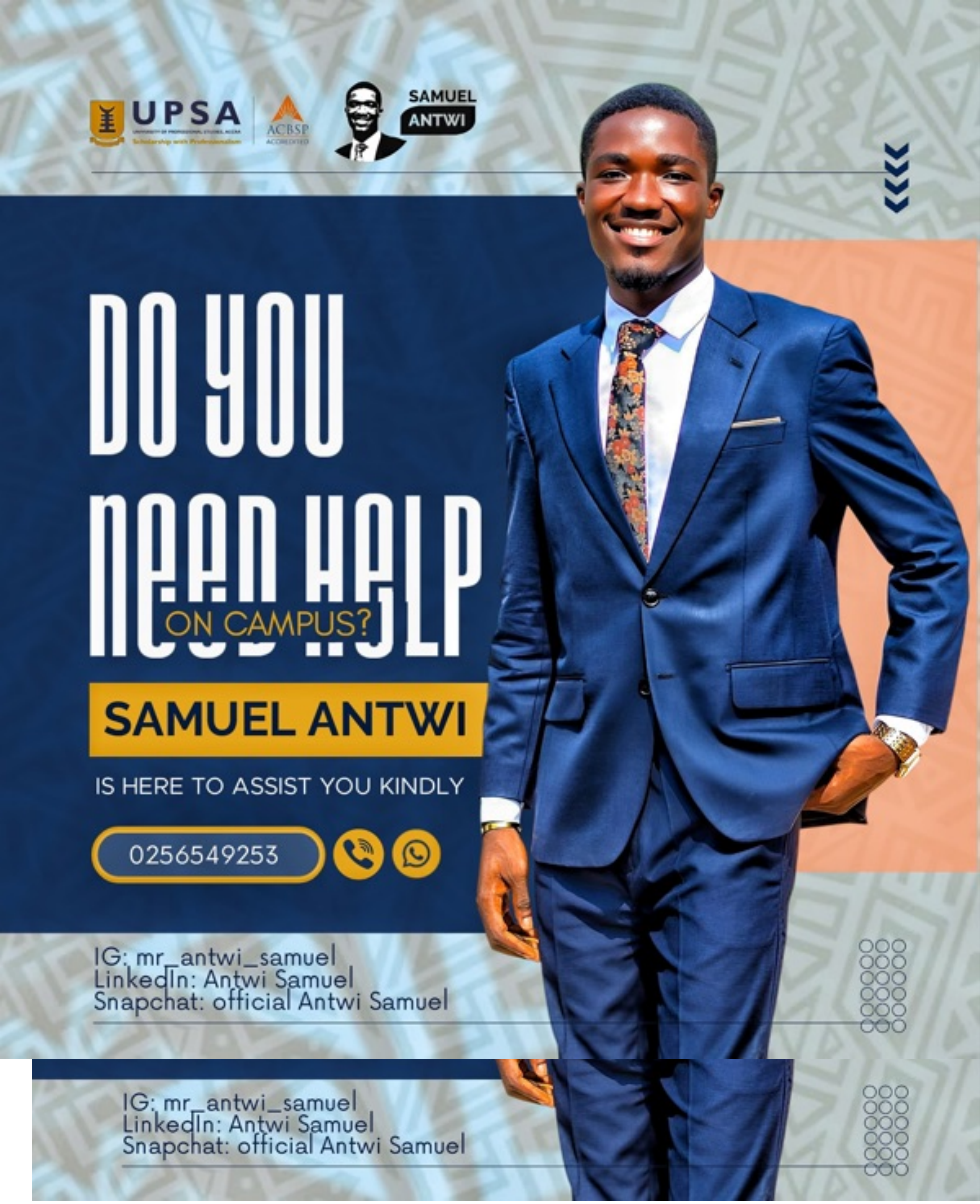
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A. COURSE DETAILS					
COURSE CODE	BCPC 105	**CREDIT HOURS**	3	**LEVEL**	100
COURSE TITLE	Introduction to Business Mathematics				
ACADEMIC YEAR	2024/2025	**SEMESTER**		First	
PROGRAMME[S]	Bachelor of Science in Accounting, Bachelor of Science in Accounting and Finance, Bachelor of Science in Business Economics				
COURSE URL: http//:					

B. COURSE INSTRUCTOR DETAILS	
NAME: COURSE LEAD	Dr. Robert Owusu Boakye
OFFICE LOCATION	Research & Consultancy Block Second Long Room
TELEPHONE	0244986200
EMAIL	boakye.robert@upsamail.edu.gh
CONTACT HOURS	
LECTURER(S)	Dr. Robert Owusu Boakye, Silas Amponsah Gyimah, Adjoa Halm-Quagraine, Ishan Mohammed and Abdul Lateef.

Course Description

This course introduces students to mathematical concepts that have practical applications in industry and business. The content of the course provides students with a basic knowledge of mathematical concepts and helps students translate questions into mathematical representations; and enhances students' analytical skills in terms of evaluating business options and personal finance matters. The course is divided into two parts. The first part covers fundamental arithmetic concepts such surds, indices, exponential and logarithmic functions, polynomials, linear and quadratic equations. The second part covers topics in financial mathematics such as simple and compound interest, matrices, price determination and break-even analysis, depreciation, annuities and amortization. The course will be delivered through class exercises, group discussions and periodic take home assignments.

Course Objectives

The objectives of the course are to:

- Explain types of equations especially from an algebraic perspective.
- Provide mathematical skills related to business applications.
- Discuss calculations related to the mathematics of financial investments.
- Explain concepts relating to time value of money such as simple and compound interest, annuities and amortization.
- Explain basic concepts in algebra, relating to linear equations, quadratic equations, linear and inequalities, indices and logarithmic functions, and its application to business.

Course Outcomes

At the end of the course, students will:

- Derive and solve mathematical equations from word problem.
- Use the various methods to calculate the rate of depreciation of assets, and time value of money mathematical finance.
- Appraise investment project using simple and compound interest approaches.
- Apply mathematical skills to solve and interpret word problems relating to business.
- Perform calculations related to the mathematics of financial investments.

Course Content

- Polynomials
- Linear Equations and Inequalities
- Quadratic Equations and Inequalities
- Rational Expressions and Partial Fractions
- Percentages, Ratios and Proportions
- Indices, Logarithms and Surds
- Systems of Linear Equations in Two and Three Variables
- Price Determination and Break-Even Analysis
- Simple and Compound Interest
- Depreciation
- Annuities and Amortization

Course Content in Weekly Format

The course outline is prepared on weekly format, describing the topics to be thought in each week.

Weeks	Course Content
Week 1	**Polynomials** Polynomials: definition, addition, subtraction and multiplication of polynomials Factoring: factoring of binomials and trinomials
Week 2	**Linear equations and inequalities** Solving simple linear equations; finding gradients and equations of straight lines; sketching linear functions; and manipulating and solving simple linear inequalities.
Week 3	**Quadratic equations and inequalities** Solving quadratic equations by various methods; sketching quadratic functions; and applying quadratic equations in Finding minimum cost and maximum profit/revenue of a firm.
Week 4	**Polynomials continues** Division of polynomials (long division method), the remainder theorem, the factor theorem Use of remainder and factor theorems to solve polynomials equations Sketching of cubic functions
Week 5	**Rational expressions** Addition, Subtraction, Multiplication and Division of Rational Fractions Decomposing Rational Fractions into Partial Fractions
Week 6	**Percentage, ratio and proportion** Percent to measure increase and decrease, ratio and proportion to calculate profit and loss.
Week 7	**Indices and logarithms, and Surds** Roots and powers; rules of powers and indexing actual figures; writing indices in logarithm form and vice-versa. Reducing a surd to its lowest form, and algebra of surds
Week 8	**Systems of linear equations in two and three variables** Using substitution and elimination methods to solve two-variable system Using the inverse of a matrix method and Crammer's rule to solve three-variable system (**But introduction the students to matrices first**: Definition, types, equal matrix, addition, subtraction, multiplication, determinants, inverse)
Week 9	**Systems of linear equations in two and three variables** Using the inverse of a matrix method and Crammer's rule to solve three-variable system continues
Week 10	**Price determination and break-even analysis** Finding equilibrium price and quantity Finding of break-even price and quantity
Week 11	**Simple and compound interest** How to Calculate Simple and Compound Interest on Loans and Other Investments
Week 12	**Depreciation**

	Calculating rate of depreciation using straight-line method and reducing balance method
Week 13	**Annuities and amortization** Discounting and Time Value for Money Loans and Mortgages

Reading Materials

1. Deitz, J. E. & Southam, J. L. (2015). Contemporary business mathematics for colleges, (17^{th} ed.). Cengage Learning.
2. Clendenen, G. & Salzman, S. A. (2014). Business mathematics, (13^{th} ed.). Pearson
3. Blyth, S. (2013). An introduction to quantitative finance, (1^{st} ed.). Oxford University Press.
4. Provost, F. & Fawcett, T. (2013). Data science for business: what you need to know about data mining and data-analytic thinking, (1^{st} ed.). O'Reilly Media.
5. Sterling, M. J. (2008). Business mathematics for dummies, (1^{st} ed.). For Dummies.
6. Leaves, C., Hobbs, M. & Noble, J. (2013). Business mathematics (10^{th} ed.). Pearson
7. ICAG (2017). Quantitative Tools in Business. Study Text. BPP Learning Media.
8. Lial, L. M, Miller, D. M. and Greenwell, R. N. (1993). Finite Mathematics and Calculus with Application (4^{th} edition). HarperCollins College Publishers

POLYNOMIALS

A polynomial is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, n is a natural number, and $a_n \neq 0$.

Examples of polynomials include:

$5x^4 + 2x^3 + 6x$, $8m^2 + 9m^2 - 6m + 3$, $10p$ and -9.

Adding and Subtracting Polynomials

Polynomials can be added or subtracted by using the distributive property, shown below.

If a, b and c are real numbers, then

$$a(b \pm c) = ab \pm ac$$

And

$$(b \pm c)a = ba \pm ca.$$

Only like terms may be added or subtracted. For example

$$12y^4 + 6y^4 = (12 + 6)y^4 = 18y^4$$

And

$$-2m^2 + 8m^2 = (-2 + 8)m^2 = 6m^2,$$

But the polynomials $8y^4 + 2y^5$ cannot be further simplified. To subtract polynomials, use the fact that

$$-(a + b) = -a - b.$$

In the next examples, we show how to add and subtract polynomials.

Example 1

Add or subtract as indicated.

(a) $(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)$
(b) $(-4x^4 + 6x^3 - 9x^2 - 12) + (-3x^3 + 8x^2 - 11x + 7)$
(c) $(2x^2 - 11x + 8) - (-7x^2 + 6x - 2)$

Answers:

(a) $11x^3 + x^2 - 3x + 8$
(b) $-4x^4 + 3x^3 - x^2 - 11x - 5$
(c) $-5x^2 - 5x + 6$

Multiplying Polynomials

The distributive property is used also when multiplying polynomials, as shown in the next example.

Example 2

Multiply the following.

(a) $8x(6x-4)$
(b) $(3p-2)(p^2+5p-1)$

Answers

(a) $48x^2-32x$
(b) $3p^3+13p^2-13p+2$

Note: When two binomials are multiplied, the FOIL method (first, outer, inner, last) is used as a shortcut. This method is shown below.

Example 3

Find $(2m-5)(m+4)$ using the FOIL method.

Solution

$2m-5)(m+4)=(2m)(m)+(2m)(4)+(-5)(m)+(-5)(4)$

$=2m^2+3m-20.$

Example 4

Find $(2k-5)^2$.

We can also expand as follows.

$(2k-5)^2=(2k)^2+(2)(2k)(-5)+(-5)^2$

$=4k^2-20k+25$

Notice that the product of the square of a binomial is the square of the first term, $(2k)^2$, plus twice the product of the two terms, $(2)(2k)(-5)$, plus the square of the last term, $(-5)^2$.

Caution:

Avoid the common error of writing $(x+y)^2=x^2+y^2$. As example 4 shows, the square of a binomial has three terms, so writing $(x+y)^2=x^2+2xy+y^2$.

Further, higher powers of a binomial also result in more than two terms. For example, verify by multiplication that

$$(x+y)^3 = (x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3.$$

Further examples

Expand the following binomials.

1. $(2x+1)^2$
2. $(1+ab)^2$
3. $(1+2y)^2$
4. $(\frac{1}{3}t - 3u)^2$
5. $(u - 3v)^2$

Factoring

Multiplication of polynomials relies on the distributive property. The reverse process, where a polynomial is written as a product of other polynomials, is called **factoring**. For example, one way to factor the number 18 is to write it as the product of $9 \,.\, 2$. When 18 is written as 9.2, both 9 and 2 are called **factors** of 18.

It is also true that $18 = 36 \,.\, {}^{1}\!/_{2}$, but 36 and ${}^{1}\!/_{2}$ are not considered factors of 18, only integers are used as factors.

The number 18 can also be written with three integer factors as $2 \,.\, 3 \,.\, 3$. The integer factors of 18 are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9$ and ± 18.

The Greatest Common Factor

To factor the algebraic expression

$$15m + 45,$$

first note that $15m$ and 45 can be divided be by 15. In fact,

$15m = 15 \,.\, m$ and $45 = 15 \,.\, 3$.

Thus, the distributive property can be used to write

$15m + 45 = 15 \,.\, m + 15 \,.\, 3 = 15(m+3)$.

Now, both 15 and $m + 3$ are factors of $15m + 45$. Since 15 divides into all terms of $15m + 45$, (and is the largest number that will do so), 15 is the **greatest common factor** for the polynomial $15m + 45$.

The process of writing $15m + 45$ as $15(m+3)$ is often called factoring the greatest common factor.

Example 1

Factor out the greatest common factor.

(a) $12p - 18q$
(b) $8x^3 - 9x^2 + 15x$

Answers

(a) $6(2p - 3q)$ or $(2p - 3q)6$
(b) $x(8x^2 - 9x + 15)$ or $(8x^2 - 9x + 15)x$

Factoring Trinomial

A polynomial that has no greatest common factor (other than 1) may still be factored. For example, the polynomial

$$x^2 + 5x + 6$$

can be factored as $(x + 2)(x + 3)$. To see that this is correct, find the product $(x + 2)(x + 3)$; you should get $x^2 + 5x + 6$. To factor a polynomial of three terms such as $x^2 + 5x + 6$, where the coefficient of x^2 is 1, proceed as shown in the following examples.

Example 2

Factorize $y^2 + 8y + 15$.

$y^2 + 8y + 15$ is of the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Steps

1. Find the product of a and c i.e. $a \times c = ac$
2. Find two factors of ac such that the product gives ac but the sum gives the coefficient of x, i.e. b.

Product	Sums
$15\,.\,1 = 15$	$15 + 1 = 16$
$5\,.\,3 = 15$	$5 + 3 = 8$

The numbers 5 and 3 have a product of 15 and a sum of 8. Thus, $y^2 + 8y + 15$ factors as

$$y^2 + 8y + 15 = (y + 3)(y + 5)$$

If the coefficient of the squared term (.i.e x^2) is not 1, work as shown below.

Example 3

(a) Factorize $21x^2 - 4x - 1$
(b) Factorize $2x^2 + 9xy - 5y^2$

Solution

a) $21x^2 - 4x - 1 = 21x^2 - (7x - 3x) - 1$
$= 7x(3x - 1) - 1(3x - 1)$
$= (3x - 1)(7x - 1)$

b) $2x^2 + 9xy - 5y^2 = 2x^2 + (10xy - xy) - 5y^2$
$= 2x(x + 5y) - y(x + 5y)$
$= (2x - y)(x + 5y)$

Special Factorizations

Four special factorizations occur so often that they are listed here for future reference.

1. Difference of two squares: $x^2 - y^2 = (x + y)(x - y)$
2. Perfect square: $x^2 + 2xy + y^2 = (x + y)^2$
3. Differences of two cubes: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
4. Sums of two cubes: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Note that a polynomial that cannot be factored is called prime polynomial.

Trial Questions

1. Perform the indicated operations.

(a) $(2k^2 - 11k + 8) - (7k^2 - 6k + 2)$
(b) $(-4q^2 - 3q + 8) - (2q^2 - 6q - 2)$
(c) $2(3h^2 + 4h + 2) - 3(-h^2 + 4h - 5)$
(d) $(3a + b)(4a - b) + (2a - 3b)(a + 2b)$
(e) $(3y - 4)(4y - 5) + (2y - 3)(y - 5)$

2. Expand and simplify

(a) $(a + 1)^2$

(b) $(x - y)^2$

(c) $(ab - bc)^2$

(d) $(\frac{1}{2}x - 2y)^2$

(e) $(m + \frac{1}{2}n)^2$

3. Expand the following

(a) $\left(\frac{2}{5}y + \frac{1}{8}z\right)\left(\frac{3}{5}y + \frac{1}{2}z\right)$
(b) $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$
(c) $\left(\frac{3q+9}{6}\right)\left(\frac{18}{5q+15}\right)$

4. Factorize

(a) $16x^2 - 25y^2z^2$
(b) $x^2 + 3x - 10$
(c) $3 - 2x - x^2$
(d) $15x^2 + 2x - 1$
(e) $3x^2 - 2xy - 8y^2$

LINEAR EQUALITIES AND INEQUALITIES

Linear Equations

Equations that can be written in the form $ax + b = 0$, where a and b are real numbers, with $a \neq 0$, are linear equations. Examples of linear equations include $5y + 9 = 16$, $8x + 4 = -5x + 32$ and $-3p + 5 = -8$. Equations that are not linear include absolute value equations such as $|x| = 4$.

Properties of Real Numbers

a. Distributive Property: For all real numbers a, b and c; $a(b + c) = ab + ac$
b. Addition Property of Equality: If $a = b$, then $a + c = b + c$ (The same number may be added to both sides)
c. Multiplication Property of Equality: If $a = b$, then $ac = bc$ (The same number may be multiplied on both sides of the equation)

Worked Examples

Solve the following equations:

1. $2x - 5 + 8 = 3x + 2(2 - 3x)$. Ans $x = {}^1\!/_5$
2. $\frac{t}{10} - \frac{2}{15} = \frac{3t}{20} - \frac{1}{5}$. Ans $t = {}^4\!/_3 = 1\,{}^1\!/_3$
3. $\frac{2}{k-3} + \frac{1}{k} = \frac{6}{k^2-3k}$. Ans Has no solution ($k \in R, k \neq 3$).

Absolute Value Equations

The absolute value of x i.e. $|x|$= (x, if x is greater than or equal to 0 also, $-x$, if x is less than 0)

Equations with Absolute value: if K is greater than or equal to 0, then $|ax + b| = k$ is equivalent to $ax + b = k \text{ or} - (ax + b) = k$

Examples:

Solve the following equations:

1. $|2x - 5| = 7$
2. $|3x + 1| = |x - 1|$
3. $\frac{\sqrt{3x^2+x}}{2\sqrt{x}} = 7$. Ans $x \neq 0, \therefore x = 65$
4. $4\sqrt{x} + 32 = 40.6718$. Ans $x \approx 4.7$

Solutions

1. $2x - 5 = 7$ or $-(2x - 5) = 7$

$2x = 12$ or $-2x + 5 = 7$

$x = 6$ or $-2x = 2 \quad \Rightarrow \quad x = -1$

2. $3x + 1 = -(x - 1)$ or $3x + 1 = x - 1$

$3x + x = 1 - 1$ or $3x - x = -1 - 1$

$4x = 0$ or $2x = -2$

$x = 0$ or $x = -1$

Linear Inequalities

Linear Inequalities: An equation states that two expressions are equal; an inequality states that they are unequal. A linear inequality is an inequality that can be simplified to the form $ax < b$ or $ax > b$ or $ax \leq b$ or $ax \geq b$.

Properties of Inequality: For all real numbers a, b and c.

1. If $a < b$, then $a + c < b + c$
2. If $a < b$, and if $c > 0$, then $ac < bc$
3. If $a < b$, and if $c < 0$, then $ac > bc$

Solving Inequalities:

1. Solve the inequality $4 - 3y \leq 7 + 2y$, and represent your answer on a number line.
2. Solve $-2 < 5 + 3m < 20$, and represent your answer on a number line.
3. Solve $-4 \leq \frac{2x-1}{3} \leq 2$, and represent your answer on a number line.

Linear Functions

Linear Function: A function f is linear if its equation can be written as $f(x) = ax + b$, for real numbers a and b. The general form of a linear equation is $y = mx + c$ where m is the slope or gradient and c is the y −intercept.

Slope of a Straight line

The slope or gradient m of a non-vertical straight line through two distinct points (x_1, y_1) and (x_2, y_2) is

$$slope = m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} \ or \ \frac{y_2 - y_1}{x_2 - x_1}$$

Example:

1. Find the slope of the line through the following pair of points.

a) $(-4, 8)$ and $(2, -3)$
b) $(2, 7)$ and $(2, -4)$
c) $(5, -3)$ and $(-2, -3)$

Equation of a line

The equation of a straight line with gradient m and passes through the point (x_1, y_1) is given by: $y - y_1 = m(x - x_1)$

Example:

Write the equation of the line;

a) through the point (-4, 1) with gradient -3
b) through which the points (-3, 2) and (2, -4)

How to Sketch a Graph

1. Sketch the graph $3x + 2y = 6$
2. Sketch the graph $y = -3x - 11$

Solutions

For instances $3x + 2y = 6$, set $x = 0$; and find y, and this gives the point $(0,3)$. Also, set $y = 0$ and find x, we have $(2,0)$.

Plots these two points in the $x - y$ plane, and draw a straight line through the points.

Parallel and Perpendicular Lines

Parallel lines

Two non-vertical lines are parallel if and only if they have the same slope or gradient.

Example:

Find the equation of the line that passes through the point (3, 5) and parallel to the line $2x + 5y = 4$.

Perpendicular lines

Two lines, neither of which is vertical, are perpendicular if and only if their slopes have a product of -1.

Example:

Find the slope of the line L perpendicular to the line having the equation $5x - y = 4$.

APPLICATIONS OF LINEAR FUNCTIONS

Linear function has a number of applications in economics, finance and commerce. Examples are the linear cost function, demand and supply functions, budget equation of producing or consuming two commodities.

Linear Cost Function

In a cost function of the form $\boldsymbol{C(x) = mx + b}$, **m** represents the marginal cost per item and **b** is the fixed cost. Remember total cost, $C(x)$ is $\boldsymbol{C(x) = VC + FC}$.

Average cost function

If $T(x)$ is the total cost to manufacture x items, then the average cost per item is given by; $\bar{T}(x) = \frac{T(x)}{x}$, where x is the number of items produced or manufactured.

Example

1. A soft-drink manufacturer can produce 1000 case of soda in a week at a total cost of GH¢$6{,}000$, and $1{,}500$ cases of soda at a cost of GH¢$8{,}500$. Fid the manufacturer's weekly fixed costs and marginal cost per case of soda.

Solution

If $c = 6{,}000$, then $x = 1{,}000$

$$1000m + b = 6000 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If $c = 8{,}500$, then $x = 1{,}500$

$$1500m + b = 8500 \ldots\ldots\ldots\ldots\ldots.. (2)$$

Solving equations 1 and 2, we have $m = 5$ and $b = 1000$.

The linear cost function is, $\boldsymbol{C(x) = 5x + 1000}$.

The fixed cost is **GH¢$\boldsymbol{1,000}$** and the marginal cost is **GH¢$\boldsymbol{5}$**

Linear Demand Function: $D(p) = mp + b, m < 0$.

Just draw a negatively sloped straight line in the $x - y$ plane to illustrate the demand function.

Linear Supply Function: $S(p) = mp + b, m > 0$.

Just draw a positively sloped straight line in the $x - y$ plane to illustrate the supply function.

Trial Questions

1. The cost of renting tuxes for the Choral Society's formal is $\$20$ down, plus $\$88$ per tux. Express the cost, C, as a function of x, the number of tuxedos rented. Use your function to answer the following questions.

a. What is the cost of renting 2 tuxes? Ans $C(2) = 88(2) + 20 = 196$.

b. What is the cost of the $2nd$ tux? Ans $C(2) - C(1) = 196 - 108 = 88$.

Assignment

1. The RideEm Bicycles factory can produce 100 bicycles in a day at a total cost of $\$10{,}500$ and it can produce 120 in a day at a total cost of $\$11{,}000$. What is the company's daily fixed costs, and marginal cost per bicycle?

2. The marginal cost of producing a certain product is GH¢12 per unit, while the cost to produce 100units is GH¢1,500.

a) Find the cost function, $C(x)$, given that it is linear.
b) Find the average cost per item to produce 50 units and 300 units.

3. When a graphing calculator is priced at $\$90$, a district manager at Office Depot observes that monthly sales average 480 units in his stores. When the price is reduced periodically for a special sale, his average monthly sales increase by 40 calculators per $\$4$ price decrease.

a) Find the linear equation which expresses this relationship.
b) Why is slope negative?
c) Find the demand for calculators when the price is set at $\$100$.

QUADRATIC FUNCTION

Quadratic functions are applied in economics, finance and commerce. The graph of a quadratic function may be used to represent the revenue/profit function or cost function of a firm.

The General Form of a Quadratic Function

A quadratic function is defined by $f(x) = ax^2 + bx + c$, where a, b and c are real numbers, with $a \neq 0$.

Graphs of Quadratic Function

The graph of a quadratic function is concave if $a < 0$ and convex if $a > 0$.

The value of x, where maximum or minimum point of a quadratic function, is given by $x = -\frac{b}{2a}$

Examples:

Sketch the following graphs

a) $y = -3x^2 - 2x$
b) $y = x^2 + 4x + 6$

Quadratic Equation

A quadratic equation is of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers with $a \neq 0$.

Solution of a Quadratic Equation

Quadratic equation can be solved using the following methods:

a) Factorization
b) Completing of squares
c) By a formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Examples:

Solve the following equations **(Please solve at least 2 questions with them)**

a) $3Q^2 - 2Q + 1 = 0$
b) $4K^2 - 12K = 9$
c) $3q^2 - 9q + 5 = 0$

How to sketch quadratic function

Sketch the following graphs **(Please, help the students to solve question (b) and allow them to solve (a) themselves)**

(a) $f(x) = 2x^2 - x - 3$
(b) $y = 2 + x - x^2$

Solutions

Steps:

1) Find the x −intercepts. That is, find the zeros of $f(x)$.
2) Find the turning point (maximum or minimum point) of the graph. That is, find the point $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
3) Draw a smooth curve through the three points.

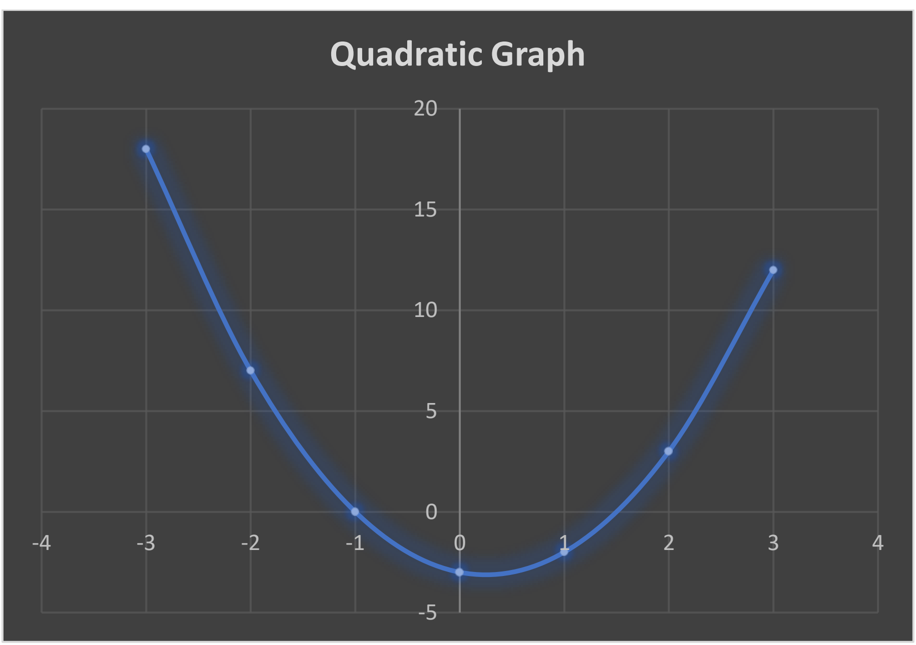
From (a), comparing $2x^2 - x - 3$ to $ax^2 + bx + c$, we have $a = 2, b = -1, c = -3$.

$2x^2 - x - 3 = (2x - 3)(x + 1)$.

Hence, the zeros of $f(x)$ are $x = \frac{3}{2}$ and $x = -1$. So, the x −intercepts are $\left(\frac{3}{2}, 0\right)$ and $(-1,0)$.

For the turning points, $x = -\frac{-1}{2(2)} = \frac{1}{4}$, and

$f\left(\frac{1}{4}\right) = -\frac{11}{4}$. Therefore, the turning point is $\left(\frac{1}{4}, -\frac{11}{4}\right)$.



Applications of Quadratic functions to Business

(Cost, Revenue and Profit Functions)

In general, the total cost (C) function is given by $\boldsymbol{C(Q) = mQ + b}$, where

C(Q) = *total cost*

$\boldsymbol{b}$ = *fixed cost*

$\boldsymbol{m}$ = *marginal cost*

Q = *number of quantities produced*

If **P** is the price of a unit of the goods produced or sold, then the total revenue function **(TR)** is given by;

$\boldsymbol{Total\ Revenue, (R) = Q.P}$

$\rightarrow \boldsymbol{R(Q) = PQ}$

It follows that the total profit, $\pi(Q)$, is given by $\boldsymbol{\pi(Q) = Total\ Revenue - Total\ cost}$

$\therefore Profit, \boldsymbol{\pi(Q) = R(\mathbf{Q}) - C(\mathbf{Q})}$

Worked Examples.

1. The demand for a certain type of cosmetic is given by

$$p = 500 - Q$$

Where p is the price in dollars when Q units are demanded.

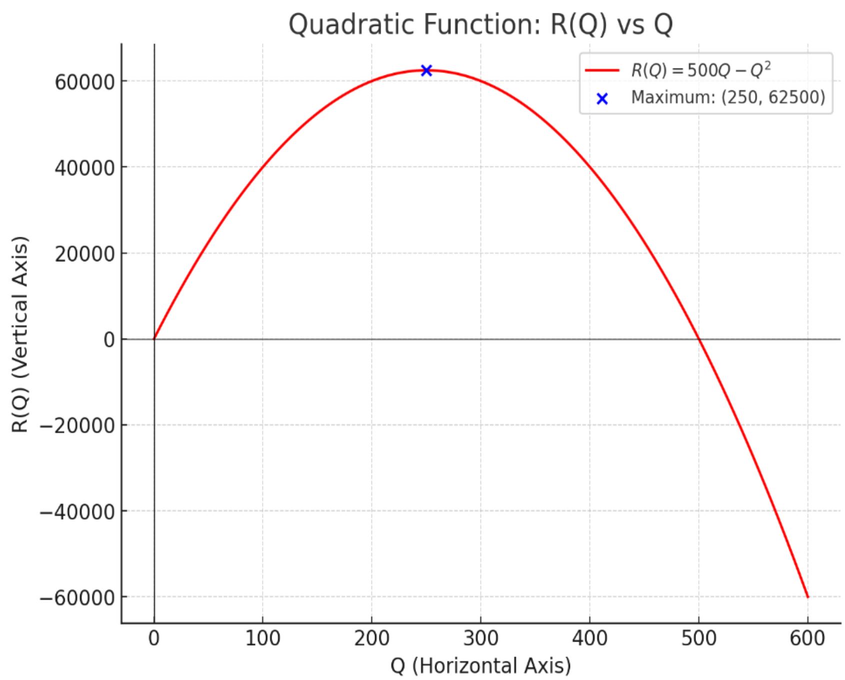
a) Find the revenue, $R(Q)$, that would be obtained at a price of p.
b) Graph the revenue function, $R(Q)$.
c) From the graph of the revenue function, estimate the price that will produce maximum revenue.
d) What is the maximum revenue?

Solution

(a) Revenue, $R(Q) = PQ$

$$R(Q) = (500 - Q)Q = 500Q - Q^2$$

(b) The graph,



(c) From the graph, the price that will produce the maximum profit is

$$Q = \frac{-b}{2a} = \frac{-(500)}{2(-1)} = 250$$

(d) $R(250) = 62500$.

2. Suppose you operate a Fitness Health Club, and you have calculated your demand equation to be:

$$\boldsymbol{q = -0.06p + 84}$$

where q is the number of members in the club and p is the annual membership fee you charge.

a. Your annual operating costs are a fixed cost of GH¢20,000 per year plus a variable cost of GH¢20 per member. Find the annual revenue and profit as functions of the membership price p.

b. At what price should you set the annual membership fee to obtain the MAXIMUM revenue? What is the maximum possible revenue?

c. At what price should you set the annual membership fee to obtain the maximum profit? What is the possible maximum profit? What is the corresponding revenue?

Solution

a. The annual revenue is given by

$$R = pq$$

$$= p(-0.06p + 84)$$

Therefore, the **annual revenue function,** $\boldsymbol{R(p) = -0.06p^2 + 84p}$**.**

The annual cost, $C(q) = 20q + 20{,}000$. But $q = -0.06p + 84$

$$C(-0.06p + 84) = 20(-0.06p + 84) + 20{,}000$$

The **annual cost function,** $\boldsymbol{C(p) = -1.2p + 21{,}680}$**.**

Thus, the profit function is given as

$$\pi(p) = R(p) - C(p)$$

$$\pi(p) = (-0.06p^2 + 84) - (-1.2p + 21{,}680)$$

Annual Profit, $\boldsymbol{\pi(p) = -0.06p^2 + 85.2p - 21{,}680}$**.**

b. From $\boldsymbol{R(p) = -0.06p^2 + 84p}$**,** $a = -0.06, b = 84, c = 0$

The maximum revenue corresponds to the highest point of the graph: the vertex, of which the p-coordinates is

$$p = -\frac{b}{2a} = -\frac{84}{2(-0.06)} \approx 710$$

$R(700) = -0.06(700)^2 + 84(700) = \mathbf{GH¢29{,}400}$.

c. From $\boldsymbol{\pi(p) = -0.06p^2 + 85.2p - 21{,}680}$**,** $a = -0.06, b = 85.2, c = -21{,}680$

$$p = -\frac{b}{2a} = -\frac{85.2}{2(-0.06)} \approx 700$$

$\pi(710) = -0.06(710)^2 + 85.2(710) - 21{,}680 =$ GH¢8,566.

The corresponding revenue is $R(710) = -0.06(710)^2 + 84(710) =$ GH¢$\mathbf{29{,}394}$**.**

Trial Examples

1. A manufacturer knows that if *x* (in hundreds) products are demanded in a particular week, the total cost function (‘000 GH¢) is $14 + 3x$ and the total revenue function (‘000 GH¢) is $19x - x^2$.
   a) derive the total profit function.
   b) find the number of *x* to be produced to achieve the maximum profit.
   c) find the greatest profit.

2. Assume a firm operating in a purely competitive market has a constant selling price of GH¢60, a fixed cost of GH¢450 and a variable cost of GH¢35 an item. Derive;
   a) the total revenue function.

b) the total cost function.
c) the profit function.

3. A firm has a fixed cost of GH¢125,000 and variable cost of GH¢685 per item manufactured. Express the firm's total cost (TC) as a function of output (x).

4. Ms. Tilden owns and operates Aunt Emma's Pie Shop. She has hired a consultant to analyze her business operations. The consultant tells her that her profit $P(x)$ from the sale of x units of pies is given by $P(x) = 120x - x^2$. How many units of pie must be sold in order to maximize the profit? What is the maximum possible profit (in dollars)?

Home Work

1. George Duda runs a Sandwich shop. By studying data concerning his past cost, he has found that the cost of operating his shop is given by: $C(x) = 2x^2 - 20x + 360$, where $C(x)$is the daily cost in dollars to make x batches of sandwiches.

a) find the number of batches George must sell to minimize the cost.
b) what is the minimum cost?
find a formula for the average daily cost per batch of sandwiches as a function of x.
c) find the average daily cost per batch of sandwiches when 7 batches are produced.

2. Cat Co. believes that demand for its product can be represented by $P = 10 - 0.003Q$, where P is the unit price in GH¢ and Q is the quantity of sales. The total cost function is (in GH¢).

$C(Q) = 1{,}000 + 3Q + 0.004Q^2$

We need to:
a) calculate the level of output and the unit price at which profit will be maximized.
b) calculate the amount of profit at this level.

CUBIC EQUATIONS AND FUNCTIONS

Cubic Equations

A cubic equation is of the form

$$ax^3 + bx^2 + cx + d = 0$$

where a, b, c and d are constants, and $a \neq 0$.

We can use the remainder and factor theorems to solve cubic equations if we can factorize the polynomial. Before we start, let us now look at how to divide polynomials using the long division method.

Division of Polynomials

A polynomial, P, can be divided by another polynomial Q, provided the degree of Q is not greater than that of P.

So, we could divide $x^3 + 3x^2 - x + 1$ (3^{rd} degree), by $x - 2$ (1^{st} degree), but not $x - 2$ by $x^3 + 3x^2 - x + 1$.

Example 1

Divide $x^3 + 3x^2 - x + 1$ by $x - 2$.

This is done by long division, using the x term of the divisor.

$$x^2 + 5x + 9$$

$$x - 2\overline{\big)\, x^3 + 3x^2 - x + 1}$$

$$\underline{-(x^3 - 2x^2)}$$

$$5x^2 - x$$

$$\underline{-(5x^2 - 10x)}$$

$$9x + 1$$

$$\underline{-(9x - 18)}$$

19 – remainder

So, the quotient is $x^2 + 5x + 9$ and the remainder is 19.

We can write

Polynomial $(\boldsymbol{P})$ = Divisor $(\boldsymbol{Q}) \times$ Quotient + Remainder $(\boldsymbol{R})$.

Hence, $x^3 + 3x^2 - x + 1 = (x - 2) \times (x^2 + 5x + 9) + 19$.

Example 2

Divide $x^3 + 3x^2 + 1$ by $x^2 - x + 1$ (Lecturer guide students).

Answer: $x^3 + 3x^2 + 1 = (x^2 - x + 1) \times (x + 4) + 3x - 3$.

Example 3

(a) Carry out the following divisions. Write each result in the form $P = (D \times Q) + R$.
   i. $(x^3 - 2x^2 + 3x - 2) \div (x - 1)$
   ii. $(4x^3 - x^2 + x - 5) \div (x^2 + x - 1)$
   iii. $(2x^3 + x^2 - 3) \div (2x - 1)$

The Remainder Theorem

Divide the function $f(x) = x^2 - 5x + 6$ by $x - 4$. You should find that the quotient is $x - 1$ and the remainder is 2. (students perform this task)

Now, evaluate $f(4)$: What do you find? $f(4) = 2$. (student perform this task)

Repeat, dividing $f(x)$ by $x - 3$: Compare your remainder with the value of $f(3)$.

Repeat again, dividing $f(x)$ by $x + 2$: Compare your remainder with the value of $f(-2)$.

These results are not coincidences. It is true that when we divide a polynomial, $\boldsymbol{f(x)}$**,** by $(\boldsymbol{x - a})$ the remainder is the value of $\boldsymbol{f(a)}$.

Similarly, if we divide $\boldsymbol{f(x)}$ by $(\boldsymbol{x + a})$**,** the remainder will be the value of $\boldsymbol{f(-a)}$**.**

The factor Theorem

Finding remainders is not of much interest, except when the remainder is zero. Then, as we saw in examples 1&2, we know that the divisor is a factor of the polynomial. Thus, the remainder theorem can be used to factorize polynomials, if it is possible. When used in this way, it is sometimes called the factor theorem.

If $f(x)$ is divided by $(x - a)$ and $f(a) = 0$, then there is no remainder. This means that $(x - a)$ is a factor of $f(x)$. The factor theorem states that

If $\boldsymbol{f(a) = 0}$, then $\boldsymbol{x - a}$ is a factor of $\boldsymbol{f(x)}$.

Applying the Remainder and Factor Theorems

Example 1

Factorize $x^3 - 2x^2 - 5x + 6$.

We try $(x + 1)$:

$$f(-1) = -1 - 2 + 5 + 6 \neq 0 \ (x + 1) \text{ is not a factor.}$$

Try $(x - 1)$:

$$f(1) = 1 - 2 - 5 + 6 = 0 \ (x - 1) \text{ is a factor.}$$

We now divide $x^3 - 2x^2 - 5x + 6$ by $x - 1$.

$$\begin{array}{r} x^2 - x - 6 \\ x - 1\sqrt{x^3 - 2x^2 - 5x + 6} \\ \underline{-(x^3 - 2x^2)} \\ -x^2 - 5x \\ \underline{-(-x^2 + x)} \\ -6x + 6 \\ \underline{-(-6x + 6)} \\ 0 \end{array}$$

0 – No remainder.

So, $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6) = (x - 1)(x + 2)(x - 3)$.

Example 2

It is known that $(x - 1)$ and $(x - 2)$ are factors of $x^3 + px^2 - 7x + q$, where p and q are constants. Find p and q and the third factor.

Let $f(x) = x^3 + px^2 - 7x + q$.

$x - 1$ is a factor of $f(x)$, $f(1) = 0$.

$$1 + p - 7 + q = 0$$

$$p + q = 6 \ldots\ldots (1)$$

Similarly, $(x - 2)$ is factor of $f(x)$, $f(2) = 0$.

$$8 + 4p - 14 + q = 0$$

$$4p + q = 6 \ldots\ldots (2)$$

Solving equations (1) and (2):

$$p = 0 \text{ and } q = 6$$

$f(x) = x^3 - 7x + 6.$

Next, we divide $x^3 + 0x^2 - 7x + 6$ by $(x-1)(x-2) = x^2 - 3x + 2$, using the long division.

Consequently, the third factor is $(x+3)$.

Solving Cubic Equations

Example

Solve the equation $2x^3 - 3x^2 - 5x + 6 = 0$.

Let $f(x) = 2x^3 - 3x^2 - 5x + 6$

Try $(x-1)$: $f(1) = 2 - 3 - 5 + 6 = 0$. This means that $(x-1)$ is a factor of $f(x)$.

Now, we divide $2x^3 - 3x^2 - 5x + 6$ by $(x-1)$. This gives $2x^2 - x - 6$.

$$(2x^2 - x - 6)(x-1) = 0$$

$$(x-1)(2x+3)(x-2) = 0$$

$$x = 1, -{}^{3}\!/_{2} \text{ or } 2$$

Cubic Functions

A typical cubic function is of the form

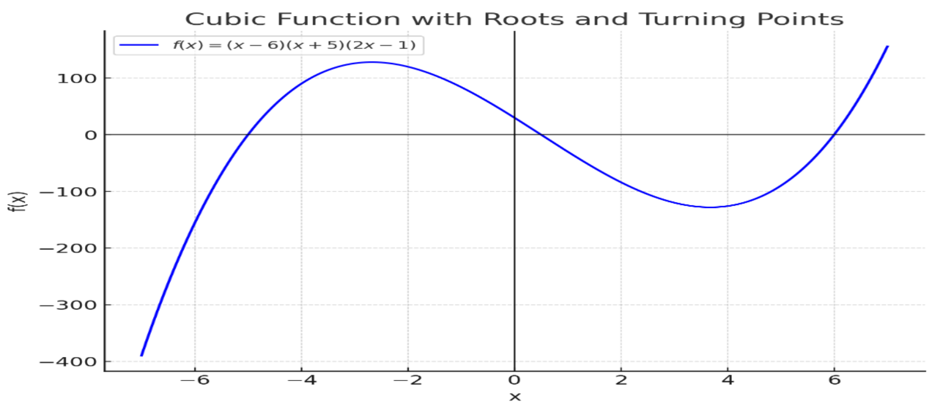
$$f(x) = ax^3 + bx^2 + cx + d$$

where a, b, c and d are constants.

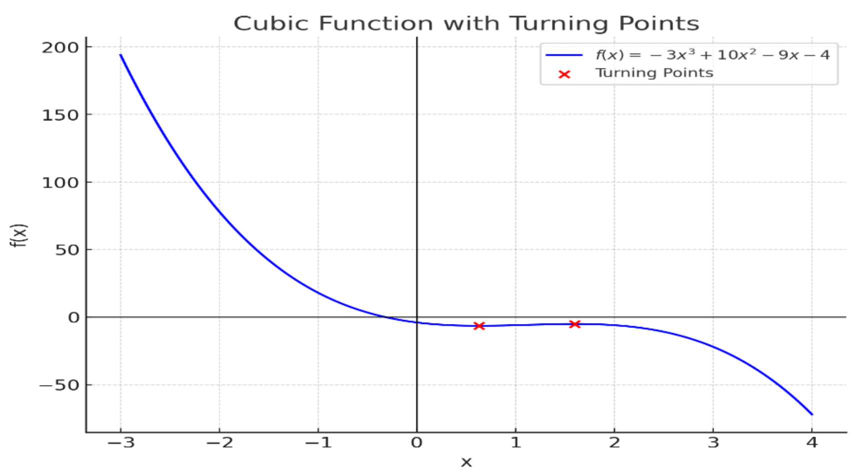
Graph of a Cubic Function

The graph of a cubic function, where

(i) $a > 0$ has the shape



(ii) $a < 0$ has the shape



Sketch the graphs of

(a) $f(x) = 2x^3 - 3x^2 - 59x + 30$

(b) $f(x) = -3x^3 - 10x^2 + 9x + 4$

For $f(x) = 2x^3 - 3x^2 - 59x + 30 = (x - 6)(x + 5)(2x - 1)$

Step 1

Finding the x −intercepts by solving

$$(x - 6)(x + 5)(2x - 1) = 0$$

$$x = 6, -5 \text{ and } {}^{1}/_{2}$$

Step 2

Find the turning points

$x = \frac{-5+0.5}{2} = -2.25$

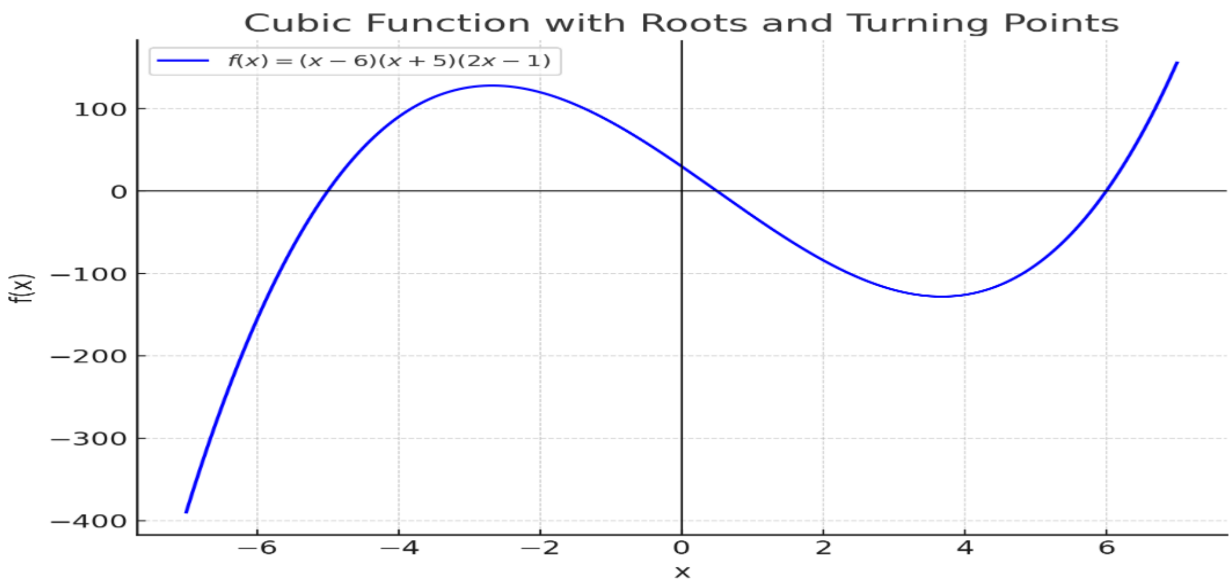
$y = (-2.25 - 6)(-2.25 + 5)(2(-2.25) - 1) = 124.8 \approx 125$

$(-2.25, 125)$

Similarly, $x = \frac{0.5+6}{2} = 3.25$

$y = (3.25 - 6)(3.25 + 5)(2(3.25) - 1) = -124.8 \approx -125$

$(3.25, -125)$



Lecturer guide students to try this:

sketch $f(x) = -3x^3 - 10x^2 + 9x + 4 = (4 + x)(1 - x)(1 + 3x)$

Home Work

1. Sketch the graph of $f(x) = 2x^3 - 3x^2 - 12x + 1$.

2. $(x + 2)$ is a factor of the polynomial $mx^3 - 2x^2 + nx - 18$. When this polynomial is divided by $(x + 1$, the remainder is 18.

(a) Find the values of m and n.

(b) With these values, find the remaining factors of the polynomial.

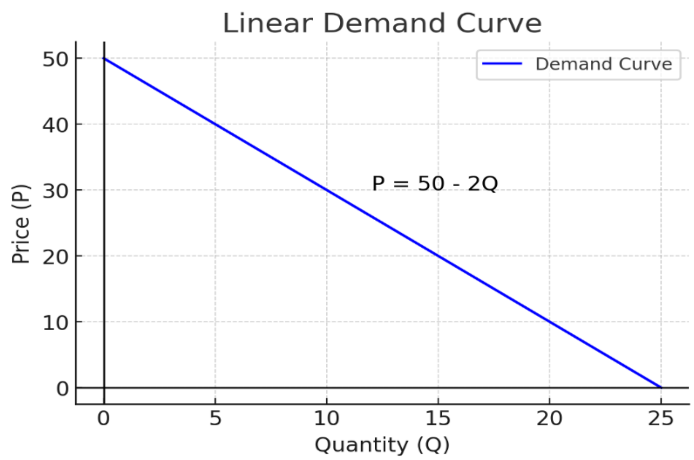
(c) Hence, solve the equation $mx^3 - 2x^2 + nx = 18$.

(d) Sketch the graph of the polynomial using the values of m and n you have found.

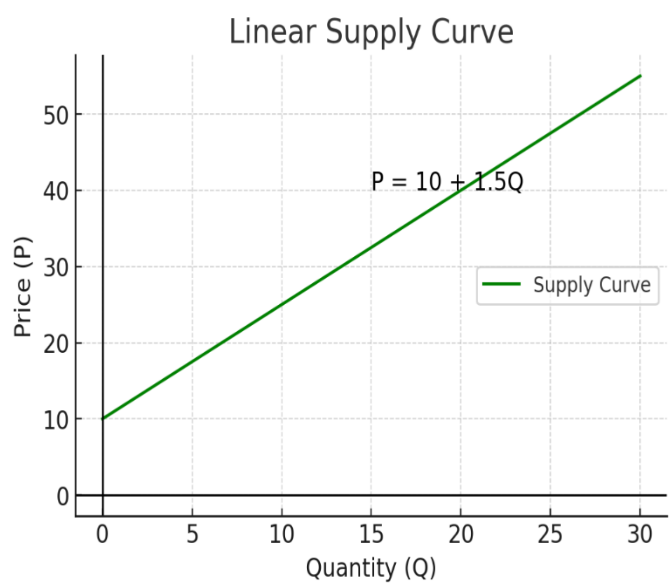
PRICE DETERMINATION

Demand and supply interact to determine to market or equilibrium price and quantity. In this unit, we learn how to estimate the market price and quantity. We also learn how to estimate the break-even quantity.

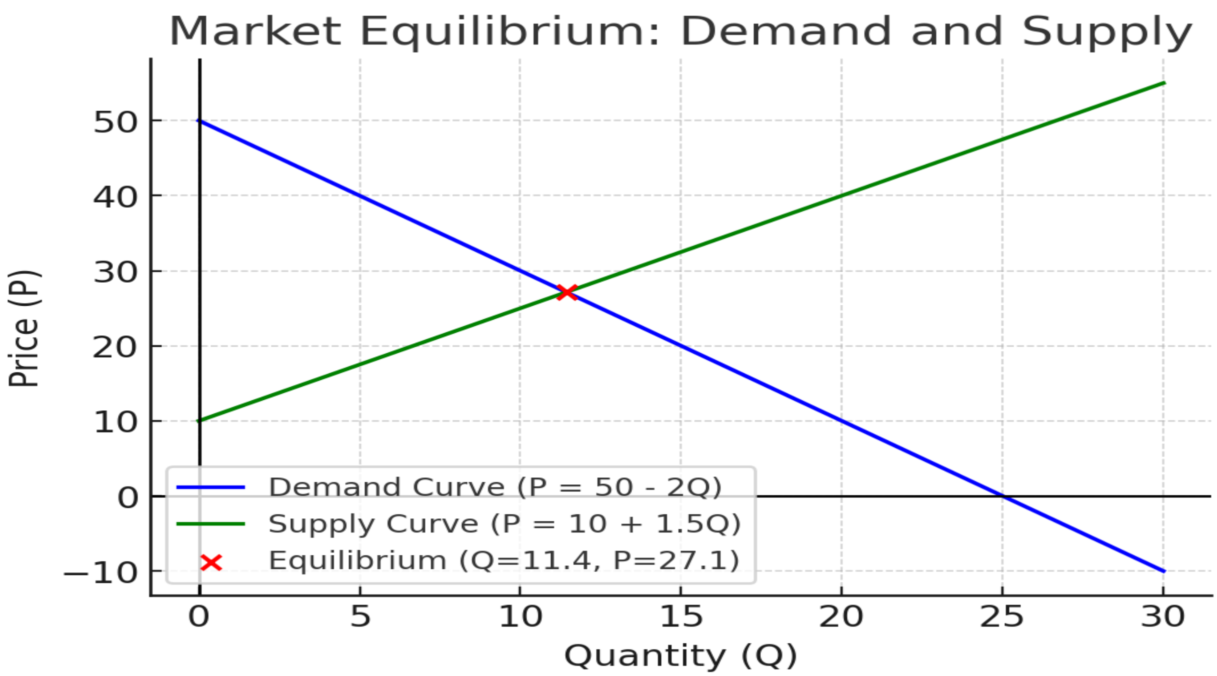
A demand equation express demand Q as a function of the unit price P. The linear demand function has the form $\boldsymbol{D(p) = mp + b}$ or $\boldsymbol{P = mQ + b}$, where $\boldsymbol{m < 0}$ measures the change in demand per unit change in price. A typical linear demand function, with equation $P = 50 - 2Q$, is shown in the figure below.



A supply equation express supply Q as a function of the unit price p. The supply function has the form $\boldsymbol{S(p) = mp + b}$ or $\boldsymbol{P = mQ + b}$, where $\boldsymbol{m > 0}$. A typical linear supply function, with equation $P = 10 + 1.5Q$, is illustrated in the figure below.



Demand and supply are said to be in equilibrium when demand equals supply. The corresponding values of P and Q are called the equilibrium price and equilibrium quantity. The figure below shows the linear demand and supply functions drawn on the same graph. The point indicated red is the equilibrium point.



To find the equilibrium price, set demand equal to supply and solve for the unit price p. To find the equilibrium demand, evaluate the demand (or supply) function at the equilibrium price.

$$D(Q) = S(Q)$$

Example 1

Suppose that Dr. Boakye, a financial economist, has studied the supply and demand for a particular type of ATL textile and has determined that the price in GH¢ per square yard, p and the quantity demanded monthly in thousands of square yards, q are related by the linear function

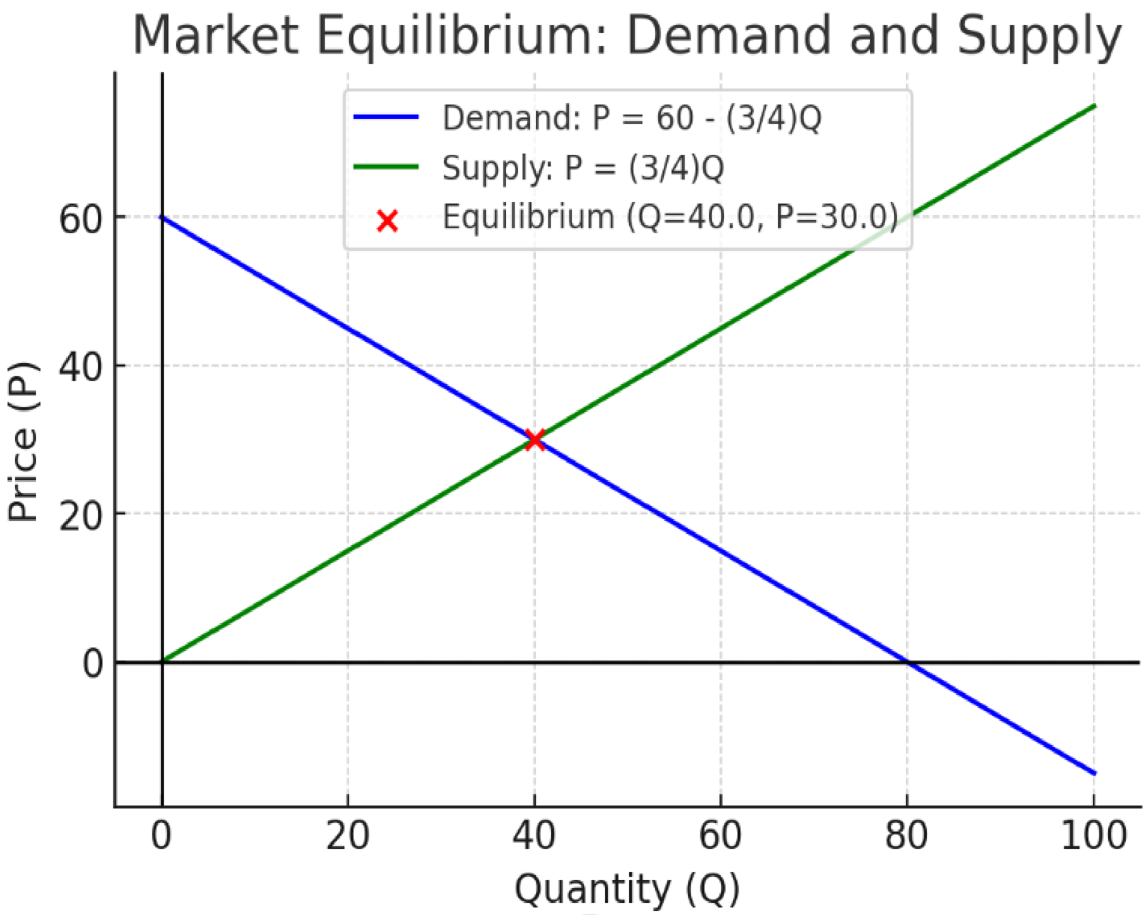
$p = D(q) = 60 - \frac{3}{4}q$, where the price p and the supply are related by $p = S(q) = \frac{3}{4}q$.

a. Find the demand at a price of GH¢45 and at a price of GH¢18.
b. Find the supply at the price of GH¢60 and at a price of GH¢12.
c. Graph both functions on the same axes.
d. Find the equilibrium price and quantity.

Solution

a) Given $p = 60 - \frac{3}{4}q$, if $P = 45$, then $q = 20$.
Also, if $p = 18$, then $q = 56$.

b) Given $p = \frac{3}{4}q$, if $p = 60$, then $q = 80$.
and if $p = 12$, $q = 16$.



c)

d) The equilibrium occurs where demand equals supply, i.e. $D(Q) = S(Q)$

$$60 - \frac{3}{4}q = \frac{3}{4}q$$

Solving the above equation, gives $Q = 40$.
Substituting $Q = 40$ into the demand or supply function gives $P = 30$.

Trial Questions

1. The demand and supply functions for two interdependent commodities are given by;

$$Qd_1 = 145 - 2P_1 + P_2$$
$$Qs_1 = -45 + P_1$$
$$Qd_2 = 30 + P_1 - 2P_2$$
$$Qs_2 = -40 + 5P_2$$

Where Qd_i, Qs_i and P_i denote the quantity demanded, quantity supplied and price of good i respectively.

Determine the equilibrium price and quantity for this two-commodity model. Are these goods substitutable or complementary?

2. The demand and supply equations of a good are given by;

$$4p = -Q_d + 240$$
$$5p = Q_s + 30$$

Determine the equilibrium price and quantity.

3. The demand and supply equations for two complementary good trousers (T) and Jacket (J) are given by;

$$Qd_T = 410 - 5P_T - 2P_J$$
$$Qs_T = -60 + 3P_T$$
$$Qd_J = 295 - P_T - 3P_J$$
$$Qs_J - 120 + 2P_J$$

Respectively where, Qd_T, Qs_T and P_T denote the quantity demanded, quantity supplied and price of Trousers, and Qd_J, Qs_J and P_J denote the quantity demanded, quantity supplied and price of Jackets. Determine equilibrium price and quantity for this two-market model.

4. Given the supply and demand functions;

$$P = Qs^2 + Qs + 32$$
$$P = -Qd^2 - 4Qd + 200$$

Calculate the equilibrium price and quantity.

Break – Even Analysis

Break-even analysis is a financial tool which helps you to determine at what stage your company or a new service or a product, will be profitable.

Break-even is a situation where the firm is neither making profit nor losing money, but all your costs have been covered. The revenue $R(Q)$ from selling Q units of a product is the product of the price per unit p and the number of units sold (demand) Q, so that

$$R(Q) = PQ$$

Then profit $\boldsymbol{\pi(Q) = R(Q) - C(Q)}$, where $C(Q)$ is the cost of selling Q units.

At the break-even point, profit is zero.

$$\boldsymbol{\pi(Q) = 0} \textbf{ or } \boldsymbol{\pi(Q) = R(Q) - C(Q)} \textbf{ or } \boldsymbol{R(Q) = C(Q)}.$$

Example 1

A firm producing poultry feed finds that the total cost of producing and selling x units is given by $C(x) = 20x + 100$. Management plans to charge GH¢24 per unit for the feed.

a. How many units must be sold for the firm to break-even?
b. What is the profit if 100 units of feed are sold?

Solution

a). At the break-even point,$R(x) = C(x)$. But $R(x) = 24x$.

So, solving, $24x = 20x + 100$, gives $x = 25$ units.

The break-even point ($x = 25$) is the break-even quantity. If the company sells more than 25 units (if $x > 25$), it makes a profit. If it sells less than 25 units, it loses money.

b). Let profit= $P(x)$, then $P(x) = R(x) = C(x)$.

$P(x) = 24x - (20x + 100)$

$P(x) = 4x - 100$

$P(100) = 4(100) - 100 = 300$. The firm will make a profit of 300GHc from the sale of 100 units of feed.

Home Work

1. Given the supply and demand functions;

$$P = Qs^2 + Qs + 32$$
$$P = -Qd^2 - 4Qd + 200$$

Calculate the equilibrium price and quantity.

2. The cost to produce x units of wire is given by $C(x) = 50x + 5000$, while revenue is given by $R(x) = 60x$. Find the break-even point.

3. If the total cost, $C = 40 + 5Q$, and the total revenue, $R = 14Q - \frac{1}{2}Q^2$, where Q is the output. Find the output at the break-even point. Also find the total revenue at that point.

MATRICES

A matrix is a rectangular array of numbers in 'm' rows and 'n' columns. A matrix with 'm' rows and 'n' columns has the matrix of dimension $(m \times n)$ or has the order $(m \times n)$.

TYPES OF MATRICES

EQUAL MATRICES

Two matrices A and B are equal if both have the same order or dimension and also elements in the corresponding positions of A and B are equal.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then $A = B$ if $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{21} = b_{21}$ and $a_{22} = b_{22}$.

SQUARE MATRICES

Two matrices A and B are called square matrices if they have the same order or dimension. For any $(m \times n)$ matrix,

if $m = n$, then that matrix is called a square.

UNIT (IDENTITY) MATRIX

A unit or Identity matrix is a square matrix with ones as leading or principal diagonal and zero elsewhere. It is usually denominated by $\boldsymbol{I}$.

Examples are

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In general, $AI = IA = A$, provided the correct form of $\boldsymbol{I}$ is used.

ALGEBRA OF MATRICES

ADDITION AND SUBSTRACTION OF MATRICES

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$Then;

i. $A+B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$

ii. $A-B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{pmatrix}$

NB:

i. $A+B=B+A$

ii. $A-B\neq B-A$

Examples:

1. Find $P+Q$ if $P=\begin{pmatrix} 5 & -6 \\ 8 & 9 \end{pmatrix}$ and $Q=\begin{pmatrix} -4 & 6 \\ 8 & -3 \end{pmatrix}$. Ans $p+Q=\begin{pmatrix} 1 & 0 \\ 16 & 6 \end{pmatrix}$

2. Given that $A=\begin{pmatrix} 4 & 3 & 2 \\ 2 & 5 & 2 \\ 2 & 3 & 4 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Find $A+B$

3. If $X=\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{pmatrix}$ and $Y=\begin{pmatrix} -2 & 3 & 0 \\ 1 & 7 & 2 \end{pmatrix}$, Find $X-Y$.

4. Evaluate $\begin{pmatrix} 2 & 8 & 12 & 0 \\ 7 & 4 & -1 & 5 \\ 1 & 2 & 0 & 10 \end{pmatrix}-\begin{pmatrix} 1 & 3 & 6 & 9 \\ 2 & -3 & -3 & 4 \\ 8 & 0 & -2 & 17 \end{pmatrix}$

MULTIPLICATION OF MATRICES

If A and B are two matrices, the matrix BA is obtained by multiplying the elements in the ith row of B by corresponding elements in the jth column of A and summing the products.

Note that, if B is $(m\times n)$ and A is $(n\times p)$ then BA is of order $(m\times p)$ $i.e.(m\times n\times n\times p)$.

Examples:

1. Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then $AB=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$

2. Find the product of CD of matrices if $C = \begin{pmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{pmatrix}$

Ans $CD = \begin{pmatrix} 32 & 4 \\ -18 & 12 \end{pmatrix}$ and $DC = \begin{pmatrix} 38 & -24 & 4 \\ 9 & 8 & 16 \\ -19 & 12 & -2 \end{pmatrix}$

3. Find BA, given $A = \begin{pmatrix} 1 & -3 \\ 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{pmatrix}$

A TRANSPOSE OF A MATRIX

A matrix A consisting of 'm' rows and 'n' columns form the matrix whose ith row and jth column of A, then also it's j[th] column is the i[th] row of A. This new matrix formed is called the transpose of A and it is denoted by A^T or A'.

If A is a square matrix such that $A = A^T$, then A is called Symmetric.

DETERMINANTS OF MATRICES

The determinant of a matrix is only defined when the matrix is a square matrix. The determinant assigns a numerical value to the matrix.

DETERMINANT OF A (2×2) MATRIX

If a matrix A is a (2×2) i.e $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then, the determinant of A is defined by the scalar

$\det(A)$ or $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Examples:

1. If $A = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}$, Find $|A|$.

2. Find the determinant of the matrix $Q = \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix}$.

Ans. $det(Q) = -8$

DETERMINANT OF A (3×3) *MATRIX*

Given the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,then

$$\det\ (A) = a_{11}\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Examples

Find the determinants of the following matrices.

a) $P = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$

$\det(P) = 3$

b) $Q = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{pmatrix}$

$\det(Q) = 5$

MATRIX INVERSES

Given the matrix A, then the inverse of A, denoted by A^{-1}, such that $AA^{-1} = I$ or $A^{-1}A = I$, where I is the identity matrix and provided the determinant of A exist.

Examples

1. Find the inverse A^{-1} of the matrix $A = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$. Ans: $A^{-1} = \begin{pmatrix} 1/6 & 2/3 \\ 1/6 & -1/3 \end{pmatrix}$

First method:

Given the matrix A, write the augmented form matrix i.e. $(A|I)$

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1/6 & 2/3 \\ 0 & 1 & 1/6 & -1/3 \end{array}\right)$$

2. Find A^{-1} if $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{pmatrix}$.

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Solution

Let R_1 be row1 and R_2 be row2 and so on.

Replace R_2 by $-2R_1 + R_2$

Replace R_3 by $-3R_1 + R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right)$$

Next

Replace R_2 by $-\frac{1}{2}R_2$

Replace R_3 by $-\frac{1}{3}R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right)$$

Replace R_2 by $-\frac{3}{2}R_3 + R_2$

Replace R_3 by $-R_1 + R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/3 \end{pmatrix}$$

Second Method: (Kindly teach this method)

Let A be a (3×3) matrix. Then, the inverse of A, A^{-1}, is given by

$A^{-1} = \frac{1}{\det A} \times$Adjoint of A

Given that $A = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix}$. To find the inverse of A, A^{-1}, (1) find the minors of A, (2) find the cofactors of A, (3) find the Adjoint of A.

Consequently,

$$\text{minors of } A = \begin{pmatrix} \begin{vmatrix} t & u \\ w & x \end{vmatrix} & \begin{vmatrix} s & u \\ v & x \end{vmatrix} & \begin{vmatrix} s & t \\ v & w \end{vmatrix} \\ \begin{vmatrix} q & r \\ w & x \end{vmatrix} & \begin{vmatrix} p & r \\ v & x \end{vmatrix} & \begin{vmatrix} p & q \\ v & w \end{vmatrix} \\ \begin{vmatrix} q & r \\ t & u \end{vmatrix} & \begin{vmatrix} p & r \\ s & u \end{vmatrix} & \begin{vmatrix} p & q \\ s & t \end{vmatrix} \end{pmatrix}$$

$$\text{cofactors of } A = \begin{pmatrix} +(tx-uw) & -(sx-uv) & +(sw-tv) \\ -(qx-rw) & +(px-rv) & (pw-qv) \\ +(qu-rt) & -(pu-rs) & +(pt-qs) \end{pmatrix}$$

Note: Adjoint of A is the transpose of the cofactors.

Example

Find A^{-1} if $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{pmatrix}$

Solution

$$\text{Minors of A}= \begin{pmatrix} \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} \end{pmatrix}$$

$$\text{Minors of } A = \begin{pmatrix} 0 & 3 & 6 \\ 0 & -3 & 0 \\ 2 & -3 & -2 \end{pmatrix}$$

$$\text{Cofactors of A}= \begin{pmatrix} 0 & -3 & 6 \\ 0 & -3 & 0 \\ 2 & 3 & -2 \end{pmatrix}$$

$$\text{Adjoint of A}= \begin{pmatrix} 0 & 0 & 2 \\ -3 & -3 & 3 \\ 6 & 0 & -2 \end{pmatrix}$$

det A$=1(0) + 0(3) + 1(6) = 6$

$$\therefore A^{-1} = \frac{1}{6}\begin{pmatrix} 0 & 0 & 2 \\ -3 & -3 & 3 \\ 6 & 0 & -2 \end{pmatrix}$$

Solving Systems of linear Equations

Solving Systems of linear Equations $(AX = B)$ using matrix inverses. Given $AX = B$, then the solution of the system is given by:

$$X = A^{-1}B$$

Alternatively, the Cramer's rule can be used to solve systems of linear equations.

Given the system below:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Then we can write:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$A_x = \begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}$$

$$A_y = \begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}$$

$$A_z = \begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}$$

If the determinant of matrix A is not zero $(\det(A) \neq 0)$, then

$x = \frac{|A_x|}{|A|}$, $y = \frac{|A_y|}{|A|}$ and $z = \frac{|A_z|}{|A|}$

Examples

1. Use the inverse of the coefficient matrix to solve the linear system.

$$2x - 3y = 4$$
$$x + 5y = 2$$

2. Use the inverse of the coefficient matrix to solve the system.

$$2x + 5y = 20$$
$$6x + 15y = 30$$

3. Use the inverse of the coefficient matrix to solve the following systems;

a. $$\begin{aligned} -x - 2y - 2z &= 9 \\ 2x + y - z &= -3 \\ 3x - 2y + z &= -6 \end{aligned}$$

b. $$\begin{aligned} -x - 2y - 2z &= 3 \\ 2x + y - z &= 3 \\ 3x - 2y + z &= 7 \end{aligned}$$

Solution of (3a) using the matrix inverse approach.

$$\begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ -3 \end{pmatrix}$$

Let $A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 9 \\ -6 \\ -3 \end{pmatrix}$

$$\text{Minors of } A = \begin{pmatrix} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$\text{Minors of } A = \begin{pmatrix} -1 & -5 & -7 \\ -6 & 5 & 8 \\ 4 & 5 & 3 \end{pmatrix}$$

$$\text{Cofactors of } A = \begin{pmatrix} -1 & -5 & -7 \\ 6 & 5 & -8 \\ 4 & -5 & 3 \end{pmatrix}$$

$$\text{Adjoint of } A = \begin{pmatrix} -1 & 6 & 4 \\ -5 & 5 & -5 \\ -7 & -8 & 3 \end{pmatrix}$$

$$\det A = -1(-1) - (-2)(5) + (-2)(-7) = 25$$

$$A^{-1} = \frac{1}{25}\begin{pmatrix} -1 & 6 & 4 \\ -5 & 5 & -5 \\ -7 & -8 & 3 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -1 & 6 & 4 \\ -5 & 5 & -5 \\ -7 & -8 & 3 \end{pmatrix}\begin{pmatrix} 9 \\ -6 \\ -3 \end{pmatrix} =$$

4. Solve the system below using Cramer's rule.

$$2x + y - z = -2$$

$$5x - 2y + 4z = 1$$

$$3x + y + z = -13$$

Solution

$$\begin{pmatrix} 2 & 1 & -1 \\ 5 & -2 & 4 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -13 \end{pmatrix}$$

Let $A = \begin{pmatrix} 2 & 1 & -1 \\ 5 & -2 & 4 \\ 3 & 1 & 1 \end{pmatrix}$

$$A_x = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & 4 \\ -13 & 1 & 1 \end{pmatrix}$$

$$A_y = \begin{pmatrix} 2 & -2 & -1 \\ 5 & 1 & 4 \\ 3 & -13 & 1 \end{pmatrix}$$

$$A_z = \begin{pmatrix} 2 & 2 & -2 \\ 5 & -2 & 1 \\ 3 & 1 & -13 \end{pmatrix}$$

$$|A| = 2\begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & 4 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix}$$

$$|A| = -12 + 7 - 11$$

$$|A| = -16$$

$$|A_x| = -2\begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ -13 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ -13 & 1 \end{vmatrix}$$

$$|A_x| = 12 - 53 + 25$$

$$|A_x| = -16$$

$$|A_y| = 2\begin{vmatrix} 1 & 4 \\ -13 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 5 & 4 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 3 & -13 \end{vmatrix}$$

$$|A_y| = 106 - 14 + 68$$

$$|A_y| = 160$$

$$|A_y| = 2\begin{vmatrix} 1 & 4 \\ -13 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 5 & 4 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 3 & -13 \end{vmatrix}$$

$$|A_y| = 106 - 14 + 68$$

$$|A_y| = 160$$

$$|A_z| = 2\begin{vmatrix} -2 & 1 \\ 1 & -13 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 3 & -13 \end{vmatrix} - 2\begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix}$$

$$|A_z| = 50 + 68 - 22$$

$$|A_z| = 96$$

$$x = \frac{-16}{-16} = 1$$

$$y = \frac{160}{-16} = -10$$

$$z = \frac{96}{-16} = -6$$

$$(x, y, z) = (1, -10, -6)$$

Home Work

Solve the following systems of linear equations using Cramer’s rule and matrix inverse methods.

1. $3x + y + 2z = 3$

$$2x - 3y - z = -3$$

$$x + 2y + z = 4$$

2. $3p + 5q - 4r = 6000$

$$2p - 3q + r = 5000$$

$$-p + 4q + 6r = 13000$$

APPLICATIONS QUESTIONS (**Not to be solved in class**)

1. A company produces three color television sets: models X, Y, and Z. Each model X requires 2 hours of electronics work, 2 hours of assembly time, and 1 hour of finishing time. Each model Y requires 1,3, and 1 hours of electronics, assembly, and finishing time, respectively. Each model Z requires 3, 2, and 2 hours of the same work, respectively. There are 100 hours available for electronics, 100 hours for assembly, and 65 hours available for finishing per week. How many of each model should be produced each week if all available time must be used?
2. A company produces two models of bicycles, model 201 and model 301. Model 201 requires 2 hours assembly time, and model 301 requires 3 hours of assembly time. The parts for model 201 costs GHS 25.00 per bike, and the parts for model 301 costs GHS 30.00 per bike. If the company has a total of 34 hours of assembly time and GHS 365.00 available per day for these two models, how many of each can be made in a day?
3. The Gonzalex Company makes ROM chips and RAM chips for computers. Both requires time on two assembly lines. Each unit of ROM chips requires 1 hour on line A and 2 hours on line B. Each unit RAM chips requires 3 hours on line A and 1 hour on line B. Both assembly lines operate 15 hours per day. How many units of each product can be produced in each day under these conditions?

INDICES, LOGARITHMS & SURDS

INDICES

Positive Integral Indices

If $64 = 4^3$, then 3 is the exponent and 4 is the base.

Zero and Negative Exponents

If a is any non-zero real number, and if n is a positive integer, then

$a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$

Example 1

(a) $6^0 = 1$
(b) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
(c) $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$

Example 2

Find the values of the following:

(a) 10^{-3} (b) $8^{\frac{1}{3}}$ (c) $(-343)^{-\frac{1}{3}}$ (d) $(27)^{-\frac{2}{3}}$ (e) $(1000)^{-\frac{5}{3}}$

Solutions

(a) $10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$
(b) $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = 2^1 = 2$
(c) $(-343)^{-\frac{1}{3}} = (-7^3)^{-\frac{1}{3}} = -7^{\left(3\times-\frac{1}{3}\right)} = -7^{-1} = -\frac{1}{7}$
(d) $(27)^{-\frac{2}{3}} = \qquad = \qquad = \qquad = \frac{1}{9}$ Try it.
(e) $(1000)^{-\frac{5}{3}} = \qquad = \qquad = \qquad = 1 \times 10^{-5}$. Try it.

Properties of Exponents

For any integers m and n, and any real numbers a and b for which the following exists.

1. $a^{m+n} = a^m \times a^n$
2. $a^{m-n} = \frac{a^m}{a^n}$
3. $(a^m)^n = a^{m\times n} = a^{mn}$
4. $(ab)^m = a^m b^n$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example

Simplify the following

(a) $9^{14} \div 9^{6}$ (b) $(2m^3)^4$ (c) $(3x)^4$ (d) $(2^{-3} \times 2^5) \div (2^4 \times 2^{-7})$ (e) $2^{-1} + 3^{-1}$

Solutions

(a) $9^{14} \div 9^{6} = \frac{9^{14}}{9^{6}} = 9^{14-6} = 9^8 = 43046721$
(b) $(2m^3)^4 = 2^4 \cdot (m^3)^4 = 16m^{12}$
(c) $(3x)^4 = 3^4 \cdot x^4 = 81x^4$
(d) $(2^{-3} \times 2^5) \div (2^4 \times 2^{-7}) = \qquad = \qquad = 32$
(e) $2^{-1} + 3^{-1} = \qquad = \qquad = \frac{5}{6}$

Simple Exponential Equations

Solve the following exponential equations

1. $3^x = 81$
2. $8^k = 0.25$
3. $2^{2x} + 4(2^x) - 32 = 0$
4. $3^{2(x-1)} - 8(3^{x-2}) = 1$

Solutions

1. $3^x = 81 \quad \Longrightarrow 3^x = 3^4 \quad \Longrightarrow \ x = 4$
2. $8^k = 0.25 \ \Longrightarrow \ 2^{3k} = 2^{-2} \ \Longrightarrow \ 3k = -2 \ \Longrightarrow \ k = -\frac{2}{3}$
3. $2^{2x} + 4(2^x) - 32 = 0$
   $(2^x)^2 + 4(2^x) - 32 = 0$
   $let\ 2^x = y$
   $y^2 + 4y - 32 = 0$, and you can now solve this quadratic equation with any of the formulas for solving quadratic equations.
4. $3^{2(x-1)} - 8(3^{x-2}) = 1$
   $3^{2x} \cdot 3^{-1} - 8(3^x \cdot 3^{-2}) = 1$
   $(3^x)^2 \times \frac{1}{3} - 8\left(3^x \times \frac{1}{9}\right) = 1$
   $let\ 3^x = y$
   $\frac{y^2}{3} - \frac{8y}{9} = 1$
   $\Longrightarrow 3y^2 - 8y = 1$, and you can now solve this quadratic equation with any of the formulas for solving quadratic equations.

Logarithms

If $y = a^x$, then $x = \log_a y$, $(a > 0)$
We define x as the logarithm of y to be base a.

Example 1

Write the following indices in logarithm form

a. $7^2 = 49$
b. $32 = 2^5$
c. $10^0 = 1$
d. $1000 = 10^3$
e. $5^{-2} = \frac{1}{25}$

Solutions

a. $7^2 = 49 \implies \log_7 49 = 2$
b. $32 = 2^5 \implies \log_2 32 = 5$
c. $10^0 = 1 \implies \log_{10} 1 = 0$
d. $1000 = 10^3 \implies \log_{10} 1000 = 3$
e. $5^{-2} = \frac{1}{25} \implies \log_5 \frac{1}{25} = -2$

Example 2

Write the following logarithms in index form and find the value of x.

a. $x = \log_2 8$
b. $\log_{10} 0.001 = x$
c. $\log_2 \frac{1}{8} = x$
d. $x = \log_5 125$
e. $\log_3 27 = x$

Solutions

a. $2^x = 8 \implies 2^x = 2^3 \implies x = 3$
b. $10^x = 0.001 \implies 10^x = 10^{-3} \implies x = -3$
c. $2^x = \frac{1}{8} \implies 2^x = 2^{-3} \implies x = -3$
d. $5^x = 125 \implies 5^x = 5^3 \implies x = 3$
e. $3^x = 27 \implies 3^x = 3^3 \implies x = 3$

Example 3

Find the values of the following logarithms

a. $\log_8 64$
b. $\log_{10} 0.01$
c. $\log_2 0.25$
d. $\log_3 1$
e. $\log_{10} 10$

Solutions

a. Let $\log_8 64 = x \implies 8^x = 64 \implies 8^x = 8^2 \quad \implies x = 2$
b. Let $\log_{10} 0.01 = x \implies 10^x = 0.01 \implies 10^x = 10^{-2} \quad \implies x = -2$
c. Let $\log_2 0.25 = x \implies 2^x = 0.25 \implies 2^x = 2^{-2} \quad \implies x = -2$
d. Let $\log_3 1 = x \implies 3^x = 1 \implies 3^x = 3^0 \quad \implies x = 0$
e. Let $\log_{10} 10 = x \implies 10^x = 10 \implies 10^x = 10^1 \quad \implies x = 1$

Common and Natural Logarithms

Logarithms to base 10 are called common logarithms, written as log_{10}or lg_{10} or just log. On the other hand, logarithms to the base e are called natural logarithms, written as ln where e is a mathematical constant.

Rules of Logarithms

1. $\ln pq = \ln p + \ln q$
   Example, $\ln 24 = \ln(4 \times 6) = \ln 4 + \ln 6$
2. $\ln\left(\frac{p}{q}\right) = \ln p - \ln q$
   Example, $\ln\left(\frac{4}{9}\right) = \ln 4 - \ln 9$
3. $\ln p^n = n \ln p$
   Example, $\ln 25 = \ln 5^2 = 2\ln 5$

Two Special Logarithms

1. $\ln 1 = 0$ or $\log_{10} 1 = 0$
2. $\ln e = 1$ or $\log_{10} 10 = 1$

Examples

Find the values of the variables in the questions below.

1. $9^x = 87$
2. $25{,}000 = 10{,}000\left(1 + \frac{0.08}{2}\right)^{2n}$
3. $\left(1 + \frac{0.05}{4}\right)^{4n} = 2$
4. $4 = e^{0.009n}$

Solutions

1. $\ln 9^x = \ln 87 \implies x \ln 9 = \ln 87 \quad \implies x = \frac{\ln 87}{\ln 9} = \frac{4.4659}{2.1972} = 2.0325$
2. $25{,}000 = 10{,}000\left(1 + \frac{0.08}{2}\right)^{2n}$
   $2.5 = \left(1 + \frac{0.08}{2}\right)^{2n}$
   $\ln 2.5 = \ln\left(1 + \frac{0.08}{2}\right)^{2n}$

$$\ln 2.5 = 2n \ln\left(1 + \frac{0.08}{2}\right)$$
$$2n = \frac{\ln 2.5}{\ln 1.04} = \frac{0.9163}{0.039}$$
$$2n = 23.4949 \quad \Longrightarrow \quad n = 11.7 \approx 12$$

Try your hand on questions 3 and 4.

SURDS

If $\sqrt[n]{a}$ is irrational, then it is called a surd of the n^{th}order. Examples of surds are $\sqrt{3}$, $\sqrt{7}$ etc.

Algebra of Surds

Addition and Subtraction

Just as we add and subtract algebraic expressions, we can also add and subtract similar surds like $3\sqrt{5}$ and $7\sqrt{5}$. However, we cannot add nor subtract unlike surds such as $3\sqrt{5}$ and $2\sqrt{3}$. The following results illustrate how we can add and subtract similar surds.

Addition of surds: $m\sqrt{a} + n\sqrt{a} = (m + n)\sqrt{a}$

Subtraction of surds: $m\sqrt{a} - n\sqrt{a} = (m - n)\sqrt{a}$

Multiplication and Division

The following results illustrate how we can multiply and divide surds of the same order.

1. $(\sqrt{a})(\sqrt{a}) = (\sqrt{a})^2 = a$
2. $(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$
3. $(a\sqrt{b})(c\sqrt{d}) = ac\sqrt{bd}$
4. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\left(\frac{a}{b}\right)}$

Worked Examples

1. Simply the following:

   (a) $\sqrt[4]{16}$ (b) $\sqrt[5]{-32}$ (c) $\sqrt[3]{1000}$ (d) $\sqrt[6]{\left(\frac{64}{727}\right)}$

Solution

(a) $\sqrt[4]{16} = (2^4)^{\frac{1}{4}} = 2^{4\times\frac{1}{4}} = 2^1 = 2$

(b) $\sqrt[5]{-32} = (-2^5)^{\frac{1}{5}} = -2^{5\times\frac{1}{5}} = -2^1 = -2$

(c) $\sqrt[3]{1000} = (10^3)^{\frac{1}{3}} = 10^{3\times\frac{1}{3}} = 10^1 = 10$

(d) $\sqrt[6]{\left(\frac{64}{727}\right)} = \quad = \quad = \quad ?$

2. Simplify the following:
   (a) $\sqrt{1000}$ (b) $\sqrt{128}$ (c) $\sqrt{288m^5}$ (d) $\sqrt{108}$ (e) $\sqrt[3]{54}$

Solution

(a) $\sqrt{1000} = \sqrt{10 \times 100} = \sqrt{10} \times \sqrt{100} = \sqrt{10} \times (10^2)^{\frac{1}{2}} = \sqrt{10} \times 10^{(2\times\frac{1}{2})} = 10\sqrt{10}$.

(b) $\sqrt{128} = \sqrt{2 \times 64} = \sqrt{2} \times \sqrt{64} = \sqrt{2} \times (8^2)^{\frac{1}{2}} = 8\sqrt{2}$

(c) $\sqrt{288m^5} = \sqrt{144 \times m^4 \times 2m} = \sqrt{144} \times \sqrt{m^4} \times \sqrt{2m} = 12m^2\sqrt{2m}$

(d) $\sqrt{108} = \quad = \quad = \quad = 6\sqrt{3}$

(e) $\sqrt[3]{54} = \quad = \quad = \quad = 3\sqrt[3]{2}$

3. Simplify the following:
   (a) $2\sqrt{45} + 3\sqrt{5}$ (b) $4\sqrt{28} - 3\sqrt{7}$ (c) $2\sqrt{18} - 5\sqrt{32}$ (d) $\sqrt{3}\left(\sqrt{12} - \frac{4}{\sqrt{75}}\right)$

Solutions

(a) $2\sqrt{45} + 3\sqrt{5} = 2\sqrt{(9 \times 5)} + 3\sqrt{5} = 2 \times \sqrt{9} \times \sqrt{5} + 3\sqrt{5} = 6\sqrt{5} + 3\sqrt{5} =$
$(6 + 3)\sqrt{5} = 9\sqrt{5}$

(b) $4\sqrt{28} - 3\sqrt{7} = 4\sqrt{(4 \times 7)} - 3\sqrt{7} = 4 \times \sqrt{4} \times \sqrt{7} - 3\sqrt{7} = 8\sqrt{7} - 3\sqrt{7} =$
$(8 - 3)\sqrt{7} = 5\sqrt{7}$

(c) $2\sqrt{18} - 5\sqrt{32} = \quad = \quad = \quad = -14\sqrt{2}$

(d) $\sqrt{3}\left(\sqrt{12} - \frac{4}{\sqrt{75}}\right) = \quad = \quad = \quad = 5\frac{1}{5}$

Rationalizing Denominators

A surd such as $\frac{\sqrt{3}}{2}$, cannot be simplified further but $\frac{2}{\sqrt{3}}$ can be written in a more convenient form. To rationalize $\frac{2}{\sqrt{3}}$, we multiply the numerator and the denominator of the fraction $\frac{2}{\sqrt{3}}$ by $\sqrt{3}$.

Example

Rationalize the following:

(a) $\frac{4}{\sqrt{3}}$ (b) $\frac{1}{2\sqrt{7}}$

Solutions

(a) $\frac{4}{\sqrt{3}} = \frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{(\sqrt{3})^2} = \frac{4\sqrt{3}}{3}$

(b) $\frac{1}{2\sqrt{7}} = \frac{1}{2\sqrt{7}} \times \frac{2\sqrt{7}}{2\sqrt{7}} = \frac{2\sqrt{7}}{4(\sqrt{7})^2} = \frac{2\sqrt{7}}{28} = \frac{\sqrt{7}}{14}$

Conjugate Surds

Given the expression $a + \sqrt{b}$, then $a - \sqrt{b}$ is called its conjugate.

Example

(a) $\frac{2\sqrt{2}-3}{2\sqrt{2}+1}$ (b) $\frac{1}{2\sqrt{3}+\sqrt{5}}$ (c) $\frac{\sqrt{2}-\sqrt{3}}{5}$

Solution

(a) $\frac{2\sqrt{2}-3}{2\sqrt{2}+1} = \frac{2\sqrt{2}-3}{2\sqrt{2}+1} \times \frac{2\sqrt{2}-1}{2\sqrt{2}-1} = \frac{4(\sqrt{2})^2-2\sqrt{2}-6\sqrt{2}+3}{4(\sqrt{2})^2-2\sqrt{2}+2\sqrt{2}-1} = \frac{11-8\sqrt{2}}{7}\ or\ \frac{11}{7} - \frac{8\sqrt{2}}{7}$

(b) $\frac{1}{2\sqrt{3}+\sqrt{5}} = \frac{1}{2\sqrt{3}+\sqrt{5}} \times \frac{2\sqrt{3}-\sqrt{5}}{2\sqrt{3}-\sqrt{5}} = \frac{2\sqrt{3}-\sqrt{5}}{4(3)^2-2\sqrt{15}+2\sqrt{15}-5} = \frac{2\sqrt{3}-\sqrt{5}}{7}\ or\ \frac{2\sqrt{3}}{7} - \frac{\sqrt{5}}{7}$

(c) $\frac{\sqrt{2}-\sqrt{3}}{5} = \qquad = \qquad = \qquad = \frac{-1}{5(\sqrt{2}+\sqrt{3})}$ (complete it).

PERCENTAGES, RATIO AND PROPORTION

PERCENTGES

Percentages are used to indicate the relative size or proportion of items, rather than their absolute size. The idea of percentages is that the whole of something can be thought of as 100%. The whole of a cake, for example, is 100%.

Example 1

If ABC Co. Ltd employs ten accountants, six secretaries and four supervisors. What will be the percentage of the total work force in each type?

Type of workers	Accountants	Secretaries	Supervisors
Number	10	6	4

Solution

Type of workers	Accountants	Secretaries	Supervisors	Total
Number	10	6	4	20
Percentage	50	30%	20%	100%

Example 2

Write the following numbers as percentages.

a) 0.16 b) $\frac{4}{5}$ c) 0.4

Solution

Numbers	0.16	$\frac{4}{5}$	0.4
Workings	0.16×100	$\frac{4}{5} \times 100$	0.4×100
Percentages	16%	80%	40%

Situations Involving Percentages

(A) Find X % of Y

Example

Find 40 % of GHS 64.

Solution

$$\frac{40}{100} \times GHS64 = GHS25.60$$

(B) Express X as a percentage of Y

Example

What $GHS16$ as a percentage of $GHS64$.

Solution

$\frac{16}{64} \times 100\% = 25\%$

(C) Find the original value of X, given that after a percentage increase of $Y\%$, it is equal to X_1.

Example

Mr. Fred Boateng's salary is now $GHS60{,}000$ per annum after an annual increase of 20%. What is his annual salary before the increase?

Solution

$120\% = 60{,}000$ (because of an increase of 20%)

$100\% = ?$

$\frac{100}{120} \times 60{,}000 = \frac{5}{6} \times 60{,}000 = GHS50{,}000.$

(D) Find the final value of A, given that after a percentage increase (or decrease) of $B\%$, it is equal to A_1.

Example

If sales receipt in year 1 are $GHS500{,}000$ and there was a percentage decrease of 10% in a year 2, what are the sales receipts in year 2?.

Solution

$100\% = GHS500{,}000$

$90\% = ?$ (because of the 10% decrease)

$\frac{90}{100} \times 500{,}000 = GHS450{,}000$

$\therefore$ sales receipts in year 2 are $GHS450{,}000$

NOTE

	Increase (%)	Decrease (%)
Original Value	100	100
Increase/Decrease	$+X$	$-X$
Final Value	$100 + X$	$100 - X$

(E) Percentage Changes

A percentage increase or decrease is calculated as (change $\div$ original) $\times$ 100%.

Percentage (%) change = $\frac{Change}{Orignal\ Value} \times 100\%$

$\therefore$ % Change = $\frac{(New\ Price - Old\ Price)}{Old\ Price} \times 100\%$

Example 1

A television set has been reduced from $GHS490.99$ to $GHS340.99$. What is the percentage reduction in price (to three decimal places)?

Solution

% Reduction= $\frac{(340.99-490.99)}{490.99} \times 100\%$

$\frac{-150}{490.99} \times 100\% = -30.551\%$

Note that the negative sign in front of -30.551% indicates a reduction.

Example 2

In 2017, the total sales of a supermarket was $GHS15.0$ million. In 2018, sales was GHS 17.7million. What was the percentage change in 2018?

Solution

% Change $= \frac{17.7-15.0}{15.0} \times 100\%$

$\frac{2.7}{15} \times 100\% = 18\%$

Application of Percentages

(A) Discount

Sometimes shops allow a reduction in the prices of certain **CASH PURCHASES** i.e., goods bought and paid for immediately. The price quoted by the shop is often called the **MARKED OR CATELOGUED PRICE**. Therefore, cash discount or cash reduction are usually calculated as a percentage of the marked price.

Example 1

A store gives 10% discount off the marked for an article bought and paid for immediately. If a lady paid $GHS2,700$ cash for a wrist watch, what was the marked price of the watch?

Solution

$90\% = 2{,}700$ (because of the 10% discount i.e. $100\% - 10\%$)

$100\% =$? (meaning the marked price should be 100%)

$\therefore$ Marked price $= \frac{100}{90} \times 2{,}700 = GHS3{,}000$

Example 2

A manufacturer allowed a retailer a trade discount of 20% on the catalogued price of a television set. If the catalogued price of the TV set was GHS$1{,}200$, how much did the retailer pay?

Solution

$100\% = 1{,}200$

$80\% =$? (because of the 20%, the retailer paid $100\% - 20\%$ for the TV set)

$\therefore$ The retailer paid $= \frac{80}{100} \times 1{,}200 =GHS960$

Try Questions

1. A travel agent is offering a 17% discount on the brochure price of a particular holiday, which is GHS$7{,}950$. What price is being offered by the travel agent? Ans: $6{,}590.85$
2. Three years ago, a retailer sold a type of toy for GHS17.50 each. At the end of the first year, he increased the price by 6% and at the end of the second year by a further 5%. At the end of the third year the selling price was GHS20.06. What was the percentage price change in year three (to the nearest number)? Ans: 3%

(B) Profit and Loss

Basically profit is calculated as:

Profit = Selling Price – Cost Price

And

Loss = Cost Price – Selling Price

Examples

(a) An article which cost GHS$2{,}500$ was sold for GHS$3{,}500$. What was the profit
(b) An article which cost GHS$6{,}000$ was sold for GHS$5{,}500$. What was the loss amount?

Solution

(a) Profit = $3{,}500 - 2{,}500 =$GHS$1{,}000$
(b) Loss = $6{,}000 - 5{,}500 =$GHS50

Profit Percent and Loss Percent

Profit Percent(%) = $\frac{Profit}{Cost\ Price} \times 100\%$

Loss Percent (%) = $\frac{Loss}{Cost\ Price} \times 100\%$

Note that the profit percent in example $1a$ is 40%.

Example 1

An article costing GHS$3{,}000$ was sold for GHS$2{,}400$. Find the loss percent.

Solution

Percentage (%) loss = $\frac{(3000-2400)}{3000} \times 100\% = 20\%$

Example 2

A gas cooker cost GHS500. If it was sold at a profit of 10%, what was the selling price?

Solution

Method 1

Profit = $\frac{10}{100} \times 500 = 50$ $\therefore$ selling price = $500 + 50 =$ GHS550.

Method 2

Profit = 10% of cost price

selling price = $(100 + 10)\%$ of cost price = 110% of cost price

$\therefore$ $\frac{110}{100} \times 500 =$ GHS550

Try Questions

1. An article which cost GHS$3{,}000$ was sold at a loss of 15%. Find the selling price. Ans ($GHS2550$)
2. An article was sold for GHS$1{,}500$ at a profit of 20%. Find the cost price. Ans ($GHS1{,}25o$)

Profit Margins and Mark-up

Generally, selling price = cost price + profit

Profit Margins (margins on sales)

If profit is expressed as a percentage of sales (margins), then the following formula can be used.

Cost of Sale $=$ selling price$-$profit

(Note that, we take the selling price as 100% if profit is expressed as margins)

Mark-up (mark-up on cost)

Mark-up is when the profit is expressed as a percentage of cost of sales. In other words, a mark-up is the amount that is added to the cost price to arrive at the selling price.

Thus,

Selling price $=$ mark-up $+$ cost price.

(Note that, we take the cost price as 100% if profit is expressed as mark-up)

Example 1

Temy's Dresses sells a dress at a 10% margin. The dress cost the shop GHS100. Calculate the profit made by Temy's Dresses.

Solution

Let the selling price be 100%. If the profit is 10%, then the cost price is 90%.

$90\% = 100$, then

$100\% = ?$

$\Longrightarrow \frac{100}{90} \times 100 = 111.11$

$\therefore$ Profit $= 111.11 - 100 =$GHS11.11

Example 2

Cruz Trousers sells a pair of trousers for GHS80 at a 15% mark-up. Calculate the profit made by Cruz Trousers.

Solution

Let the cost price be 100%. If the profit is 15%, then selling price is 115%.

$115\% = 80$

$100\% = ?$

$\Longrightarrow \frac{100}{115} \times 80 =GHS69.57$

$\therefore$ Profit $= 80 - 69.57 =$GHS10.43

Try Question

A skirt which cost the retailer GHS75 is sold at a profit of 25% on the selling price. What is the profit? Ans: GHS25.0

(C) Commission

Sometimes a company may employ an agent to sell or buy goods or property on its behalf. Selling agents are called salesmen or sales representatives. The agent is paid a fee for his/her services. This fee is called a commission. The commission paid to the agent is usually a percentage of the total sales or purchases made by the agent.

Example

Ama sells bread. She earns 20% commission on every loaf of bread she sells. If she sell GHS500 worth of bread in one day, how much commission does she earns?

Solution

$\frac{20}{100} \times 500 = 100$

$\therefore$ Ama earns GHS100 a day.

Try Question

A bookseller receives GHS800 on a GHS$10{,}000$ sale of books on behalf of a publisher. What is his rate of commission?

PROPORTION

Proportion means writing percentage as a proportion of 1 (that is, writing it as a decimal). Hundred percent can be thought of as a whole, or 1. Fifty percent is half of that, or 0.5.

Example 1

Suppose there are 14 women in an audience of 70. What proportion of the audience are men?

Solution

Number of men = $70 - 14 = 56$

$\therefore \quad \frac{56}{70} = 0.8 = 80\%$

Note that, fraction of men = $\frac{8}{10}$ or $\frac{4}{5}$; percentage of men = 80%, and decimal is 0.8.

Example 2

There are 30 students in a classroom, 17 of whom have short hair. What is the proportion of students (to four decimal places) do not have short hair?

Solution

$\frac{30-17}{30} = \frac{13}{30} = 0.4333$ (or 43.33%)

RATIO

Ratios show relative share of a whole. Suppose Tom has GHS12 and Dick has GHS8. The ratio of Tom's cash to Dick's is $12: 8$ or $3: 2$.

Example 1

Suppose Kwame and Yaw wish to share GHS$2{,}000$ out in a ratio $3: 2$. How much will each receive?

Solution

Total ratio = $3 + 2 = 5$

So, $\frac{2000}{5} = 400$, then

Kwame's share = 3×400 =GHS$1{,}200$

Yaw's share = $2 \times 400 =$ GHS800

Example 2

A, B, C, and D wish to share GHS600 in the ratio $6: 1: 2: 3$. How much will each receive?

Solution

Total ratio = $6 + 1 + 2 + 3 = 12$

So, $\frac{600}{12} = 50$, then

A's share = $6 \times 50 =$ GHS300

B's share = $1 \times 50 =$ GHS50

C's share = $2 \times 50 =$ GHS100

D's share = $6 \times 50 =$ GHS150

Try Question

Tom, Dick and Harry wish to share GHS$8{,}000$. Calculate how much each would receive if they use the ratios

(a) $3: 2: 5$
(b) $5: 3: 2$
(c) $3: 4: 2$

RATIONAL FUNCTIONS AND PARTIALS FRACTIONS

Simplification of Rational Fractions

The function $f: x \to \frac{x+1}{x^2-2x+3}$ is called rational function. It is defined by $f(x) = \frac{x+1}{x^2-2x+3}$ or $y = \frac{x+1}{x^2-2x+3}$, where $y = f(x)$.

Examples

Simplify the following rational fractions

a) $\frac{2x+4}{6x+14}$ b) $\frac{x+3}{x^2+2x-3}$ c) $\frac{2x^2+x-3}{2x^2+5x+3}$

Solutions

a) $\frac{2x+4}{6x+14} = \frac{2(x+2)}{2(3x+7)} = \frac{x+2}{3x+7}, x \neq -\frac{7}{3}$

b) $\frac{x+3}{x^2+2x-3} = \frac{x+3}{(x+3)(x-1)} = \frac{1}{x-1}, x \neq -3,1.$

c) $\frac{2x^2+x-3}{2x^2+5x+3} = \frac{(2x+3)(x-1)}{(2x+3)(x-1)} = \frac{x-1}{x+1}, x \neq -1.$

Addition and Subtraction of Rational Fractions

Add

a) $\frac{2}{x+1}$ and $\frac{3}{x-1}$ b) $\frac{1}{x(x+1)}$ and $\frac{1}{(x+1)^2}$

Solutions

a) $\frac{2}{x+1} + \frac{3}{x-1} = \frac{2(x-1)+3(x+1)}{(x+1)(x-1)} = \frac{2x-2+3x+3}{(x+1)(x-1)} = \frac{5x+1}{(x+1)(x-1)}, x \neq -1,1.$

b) $\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} = \frac{(x+1)+x}{x(x+1)^2} = \frac{2x+1}{x(x+1)^2}, x \neq 0, -1.$

Subtract

$\frac{2x-1}{4x+2}$ from $\frac{x-1}{2x-1}$

Solution

$$\frac{x-1}{2x-1} - \frac{2x-1}{4x+2} = \frac{(x-1)(4x+2)-(2x-1)(2x-1)}{(2x-1)(4x+2)} = \frac{(4x^2-2x-2)-(4x^2-4x+1)}{(2x-1)(4x+2)} = \frac{2x-3}{(2x-1)(4x+2)}, x \neq -\frac{1}{2}, \frac{1}{2}$$

Multiplication and Division of Rational Fractions

Find the product of

a) $\frac{x+1}{x-1}$ and $\frac{x+2}{x-3}$ b) $\frac{(x+1)}{(x-1)^2}$ and $\frac{(x-1)^3}{(x+1)^2}$

Solutions

a) $\frac{x+1}{x-1} \cdot \frac{x+2}{x-3} = \frac{(x+1)(x+2)}{(x-1)(x-3)} = \frac{x^2+3x+2}{x^2-4x+3}, x \neq 1,3$

b) $\frac{(x+1)}{(x-1)^2} \cdot \frac{(x-1)^3}{(x+1)^2} = (x+1)^1(x-1)^{-2}(x-1)^3(x+1)^{-2} = (x+1)^{-1}(x-1)^1$
$= \frac{x-1}{x+1}, x \neq -1$

Divide

$\frac{x}{x+1}$ by $\frac{x-1}{x+3}$, where $x \neq -1, -3$

Solution

$\frac{x}{x+1} \div \frac{x-1}{x+3} = \frac{x}{x+1} \cdot \frac{x+3}{x-1} = \frac{x(x+3)}{(x+1)(x-1)}, x \neq -1,1$

TRY

$\left(\frac{1}{x+1} + \frac{x+1}{x}\right) \div \left(\frac{3}{x+1} - \frac{1}{x-1}\right)$ ans: $\frac{(x^2+3x+1)(x-1)}{2x(x-2)}, x \neq 0,2$

Partial Fractions

Under ‘Addition and Subtraction of Rational Fractions’, we saw that the fractions $\frac{2}{x+1}$ and $\frac{3}{x-1}$ are added as follows:

$$\frac{2}{x+1} + \frac{3}{x-1} = \frac{2(x-1)+3(x+1)}{(x+1)(x-1)} = \frac{2x-2+3x+3}{(x+1)(x-1)} = \frac{5x+1}{(x+1)(x-1)}, x \neq -1,1.$$

In partial fractions, we do the reverse. That is, given a fraction like $\frac{5x+1}{(x+1)(x-1)}$, how can we express it in the form,

$\frac{5x+1}{(x+1)(x-1)} = \frac{2}{x+1} + \frac{3}{x-1}$?

In resolving or decomposing rational fractions into partial fractions, we shall consider four (4) cases for the fraction $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

Case 1: Distinct Linear Factors

If $q(x)$ has a distinct linear factor, then the numerator of the partial fraction corresponding to this factor is a constant.

Example

Decompose $\frac{3x+2}{x^2+x-2}$ into partial fractions.

Solution

$\frac{3x+2}{(x+2)(x-1)} \equiv \frac{A}{x+2} + \frac{B}{x-1} \equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

$3x + 2 = A(x - 1) + B(x + 2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$

We give two methods for finding the values of A and B.

First Method

We can substitute any real value of x in equation (1). However, it is convenient to substitute values of x which make more of the terms on the right-hand of (1) equal to zero.

When we substitute $x = -2$ into (1), we have:

$$3(-2) + 2 = A(-2 - 1) + 0$$

$$-4 = -3A$$

$$A = \frac{4}{3}$$

When we substitute $x = 1$ into (1), we have:

$$3(1) + 2 = 0 + B(1 + 2)$$

$$5 = 3B$$

$$B = \frac{5}{3}$$

Second Method

We can further expand the right-hand side of equation (1) to give,

$3x + 2 = A + B(x) + (2B - A) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (2)$

Equating the coefficients of x in (2), we have: $A + B = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (3)$

Equating the term constant terms in (2), we have: $-A + 2B = 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

Solving equations (3) and (4) simultaneously, we have:

$A = \frac{4}{3}$ and $B = \frac{5}{3}$

Substituting the values of A and B in $\frac{3x+2}{(x+2)(x-1)} \equiv \frac{A}{x+2} + \frac{B}{x-1}$, we obtain

$$\frac{3x + 2}{(x + 2)(x - 1)} = \frac{4}{3(x + 2)} + \frac{5}{3(x - 1)}$$

Distinct Quadratic Factors

If $q(x)$ has a distinct quadratic factor, then the numerator of the partial fraction corresponding to this factor is a linear function.

Example

Express $\frac{x^2-2x-3}{(x-1)(x^2+4)}$ in partial fractions.

Solution

The numerator of the partial fraction corresponding to $(x-1)$ is A, a constant, and the numerator of the partial fraction corresponding to (x^2+4) is $(Bx+C)$, where B and C re constants. Thus,

$$\frac{x^2-2x-3}{(x-1)(x^2+4)} \equiv \frac{A}{x-1} + \frac{Bx+c}{x^2+4} \equiv \frac{A(x^2+4)+(x-1)(Bx+C)}{(x-1)(x^2+4)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$x^2-2x-3 = A(x^2+4)+(x-1)(Bx+C) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$x^2-2x-3 = (A+B)x^2+(C-B)x+(4A-) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Substituting $x=1$ in (2), we obtain

$$1^2-2(1)-3 = A(1^2+4)+0$$

$$-4 = 5A$$

$$A = -\frac{4}{5}$$

Equating the coefficients of x^2 in (3), we have

$$1 = A+B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Substituting $A=-\frac{4}{5}$ in (4),

$$1 = -\frac{4}{5}+B$$

$$B = 1+\frac{4}{5} = \frac{9}{5}$$

Equating the coefficients of x in (3), we have

$$-2 = C-B$$

But $B = \frac{9}{5}$

$$C = -2+\frac{9}{5} = -\frac{1}{5}$$

Hence, substituting the values of A, B and C in (1),

$$\frac{x^2-2x-3}{(x-1)(x^2+4)} = \frac{-4}{5(x-1)} + \frac{\frac{9}{5}x-\frac{1}{5}}{x^2+4} = -\frac{4}{5(x-1)} + \frac{9x-1}{5(x^2+4)}.$$

Repeated Linear Factors

If $q(x)$ has a repeated linear factor such as $(x-a)^2$, then the partial fraction corresponding to this factor can be expressed in the form

$$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$$

where A and B are constants.

Example

Express $\frac{1}{(x+2)(x-1)^2}$ in partial fractions.

Solutions

$\frac{1}{(x+2)(x-1)^2} \equiv \frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x+1)^2} \equiv \frac{A(x-1)^2+B(x-1)(x+2)+C(x+2)}{(x+2)(x-1)^2} \dots\dots\dots\dots\dots\dots\dots\dots (1),$

where A, B and C are constants. Multiplying (1) by $(x+2)(x-1)^2$, we have

$$1 = A(x-1)^2 + B(x-1)(x+2) + C(x+2) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. (2)$$

Substituting $x = 1$ in (2),

$$1 = 0 + 0 + C(1+2)$$

$$C = \frac{1}{3} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

Substituting $x = 2$ (2),

$$1 = A(-2-1)^2 + 0 + 0$$

$$A = \frac{1}{9}$$

Equating the coefficients of x^2 in (2)

$0 = A + B$

But $A = \frac{1}{9}$

$\Longrightarrow B = -\frac{1}{9}$

Substituting the values of A, B nd C in (1)

$$\frac{1}{(x+2)(x-1)^2} = \frac{1}{9(x+2)} - \frac{1}{9(x-1)} + \frac{1}{3(x+1)^2}$$

Improper Fraction

We now consider the case where the degree of $p(x)$ is greater than or equal to that of $q(x)$. The fraction $\frac{p(x)}{q(x)}$ is then called an improper fraction. To express such a fraction in partial fractions, we first divide $p(x)$ by $q(x)$ to obtain a quotient and a proper fraction. We can then express the proper fraction in partial fractions. The following example illustrate the procedure.

Example 1

Express $\frac{2x^2+1}{(x-1)(x+2)}$ in partial fraction.

Solution

Since the degree of the numerator is equal to that of the denominator, we first divide the numerator by the denominator. This gives

$$\frac{2x^2+1}{(x-1)(x+2)} = \frac{2x^2+1}{x^2+x-2} = \frac{2(x^2+x-2)-2x+4+1}{x^2+x-2} = \frac{2(x^2+x-2)}{x^2+x-2} + \frac{(-2x+5)}{x^2+x-2}$$

$$= 2 + \frac{5-2x}{x^2+x-2} = 2 + \frac{5-2x}{(x-1)(x+2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We now express $\frac{5-2x}{x^2+x-2}$ in partial fractions. We write

$$\frac{5-2x}{x^2+x-2} \equiv \frac{A}{x-1} + \frac{B}{x+2} \equiv \frac{A(x+2)+B(x-1)}{(x-1)(x+2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equating the numerators, we have

$$5-2x = A(x+2)+B(x-1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Substituting $x = 1$ in (3),

$$5-2(1) = 3A + 0$$

$$\Longrightarrow A = 1$$

Substituting $x = -2$ in (3),

$$5-2(-2) = 0 + B(-2-1)$$

$$\Longrightarrow B = -3$$

Hence, $\frac{2x^2+1}{(x-1)(x+2)} = 2 + \frac{1}{x-1} - \frac{3}{x+2}$.

Example 2

Determine the values of the constants, a, b and c so that

$$\frac{x^2}{(x-1)(x-2)} = a + \frac{b}{x-2} + \frac{c}{x-1}$$

for all real values of x, except $x = 1$ and $x = 2$.

Solution

We proceed as follows

$$\frac{x^2}{(x-1)(x-2)} \equiv a + \frac{b}{x-2} + \frac{c}{x-1} \equiv \frac{a(x-2)(x-1) + b(x-1) + c(x-2)}{(x-2)(x-1)} \ldots\ldots\ldots\ldots..(1)$$

Equating the numerators, we have

$$x^2 = a(x-2)(x-1) + b(x-1) + c(x-2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Substituting $x = 1$ in (2),

$$1 = 0 + 0 + c(1-2)$$

$$\Longrightarrow \;\; c = -1$$

Substituting $x = 2$ in (2),

$$4 = 0 + b(2-1) + 0$$

$$\Longrightarrow \;\; b = 4$$

Expanding and equating the coefficients of x^2 in (2), we obtain $a = 1$.

Hence, $\frac{x^2}{(x-1)(x-2)} = 1 + \frac{4}{x-2} - \frac{1}{x-1}$.

MATHEMATICS OF FINANCE

A. SIMPLE INTEREST

The simple interest I on a principal of P invested of a rate of interest r for a time t in years is given by: $I = PRT$

Examples

1. To buy furniture for a new apartment Miss Boateng borrowed GH¢5,000 at 11% simple interest for 11 months. How much interest will she pay?
2. A sum of GH¢2,500 is invested at a rate of 12 percent per annum for five years. How much interest would this yield?

FUTURE OR MATURITY VALUE OF SIMPLE INTEREST

The future or maturity value of F of P amount invested at a rate of interest r for t years is

$$F = P(1 + RT)$$

Examples

1. Find the maturity value of each of the following loans at simple interest.
   a. A loan of GH¢2500 to be paid in 8 months with interest of 12.1%
   b. A loan of $11,280 for 85days at 11%.

NB: Interest found using a 360-days a year is called ordinary interest and interest found using a 365-days a year is called exact interest.

PRESENT VALUE OF SIMPLE INTEREST

The present value P of a future amount F amount invested at a simple interest rate r for t years is: $P = \frac{F}{1+RT}$

Examples

1. Find the present value of GH¢10,000 invested for 1 year at 8% simple interest.
2. Suppose that in order to borrow GH¢8,000 today, you must agree to pay GH¢8380 in 6months to repay loan with interest. What is the simple interest rate?

B. COMPOUNT INTEREST

If the interest gained each year is added to the sum saved, we are looking at compound interest.

If P amount is deposited for a compounding periods at a rate of interest r per period, the compound amount A is given by:

$$A = P(1 + r)^n$$

Examples

1. Suppose GH¢1,000 is deposited for 6 years in an account paying 8% per year compounded annually.
a. Find the compound amount.
b. Find the actual amount of interest earned.
2. Find the amount of interest earned by a deposit of GH¢1,000 for 6 years at 6% compounded quarterly.

NB: The compound amount, $A = P\left(1+\frac{r}{f}\right)^{n\times f}$ where f is the frequency of compounding other item annually.

3. Suppose GH¢24,000 is deposited at 8% for 9 years. Find the interest earned by:
a. Daily and
b. Hourly compounding.

PRESENT VALUE FOR COMPOUND INTEREST (DISCOUNTING)

The present value P, of A amount invested, compounded at an interest rate r for n periods is

$$P = \frac{A}{(1+r)^n} = A(1+r)^{-n}$$

Examples:

1. Find the present value of GH¢16,000 in 9 years if money can be deposited at 6% compounded semi-annually.
2. How long will it take to increase an amount of GH¢10,000 to GH¢25,000 when compounded semi-annually at 8%?
3. How long will it take your money to double at 5% interest rate when compounded quarterly?

C. <u>CONTINUOUS COMPOUNDING</u>

It is possible for interest to be compounded continuously. If P amount is invested at an annual interest rate r compounded continuously the accumulated amount after t years is:

$A(t) = Pe^{rt}$, where e is a mathematical constant.

Examples

1. If GH¢1,000 is invested in an account that bears 15% interest compounded continuously at the end of 10years, what will be the worth of the investment?
2. Your friend has just invested $110,000 in constant growth fund, whose stocks are continuously declining at a rate of 6%. How much will her investment be worth in five years?
3. Find the present value of GH¢750 to be paid 4 years from now when the prevailing interest rate is 10% if interest is compounded continuously.

4. Find the present value of GH¢5000 in 36 months at 8% when interest is compounded continuously.

EFFECTIVE RATE

If GH¢1 is deposited at 4% compound quarterly, a calculator can be used to find that at the end of one year, the compound amount is GH¢1.0406, an increase of 4.06% over the original GH¢1. The actual increase of 4.06% in money is somewhat higher than the stated increase of 4%. To differentiate between these two numbers, 4% is called the nominal or stated rate while 4.06% is called the effective rate.

The effective rate (r_e) corresponding to a stated rate of interest r compounding m times per year is given by; $r_e = \left(1 + \frac{r}{m}\right)^m - 1$

Examples

1. A bank pays interest of 5% compounded monthly. Find the effective rate.
2. Find the effective rate corresponding to each of the following nominal rates.
   a. 8% compounding quarterly.
   b. 10% compounding semi-annually.

Annuity

A sequence of equal payments (or receipts) made at equal periods of time. If the frequency of payment is the same as the frequency of compounding interest periods, the annuity is called Ordinary Annuity.

Future Value of an Ordinary Annuity

The future value, F, of an annuity of n payments of R cedis each at the end of each consecutive interest periods, with interest compounded at a rate r per period is given by

$$F = R\left[\frac{(1+r)^n - 1}{r}\right]$$

Example

What is the amount of an annuity if the size of each payment is GHc 1,000, payable at the end of each quarter for 3 years at an interest rate of 4% compounded quarterly?

Present Value of an Ordinary Annuity

The present Value P of an annuity of n payments of R cedis each at the end of each consecutive interest periods with interest compounded at a rate of interest r per period is given by

$$P = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

Example

What is the present value of an annuity if the size of each payment is 500 cedis payable at the end of each quarter for 2 years at an interest rate of 4% compounded quarterly.

Examples

1. How much would we need to set aside at the end of each of the following 5 years to accumulate GH¢20,000, given an interest rate of 12% per annum compound?

2. You require GH¢4,000 in five (5) years' time. How much will you have to invest at the end of each year if interest charged is 15% per annum compounded?

AMORTIZATION

A loan is amortized if both the principal and interest are paid by a sequence of equal periodic payments.

Amortization Payments

A loan of P cedis at interest rate r per period may be amortized in n equal period payments R cedis made at the end of each period.

$$R = \frac{P.r}{[1-(1+r)^{-n}]}$$

Example

1. The Andoh Family buys a house for GH¢94,000 with a down payment of GH¢16,000. They take out a 30-year mortgage for GH¢78,000 at an annual interest rate of 9.6%.
   a. Find the amount of the monthly payment to amortize this loan.
   b. Find the total amount of interest paid when the loan is amortized after 30years.

2. Compute the monthly payment on a GH¢10,000 loan at a 6% annual interest rate which is amortized over 15years.

3. Find the monthly payments of an auto loan of GH¢20,000 to be amortized over a 5-year period at a rate of 9%.

DEPRECIATION

1. **Straight line method (linear depreciation)**
   If we assume that a constant level of value is lost each year, then the straight line method of depreciation is calculated by dividing the total loss of value by the number of years. i.e.
   Depreciation per year; $d = \frac{A_0 - A_t}{n}$, where
   $A_0 = initial\ value\ to\ the\ asset$
   $A_t = Book\ or\ savage\ or\ scrap\ value$
   $n = the\ number\ of\ years$

   Examples:
   1. The cost of a particular asset is GH¢20,000 and its salvage value is GH¢8,000 after five years. Determine the annual depreciation using the linear method.
   2. A truck was purchased by Sammy Ventures on September 30, 2019. The initial cost of the truck was GH¢9,000 but some new parts were bought in addition at a cost of GH¢800. Under the straight-line method. What is the depreciation to the nearest GH¢ pesewa at the end of 2019, if the estimated useful life of the truck is 5 years and the scrap value is GH¢1,200?

2. **Service hours methods**
   It relates the number of useful service hours of the asset $d = \frac{A_0 - A_t}{n}$, where n is the service hours.
   Example;
   A machine was purchased for GH¢5,000 with a book value of GH¢150 has an estimated useful life of 20,000 service hours and the actual number of hours spent on production each year is as follows:

1st year	5,000 service hours
2nd year	4,500 service hours
3rd year	4,200 service hours
4th year	3,400 service hours
5th year	2,900 service hours

   Use the service hours method to calculate the depreciation charges for each year and construct a depreciation schedule.

3. **Product unit methods**
   This method relates depreciation estimation using the number of units that will be produced by an asset during its useful life. i.e.
   Annual depreciation, $d = \frac{A_0 - A_t}{n}$, where $n = number\ of\ product\ units.$
   Example:

A machine was purchased for GH¢5,000 with a book value of GH¢150 has an estimated useful life of 48,500 product units and the number of units produced each year is as follows:

1st year	11,500 units
2nd year	10,500 units
3rd year	9,500 units
4th year	9,000 units
5th year	8,000 units

Use the product unit method to find the depreciation charges for each year and construct a depreciation schedule.