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Discrete mathmatics

Propositional logic

* Def: A Proposition is a declarative Sentence (declaring a fact) that is either a true or False but not both

ex:-

- cairo is the capital of egypt → True
- $5 + 3 = 6$ → false

False examples:-

* $x + 1 = 12$

← لاحظ ان اي حاجة الـ output بتاعها boolean
هتكون proposition غير كده لا

★ هنرمز للـ proposition بـ "P"

تعالى نتعرف على
شوية compound statements

Topic: $P \oplus q$ — Xor | Truth Table | $\sim P$ $\overline{P}$ $P \wedge q$ | $P \vee q$

Connectives

Truth Table

P	q	$P \oplus q$
T	T	F
T	F	T

* $\sim P$, $\overline{P}$ → Not
* $P \wedge q$ → and
* $P \vee q$ → or
* $P \oplus q$ → Xor → لازم واحدة بس تكون True

Conditional statements

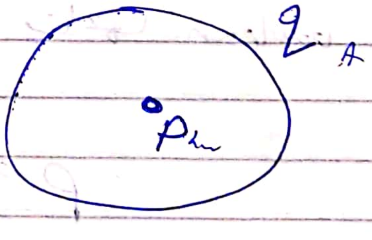
* Def: let "P" and "q" be propositions. "$P \to q$" معناها if "P", Then "q". Also,

احل السؤال ← "P" → is called The hypothesis الفرضية

A أجيب ← "q" → is called The conclusion

- The conditional statement $P \to q$ is False when "P" is True and "q" is False, and True otherwise.

انا ممكن أكون جبت A متغير مكون بلمت السؤال
لكن معنى انى بلمت السؤال انى جبت A



افتكارة مهمة

"q unless $\sim P$" (if you exclude The case That $\sim P$, Then "q" is true)

not P

* "P only if q"
* "q" is necessary for "P"

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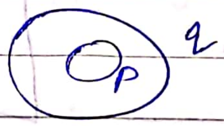
truth Table for conditional statement

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

ContraPositive, Converse and Inverse



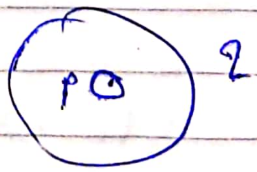
~~If~~ *: $p \to q$

- ContraPositive : $\sim q \to \sim p$ (دي نفس القيمة = equivalent ; دي)
- Converse : $q \to p$ (non-equivalent)
- inverse : $\sim p \to \sim q$ (non-equivalent)

← "P" كافية لاستنتاج q

"q" ممكن يتحقق "P" متغير - * "P" is sufficent For q

* "P" is not necessary for "q"



" if and only if "

Bi conditionals

* Def. let "p" and "q" be propositions. The biconditional statement "$p \leftrightarrow q$" is the proposition "p if and only if q". It is true when "p" and "q" have the same truth value and is false otherwise. It is also called "bi-implications"

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

* ممكن نقولها ازاي ؟ -

- "p iff q"
- "p is necessary and sufficient for q"
- if "p" then "q" and conversely

الأولوية Precedence of logical operators

1- $\neg$
2- $\wedge$
3- $\vee$
4- $\rightarrow$
5- $\leftrightarrow$

Translating english sentence

- ex:- "You can access wifi only if you are CS or I love you"

a: you can access wifi
b: you are a CS
c: I love you

$a \rightarrow (b \vee c)$

- ex:- "You can't ride the train if you are under 4 feet tall unless you are older than 16 years old".

a: You can ride the train
b: You are under 4 feet tall
c: You are older than 16 years old

$\neg a \rightarrow (b \vee c)$

4

Consistent → كل الشروط الثلاثة تتحقق

. Solution $(b \wedge \neg c) \rightarrow \neg a$

• الحالة اللى هتخلينى أوصل لـ " not a " هى لو انا مش أكبر من 16
أو أقصر من 4 feet Tall وبالتالى

" you are under 4 feet tall unless you are older 16

انا فى الـ relation بما انه قالى فأول الكلام " You can't ride the train "
هنا انا عايز الحاجات اللى هتجيبلى " not a "

فبالمنطق

انت مش هتركب القطر لو :

1. انت أقصر من 4 feet tall
and
2. انت أصغر من 16 سنة

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Propositional equivalence

* الأول، عايزين نعرف شوية حاجات :- *

① tautology : A Compound Proposition that is always true

مثلاً

p	q	$p \leftrightarrow q$
T	T	T
T	F	T
F	T	T
F	F	T

② Contradiction : A Compound that is always False

③ Contingency : neither Tautology or contradiction

* ex:- The Compound Proposition "p" and "q" are called logically equivalent if $p \leftrightarrow q$ is a tautology, $p \equiv q$ (They Produce same truth table)

الخلاصة

$\neg(p \vee q)$ and $\neg p \wedge \neg q$

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p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

بما انهم عملوا نفس ال Truth Table و

$\neg p \wedge \neg q \equiv \neg(p \vee q)$

بس الحالة دي تعتبر قانون

De Morgan's laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

لو عايز تدخل " $\neg$ " على bracket
اعكس اللي جواه ووزع ال " $\neg$ "

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* Imp note :-

→ if you have 3 Propositions P, q and r
Then you will make 2^3 rows

t	t	t
t	t	f
t	f	t
t	f	f
f	t	t
f	t	f
f	f	t
f	f	f

بنفس
الشكل ده كده

with you step by step

I wanted To ... of taking to instance

Predicates & Quantifiers

① Predicates :-

* A Predicate is a statement involving variable. The Predicate $P(x)$ takes a Propositional Value (true or false) after assigning a value to "x"

ex:- Let $P(x)$ denotes the statement "$x > 3$" so:

$P(4)$ "$4 > 3$" → true

$P(2)$ "$2 > 3$ → false

- In Programming all conditions are Predicates

Quantifiers

- to Scope a Predicate over a range of values domain, universe of discourse
- "universal quantification" → دى عبارة عن symbol ($\forall$) That indicates That a Certain statement holds true for all elements within a specified or domain

*An element for which $P(x)$ is false is called a Counterexample of "$\forall x\, P(x)$"

- "existential quantification"

عبارة عن symbol "$\exists$"

it indicates that there exists at least one element within a specified set or domain for which a certain statement holds true

يعني الرمز ده بيعبر ان في على الأقل قيمة واحدة للـ variable اللي هو "x" ضمن range معين بتحقق statement معينة

ex:- There exist a student in the class who has scored 100% on the exam

Translation

$$\exists x \in Students : Score(x) = 100\%$$

- "$\exists x$" → There exist an "x"
- "$\in$" → belongs to
- "$Score(x) = 100\%$" → represent the statement that the score of student "x" = 100%

Universal quantification refers to statements that are true for all elements in a set, while existential quantification that are true for at least one element in a set

ملحوظة: الحته دي كوه اسمها Predicate عشان
هي statement مستنية تاخد Preposition
الـ Variable = x

Ex:-
- $P(x)$: "$x^2 > 0$" and The domain is all integers

ترجمتها "$\forall x\, P(x)$" is False by counterexample $x = 0$

لو قولت ان "$\forall x P(x)$" غلط فلازم تجيب counterexample

Quantifiers with Restricted domains

* Ex:- $\forall (x < 0)\ (x^2 > 0)$ ~~and The domain is negative no.~~

بدل ما كتب كده اختصرتها كده

يعني

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* All quantifiers have higher Presedence than Logical operators.

ex:- $\forall x\, P(x) \vee Q(x) \equiv (\forall x\, P(x)) \vee Q(x)$

$\not\equiv \forall x\, (P(x) \vee Q(x))$

— // — // — // — // —

Binding Variables

quantifier

• مبدئيًا عايز افهمك حاجة

$\forall x\, (P(x))$

كل ما هو بين الـ bracket

" Scope of a quantifier = يسمى

* A Variable "x" is free if and only if the Variable does not occur in the Scope of a quantifier

* A Variable "x" is bound if and only if The Variable occurs in the Scope of a quantifier

" bound Variable =

ex:- $\exists x\, (P(x) \wedge D(x)) \wedge Q(x)$ Free Variable

طبعًا انك فاكر قانون الـ Presedence بتاع الـ quantifiers فانت عارف ان الـ ∃x هتطبق على كل ما هو قبل الـ and

وان الـ x اللى بعد الـ and ملهاش اى علاقة

بالـ "∃x" اللى قبل الـ and

" iff " → if and only if

☆ A sentence is well formed iff there are no occurence of free Variables in a sentence, and it follows all other synattic rules → ملناش دعوة بيها دلوقتي

← هو بيقولك ان المفروض متسيبش دول Variables عشان تكون برة ال scope بتاع اي Quantifier

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T T T only case

الشمال لليمين $A \longrightarrow B$ (sufficient)

اليمين للشمال $B \longrightarrow A$ (necessary)

Logical equivalence involving quantifiers

* $\forall x \in S\, P(x) \equiv \forall x\,((x \in S) \rightarrow P(x)) \not\equiv \forall x\,((x \in S) \land P(x))$
* $\exists x \in S\, P(x) \equiv \exists x\,((x \in S) \land P(x)) \not\equiv \exists x\,((x \in S) \rightarrow P(x))$

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Negating quantified expressions

* Q: "Every student in my class has successed"

↳ $P(x)$ "student "x" in my class has successed".

↳ $\underbrace{\neg \forall x P(x)}$, where the domain is the students in my class

الترجمة ↳ It's not the case that every student in my class has successed

which means . . . .

There is a student in my class hasn't successed

بالتالي

$$\neg \forall x \, P(x) \equiv \exists x \, \neg P(x)$$

$\forall x > 0 \; P(x)$

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* Proof $\neg\forall x\, P(x) \equiv \exists x \neg P(x)$

الشمال بيساوي اليمين

- Sufficiency: suppose that $\neg\forall x P(x)$ is true which means "$\forall x P(x)$" is False which means that $\exists x \neg P(x)$ is true.

- Necessity: suppose that $\exists x \neg P(x)$ is true So there is at least $x = a$ where $P(a)$ is false, which means $\neg\forall_x P(x)$ is true

De morgen's laws (quantifiers)

* $\neg\forall x\, P(x) \equiv \exists x \neg P(x)$

* $\neg\exists x\, P(x) \equiv \forall x \neg P(x)$

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(logic law)

De morgen's laws

* we can use logical equivalences to reduce complex Formulas into simpler ones

* Identity laws :-

* $P \wedge t \leftrightarrow P$
* $P \vee F \leftrightarrow P$

small example

$(P \vee F) \wedge (q \vee t)$

$\downarrow$ Id $\quad \downarrow$ Dom

$P \wedge t$

$\downarrow$ Id.

P

* Domination laws :-

* $P \vee t \leftrightarrow t$
* $P \wedge F \leftrightarrow F$

* $\neg\neg P \leftrightarrow P$ "Double Negation"

* Demorgan's law :-

* $\neg(P \wedge q) \leftrightarrow \neg P \vee \neg q$
* $\neg(P \vee q) \leftrightarrow \neg P \wedge \neg q$

small ex :-

$\neg(\neg P \wedge \neg q)$

$\neg\neg P \vee \neg\neg q$

$\rightarrow P \vee q$

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* Distributive laws :-

* $p \wedge (q \vee r) \longleftrightarrow (p \wedge q) \vee (p \wedge r)$

* $p \vee (q \wedge r) \longleftrightarrow (p \vee q) \wedge (p \vee r)$

* Absorption Laws :-

* $p \wedge (p \vee q) \longleftrightarrow p$

* $p \vee (p \wedge q) \longleftrightarrow p$

→ ex:- $\neg\neg p \vee ((p \vee F) \wedge \neg\neg q)$

$\downarrow$

$p \vee ((p \vee F) \wedge q)$ 2x DN (double negation)

$\downarrow$

$p \vee (p \wedge q)$ identity

$\downarrow$

p absorbtion

Date: / / Subject:

ودول (V و ∧) هنا هنتكلم على

* Commutativity & Associativity :-

← ممكن تبدل أما كنعوم ولو اتنين جوا bracket وواحده بره ممكن تدخل اللي بره لجوه وتطلع أي واحده بره الها عادي شرط ان انت تكون متعامل مع "∧" بس أو "V" بس

- ex:- $p \wedge q \longleftrightarrow q \wedge p$
  $p \vee q \longleftrightarrow q \vee p$ (Commutative)

- ex:- $p \wedge (q \wedge r) \longleftrightarrow (p \wedge q) \wedge r$

لاحظ ان الاتنين لازم يكونوا "∧"
وإلا هتشتغل بقانون الdistributive

Inverse laws

* $p \wedge \neg p \longleftrightarrow F$ (contradiction)
* $p \vee \neg p \longleftrightarrow T$ (tautology)

* Conditional law :-

* $p \rightarrow q \longleftrightarrow \neg p \vee q$

لأن لو "¬p" فالنتيجة هتبقى True على طول

لو "q" اتحققت يبقى الstatement صح

with you step by step

$\neg(p \wedge q) \wedge q$

$\downarrow$

$(\neg p \vee \neg q) \wedge q$ De morgen's

$\downarrow$

$(\neg p \wedge q) \vee (\neg q \wedge q)$ Distributive

$\downarrow$

$(\neg p \wedge q) \vee$ False inverselaw

$\downarrow$

$(\neg p \wedge q)$ identity laws

Follow Negating quantified expressions

* $\forall_x P(x) \equiv (P(x_1) \land P(x_2) \lor \ldots P(x_n))$
* $\exists x P(x) \equiv (P(x_1) \lor (P x_2) \lor P(x_n))$

$\neg \forall x P(x) \longrightarrow \exists x \neg P(x)$

Nested Quantifiers

* def: one or more quantifiers are placed inside the scope of another quantifier

* ex:- $\forall x \exists y (x+y=0) \equiv \forall x (\exists y (x+y=0))$

↳ for all "x" there exist "y"

Rules of inference

* Valid Argument in Propositional logic :-

* An "Argument" in Propositional logic is a sequence of Propositions

* All but the final proposition in the argument are called "Premises" the final is called "Conclusion"

* ال argument يعتبر Valid لو ال truth value بتاع كل ال Premises True
بـ imply ال Conclusion انها True حتى لو ال order بتاع ال Premises
اتبدل

ex:- $(P_1 \wedge P_2 \wedge \dots P_n) \rightarrow q$ (is a tautology)
← لازم يكون كله True

ex:- "if you are tall, you can score"
"You are tall"
Therefore,
"You can score"

Argument form,

$$\begin{array}{l} P \rightarrow q \\ \underline{P} \\ q \end{array}$$

→ الخطوه دي

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* **Modus Ponens :-** → دە شكل من ال forms بتاعة ال argument :

- if P, then q.

P.
Therefore, q.

$((P \to q) \land P) \to q$ (tautology)

$\therefore ((\neg P \lor q) \land P) \to q$

$\therefore \underbrace{(\neg P \land P)}^{\text{false}} \lor (P \land q) \to q$

$\therefore \neg P \lor \neg q \lor q$

$\therefore \neg P \lor T$ ↓ dominance law

$\therefore T$

* ممكن يبقى ال Conclusion بـ False وال Argument بتاعك يبقى Valid عادى بدل لو ال P or q بـ false وال q بـ false فالـ statement بـ True عادى جدًا

* ممكن يبقى ال Argument مش Valid وال Conclusion بـ True

ex:-
$P \to q$
$\neg P$
$\overline{\neg q}$ → غلط

Argument is Valid if Premises logically entail The conclusion

ممكن في ال Truth Table ال q → تطلع True مرة لكن ال Logic الكانت شغال بيه مش سليم

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طيب يعني نفتش تانية لل argument forms غير ال Modus Ponens.

ال Modus Ponens هو عبارة عن law

من ال inference laws ← استنتاج او استدلال

Inference laws ← بنستعملها عشان نعمل reasoning & نطلع conclusion من ال premises

① Modus Ponens (MPP)

② Modus tollens :- (MTT)

contra

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \neg p \end{array}$$

Proof → $\equiv \neg q \rightarrow \neg p$ ($p \rightarrow q$)

(contrapositive)

③ Hypothetical syllogism :- (HS)

القياس الافتراضي

المنطقي → القياس المنطقي

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

④ Disjunctive syllogism :- (DS)

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

with you step by step

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(5) Addition ($\vee I$)

$$\frac{p}{\therefore p \vee q}$$

→ لو P = True
يبقى (P أو أي حاجة) = True

حلال

(6) Simplification ($\wedge E$)

$$\frac{p \wedge q}{\therefore p \quad \therefore q}$$

→ لا تعليق

(7) Conjunction ($\wedge I$)

$$\frac{p \quad q}{\therefore p \wedge q}$$

→ WOW !!

اختصارات سريعة

(1) MPP $\frac{p \to q \quad p}{\therefore q}$

(2) MTT $\frac{p \to q \quad \neg q}{\neg p}$

(3) الطريق السريع (اول ما syllogism) $p \to q$

(4) لو انا وانت وحش انا يبقى انت

(5) addition (انا جامد) ← ~~لو انا يبقى انا اذى~~

(6) simplification كلنا كل واحد فينا

(7) conjunction لو انا وانت يبقى انا وانت

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search كلمه اكتر

⑧ Resolution

$$\frac{\begin{array}{l} p \vee q \\ \neg p \vee r \end{array}}{q \vee r}$$

لان يا اما P هتحصل أو مش هتحصل

اضرب

* Fallacies → تعمل غلطة = وانت بتعمل prove بال arguments وكده

ex:-

$$\frac{p \to q}{q \to p}$$

→ دي كده fallacie

Inference laws involving Quantifiers

① Universal instantiation:- (UI)

استنباط شئ

- $$\frac{\forall x \; P(x)}{\therefore P(c)}$$

"c" جوا ال domain

② Universal Generalization:- (UG)

- $$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x \; P(x)}$$

يعني لو في صفة معينة
بتنطبق على بعض أفراد مجموعة
هنا ممكن أفترض ان المجموعة كلها
كده

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③ Existential instantiation: (EI)

- $\exists x\ P(x)$

$P(c)$ for some c

④ Existential Generalization :-

- $P(c)$ for some element c

$\therefore \exists x\ P(x)$

⑤ Universal Modus Ponens :-

- $\forall x\ (P(x) \rightarrow Q(x))$

$P(a)$ where "a" is in domain

$\therefore Q(a)$

ex:- "Oliver

Proofs

* a Proof is a Valid argument

* Formal Proof →

$$\begin{array}{l} P \to E \\ P \\ \hline E \end{array}$$

* informal Proof → if "P" Then "E"

شوية مصطلحات مهمة

① ProPosition → Statement That is either True or False

② Axiom → Statement which we assume to be True without a Proof

③ Lemma → a small result that has been proved. it is used To Prove a Theorem

(4) Theorem → an important result that has been proven

(5) Corollary → a Theorem that follows on from another Theorem

Theorem معتمدة في إثباتها
على Theorem تانية

(6) Conjecture → a Statement that someone guesses to be true, although they are not yet able to prove or disprove it

بذاتها

* الـAxiom هي حاجة بنفترض انها صح وبانبني عليها قوانين
زي مثلاً ان طول الـst. line ممكن يبقى infinity

بناءً على شيء معين و فكرة معينة

* لكن الـConjecture هو مجرد فرضية ~~بدليل~~ منتظر
ابد اثبات او النفي

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$P \rightarrow Q$

antecedent consequent

* Methods of Proof :-

① direct Proof :

→ we assume the antecedent is true, Then use rules of inference, axioms, definitions and / or previously proven Theorems to show the consequent is true

ex:- Prove : If "n" is an odd integer, then n^2 is odd

* طيب مبدئياً هنتفق على شوية حاجات

← الـ even والـ odd بيكتبوا كده ←

- Even integer: $n = 2k, k \in \mathbb{Z}$
- odd integer: $n = 2k+1, k \in \mathbb{Z}$

* Assume "n" is an odd No. Then $n = 2k+1, k \in \mathbb{Z}$ by def. of an odd integer.

$$(n)^2 = (2k+1)^2$$

$$\therefore n^2 = 4k^2 + 4k + 1$$

$$\therefore n^2 = 2(2k^2 + 2k) + 1$$

Let $u = 2k^2 + 2k$

$$\therefore n^2 = 2u + 1$$

$\therefore n^2$ is odd $\therefore$ □ ◁ QED

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ex. 2 Prove "The sum of 2 even integers is even"

وأحسن 'a', 'b' are even integers

→ Assume that "n" is an even number, then $n = 2K$, $K \in \mathbb{Z}$. by def. of even num.

$$n + n = 2K + 2K$$
$$\therefore n + n = 4K$$
$$\therefore n + n = 2(2K)$$
$$\text{Let } u = 2K$$
$$\therefore n + n = 2u$$
$$\therefore n + n \text{ is even}$$

هو عايزك تثبت ان الsum بتاع رقمين even مختلفين even فالأصح انك تكتبه كده

Assume a , b even

$$a + B = 2m + 2Z$$
$$\therefore a + B = 2(m + Z)$$
$$\text{let } u = m + Z$$
$$\therefore a + B = 2u +$$

even

with you step by step

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② Contra Position proof

* if direct proof reaches dead end
* ~~if~~ $P \rightarrow q$: $\neg q \rightarrow \neg P$

بفترض

لو

وهوصل لـ

• ex:- Prove That if $3n+2$ is odd, where n is an integer Then n is odd

لو بالـ direct

$3n + 2 = 2k+1$

$n = (2k-1)/3$ (???)

هقفلها

بـ

لو بالـ ContraPosition

if n is even :

$n = 2k$

$\therefore 3n+2 = 6k+2$

$\therefore 3n+2 = 2(3k+1)$

Let $u = 3k+1$

$\therefore 3n+2 = 2u$

$\therefore \neg q \rightarrow \neg P$

$\therefore P \rightarrow q$

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③ Vacuous and trivial Proofs :-

* $P \to q$ is true when P is false (Vacuous)
* $P \to q$ is true when q is true (trivial)

ex:- Let $P(n)$: $\forall n$ if $n > 1$, Then $n^2 > n$ where the domain is $\mathbb{Z}$

Vacuosly, $P(0)$ is true since $0 > 1 \to 0 > 0$ is true

ex:- if a and b are Positive integers with $a \geq b$ Then $a^n \geq b^n$

trivially, $P(0)$ is true since $a^0 \geq b^0$ regardless
$1 \geq 1$ to the hypothesis

with you step by step

$\neg p$ $\neg p$ — $\neg q$ $a = 2r$

④ Proof by Contradiction :- $\neg P$ P

→ to prove 'P' is true, we assume 'P' is False. then use That hypothesis to derive a Falsehood

ex:- Prove $\sqrt{2}$ is irrational

→ let's assume that $\sqrt{2}$ is a rational number

$\therefore \sqrt{2} = \frac{a}{b}$

$\therefore 2 = \frac{a^2}{b^2}$

$\therefore 2b^2 = a^2$

$\therefore$ a is even

$\therefore a = 2c$

$\therefore a^2 = 4c^2$

$\therefore 2b^2 = 4c^2$

$\therefore b^2 = 2c^2$

$\therefore$ b is even

$\therefore \left[\frac{a}{b}\right]$ is not in the simplest form
\# contradiction

$\therefore \sqrt{2}$ is irrational QED

Prove is done

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Proof of equivalence

* prove that $p \to q$
  and $q \to p$

* ex1- prove that

$$\begin{tikzcd} P_1 \arrow[r] & P_2 \arrow[dl] \\ P_3 \arrow[u] & \end{tikzcd}$$

كده نتاثبت ان كلهم equivalent

* (Dis) Proof by Counterexample :-

- this is to show that $\forall x\, P(x)$ is false

→ in order to do that you need to find counter example

★

$4 - 4 + 4 - 4$

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* Mistakes in Proofs :-

① Dividing by zero

يـ الناس الحلوة اللي بتقولك انا هثبتلك
ان واحد بيساوي اتنين 😄

ex:

$$4-4 = 4-4$$
$$\therefore 2(\cancel{2-2}) = (\cancel{2-2})(2+2)$$
$$\therefore 2 = 4$$
$$\therefore 1 = 2$$

المشكلة هنا انه
قسم على zero وده غلط

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Induction

- Induction axiom : Let $P(n)$ be a predicate. If $P(0)$ is True and $\forall n \in N \; (P(n) \Rightarrow P(n+1))$ is true. then $\forall n \in N \; P(n)$

- يعني لو $P(1) \to P(2)$ ، $P(2) \to P(3)$ ، $P(3) \to$ ...

يبقى $P(n)$ معتبر true لكل ال Values بتاعة ال n

* How to Prove by induction :-

1) Prove Base case
2) induction step → $\underbrace{\text{if } K(n) \text{ Then } K(n+1)}$

ده Prove ني
عن طريق اني افترض ان $K(n)$ صح
واثبت بعدها ان $K(n+1)$ صح

ex:-

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Ex:- Prove That $1+2+3+4+5+\dots n = \frac{n(n+1)}{2}$

Base case

$P(0)$ is true

$$\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$$

Inductive steps

for $n \geq 0$, show $P(n) \Rightarrow P(n+1)$ is true.

→ Assume $P(n)$ is true for purpose of induction

we need to show $1+2+\dots(n+1) = \frac{(n+1)(n+2)}{2}$

$$1+2+3\dots(n+1)$$

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n^2+n+2n+2}{2} = \frac{(n+1)(n+2)}{2}$$

$$\frac{n^2+n}{2} + \frac{n+1}{1}$$

$$\frac{n^2+n+2n+2}{2} =$$

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EX.2 Prove $\forall n \in \mathbb{N}$ $3 \mid (n^3 - n)$

- base case $P(0)$ is true

→ مش محتاج اعملها

- Inductive step

Assume That $P(n)$ is true for Purpose of induction
i.e. $3 \mid (n^3 - n)$

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1)$$
$$= n^3 + 3n^2 + 2n$$
$$= (n^3 - n) + 3n^2 + 3n \quad \text{(by adding and subtracting "n")}$$

← ده انا افترضت انه divisible لـ 3 اصلا

دول [illegible] عليهم

خد بالك

- Base case $P(b)$ is true

- Induction step ~~$\forall n \in \mathbb{N}$~~ $\forall n \geq b$ $P(n) \to P(n+1)$

- Conclude

$\forall_{n \geq b}$ ($P(n)$ is true)

with you step by step

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q → necessary

* invariance property: It's a special quality about something that doesn't change, no mmatter what you do to it.

~~Number~~ Theory

Sets

* def: A set is an unordered collection of objects

* two Sets are equal if They have the same elements

$A = B \equiv$ if $\forall x (x \in A \longleftrightarrow x \in B)$ is True

* Repitition has no Value in set

$x = \{1, 3, 3\} \rightarrow \{1, 3\}$

* $\phi = \{ \}$

* ~~Singel~~
* Singleton set (one element set) $\rightarrow \{a\}$; $\{\{a, b, c\}\}$

* The set "A" is said to be a subset of B $A \subseteq B$

$\forall x (x \in A \rightarrow x \in B)$ لو

مش ⟷ لا لأن عايز تقول ان ال A

تبقى subset ال B مش ان ال B subset ال A

if $\neg(A \in B)$

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~~$A \subseteq B$~~ $A \subset B$

* The set "A" is said to be a <u>Proper subset</u> of B

لو $\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$

يعني لو كل الحاجات اللي في "A" موجوده في "B" ومش كل
الحاجات اللي في "B" موجوده في "A" او A يكون عندها .

* null "{ }" is considered to be a subset of every set :

Let's Prove it :-

definition of subset

$\forall x(x \in \phi \rightarrow x \in S)$

الطريقة الأسهل

- Proof by contraposition.

  هي طريقة $\neg q \rightarrow \neg p$

  $x \notin S \rightarrow x \notin \phi$

  $\rightarrow x \notin S$ دي معايا

  ~~$\rightarrow x$~~ وانا محتاج اثبت ان $x \notin \phi$

  $x \notin \phi$ is $T \Rightarrow$ Vacuous دي صح

  $\therefore x \in \phi \rightarrow x \in S$

$\forall x(x \in \phi \rightarrow x \in S)$

$\because x \in \phi$ is always false

$\therefore \forall_x(x \in \phi \rightarrow x \in S)$ is always True

ليه ϕ مش proper subset لأي set ؟

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* Cardinality of $|S|$
where $S = \{1, 2, 3\}$ is $3 \rightarrow$ الـ size الي هو الـ عدد الـ elements

* $|\phi| = 0$

• The Power Set :-

* The Power set of the set S ($P(S)$) is The set of all Subsets of the set S. (يعني لازم يكونوا sets)

• We will prove that if $|S| = n$, then
$$|P(S)| = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$$

وخد بالك ان الـ sets مبتعترفش بالـ order او التكرار

ex:-
$S = \{1, 2, 3\}$
$P(S) = \{\phi, \{3\}, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

خد بالك ان اي $P(S)$ بيبقى فيها الـ S نفسها والـ ϕ.

$$|P(S)| = \underset{\phi}{1} + \underset{{}^3C_1}{3} + \underset{{}^3C_2}{3} + \underset{\{1,2,3\}}{1} = 2^3 = 8$$

cardinality

$|P|$

* $P(\phi) = \{\phi\}$
* $P(\{\phi\}) = \{\phi, \{\phi\}\}$

with you step by step

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* if $s = \{1, 2\}$

So :-

- $\phi \in P(s)$
- $s \in P(s)$
- $\{\phi\} \subseteq P(s)$
- $\{s\} \subseteq P(s)$

$\rightarrow$ $P(s) = \{\phi, 1, 2, s\}$

Cartesian Products

* def: Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$

$\rightarrow$ $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

ع الترجمة: ~~حط~~ كل ال (a, b) where ان
$b \in B$ و $a \in A$

Truth Sets of Predicates

* if The domain is "D", Then The truth set of The Predicate $P(x)$ is $\{x \in D \mid P(x)\}$

ع وبنقول عليها ال solution set

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Set operations

* union $(A \cup B)$

• $A \cup B = \{x \mid x \in A \vee x \in B\}$

* intersection $(A \cap B)$

• $A \cap B = \{x \mid x \in A \wedge x \in B\}$

* Disjoint :-

* A and B are disjoint if $A \cap B = \phi$

* Difference:-

• $A - B = \{x \mid x \in A \wedge x \notin B\}$

* Complement

• $\bar{A} \rightarrow$ elements which are not in set A

$\bar{A} = \{x \mid \neg(x \in A)\}$

De morgen's for sets : ال cartesian مش commutative

- **Generalized unions and intersections :-**

$$* \quad A_1 \cup A_2 \cup A_3 \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

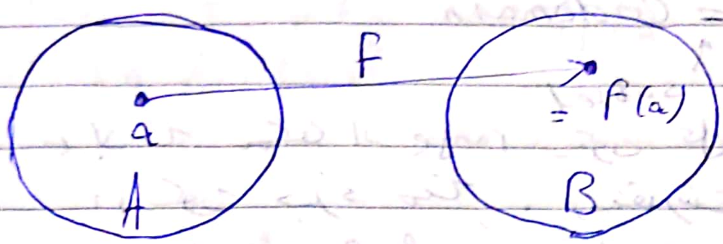
$$* \quad A_1 \cap A_2 \cap A_3 \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

- **Superset :-**

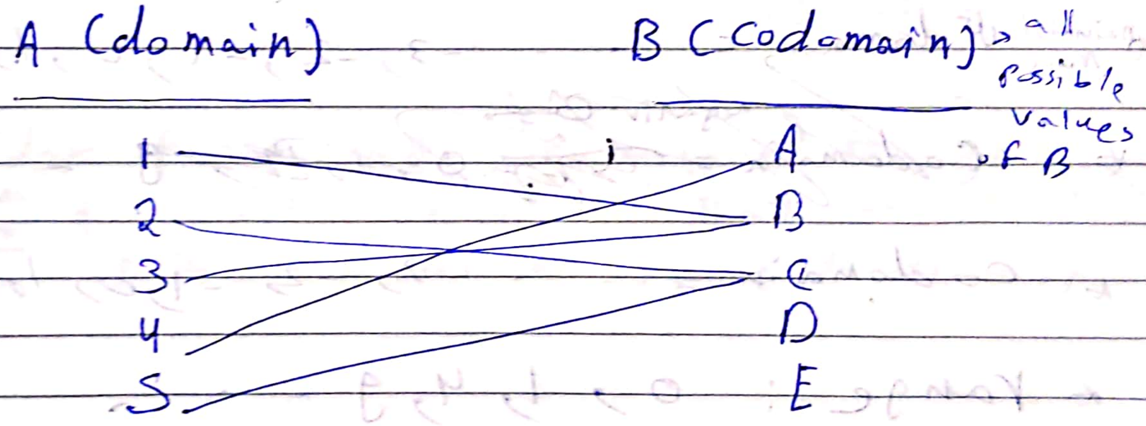
$x \subseteq Y$

$\therefore$ Y is a super set of x

Functions



* def: Let A and B be nonempty sets. A function "f" from A to B is an assignment of exactly one element of B to each element of A. We write f : A → B (assign) (also called mapping or transformation or f maps A to B). The range of f is the set of all images of elements of A.



• "f" maps {1, 2, 3, 4, 5} to {A, B, C}.

• Terminologies:-

* Image: {A, B, C} is the image of A under f
* Pre image: {1, 3} the Preimage of B under f
* range: {A, B, C} all ~~possible~~ values of B that represent images to A - element.

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* codomain → كل الإحتمالات الممكنة
* range → الإحتمالات اللي بتحقق images فعلاً

* So, range $\subseteq$ codomain
  ↑ subset

بلاش ممكن الـ range يكون كل اللي في الـ codomain أو يكون جزء منه . ومنقدرش نقول ان الـ range يعتبر proper subset من الـ codomain لإن جايز الـ codomain range ; ميبقاش عنده أي حاجة مش موجوده في الـ range

ex:-
$$F(x) = \underbrace{x^2}_{\text{domain}} \rightarrow \text{range}$$
where $F: \underset{\text{domain}}{Z} \rightarrow \underset{\text{codomain}}{Z} \rightarrow$ natural No.s

* domain = كل الـ natural No.s $\{ \dots -3, -2, -1, 0, 1, 2, 3 \dots \}$

* ~~codomain = {0, 1, ... 9}~~ negative مفيش

* codomain = $\dots -3, -2, -1, 0, 1, 2, 3 \dots$

* range : $0, 1, 4, 9 \dots$

واضح هنا ان الـ range يعتبر subset من الـ codomain

• Some info :-

* Let f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to R

ex :-

$$f_i : R \to R,\ \underbrace{i = 1, 2}_{}\ ,\ f_1(x) = x^2\ ,\ f_2(x) = (x - x^2)$$

2 functions ای

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = \underbrace{x^2 + (x - x^2) = x}_{}$$

هدي جوا ال R
عادي

$$(f_1 f_2)(x) = f_1(x) \times f_2(x) = x^2(x - x^2) = \underbrace{x^3 - x^4}_{}$$

جوه ال R عادي

* Consider $f : A \to B$. The image of the set $S \subseteq A$ is the subset of B that consists of images of the elements of S; we denote it by $f(S)$.

Formula : $f(S) = \{t \mid \exists s \in S\ (t = f(s))\}$

$= \{f(s) \mid s \in S\}$

هو ال set بتاعة ال images بتاعة ال elements اللي موجودة في S

بتكون من $\underbrace{t\ \text{elements}}_{}$ ← اللي هي ال image بتاعة ال elements

اللي موجودين في S ، و S دي تعتبر sub set من A

* one - to - one function
—— " —— " —— " ——

- def : A function is said to be one-to-one function (infective function) if $f(a) = f(b)$ implies that $a = b$ for all a, b in the domain of f

- Formal: $\forall a \forall b \, (f(a) = f(b) \rightarrow a = b)$

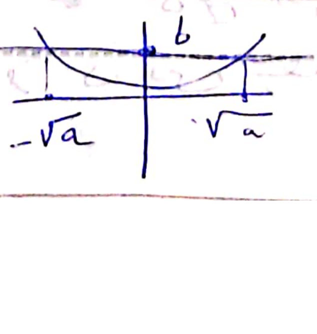
- كل الـ element في الـ set A ليه image واحدة في الـ set B .

ex :- $f(x) = x^2$

elements of A $a = b^2$ elements of b

$\therefore b = \pm\sqrt{a}$

ده كده مش one-to-one function علشان الـ elements الموجودة في b جايز يكون ليها 2 pre image في A



مش one to one

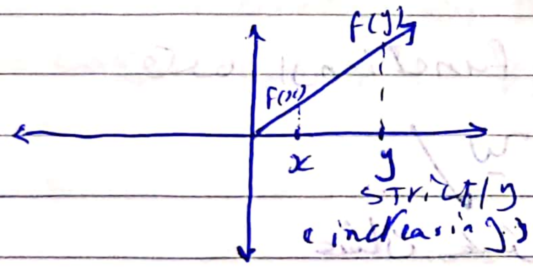
* Increasing & decreasing functions :-

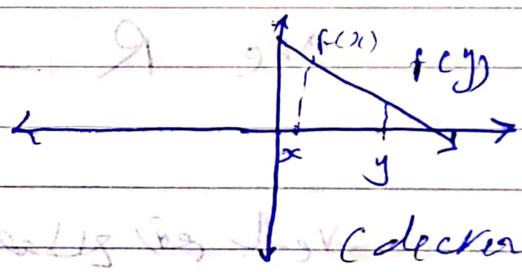
• A function f, whose domain and codomain are subsets of $\mathbb{R}$ is called increasing if $f(x) \le f(y)$, and strictly increasing if $f(x) < f(y)$, whenever $x < y$ and x, y belong to domain

التعريف علمياً
formal

$$\forall x \forall y \, (x < y \rightarrow f(x) \le f(y)) \text{ (increasing)}$$

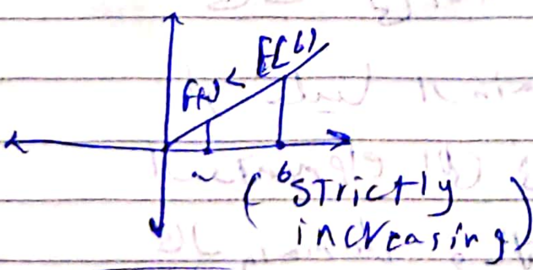
$$\forall x \forall y \, (x < y \rightarrow f(x) < f(y)) \text{ (strictly increasing)}$$



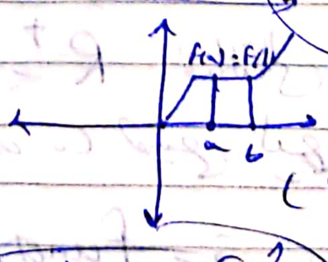


and vice versa

* لو ال function كان strictly increasing يبقى معناه انه اكيد



One-to-one



one-to-one function

مش one-to-one

* onto functions :- (Surjective functions)

- the function $f: A \to B$ is called "onto" or (surjective) if $\forall_{b \in B} \exists a \in A \, (f(a) = b)$

  each element جوا الـ codomain ليها pre-image في الـ domain

  وبالتالي

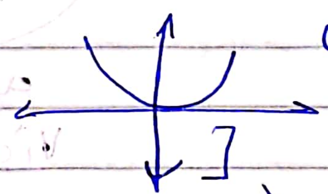
range = Codomain.

ex: $f(x) = x^2$, both domain and codomain are R → يكون الـ function دي مش "onto"

ليه؟

عشان مفيش x^2 بالسالب هيطلع رقم بالـ -ve

وبالتالي



كل الـ negative No.s اللي تحت هنا متعتبرش ليها image في حاجة في الـ domain (الـ x-axis). لكن مثلاً لو خلينا الـ codomain = R^+ فبالتالي عندي كل الـ elements اللي في الـ codomain هتحقق ليهم pre images في الـ domain وبالتالي الـ function هتبقى onto.

* if The function is both surjective and injective. we call it **bijective** or (one-to-one correspondence)

ex:- Let "S" be a set with $|S| = m$ (cardinality), Then there is a bijection between "S" and the set $\{1, 2, \dots, m\}$ → let's call it "t"

proof: $S = \{s_1, s_2, \dots, s_m\}$; $t = \{1, 2, \dots, m\}$

$f : s_i \rightarrow i$ where $i = 1, 2, 3, \dots, m$

injection: $s_i = s_j \rightarrow i = j$

surjection: $\forall i \in t \; \exists s_i \; (f(s_i) = i)$

☆ Inverse function:-

• def: Let $f : A \rightarrow B$ be a one-to-one correspondence (bijection). The inverse function of f is The function that assigns every element $b \in B$ to a unique element $a \in A$,

يعني العكس

$f(a) \rightarrow b$

$f^{-1}(b) \rightarrow a$

• الـ Preimage بتاعة الـ element اللي موجودة في الـ codomain هي نتيجة اني اعمل mapping من set B لـ set A باستعمال الـ function ← f^{-1}

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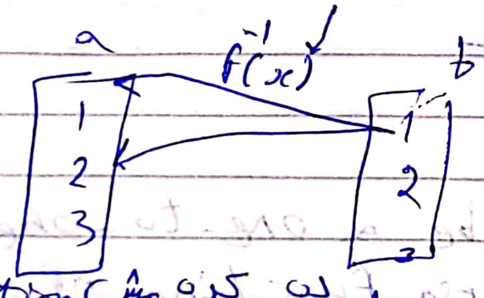
• an inverse function can't be obtained unless :

- The main function was one-To-one

عشان تجيب الـ inverse function لازم (f) يكون element واحدة من الـ A

طب هو ليه لازم يكون element واحده من الـ A

عشان يا ذكي تعريف الـ function انا انا
هو اني assign حاجة أو اكتر من الـ set
اللي انا بدأت عندها بحاجة واحدة
تانية في set مختلفة



لو كده مش function
انا انا

if not one-to-one
Then $f^{-1}(x)$ is
not a function

- The main function was surjective (onto)

عشان كل الحاجات اللي في الـ co domain
لازم يكون ليها
preimage ~~inverse~~
وإلا هيبقى عندي
elements في B
لما اجي اعمل mapping
باستخدام الـ set A لـ
الـ function f مش موجودة
function لأن مش هتبقى
في A بيوصل ليها

if not onto
Then $f^{-1}(x)$ has
no meaning for
some values

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• ويسمى كده

bijection is also called invertible

انفرز اكسبد

* Composite of a function :-

• def: Consider $g: A \to B$ and $f: B \to C$. The Composition of f and g, denoted by $f \circ g$, is defined as

$$(f \circ g)(x) = f(g(x))$$

* Identity function :-

• $i_A : A \to A$, $i(x) = x$

هي عبارة عن assign لكل ال elements اللي في ال set A لنفسها

تاني بحيث ان ال image بتاعة ال A هي ال A نفسها

• We can obtain I_A by $f^{-1} \circ f$, $\forall A$