Assignment 1

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Problem 1

$$f' = \frac{f(x+\delta) - f(x-\delta)}{2\delta}$$

$$f' = \frac{f(x+2\delta) - f(x-2\delta)}{4\delta}$$

$$\Rightarrow f(x\pm\delta) = f(x) \pm f'(x)\delta x + \frac{f''(x)}{2}\delta x^2 \pm \frac{f'''(x)}{6}\delta x^3 + \frac{f^{(4)}(x)}{24}\delta x^4 \pm \frac{f^{(5)}(x)}{120}\delta x^5$$

$$\Rightarrow f(x\pm 2\delta) = f(x) \pm 2f'(x)\delta x + 2f''(x)\delta x^2 \pm \frac{4}{3}f'''(x)\delta x^3 + \frac{2}{3}f^{(4)}(x)\delta x^4 \pm \frac{4}{15}f^{(5)}(x)\delta x^5$$

$$a + \underline{b\delta x^2} + c\delta x^4$$

Need a linear combination of both derivatives such as:

$$-a\left(\frac{f(x+2\delta) - f(x-2\delta)}{4\delta x}\right) + b\left(\frac{f(x+\delta) - f(x-\delta)}{2\delta x}\right) = f'(x)\delta x + cf^{(5)}(x)\delta x^5$$

$$\Rightarrow a\left(\frac{4f'(x)\delta x + \frac{8}{3}f'''(x)\delta x^3 + \frac{8}{15}f^{(5)}(x)\delta x^5 + \dots}{4\delta x}\right) + b\left(\frac{2f'(x)\delta x + \frac{1}{3}f'''(x)\delta x^3 + \frac{1}{60}f^{(5)}(x)\delta x^5 + \dots}{2\delta x}\right)$$

$$\Rightarrow a\left(f'(x) + \frac{2}{3}f'''(x)\delta x^2 + \frac{8}{60}f^{(5)}(x)\delta x^4 + \dots\right) + b\left(f'(x) + \frac{1}{6}f'''(x)\delta x^2 + \frac{1}{120}f^{(5)}(x)\delta x^4 + \dots\right)$$

Need to find a, b, c s.t. the term $f'''(x)$ cancels out

$$\Rightarrow a = -\frac{1}{3},\ b = \frac{4}{3}$$

$$\Rightarrow -\frac{1}{3}\left(f'(x) + \frac{2}{3}f'''(x)\delta x^2 + \frac{2}{15}f^{(5)}(x)\delta x^4 + \dots\right) + \frac{4}{3}\left(f'(x) + \frac{1}{6}f'''(x)\delta x^2 + \frac{1}{120}f^{(5)}(x)\delta x^4 + \dots\right)$$

$$\Rightarrow f'(x) - \frac{1}{30}f^{(5)}(x)\delta x^4$$

$$f'(x) - \frac{1}{10}f^{(5)}(x)\delta x^4$$

$$\Rightarrow f'(x) = \underbrace{\frac{8\left(f(x+\delta x) - f(x-\delta x)\right) - \left(f(x+2\delta x) - f(x-2\delta x)\right)}{12\delta x}}_{\text{derivative}} + \overbrace{\frac{1}{30}f^{(5)}(x)\delta x^4}^{\text{error}}$$

b) To compute the optimal δx, we first need to look at the error coming from the representation of the number in binary. We saw in class that,

$$\bar{F}(x) = f(x) + \epsilon f(x)$$

binary representation

true value

Accounts for the difference in both

Therefore, using the results from a) and computing the difference between $f'(x)$ and $\bar{f}'(x)$

$$\Delta f = f'(x) - \frac{8(\bar{f}(x+\delta x) - \bar{f}(x-\delta x)) - (\bar{f}(x+2\delta x) - \bar{f}(x-2\delta x))}{12\delta x}$$

$$= f'(x) - \frac{8(f(x+\delta x) - f(x-\delta x)) - (f(x+2\delta x) - f(x-2\delta x))}{12\delta x}$$

$$+ \frac{8(\epsilon f(x+\delta x) - \epsilon f(x-\delta x)) - (\epsilon f(x+2\delta x) - \epsilon f(x-2\delta x))}{12\delta x}$$

$$\Rightarrow \frac{\delta x^4}{30} f^{(5)}(x) + \frac{\epsilon}{\delta x}\frac{18}{12} f(x)$$

$$\Rightarrow \Delta f = \frac{\delta x^4}{30} f^{(5)}(x) + \frac{3\epsilon}{2\delta x} f(x)$$

To find the minimum, we differentiate Δf with respect to δx and equate to 0. We find that:

$$\boxed{\delta = \left(\frac{45}{5}\frac{f(x)}{f^{(5)}(x)}\epsilon\right)^{1/5}}$$

See Problem_1b for testing.