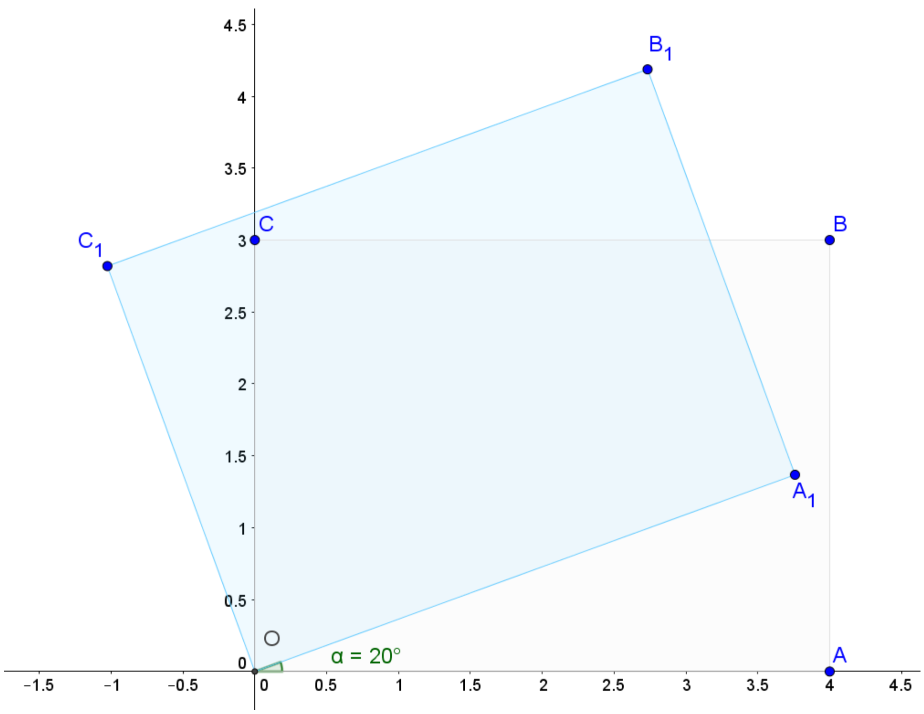

Figure 3: Image of Rectangle OABC Rotated 20°

Figure 3 shows the rectangle OABC rotated 20° from the x-axis about the origin O, giving the image $OA_1B_1C_1$.

I will calculate the coordinates of the image in terms of sines and cosines in order to deduce general expressions for calculating the coordinates.

2.1 Calculating A_1

Let $A_1(X_1, Y_1)$. In order to calculate A_1, I isolated the segment OA_1 from the image to form a right-angled triangle with the x-axis. This is demonstrated in the following diagrams:

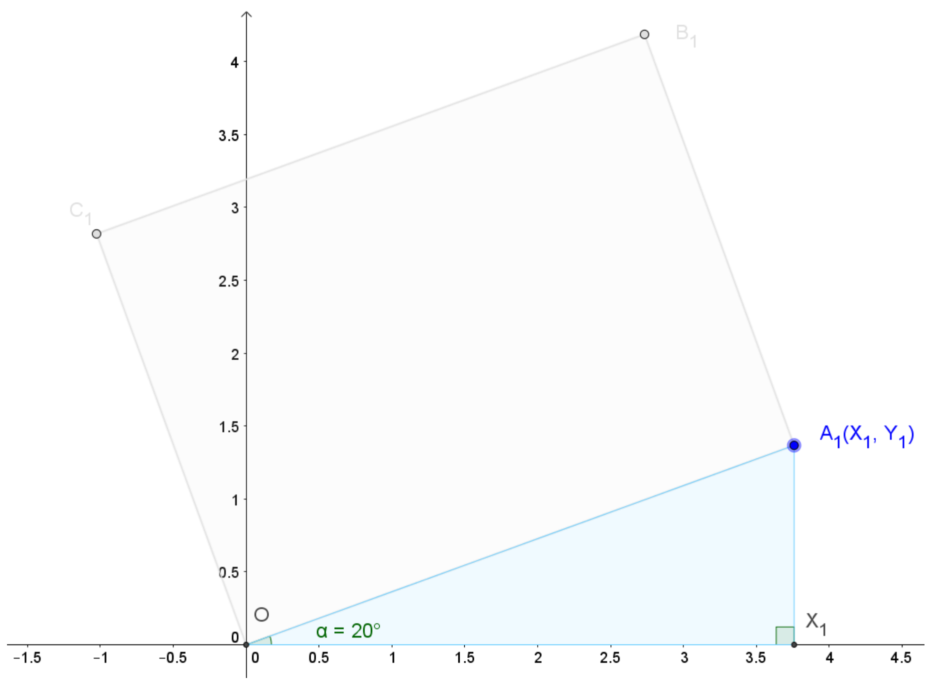

Figure 4: Right-angled Triangle Formed By OA_1

Then, I treated the hypotenuse (the length of the rectangle $OA_1 = 4$) as the radius of a unit circle multiplied by the length OA_1, i.e. a circle with radius $r = 4$.

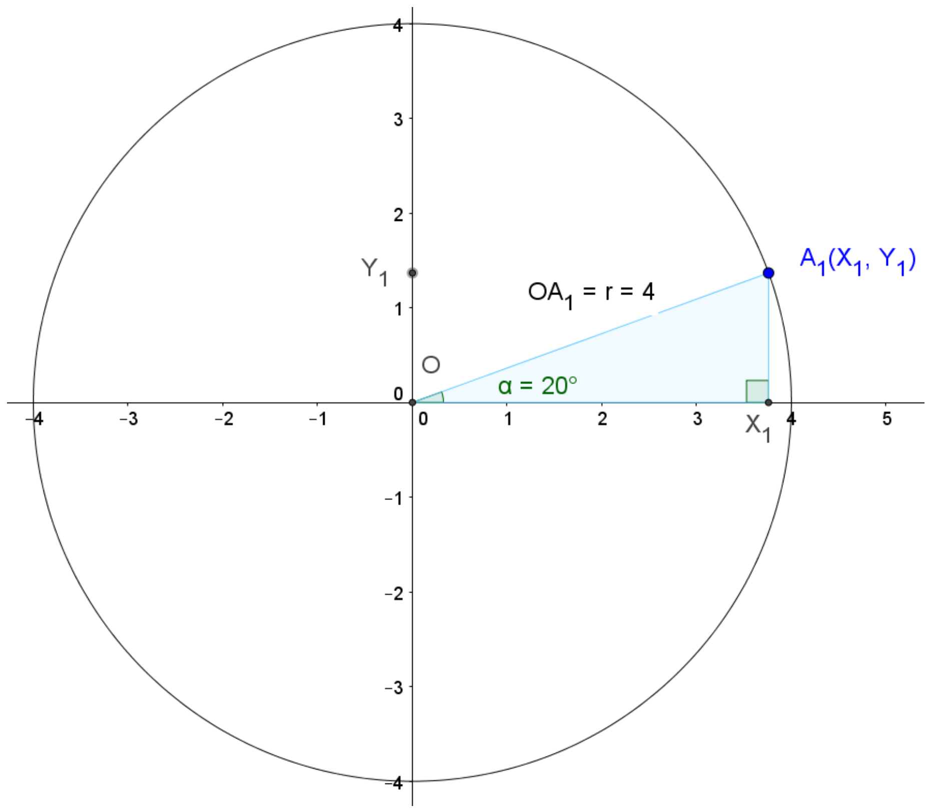

Figure 5: Right-angled Triangle Formed By OA_1

Recall that the height (y-value) of a point on the circumference of a unit circle equals to $\sin\alpha$; the width (x-value) of a point on the circumference of a unit circle equals to $\cos\alpha$. This is because (a is adjacent side, b is opposite, r is radius, also the hypotenuse):

$$
\begin{aligned}
\cos\alpha &= \frac{adjacent}{hypotenuse} \\
\cos\alpha &= \frac{a}{r} \\
adjacent \Rightarrow a &= r\cos\alpha = \cos\alpha \\
\therefore x = a &= \cos\alpha \\
\sin\alpha &= \frac{opposite}{hypotenuse} \\
\sin\alpha &= \frac{b}{r} \\
opposite \Rightarrow a &= r\sin\alpha = \sin\alpha \\
\therefore y = b &= \sin\alpha
\end{aligned}
$$

In the case of the image of the rectangle, the r value is no longer 1, but instead the length (or width for other coordinates) of the rectangle. For A_1, $r = length = 4$. As such, the values for X_1 and Y_1 can be obtained as follows:

$$X_1 = r\cos\alpha = 4\cos 20^\circ$$
$$Y_1 = r\sin\alpha = 4\sin 20^\circ$$

Therefore, A_1 have the coordinates:

$$A_1(4\cos 20^\circ, 4\sin 20^\circ) \tag{7}$$

2.2 Calculating C_1

C_1 can be calculated using the same method as A_1, just that the angle and radius is different. Let $C_1(X_2, Y_2)$.

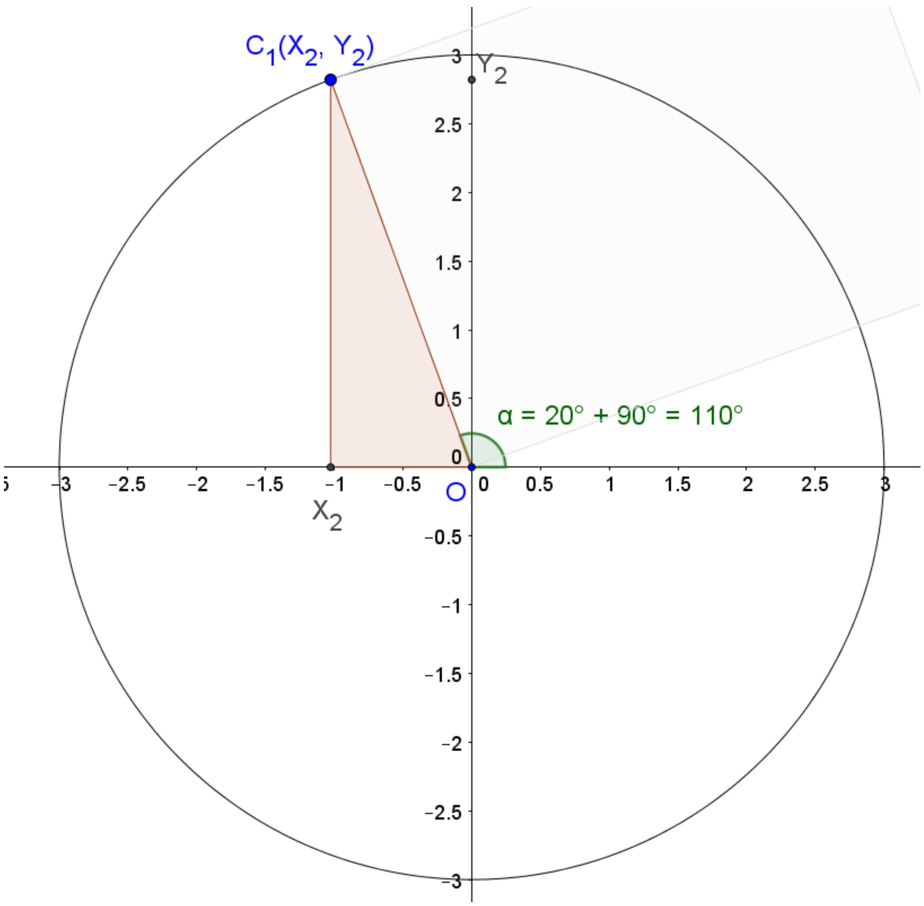

Figure 6: Right-angled Triangle Formed By OC_1

In Figure 6, the radius, i.e. the hypotenuse of the triangle is 3 units (OC_1). The angle $\alpha = 90° + 20° = 110°$. Using the expressions found previously, X_2 and Y_3 can be calculated as follows:

$$\begin{aligned} X_2 &= r \cos \alpha \\ &= 3 \cos (90° + 20°) \end{aligned}$$

Notice the presence of two angles here, 90° and 20°. X_2 can thus be calculated using the composite angle identity for cosine (see equation **(5)**):

$$\begin{aligned} X_2 &= 3\cos(90^\circ + 20^\circ) \\ &= 3[\cos 90^\circ \cos 20^\circ - \sin 90^\circ \sin 20^\circ] \\ &= 3[0 \times \cos 20^\circ - 1 \times \sin 20^\circ] \\ &= 3[-\sin 20^\circ] \\ &= -3\sin 20^\circ \end{aligned}$$

Y_2 can be calculated in a similar fashion:

$$\begin{aligned} Y_2 &= r\sin\alpha \\ &= 3\sin(90^\circ + 20^\circ) \\ &= 3[\sin 90^\circ \cos 20^\circ + \cos 90^\circ \sin 20^\circ] \\ &= 3[1 \times \cos 20^\circ + 0 \times \sin 20^\circ] \\ &= 3[\cos 20^\circ] \\ &= 3\cos 20^\circ \end{aligned}$$

Therefore, C_1 have the coordinates:

$$C_1(-3\sin 20^\circ, 3\cos 20^\circ) \tag{8}$$

2.3 Calculating B_1

Let $B_1(X_3, Y_3)$. B_1 can be obtained in two ways: B_1 can be obtained via the solutions to two simultaneous equations from the distances C_1B_1 and A_1B_1 (graphical method), or, obtained by finding OA_1 and angle $\widehat{AOB}$ (trigonometrical method).

2.3.1 Calculating B_1 (Graphical Method)

The distance C_1B_1 is equivalent to the length of the rectangle [4 units], and the distance A_1B_1 is equivalent to the width of the rectangle [3 units]. As such, two equations can be formed. In addition, OB_1 is also the diagonal length of the rectangle; it is also the hypotenuse of the triangle OB_1X_3.

Using the Pythagorean theorem, OB_1 is equivalent to:

$$\begin{aligned} OB_1^2 &\Leftrightarrow X_3^2 + Y_3^2 \Leftrightarrow 3^2 + 4^2 = 25 \\ \therefore OB_1 &= \sqrt{25} = 5 \end{aligned}$$

Equation 1 This is the distance, i.e. length of the segment C_1B_1, which in turn is equivalent to the length of the rectangle, 4 units. See **(8)** for the coordinates of C_1.

$$
\begin{aligned}
C_1B_1 = 4 = \sqrt{(-3\sin 20^\circ - X_3)^2 + (3\cos 20^\circ - Y_3)^2} \\
16 &= (-3\sin 20^\circ - X_3)^2 + (3\cos 20^\circ - Y_3)^2 \\
16 &= 9\sin^2 20^\circ + 6X_3\sin 20^\circ + X_3^2 + 9\cos^2 20^\circ - 6Y_3\cos 20^\circ + Y_3^2
\end{aligned}
$$

Rearranging, gives:

$$
\begin{aligned}
16 &= 9\sin^2 20^\circ + 6X_3\sin 20^\circ + X_3^2 + 9\cos^2 20^\circ - 6Y_3\cos 20^\circ + Y_3^2 \\
16 &= 9\sin^2 20^\circ + 9\cos^2 20^\circ + X_3^2 + Y_3^2 + 6X_3\sin 20^\circ - 6Y_3\cos 20^\circ \\
16 &= 9[\cos^2 20^\circ + \sin^2 20^\circ] + [X_3^2 + Y_3^2] + 6X_3\sin 20^\circ - 6Y_3\cos 20^\circ
\end{aligned}
$$

Recall that $\cos^2\theta + \sin^2\theta \equiv 1$, as per the first Pythagorean identity [see **(1)**]. Also, $OB_1 = X_3^2 + Y_3^2 = 5$, as per the previous calculation for the length OB_1. Substituting these values in, giving:

$$
\begin{aligned}
16 &= 9 + 25 + 6X_3\sin 20^\circ - 6Y_3\cos 20^\circ \\
-18 &= 6X_3\sin 20^\circ - 6Y_3\cos 20^\circ \\
-3 &= X_3\sin 20^\circ - Y_3\cos 20^\circ
\end{aligned}
$$

And thus, the first equation is:

$$
-3 = X_3\sin 20^\circ - Y_3\cos 20^\circ \tag{9}
$$

Equation 2 This is the distance, i.e. length of the segment A_1B_1, which in turn is equivalent to the width of the rectangle, 4 units. See **(7)** for the coordinates of A_1.

$$
\begin{aligned}
A_1B_1 = 3 = \sqrt{(4\cos 20^\circ - X_3)^2 + (4\sin 20^\circ - Y_3)^2} \\
9 &= (4\cos 20^\circ - X_3)^2 + (4\sin 20^\circ - Y_3)^2 \\
9 &= 16\cos^2 20^\circ - 8X_3\cos 20^\circ + X_3^2 + 16\sin^2 20^\circ - 8Y_3\sin 20^\circ + Y_3^2
\end{aligned}
$$

Rearranging, gives:

$$
\begin{aligned}
9 &= 16\cos^2 20^\circ - 8X_3\cos 20^\circ + X_3^2 + 16\sin^2 20^\circ - 8Y_3\sin 20^\circ + Y_3^2 \\
9 &= 16\cos^2 20^\circ + 16\sin^2 20^\circ + X_3^2 + Y_3^2 - 8X_3\cos 20^\circ - 8Y_3\sin 20^\circ \\
9 &= 16[\cos^2 20^\circ + \sin^2 20^\circ] + [X_3^2 + Y_3^2] - 8X_3\cos 20^\circ - 8Y_3\sin 20^\circ \\
9 &= 16 + 25 - 8X_3\cos 20^\circ - 8Y_3\sin 20^\circ \\
-32 &= -8X_3\cos 20^\circ - 8Y_3\sin 20^\circ \\
4 &= X_3\cos 20^\circ + Y_3\sin 20^\circ
\end{aligned}
$$

And thus, the second equation is:

$$
4 = X_3\cos 20^\circ + Y_3\sin 20^\circ \tag{10}
$$

Solving for X_3 X_3 can be found by making Y_3 the subject of both equations **(9)** and **(10)**, and then equating them.

Rearranging (9) Making Y_3 the subject of the equation:

$$\begin{aligned} -3 &= X_3 \sin 20^\circ - Y_3 \cos 20^\circ \\ Y_3 \cos 20^\circ &= X_3 \sin 20^\circ + 3 \\ \therefore Y_3 &= \frac{X_3 \sin 20^\circ + 3}{\cos 20^\circ} \end{aligned}$$

Rearranging (10) Making Y_3 the subject of the equation:

$$\begin{aligned} 4 &= X_3 \cos 20^\circ + Y_3 \sin 20^\circ \\ Y_3 \sin 20^\circ &= 4 - X_3 \cos 20^\circ \\ \therefore Y_3 &= \frac{4 - X_3 \cos 20^\circ}{\sin 20^\circ} \end{aligned}$$

Equating Y_3 Equations Equating the two equations where Y_3 is the subject can give the value for X_3.

Since:

$$Y_3 = \frac{X_3 \sin 20^\circ + 3}{\cos 20^\circ} \Leftrightarrow \frac{4 - X_3 \cos 20^\circ}{\sin 20^\circ} = Y_3$$

Therefore:

$$\frac{X_3 \sin 20^\circ + 3}{\cos 20^\circ} = \frac{4 - X_3 \cos 20^\circ}{\sin 20^\circ}$$

Cross-multiplying, gives:

$$\begin{aligned} \frac{X_3 \sin 20^\circ + 3}{\cos 20^\circ} &= \frac{4 - X_3 \cos 20^\circ}{\sin 20^\circ} \\ \sin 20^\circ [X_3 \sin 20^\circ + 3] &= \cos 20^\circ [4 - X_3 \cos 20^\circ] \\ X_3 \sin^2 20^\circ + 3 \sin 20^\circ &= 4 \cos 20^\circ - X_3 \cos^2 20^\circ \\ X_3 [\sin^2 20^\circ + \cos^2 20^\circ] &= 4 \cos 20^\circ - 3 \sin 20^\circ \end{aligned}$$

Notice that $\sin^2 20^\circ + \cos^2 20^\circ \equiv 1$ as according to the first Pythagorean identity [see **(1)**]. Therefore:

$$\begin{aligned} X_3 [\sin^2 20^\circ + \cos^2 20^\circ] &= 4 \cos 20^\circ - 3 \sin 20^\circ \\ \therefore X_3 &= 4 \cos 20^\circ - 3 \sin 20^\circ \end{aligned}$$

Solving for Y_3 Y_3 can be found by making X_3 the subject of both equations **(9)** and **(10)**, and then equating them.

Rearranging (9) Making X_3 the subject of the equation:

$$\begin{aligned} -3 &= X_3 \sin 20^\circ - Y_3 \cos 20^\circ \\ X_3 \sin 20^\circ &= Y_3 \cos 20^\circ - 3 \\ \therefore X_3 &= \frac{Y_3 \cos 20^\circ - 3}{\sin 20^\circ} \end{aligned}$$

Rearranging (10) Making X_3 the subject of the equation:

$$\begin{aligned} 4 &= X_3 \cos 20^\circ + Y_3 \sin 20^\circ \\ X_3 \cos 20^\circ &= 4 - Y_3 \sin 20^\circ \\ \therefore X_3 &= \frac{4 - Y_3 \sin 20^\circ}{\cos 20^\circ} \end{aligned}$$

Equating X_3 Equations Equating the two equations where X_3 is the subject can give the value for Y_3.

Since:

$$X_3 = \frac{Y_3 \cos 20^\circ - 3}{\sin 20^\circ} \Leftrightarrow \frac{4 - Y_3 \sin 20^\circ}{\cos 20^\circ} = X_3$$

Therefore:

$$\frac{Y_3 \cos 20^\circ - 3}{\sin 20^\circ} = \frac{4 - Y_3 \sin 20^\circ}{\cos 20^\circ}$$

Cross-multiplying, gives:

$$\begin{aligned} \frac{Y_3 \cos 20^\circ - 3}{\sin 20^\circ} &= \frac{4 - Y_3 \sin 20^\circ}{\cos 20^\circ} \\ \cos 20^\circ [Y_3 \cos 20^\circ - 3] &= \sin 20^\circ [4 - Y_3 \sin 20^\circ] \\ Y_3 \cos^2 20^\circ - 3 \cos 20^\circ &= 4 \sin 20^\circ - Y_3 \sin^2 20^\circ \\ Y_3 \cos^2 20^\circ + Y_3 \sin^2 20^\circ &= 4 \sin 20^\circ + 3 \cos 20^\circ \\ Y_3 [\cos^2 20^\circ + \sin^2 20^\circ] &= 4 \sin 20^\circ + 3 \cos 20^\circ \end{aligned}$$

Notice that $\cos^2 20^\circ + \sin^2 20^\circ \equiv 1$ as according to the first Pythagorean identity [see **(1)**]. Therefore:

$$\begin{aligned} Y_3 [\cos^2 20^\circ + \sin^2 20^\circ] &= 4 \sin 20^\circ + 3 \cos 20^\circ \\ \therefore Y_3 &= 3 \cos 20^\circ + 4 \sin 20^\circ \end{aligned}$$

Coordinates for B_1 The values for X_3 and Y_3 have been found, and thus B_1 have the coordinates:

$$B_1(4 \cos 20^\circ - 3 \sin 20^\circ, 3 \cos 20^\circ + 4 \sin 20^\circ) \tag{11}$$

2.3.2 Calculating B_1 (Trigonometrical Method)

B_1 can be calculated in the same way as A_1 and C_1.

Calculating OB The length of OB is required as it is the same as OB_1. Both are the diagonal length of the rectangle. To calculate, divide the rectangle into two equivalent triangles via OB. OB then becomes the hypotenuse of the right-angled triangle $\triangle AOB$:

$$\begin{aligned} OB^2 &= OA^2 + AB^2 \\ OB^2 &= 4^2 + 3^2 \\ OB^2 &= 25 \\ \therefore OB &= 5 \end{aligned}$$

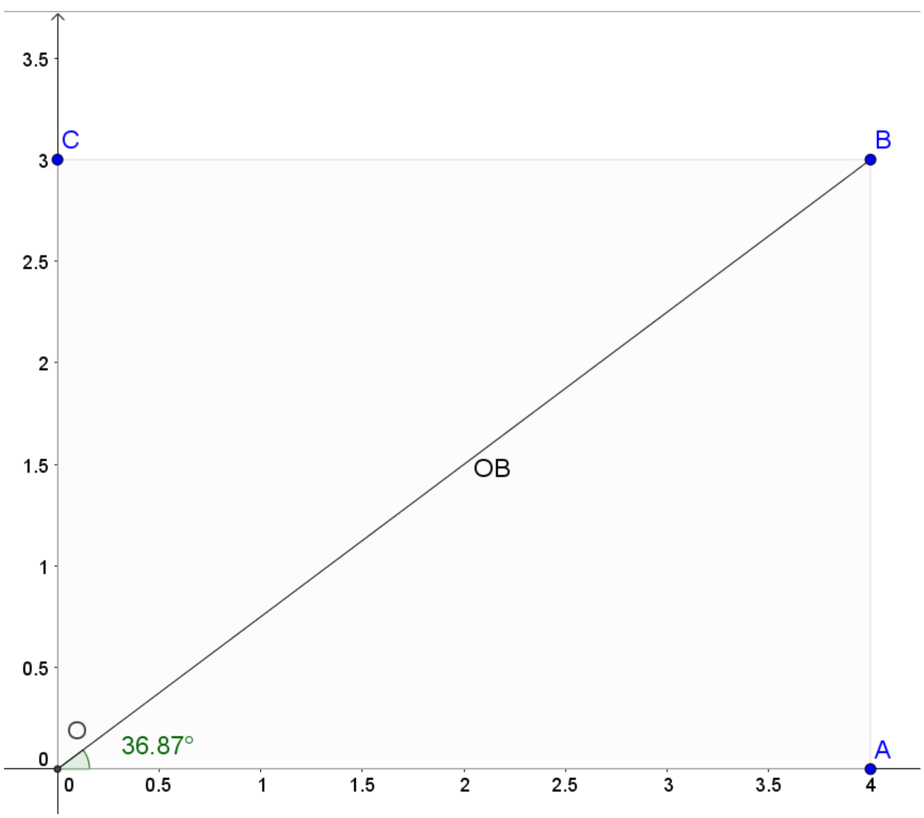

Figure 7: Right-angled Triangle Formed By OB

And thus,

$$OB_1 \Leftrightarrow OB = 5 \tag{12}$$

Calculating $\widehat{AOB}$ $\widehat{AOB}$ is important as it forms part of the composite angle $\widehat{AOB_1}$, which is the angle that will be used to calculate the coordinates.

$$\begin{aligned}\tan\widehat{AOB} &= \frac{opposite}{adjacent}\\ \tan\widehat{AOB} &= \frac{3}{4}\\ \therefore \widehat{AOB} &= \tan^{-1}\left(\frac{3}{4}\right)\end{aligned}$$

Calculating $\widehat{AOB_1}$ AOB_1 is calculated by the 20° rotation angle plus the angle formed by OB_1 and OA_1 which is equivalent to that of $\widehat{AOB}$:

$$\widehat{AOB_1} = 20^\circ + \tan^{-1}\left(\frac{3}{4}\right)$$

Calculating X_3 X_3 can then be calculated in the fashion as A_1. The radius is OB, which is 5 units. The angle is AOB_1, which equals to $\tan^{-1}\left(\frac{3}{4}\right)$:

$$X_3 = 5\cos(20^\circ + \tan^{-1}\left(\frac{3}{4}\right))$$

Using the composite angle identity for cosine [see **(5)**], X_3 becomes:

$$\begin{aligned}X_3 &= 5\cos(20^\circ + \tan^{-1}\left(\frac{3}{4}\right))\\ &= 5\left[\cos 20^\circ \cos\tan^{-1}\left(\frac{3}{4}\right) - \sin 20^\circ \sin\tan^{-1}\left(\frac{3}{4}\right)\right]\end{aligned}$$

$\cos\tan^{-1}\left(\frac{3}{4}\right)$ and $\sin\tan^{-1}\left(\frac{3}{4}\right)$ can be calculated in the following way:

$$\text{For} \cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$\begin{aligned}\text{Let } \theta &= \tan^{-1}\left(\frac{3}{4}\right)\\ \Rightarrow \tan\theta &= \frac{3}{4}\end{aligned}$$

Since $\tan\theta = \frac{opposite}{adjacent}$, $opposite = 3$ and $adjacent = 4$. Therefore, $hypotenuse = \sqrt{3^2+4^2} = 5$. And thus,

$$\begin{aligned}
\because \tan\theta &= \frac{3}{4} \\
\Rightarrow \cos\theta &= \frac{adjacent}{hypotenuse} = \frac{4}{5} \\
\therefore \cos\tan^{-1}\left(\frac{3}{4}\right) &= \frac{4}{5} \\
\Rightarrow \sin\theta &= \frac{opposite}{hypotenuse} = \frac{3}{5} \\
\therefore \sin\tan^{-1}\left(\frac{3}{4}\right) &= \frac{3}{5}
\end{aligned}$$

Substituting these values back to the composite angle calculation:

$$\begin{aligned}
X_3 &= 5\left[\cos 20^\circ \cos\tan^{-1}\left(\frac{3}{4}\right) - \sin 20^\circ \sin\tan^{-1}\left(\frac{3}{4}\right)\right] \\
&= 5\left[\cos 20^\circ \times \left(\frac{4}{5}\right) - \sin 20^\circ \times \left(\frac{3}{5}\right)\right] \\
&= 4\cos 20^\circ - 3\sin 20^\circ
\end{aligned}$$

Which is consistent with the X_3 value obtained via the Graphical method.

Calculating Y_3 Y_3 can then be calculated in the fashion as A_1. The radius is OB, which is 5 units. The angle is AOB_1, which equals to $\tan^{-1}\left(\frac{3}{4}\right)$:

$$X_3 = 5\sin(20^\circ + \tan^{-1}\left(\frac{3}{4}\right))$$

Using the composite angle identity for cosine [see **(5)**], X_3 becomes:

$$\begin{aligned}
X_3 &= 5\sin(20^\circ + \tan^{-1}\left(\frac{3}{4}\right)) \\
&= 5\left[\sin 20^\circ \cos\tan^{-1}\left(\frac{3}{4}\right) + \cos 20^\circ \sin\tan^{-1}\left(\frac{3}{4}\right)\right]
\end{aligned}$$

$\sin\tan^{-1}\left(\frac{3}{4}\right)$ and $\cos\tan^{-1}\left(\frac{3}{4}\right)$ can be calculated in the following way:

$$\begin{aligned}
\text{For}\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right) & \\
\text{Let } \theta &= \tan^{-1}\left(\frac{3}{4}\right) \\
\Rightarrow \tan\theta &= \frac{3}{4}
\end{aligned}$$

Since $\tan\theta = \frac{opposite}{adjacent}$, $opposite = 3$ and $adjacent = 4$. Therefore, $hypotenuse = \sqrt{3^2 + 4^2} = 5$. And thus,

$$
\begin{aligned}
\because \tan\theta &= \frac{3}{4} \\
\Rightarrow \sin\theta &= \frac{opposite}{hypotenuse} = \frac{4}{5} \\
\therefore \sin\tan^{-1}\left(\frac{3}{4}\right) &= \frac{3}{5} \\
\Rightarrow \cos\theta &= \frac{adjacent}{hypotenuse} = \frac{3}{5} \\
\therefore \cos\tan^{-1}\left(\frac{3}{4}\right) &= \frac{4}{5}
\end{aligned}
$$

Substituting these values back to the composite angle calculation:

$$
\begin{aligned}
X_3 &= 5\left[\sin 20^\circ \cos\tan^{-1}\left(\frac{3}{4}\right) + \cos 20^\circ \sin\tan^{-1}\left(\frac{3}{4}\right)\right] \\
&= 5\left[\sin 20^\circ \times \left(\frac{4}{5}\right) + \cos 20^\circ \times \left(\frac{3}{5}\right)\right] \\
&= 4\sin 20^\circ + 3\cos 20^\circ \\
&= 3\cos 20^\circ + 4\sin 20^\circ
\end{aligned}
$$

Which is also consistent with the Y_3 value obtained via the Graphical method.

2.4 Approaching a General Formula

I have drawn several images of the rectangle rotating in different angles, quadrants, as well as one image in a clockwise direction.

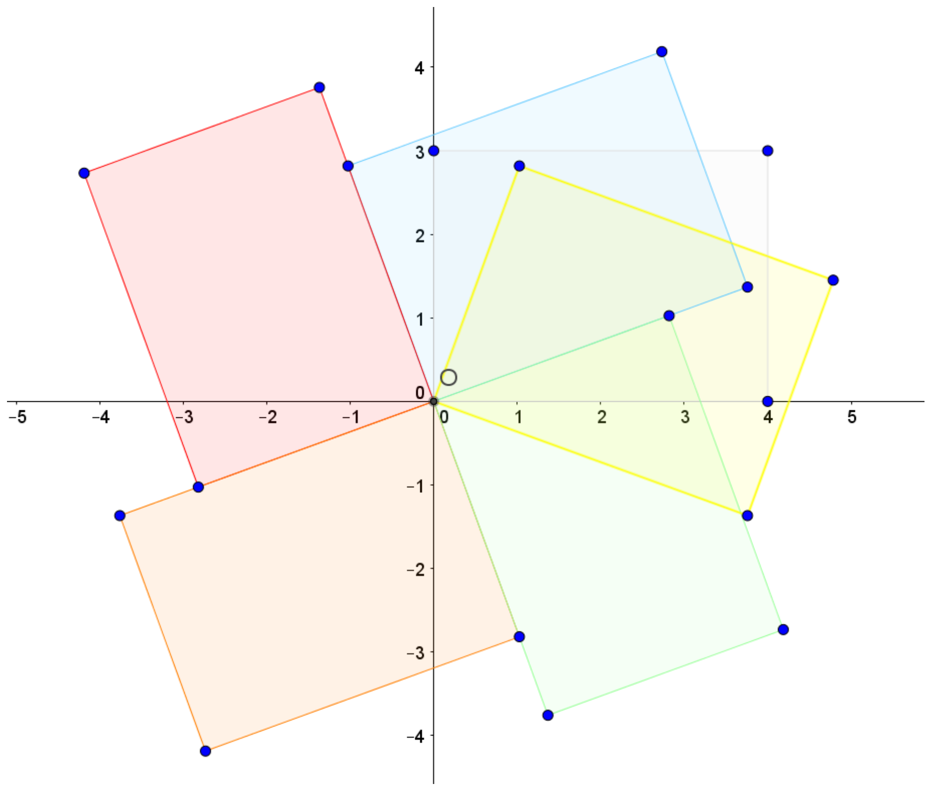

Figure 8: Rotated Rectangles

Figure 8 shows images of rectangles rotated by different angles, including an image that has been rotated in the clockwise direction.

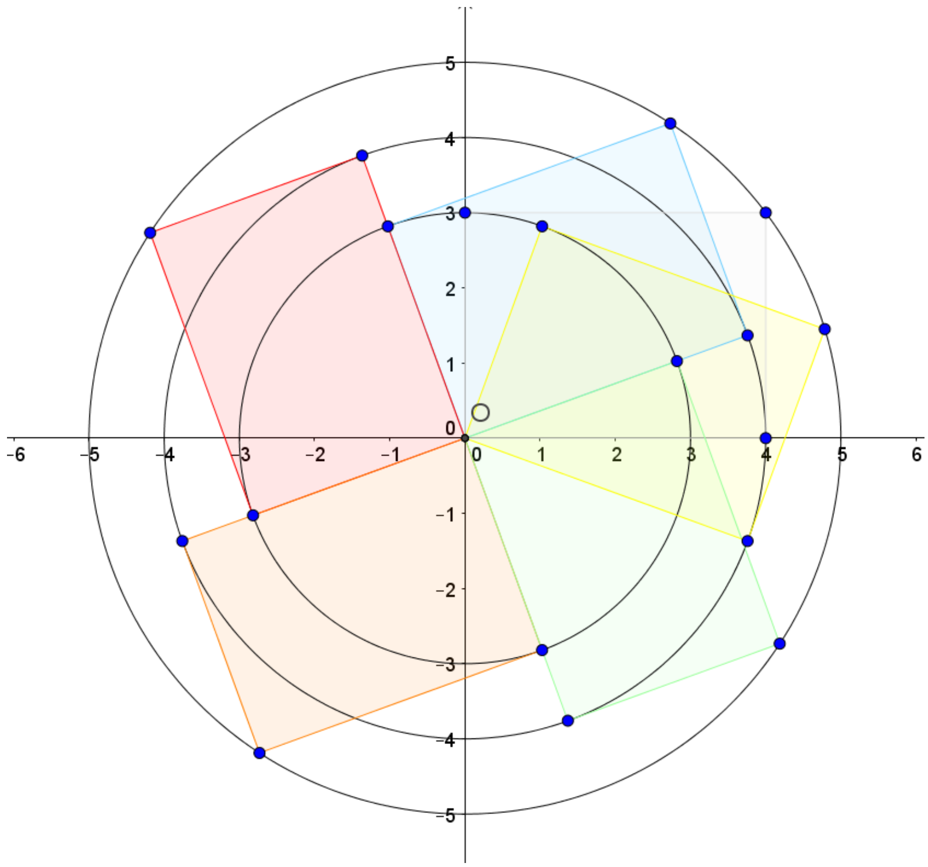

Figure 9: Rotated Rectangles With Circles

As shown in Figure 9, the general method to calculate x and y coordinates should still hold, as the lengths, widths as well as diagonal lengths do not vary. They can still be calculated via the method used to calculate A_1 and B_1 and C_1.

2.5 Finding General Expressions For Calculating Coordinates of An Image of a Rectangle

Let the length of the rectangle be a and the width be b, and the angle of rotation be θ.

Let the points A', B' and C' be the respective points of the image after the rectangle with the points A, B and C has been rotated θ degrees (or radians) [excluding the origin]. X and Y are the x and y coordinates of each point.

$$A'(X_A, Y_A)$$
$$B'(X_B, Y_B)$$
$$C'(X_C, Y_C)$$

2.5.1 Finding General Expression for A'

Using the case study of the image of rectangle (length 4, width 3) rotated 20° from the x-axis about the origin, A' can be calculated as:

$$A_1(4\cos 20^\circ, 4\sin 20^\circ)$$
$$\because 4 = length, length = a,$$
$$\because 20^\circ = angle, angle = \theta$$
$$A_1(4\cos 20^\circ, 4\sin 20^\circ) \Rightarrow A_1(a\cos\theta, a\sin\theta) \Rightarrow A'(a\cos\theta, a\sin\theta)$$
$$\therefore A'(a\cos\theta, a\sin\theta)$$

2.5.2 Finding General Expression for C'

Using the case study of the image of rectangle (length 4, width 3) rotated 20° from the x-axis about the origin, C' can be calculated as:

$$C_1(-3\sin 20^\circ, 3\cos 20^\circ)$$
$$\because 3 = width, width = b,$$
$$\because 20^\circ = angle, angle = \theta$$
$$C_1(-3\sin 20^\circ, 3\cos 20^\circ) \Rightarrow C_1(-b\sin\theta, b\cos\theta) \Rightarrow C'(-b\sin\theta, b\cos\theta)$$
$$\therefore C'(-b\sin\theta, b\cos\theta)$$

2.5.3 Finding General Expression for B'

Using the case study of the image of rectangle (length 4, width 3) rotated 20° from the x-axis about the origin, B' can be calculated as:

$$B_1(4\cos 20^\circ - 3\sin 20^\circ, 3\cos 20^\circ + 4\sin 20^\circ)$$
$$\because 4 = length, length = a,$$
$$\because 3 = width, width = b,$$
$$\because 20^\circ = angle, angle = \theta$$
$$B_1(4\cos 20^\circ - 3\sin 20^\circ, 3\cos 20^\circ + 4\sin 20^\circ) \Rightarrow B_1(a\cos\theta - b\sin\theta, b\cos\theta + a\sin\theta)$$
$$\Rightarrow B'(a\cos\theta - b\sin\theta, b\cos\theta + a\sin\theta)$$
$$\therefore B' a\cos\theta - b\sin\theta, b\cos\theta + a\sin\theta)$$

2.6 General Expressions

These are the three expressions found above for calculating coordinates of an image of rectangle rotated.

2.6.1 A'

$$A'(a\cos\theta, a\sin\theta) \tag{13}$$

2.6.2 B'

$$B'(a\cos\theta - b\sin\theta, b\cos\theta + a\sin\theta) \tag{14}$$

2.6.3 C'

$$C'(-b\sin\theta, b\cos\theta) \tag{15}$$

3 Further Investigation

This part of the paper focuses more on the nature of trigonometric functions.

3.1 Investigating Functions with Composite Angles

Combining sine and cosine function A function with both cosine and sine, $a\sin\theta + b\cos\theta$, can be rewritten in the form $r\sin(\theta+\alpha)$, such that the expression only has sine and not both sine and cosine.

Expressing r and α in terms of a and b r and α can be calculated using a and b:

$$\begin{aligned}
a\sin\theta + b\cos\theta &\Leftrightarrow r\sin(\theta+\alpha)\\
\text{Let } a\sin\theta + b\cos\theta &= r\sin(\theta+\alpha)\\
a\sin\theta + b\cos\theta &= r[\sin\theta\cos\alpha + \cos\theta\sin\alpha]\\
a\sin\theta + b\cos\theta &= [r\cos\alpha]\sin\theta + [r\sin\alpha]\cos\theta
\end{aligned}$$

Equating both sides, such that:

$$\begin{aligned}
a\sin\theta + b\cos\theta &\Leftrightarrow [r\cos\alpha]\sin\theta + [r\sin\alpha]\cos\theta\\
\therefore a &= [r\cos\alpha]\\
\therefore b &= [r\sin\alpha]
\end{aligned}$$

Giving a and b in terms of r and α:

$$a = r\cos\alpha \tag{16}$$
$$b = r\sin\alpha \tag{17}$$

Squaring a and b, giving:

$$a^2 = r^2\cos^2\alpha \tag{18}$$
$$b^2 = r^2\sin^2\alpha \tag{19}$$

Adding **(16)** and **(17)**, giving:

$$\begin{aligned}
a^2 + b^2 &= r^2\cos^2\alpha + r^2\sin^2\alpha\\
a^2 + b^2 &= r^2[\cos^2\alpha + \sin^2\alpha]\\
r^2 &= a^2 + b^2\\
\therefore r &= \sqrt{a^2+b^2}
\end{aligned}$$

And thus giving r in terms of a and b:

$$r = \sqrt{a^2+b^2} \tag{20}$$

α can be expressed via a and b via rearranging **(16)** and **(17)**:

$$\begin{aligned}
a &= r\cos\alpha \\
&\Rightarrow \cos\alpha = \frac{a}{r} \\
&(\Rightarrow \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}) \\
b &= r\sin\alpha \\
&\Rightarrow \sin\alpha = \frac{b}{r} \\
&(\Rightarrow \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}) \\
\tan\alpha &= \frac{\sin\alpha}{\cos\alpha} \\
&\Rightarrow \tan\alpha = \frac{\frac{b}{r}}{\frac{a}{r}} \\
&\Rightarrow \tan\alpha = \frac{b}{a} \\
\therefore \alpha &= \tan^{-1}\left(\frac{b}{a}\right)
\end{aligned}$$

As such α can be expressed as:

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) \tag{21}$$

Similarity, r and α of the combined function $r\cos(\theta+\alpha)$ of $a\cos\theta - b\sin\theta$ can be given in terms of a and b.

Expressing r and α in terms of a and b r and α can be calculated using a and b:

$$\begin{aligned}
a\cos\theta - b\sin\theta &\Leftrightarrow r\cos(\theta+\alpha) \\
\text{Let } a\cos\theta - b\sin\theta &= r\cos(\theta+\alpha) \\
a\cos\theta - b\sin\theta &= r[\cos\theta\cos\alpha - \sin\theta\sin\alpha] \\
a\cos\theta - b\sin\theta &= [r\cos\alpha]\cos\theta - [r\sin\alpha]\sin\theta
\end{aligned}$$

Equating both sides, such that:

$$\begin{aligned}
a\cos\theta - b\sin\theta &\Leftrightarrow [r\cos\alpha]\cos\theta - [r\sin\alpha]\sin\theta \\
\therefore a &= [r\cos\alpha] \\
\therefore b &= [r\sin\alpha]
\end{aligned}$$

Giving a and b in terms of r and α:

$$a = r\cos\alpha \tag{22}$$
$$b = r\sin\alpha \tag{23}$$

Squaring a and b, giving:

$$a^2 = r^2\cos^2\alpha \tag{24}$$
$$b^2 = r^2\sin^2\alpha \tag{25}$$

Adding **(16)** and **(17)**, giving:

$$\begin{aligned} a^2 + b^2 &= r^2\cos^2\alpha + r^2\sin^2\alpha \\ a^2 + b^2 &= r^2[\cos^2\alpha + \sin^2\alpha] \\ r^2 &= a^2 + b^2 \\ \therefore r &= \sqrt{a^2 + b^2} \end{aligned}$$

And thus giving r in terms of a and b:

$$r = \sqrt{a^2 + b^2} \tag{26}$$

α can be expressed via a and b via rearranging **(22)** and **(23)**:

$$\begin{aligned} a &= r\cos\alpha \\ &\Rightarrow \cos\alpha = \frac{a}{r} \\ &(\Rightarrow \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}) \\ b &= r\sin\alpha \\ &\Rightarrow \sin\alpha = \frac{b}{r} \\ &(\Rightarrow \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}) \\ \tan\alpha &= \frac{\sin\alpha}{\cos\alpha} \\ &\Rightarrow \tan\alpha = \frac{\frac{b}{r}}{\frac{a}{r}} \\ &\Rightarrow \tan\alpha = \frac{b}{a} \\ \therefore \alpha &= \tan^{-1}\left(\frac{b}{a}\right) \end{aligned}$$

As such α can be expressed as:

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) \tag{27}$$

Predicting the Amplitude, Phase Shift and Period Of Trigonometric Function Containing Both Cosine and Sine Using the composite function found above, a function containing both sine and cosine can thus be combined into one and makes determining the amplitude, phase shift and period easier. The r found above is known as the amplitude, while α is the phase shift.

Example 1 $f : \theta \mapsto 3\sin\theta + 4\cos\theta$

Equate $3\sin\theta + 4\cos\theta$ with $a\sin\theta + b\cos\theta$, giving:

$$\begin{aligned} 3\sin\theta + 4\cos\theta &\Leftrightarrow a\sin\theta + b\cos\theta \\ \therefore a &= 3 \\ \therefore b &= 4 \end{aligned}$$

The amplitude of the function can be calculated as:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

The phase shift of the function can also be calculated as:

$$\begin{aligned} \alpha &= \arctan\frac{b}{a} \\ &= \arctan\frac{4}{3} \\ (&= 53.1^\circ[1.d.p]) \end{aligned}$$

The period of the function is determined by the coefficient of the theta variable. Normally, i.e. when the coefficient is 1, the period is 2π. That is, if the parameter of function f becomes 2θ, the period becomes $\frac{2\pi}{2} = \pi$. As such, if the parameter supplied is $k\theta$, $k \in \mathbb{R}$, the period then becomes $\frac{2\pi}{k}$

Example 2 $f : \theta \mapsto 3\sin\theta - 4\cos\theta$

Equate $3\sin\theta - 4\cos\theta$ with $a\sin\theta + b\cos\theta$, giving:

$$\begin{aligned} 3\sin\theta - 4\cos\theta &\Leftrightarrow a\sin\theta + b\cos\theta \\ \therefore a &= 3 \\ \therefore b &= -4 \end{aligned}$$

The amplitude of the function can be calculated as:

$$
\begin{aligned}
r &= \sqrt{a^2 + b^2} \\
&= \sqrt{3^2 + (-4)^2} \\
&= 5
\end{aligned}
$$

The phase shift of the function can also be calculated as:

$$
\begin{aligned}
\alpha &= \arctan \frac{b}{a} \\
&= \arctan \frac{-4}{3} \\
(&= -53.1^\circ [1.d.p])
\end{aligned}
$$

The period is the same situation as example 1.

Example 3 $f : \theta \mapsto 2\cos\theta - \sin\theta$

Equate $2\cos\theta - \sin\theta$ with $a\cos\theta - b\sin\theta$, giving:

$$
\begin{aligned}
2\cos\theta - \sin\theta &\Leftrightarrow a\cos\theta - b\sin\theta \\
\therefore a &= 2 \\
\therefore b &= 1
\end{aligned}
$$

The amplitude of the function can be calculated as:

$$
\begin{aligned}
r &= \sqrt{a^2 + b^2} \\
&= \sqrt{1^2 + 2^2} \\
&= \sqrt{5}
\end{aligned}
$$

The phase shift of the function can also be calculated as:

$$
\begin{aligned}
\alpha &= \arctan \frac{b}{a} \\
&= \arctan \frac{1}{2} \\
(&= 26.6^\circ [1.d.p])
\end{aligned}
$$

Example 4 $f : \theta \mapsto \sqrt{5}\sin\theta + \sqrt{11}\cos\theta$

Equate $\sqrt{5}\sin\theta + \sqrt{11}\cos\theta$ with $a\sin\theta + b\cos\theta$, giving:

$$
\begin{aligned}
\sqrt{5}\sin\theta + \sqrt{11}\cos\theta &\Leftrightarrow a\sin\theta + b\cos\theta \\
\therefore a &= \sqrt{5} \\
\therefore b &= \sqrt{11}
\end{aligned}
$$

The amplitude of the function can be calculated as:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(\sqrt{5})^2 + (\sqrt{11})^2} \\ &= \sqrt{5 + 11} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

The phase shift of the function can also be calculated as:

$$\begin{aligned} \alpha &= \arctan \frac{b}{a} \\ &= \arctan \frac{\sqrt{11}}{\sqrt{5}} \\ (&= 56.0^\circ [1.d.p]) \end{aligned}$$

4 Appendix

4.1 TeX Source Code

The TeX Source Code for this document is available at https://github.com/Joe-X/Math-IA-RotatingRectangles.

5 Works Cited

References

Exact Values on the Unit Circle. 2005. URL: `http://www.onlinemathlearning.com/math-trick-unit-circle.html`.

Institute, International GeoGebra. *GeoGebra*. Software. Version 5.0.297.0-3D. URL: `http://www.geogebra.org/`.

LaTeX Tutorial. URL: `https://www.latex-tutorial.com/`.

Zander, Jan Sundermeyer, Daniel Braun, Tim Hoffmann, Benito van der. *TeXstudio*. Software. Version 2.11.2 (hg 6192:72f68414a729). URL: `http://texstudio.sourceforge.net/`.