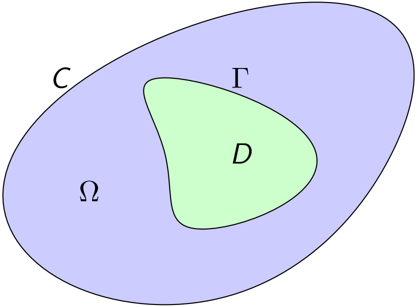
Gebietserkennung

Verena Treitz, Stephan Hilb

25. Januar 2014

Die Aufgabenstellung



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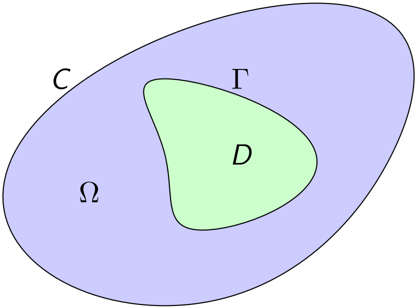

- homogenes Gebiet Ω
- unbekannte inner Inhomogenität D
- Rekonstruktion des Randes Γ von D

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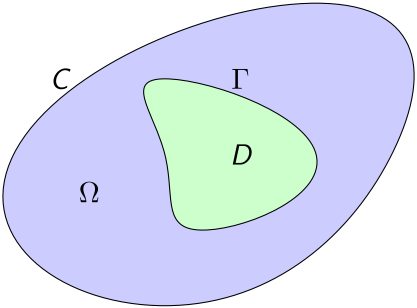
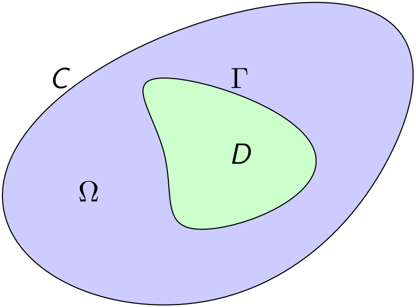


- homogenes Gebiet Ω
- unbekannte inner Inhomogenität D
- Rekonstruktion des Randes Γ von D
- Messdaten nur auf äußerem Rand C
  - angelegte Spannung f
  - gemessener Strom g

Mathematische Formulierung



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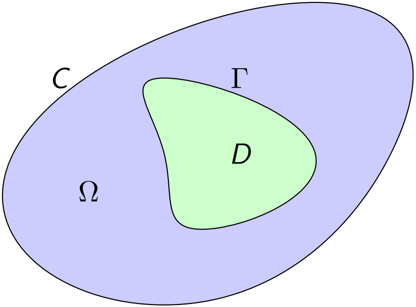

$$\Delta u = 0 \quad \text{auf } \Omega \setminus D$$
$$u|_\Gamma = 0$$
$$u|_C = f$$
$$\frac{\partial u}{\partial \nu}|_C = g$$

Das Randwertproblem

$$\Delta u = 0 \quad \text{auf } \Omega \setminus D$$

- Dirichlet-Bedingungen: $u|_\Gamma, u|_C$
- Neumann-Bedingungen: $\frac{\partial u}{\partial \nu}|_\Gamma, \frac{\partial u}{\partial \nu}|_C$
- Lösbar, falls auf Γ und C jeweils eine Bedingung gegeben ist

Das Inverse Problem

$$\begin{aligned} \Delta u &= 0 \quad \text{auf } \Omega \setminus D \\ u|_\Gamma &= 0 \\ u|_C &= f \\ \frac{\partial u}{\partial \nu}|_C &= g \end{aligned}$$

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Welches Γ erfüllt dies?

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Ansatz:

- Nutze Vorwärtsproblem

Das Inverse Problem

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Welches Γ erfüllt dies?

Ansatz:

- Nutze Vorwärtsproblem
- Löse $f = F(\Gamma)$

Das Vorwärtsproblem $F(\Gamma)$

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\Delta u &= 0 \quad \text{auf } \Omega \setminus D \\
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\tfrac{\partial u}{\partial \nu}|_C &= g \\
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- Jede Auswertung löst ein RWP $\implies$ teuer

Das Vorwärtsproblem $F(\Gamma)$

$$\begin{aligned} \Delta u &= 0 \quad \text{auf } \Omega \setminus D \\ u|_\Gamma &= 0 \\ \tfrac{\partial u}{\partial \nu}|_C &= g \\ \implies u|_C &= f =: F(\Gamma) \end{aligned}$$

- Jede Auswertung löst ein RWP $\implies$ teuer
- Was ist mit $F'(\Gamma)$?

Das Vorwärtsproblem $F'(\Gamma) \cdot h$

$$
\begin{aligned}
\Delta w &= 0 \quad \text{auf } \Omega \setminus D \\
w|_\Gamma &= \tfrac{\partial u}{\partial \nu} h_\nu \\
\tfrac{\partial w}{\partial \nu}|_C &= 0 \\
\Longrightarrow \quad w|_C &=: F'(\Gamma) \cdot h
\end{aligned}
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- Jacobi-Matrix berechenbar

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- Jacobi-Matrix berechenbar
- Jede Auswertung $F'(\Gamma) \cdot h$ löst ein RWP
- Auch teuer, aber besser als FDs

Diskretisierung

Diskretisierung

- Kurven, z.B. C, Γ:
  - kartesisch: $n \times 2$-Matrix, $[x, y]$,
  - oder äquidistant radial: n-Vektor
  - Comsol: `InterpolationCurve`

Diskretisierung

- Kurven, z.B. C, Γ:
  - kartesisch: $n \times 2$-Matrix, $[x, y]$,
  - oder äquidistant radial: n-Vektor
  - Comsol: `InterpolationCurve`
- Dirichlet-/Neumann Randdaten
  - kartesisch: $n \times 3$-Matrix, $[x, y, f(x, y)]$,
  - Comsol: `Function Interpolation`

Algorithmen

Löse $H(x) := F(x) - f = 0$ durch Minimierung der Fehlerquadrate

- Gauß-Newton mit berechneter Jacobimatrix
- Levenberg-Marquardt (`fsolve`) mit finiten Differenzen
- Levenberg-Marquardt (`fsolve`) mit berechneter Jacobimatrix

Gauss-Newton

- Minimiere
$$\|H(x)\|^2 \approx \|H(x_k) + H'(x_k)(x - x_k)\|^2$$
- $x_{k+1} := x_k - \left(H'(x_k)^T H'(x_k)\right)^{-1} H'(x_k)^T H(x_k)$

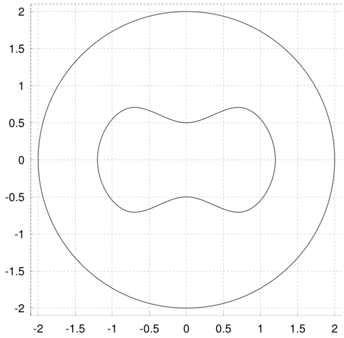
Levenberg-Marquardt

- Minimiere die Summe

$$\|H(x_k) + H'(x_k)(x - x_k)\|^2 + \mu\|x - x_k\|^2$$

- $x_{k+1} := x_k - \left(H'(x_k)^T H'(x_k) + \mu I\right)^{-1} H'(x_k)^T H(x_k)$

Testumgebung



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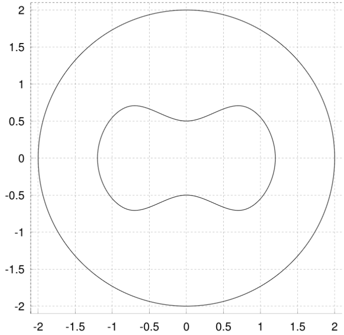
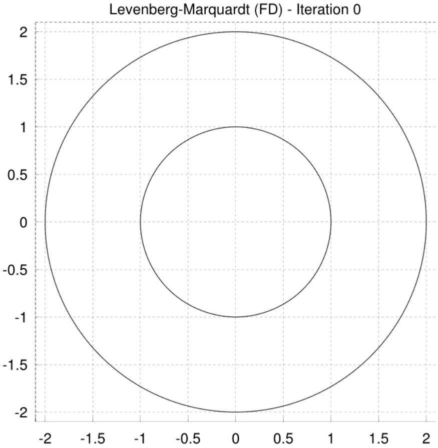


- C: Kreis mit Radius 2
- Γ: Erdnussform, 8 Stützstellen (radial gegeben)
- $f = 1$, 30 Stellen, äquidistant auf C
- g simuliert, 30 Stellen, äquidistant auf C
- Γ_0: Einheitskreis

Bilder, Levenberg-Marquard FD



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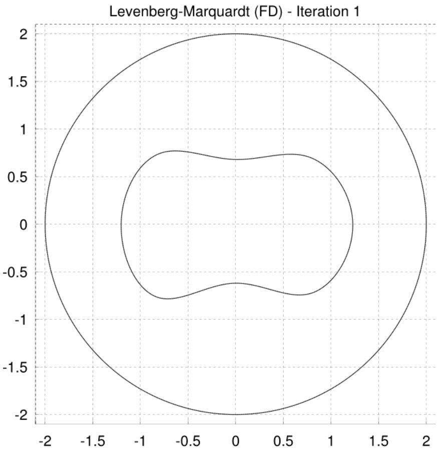

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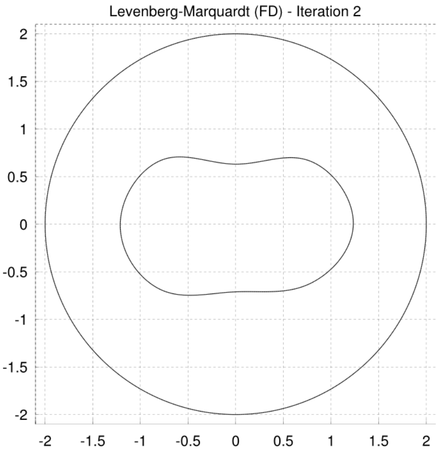

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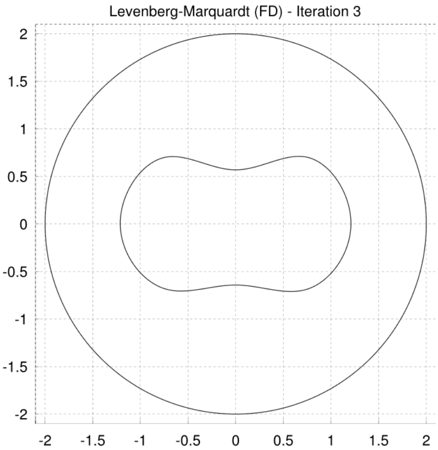

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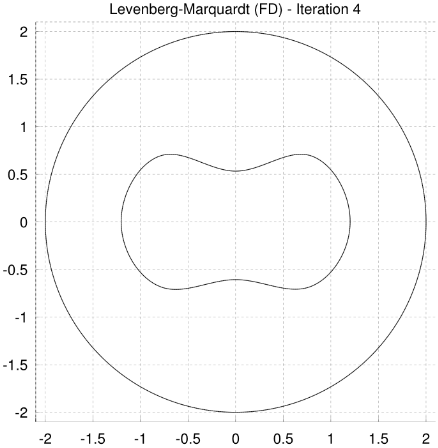

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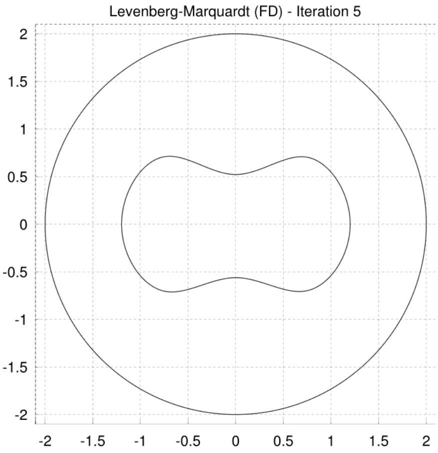

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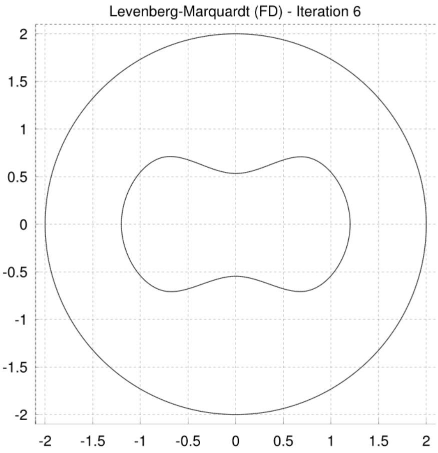

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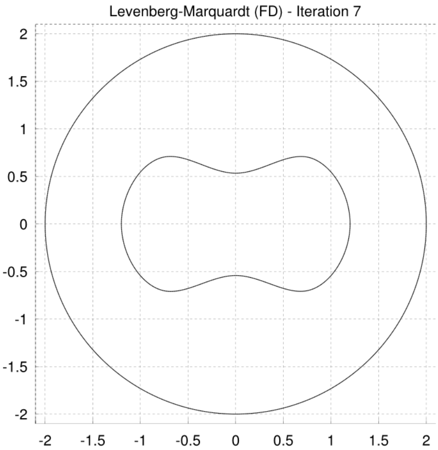

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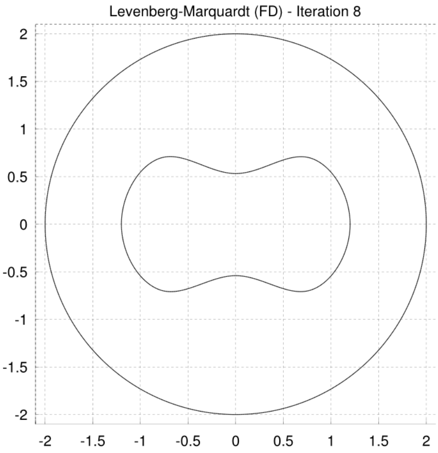

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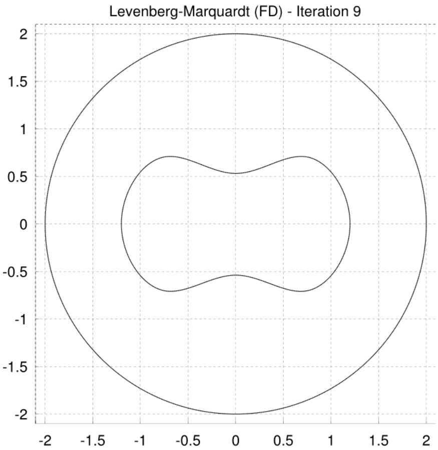

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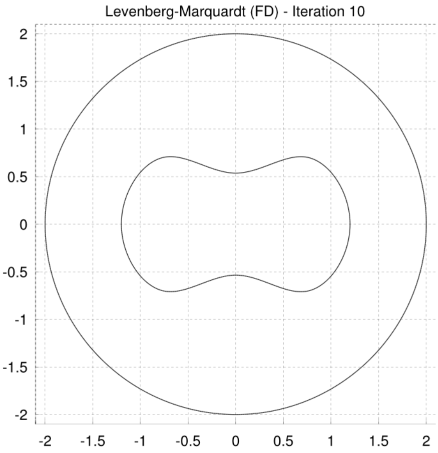

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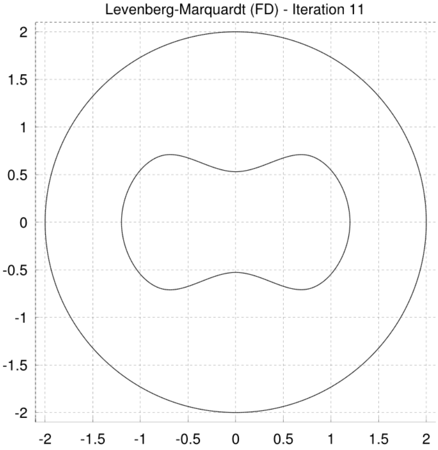

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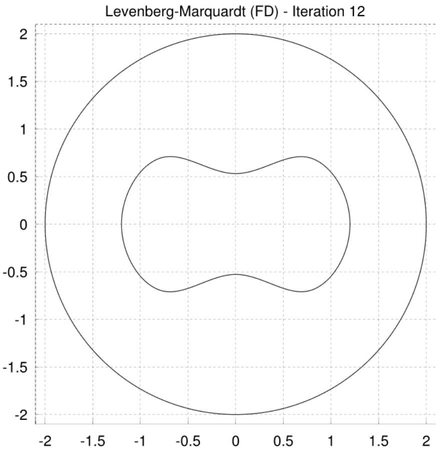

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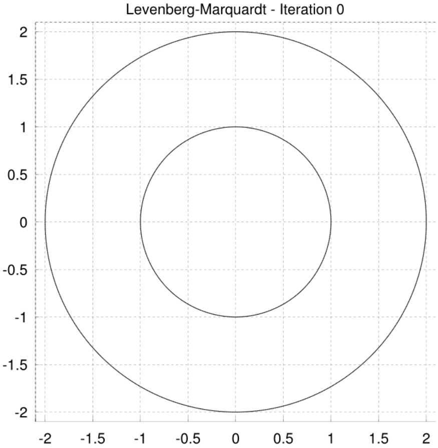

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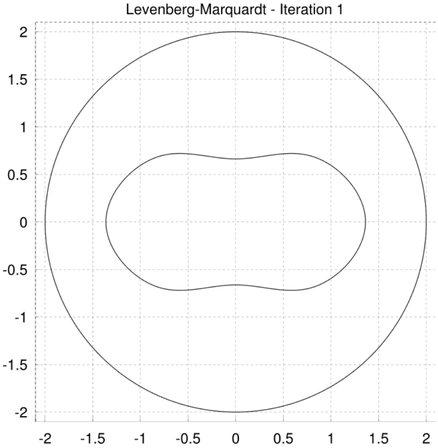

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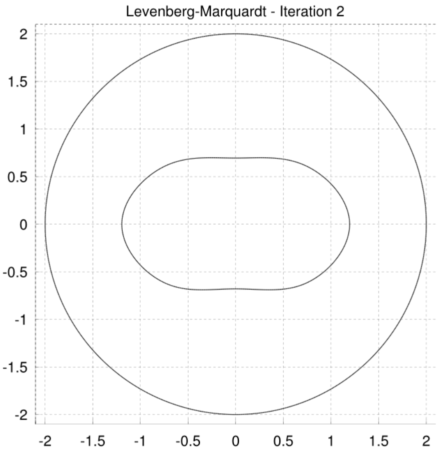

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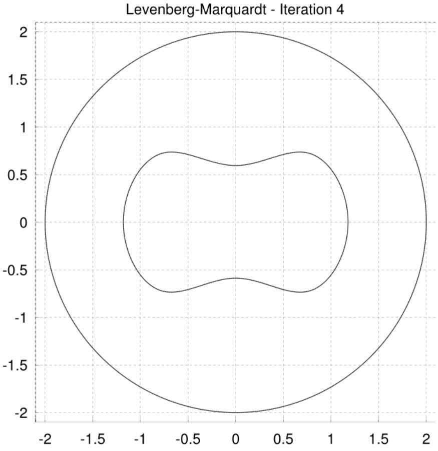

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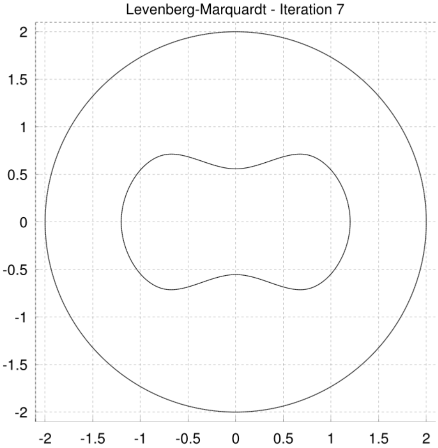

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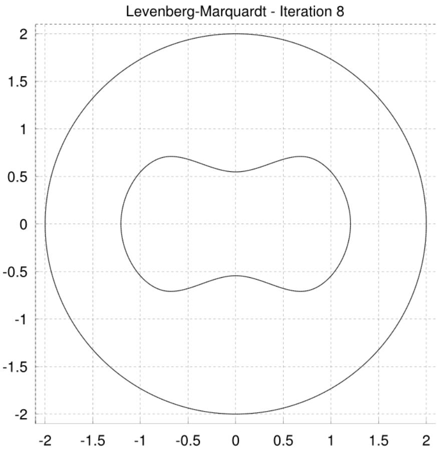

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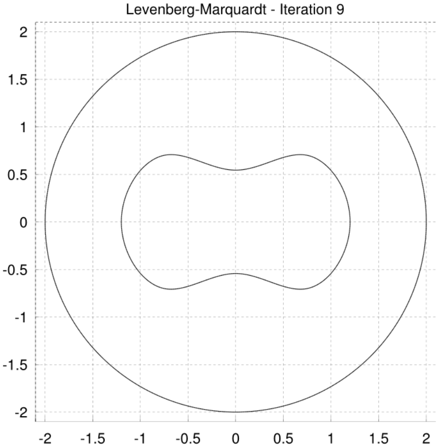

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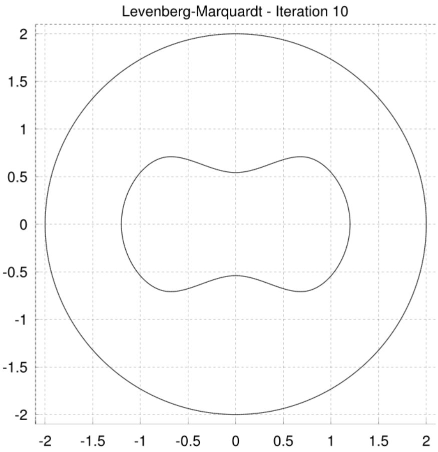

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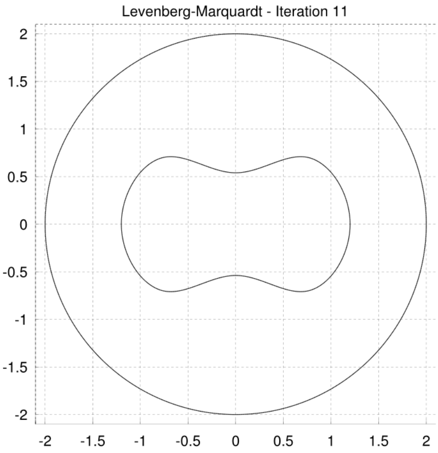

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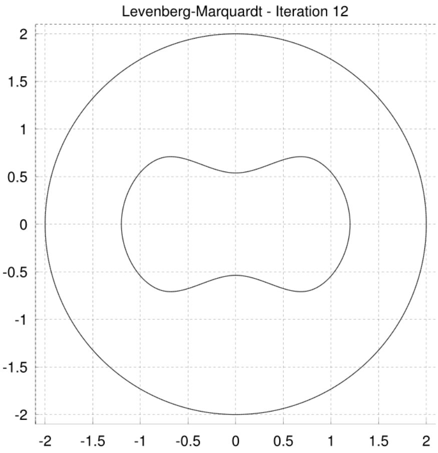

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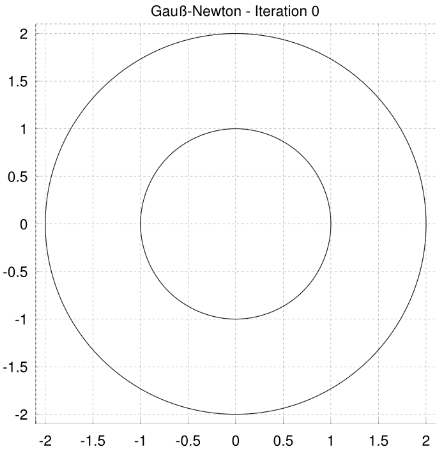

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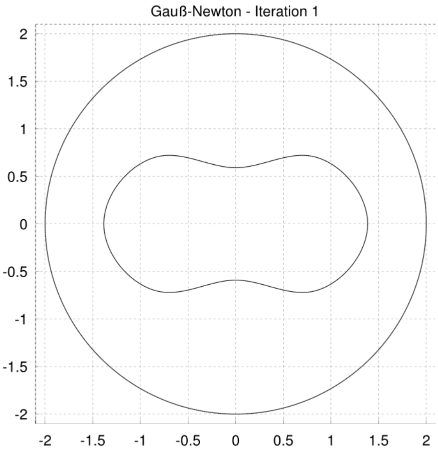

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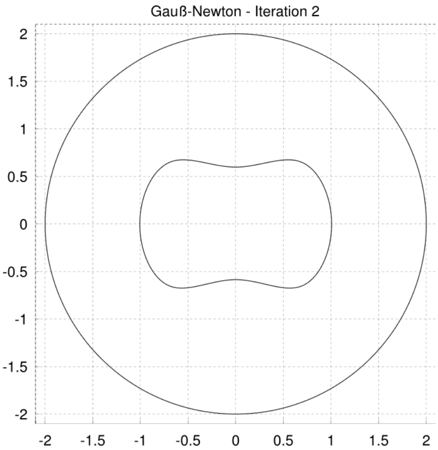

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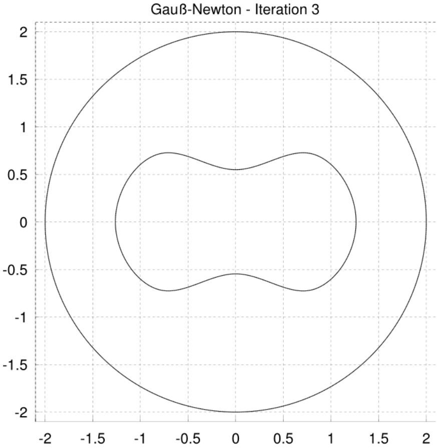

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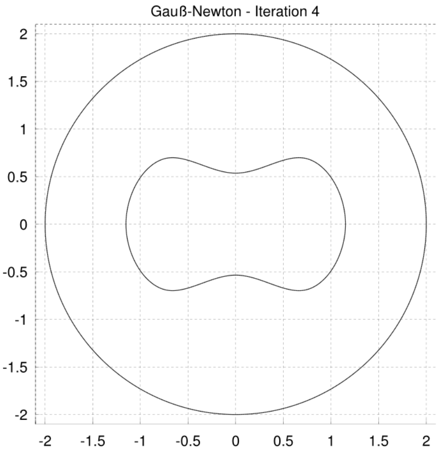

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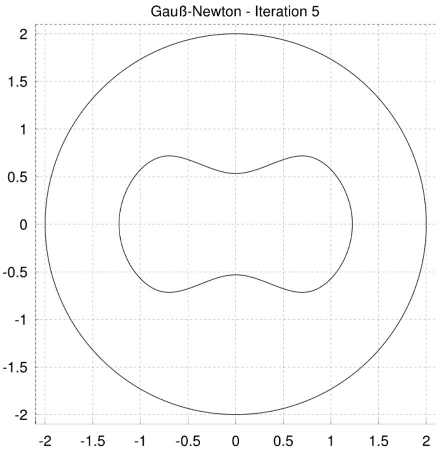

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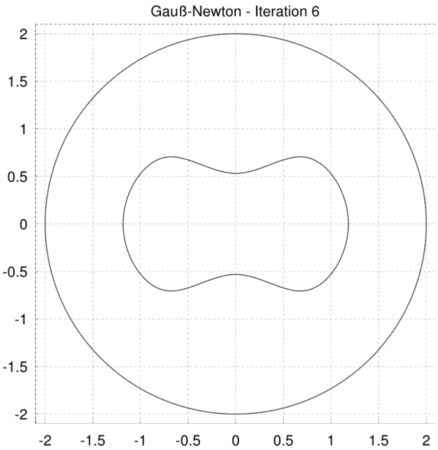

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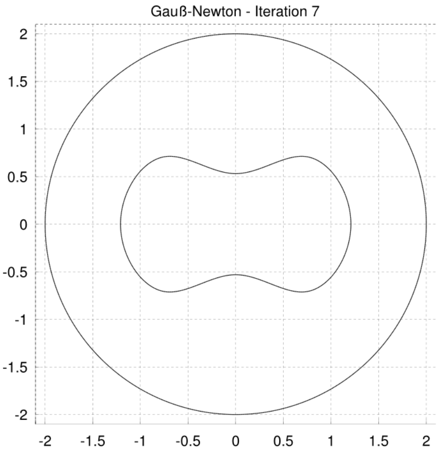

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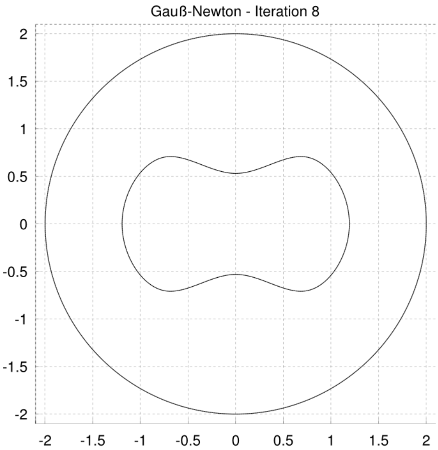

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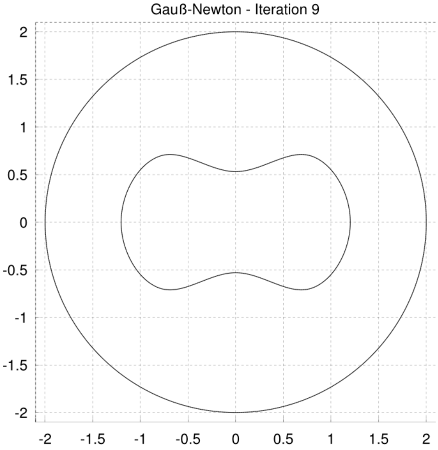

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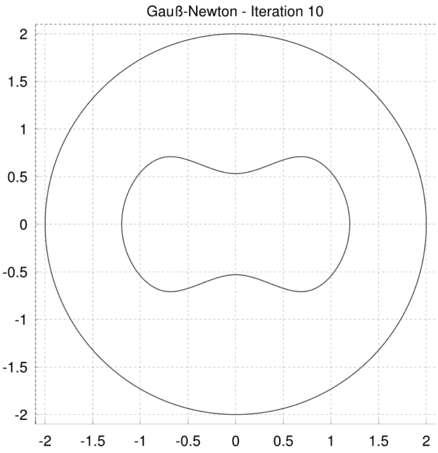

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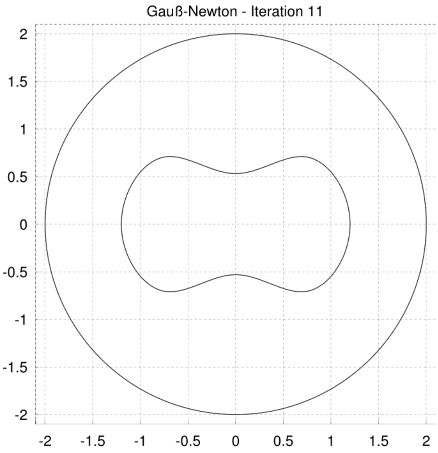

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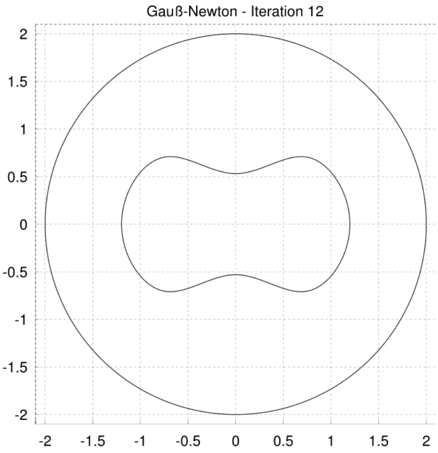

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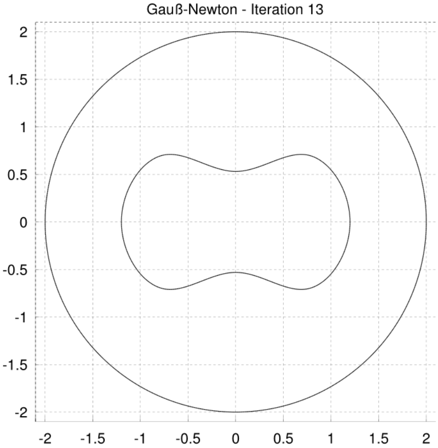

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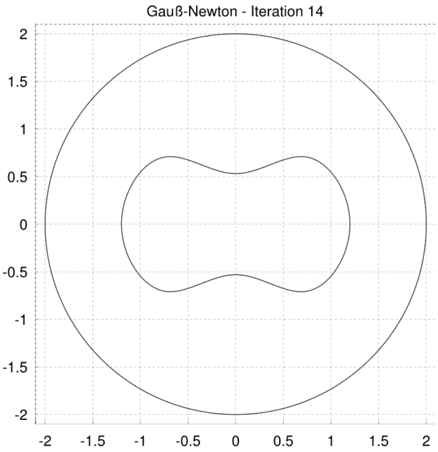

Vergleich, $\|x_k - x\|$

k	levmarqfd	levmarq	gsnewt
00	0.7616	0.7616	0.7616
01	0.2544	0.3245	0.2931
02	0.2585	0.3100	0.3291
03	0.1603	0.1881	0.1243
04	0.1122	0.1505	0.0897
05	0.0664	0.1147	0.0629
06	0.0576	0.0919	0.0517
07	0.0544	0.0802	0.0478
08	0.0518	0.0654	0.0451
09	0.0500	0.0617	0.0448
10	0.0508	0.0587	0.0442
11	0.0418	0.0556	0.0443
12	0.0426	0.0532	0.0441
13			0.0442
14			0.0441

Implementierung

- Referenzierung von geometry-features
- Funktioneninterpolation in 2D benötigt Daten in externer Datei
- Funktioneninterpolation in 2D nur linear
- Punkte auf interpolierten Kurven generieren?

Quellen

- William Rundell, 2008, Recovering an obstacle and a nonlinear conductivity from Cauchy data; Inverse Problems 24
- F. Hettlich, W. Rundell, 1998, The determination of a discontinuity in a conductivity from a single boundary measurement; Inverse Problems 14 67-82
- http://www.mathematik.uni-wuerzburg.de/~borzi/proj_elliptic.pdf
- Comsol Modelle
- Skript der Vorlesung "Einführung in die Optimierung"von Prof. Dr. B. von Harrach, WS 2013/14