NON-CONVEX SHREDDED SIGNAL RECONSTRUCTION VIA SPARSITY ENHANCEMENT

Arindam Bose and Mojtaba Soltanalian*

Department of Electrical and Computer Engineering
University of Illinois at Chicago
Chicago, USA

ABSTRACT

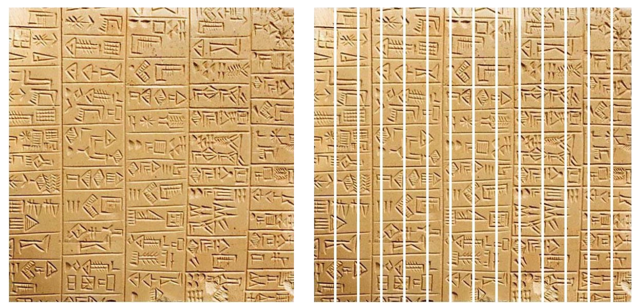
Restoration of shredded signals remains a relevant and significant challenge in archaeological and forensic efforts. In this work, we present a novel approach for reconstruction of shredded signals (including text documents and images) within a context of general multidimensional sparse signals. To this end, we present a generic efficient non-convex optimization method that employs iterative sparsity enhancement of the observed signal. A key component of the design follows from the observation that most natural signals are sparse in a given representation domain. Computational results portrait the potential of our suggested method in several practical cases of signal reconstruction.

***Index Terms*—** Non-convex optimization, signal reconstruction, sparsity, strip-shredded documents.

1. INTRODUCTION

In the past few years, there has been a growing interest in the field of document analysis (see, e.g. [1–5]). Recovering lost information by reconstruction of shredded documents is one of the important and interesting fields of research with significant applications in forensic and investigative sciences. Shredding of documents is often practiced to destroy potentially incriminating evidences, and it can be essential to the investigating authorities to recover the lost documents. There also exists a great level of interest in such research problems by archaeologists in reassembling historical documents as well as putting together clay or other form of fragments containing ancient texts and paintings.

The documents are typically shredded efficiently by using mechanical shredding devices producing thin strips (often termed as "spaghetti" as depicted in Fig. 1), smaller rectangular pieces, or even some other complex geometrical shapes such as circular fragments (named as "confetti") or hexagons. However, the problem of shredded document recovery requires enormous amount of time and effort when done manually, essentially fabricating an extended jigsaw puzzle, with the added difficulty of all pieces being identical in shape and size. Therefore there is a conspicuous motivation to automate the process as much as possible, with the potential to increase both accuracy and speed of reconstruction.



(a) (b)

Fig. 1. Rectangular strip-shredded Sumerian inscription: (a) original and (b) "spaghetti" fragments.

In some recent efforts, the problem of reconstruction of shredded documents has been regarded as a particular case of *jigsaw puzzle*. Interestingly, a number of computer vision methods for the solution of jigsaw puzzles have been proposed since 1963 [6–9]. Moreover, for the computer-based reconstruction of two-and three-dimensional fragmented objects, jigsaw puzzle-based approaches have been espoused in the research of archeology and historical art conservation [10–12]. In the literature, on one hand, much of the solutions proposed for restoration of jigsaw puzzles, are based on a model for the piece contour shape and this problem is titled as "apictorial jigsaw puzzle" [6]. On the other hand, in some other contributions, the research on the assembly of fragmented pieces has considered the fact that a human being does not only consider just the information on the piece contour or shape, rather also tries to find the optimum matching pieces with respect to their contents, color or texture appearance (see [13–20] and the references therein). Nonetheless, the practical problem of shredded signal reconstruction remains particularly difficult (if not impossible) to solve in a reasonable time when the dimensions grow large, and hence, efficient optimization approaches are crucial to make such tasks viable.

*Please address all the correspondence to Arindam Bose, Phone: (+1) 312-355-0854; Email: abose4@uic.edu

Table 1. Notations

Notation	Description
$\boldsymbol{x}_k$	the k^{th} entry of the vector $\boldsymbol{x}$
$\boldsymbol{x}^T$	the transpose of the vector $\boldsymbol{x}$
$\boldsymbol{x}^H$	the conjugate transpose of the vector $\boldsymbol{x}$
$\Vert\boldsymbol{x}\Vert_p$	the l_p-norm of $\boldsymbol{x}$, defined as $\left(\sum_k \vert\boldsymbol{x}(k)\vert^p\right)^{\frac{1}{p}}$
$\boldsymbol{I}_N$	the identity matrix for order N
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{1}$	the unit column matrix with all elements as 1
$\otimes$	the Kronecker product
$\odot$	the Hadamard product

In this paper, we devise an efficient non-convex optimization approach for the reconstruction of shredded signal (be it an image or audio signal, one or multi-dimensional). The proposed approach relies on the fact that most *natural* signals we deal with are sparse in given representation domains. The suggested method can be categorized as an *alternate minimization* technique, where the optimization is carried out with some variables held fixed and in a cyclic manner [21, 22].

The rest of the paper is organized as follows. The non-convex problem formulation is presented in Section 2, while Section 3 describes the reconstruction approach. Section 4 describes how our method can be extended to handle multi-dimensional signals, and images in particular. Section 5 is dedicated to the numerical results and relevant discussions. Finally, Section 6 concludes the paper.

Notation: We use bold lowercase letters for vectors and bold uppercase letters for matrices. The reader may refer Table 1 for other notations used throughout this paper.

2. PROBLEM FORMULATION

We begin our study with signals lying in a one-dimensional space. Our goal is to reconstruct a finite-length discrete-time signal denoted by $\boldsymbol{x} \in \mathbb{C}^{MN}$, where M and N are the number of shredded parts and the length of each part, respectively. We represent the *shredded signal* $\boldsymbol{y}$ as

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1^T & \boldsymbol{y}_2^T & \cdots & \boldsymbol{y}_M^T \end{pmatrix}^T.$$

Similar to the original signal $\boldsymbol{x}$, the shredded signal $\boldsymbol{y}$ contains M partitions with each partition having a length of N. Note that, for convenience in formulation, we have assumed that all signal partitions (or the shredded parts) are of the same size, namely $\boldsymbol{x}_m, \boldsymbol{y}_m \in \mathbb{C}^N$.

Moreover, we assume that the original signal $\boldsymbol{x}$ is *sparse* in a given representation domain [23, 24]. Note that this is a practical assumption for many *natural* signals including images, text documents, and audio. In the following, we consider to deal with the Discrete Fourier Transform (DFT) domain sparsity although the main idea is general. The order of sparsity may be unknown.

It is well-known that any signal in $\mathbb{C}^N$ can be represented in terms of its orthonormal Fourier basis denoted as $\{\boldsymbol{\psi}_m\}_{m=1}^{MN}$, forming the $MN \times MN$ basis matrix $\boldsymbol{\Psi} := [\boldsymbol{\psi}_1|\boldsymbol{\psi}_2|\cdots|\boldsymbol{\psi}_{MN}]$. Let $\boldsymbol{v} \in \mathbb{C}^{M\times N}$ denote the representation of $\boldsymbol{x}$ in the DFT domain, i.e. the signal $\boldsymbol{x}$ can be expressed as,

$$\boldsymbol{x} = \boldsymbol{\Psi}^H \boldsymbol{v} \tag{1}$$

where $\boldsymbol{v}$ is an $MN \times 1$ column vector of DFT coefficients. The DFT matrix $\boldsymbol{\Psi}$ is given by

$$[\boldsymbol{\Psi}]_{l,p} = \frac{1}{\sqrt{MN}} \exp\left\{\frac{j2\pi lp}{MN}\right\}, \qquad l, p = 1, 2, \cdots, MN.$$

Let $\mathcal{X}_s$ be the set of all vectors with at most s non-zero values. The sparsity assumption thus implies that $\boldsymbol{v}$ belongs to $\mathcal{X}_s$ for some $s \ll MN$.

Next note that the desired signal partitions $\{\boldsymbol{x}_m\}$ can be obtained via a permutation of $\{\boldsymbol{y}_m\}$ through (to be determined) permutation matrix $\boldsymbol{P} \in \mathbb{R}^{M\times M}$, viz.

$$\boldsymbol{x} = (\boldsymbol{P} \otimes \boldsymbol{I}_N)\boldsymbol{y}. \tag{2}$$

Consequently, from (1) and (2) it is evident that in order to find $\boldsymbol{x}$, one may solve the following optimization problem,

$$\begin{aligned} \min_{\boldsymbol{P},\boldsymbol{v}} \quad & \left\Vert(\boldsymbol{P} \otimes \boldsymbol{I}_N)\boldsymbol{y} - \boldsymbol{\Psi}^H\boldsymbol{v}\right\Vert_2 \\ \text{s.t.} \quad & \boldsymbol{P} \text{ is a permutation matrix of size } M, \\ & \boldsymbol{v} \in \mathcal{X}_s, \\ & \Vert\boldsymbol{v}\Vert_2 = \Vert\boldsymbol{y}\Vert_2, \end{aligned} \tag{3}$$

while s remains unknown.

3. THE RECONSTRUCTION APPROACH

In the following, we propose an efficient method to tackle (3). Our approach relies on the following key observations:

- **Observation 1:**

In a case where *the sparsity order s is given*, the optimization problem in (3) can be tackled rather efficiently using cyclic minimization— as discussed below.

For fixed $\boldsymbol{P}$: One only needs to tackle the nearest-vector problem to find the *optimal* $\boldsymbol{v}$:

$$\begin{aligned} \min_{\boldsymbol{v}} \quad & \Vert\boldsymbol{v} - \boldsymbol{\Psi}(\boldsymbol{P} \otimes \boldsymbol{I}_N)\boldsymbol{y}\Vert_2 \\ \text{s.t.} \quad & \boldsymbol{v} \in \mathcal{X}_s, \\ & \Vert\boldsymbol{v}\Vert_2 = \Vert\boldsymbol{y}\Vert_2. \end{aligned} \tag{4}$$

To solve the above problem, let $\tilde{\boldsymbol{v}} \triangleq \boldsymbol{\Psi}(\boldsymbol{P} \otimes \boldsymbol{I}_N)\boldsymbol{y}$, and note that for the optimal $\boldsymbol{v}$, $\arg(\boldsymbol{v}) = \arg(\tilde{\boldsymbol{v}})$. Therefore, we can exclude the phase variables from the analysis in the complex-valued case and solely compute the absolute values of the entries of $\boldsymbol{v}$. As a result, without any loss of generality, we can assume that both $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are real-valued and non-negative. It may be observed that,

$$\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_2^2 = c - 2\boldsymbol{v}^T\tilde{\boldsymbol{v}} \tag{5}$$

where $c = \|\boldsymbol{v}\|_2^2 + \|\tilde{\boldsymbol{v}}\|_2^2 = 2\|\boldsymbol{y}\|_2^2$ is constant. According to the theorem by Hardy, Littlewood and Pólya [25], the inner product of $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ (i.e. $\boldsymbol{v}^T\tilde{\boldsymbol{v}}$) can be maximal if and only if the elements of $\boldsymbol{v}$ are sorted in the same order of *magnitude* as in $\tilde{\boldsymbol{v}}$ [26, 27]. Consider the s entries of $\tilde{\boldsymbol{v}}$ with maximum absolute values and let $\boldsymbol{\mu}$ be a binary vector showing their *support*. The optimal $\boldsymbol{v}$ of (4) is thus simply given as

$$\boldsymbol{v}_{opt} = \|\boldsymbol{y}\|_2 \left(\frac{\tilde{\boldsymbol{v}} \odot \boldsymbol{\mu}}{\|\tilde{\boldsymbol{v}} \odot \boldsymbol{\mu}\|_2} \right). \tag{6}$$

Furthermore, note that $\tilde{\boldsymbol{v}}$ can be computed very efficiently usig Fast Fourier Transform (FFT) operations, leading to a low-cost computation of $\boldsymbol{v}_{opt}$.

For fixed $\boldsymbol{v}$: One may simplify (3) as,

$$\min_{\boldsymbol{P}} \quad \sum_{m=1}^{M}\sum_{l=1}^{N}\left|\sum_{k=1}^{M} p_{m,k}\cdot y_{k,l} - \widehat{v_{m,l}}\right|^2 \tag{7}$$

where $p_{m,k}$ is the entry in m^{th} row and k^{th} column in permutation matrix $\boldsymbol{P}$ and $y_{k,l}$ is the l^{th} entry in k^{th} partition in observed signal $\boldsymbol{y}$ and also $\widehat{\boldsymbol{v}} \triangleq \{\widehat{v_{m,l}}\}_{m=1,l=1}^{M,N} = \boldsymbol{\Psi}^H\boldsymbol{v}$. As $\boldsymbol{P}$ only consists of $\{0,1\}$ values, we have that

$$\begin{aligned} &\sum_{k=1}^{M} p_{m,k}\cdot y_{k,l} = y_{\pi_{\overline{m}}}, \\ &m = 1, 2, \cdots, M, \\ &l = 1, 2, \cdots, N, \end{aligned}$$

where $\pi_{\overline{m}}$ is the only column in $\overline{m}^{\text{th}}$ row of matrix ($\boldsymbol{P} \otimes \boldsymbol{I}_N$) where the respective entry is 1. Hence, the optimization problem in (7) can simply be written as,

$$\min_{\{\pi_{\overline{m}}\}} \quad \sum_{\overline{m}=1}^{MN} |y_{\pi_{\overline{m}}} - \widehat{v_{\overline{m}}}|^2 \tag{8}$$

As a result, to find the optimal permutation matrix $\boldsymbol{P} = \boldsymbol{P}_{opt}$ of (3), instead of minimizing (8) with respect to all $\pi_{\overline{m}}$, we may consider finding an M-sized subset that covers all the partitions and also has the lowest cost. To accomplish the mentioned task, we build a matrix $\boldsymbol{U}$ of size $M \times M$ such that,

$$\begin{aligned} &\boldsymbol{U}_{k,l} \triangleq \|\boldsymbol{y}_k - \widehat{\boldsymbol{v}}_l\|_2^2, \\ &k, l = 1, 2, \cdots, M \end{aligned} \tag{9}$$

Table 2. Algorithm for Shredded Signal Reconstruction

Step 0: Set $s = 1$.
Step 1: Monotonically decrease the objective of (3) via cyclic minimization until convergence using (6) and (10).
Step 2: Set $s \leftarrow s + 1$.
Step 3: Repeat Step 1 until the decrease in the objective of (3) is *negligible*.

where each partition of $\boldsymbol{y}_k$ and $\widehat{\boldsymbol{v}}_l$ has size $N \times 1$ and k and l denotes the respective row and column in $\boldsymbol{U}$. The minimization problem for finding the optimal permutation matrix $\boldsymbol{P}_{opt}$ can be recast as

$$\boldsymbol{P}_{opt} = \arg\min_{\boldsymbol{P}} \; \left[\mathbb{1}^T(\boldsymbol{P} \odot \boldsymbol{U})\mathbb{1}\right] \tag{10}$$

where $\boldsymbol{U}$ is the associated cost matrix. Note that the above problem is in fact an *Assignment Problem* that can be solved efficiently using the *Hungarian Algorithm* [28] with an $\mathcal{O}(M^2)$ computational cost.

- **Observation 2:**

Another key observation is the inclusion

$$\mathcal{X}_1 \subset \mathcal{X}_2 \subset \mathcal{X}_3 \subset \cdots \tag{11}$$

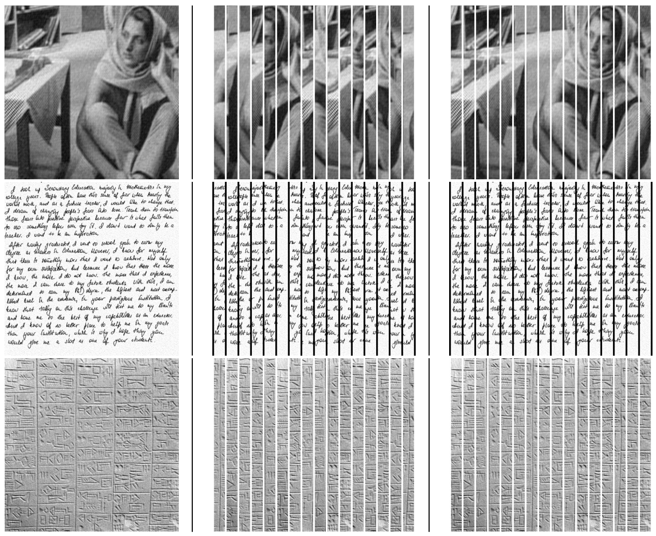
which implies that, while by increasing s we expand the search space of $\boldsymbol{v}$, we can always use the appropriate values of $\boldsymbol{v}$ obtained for a smaller s to search for an updated $\boldsymbol{v}$ as we increase s.

- **The Approach:**

In light of the above, one can use the *sparsity enhancement* procedure summarized in Table 2 to reconstruct the shredded signal. Note that both steps 1 and 2 in the proposed algorithm can only decrease the criterion in (3). This guarantees the convergence of the algorithm, given the fact that the objective of (3) is lower bounded at zero.

4. EXTENSIONS OF THE RECONSTRUCTION ALGORITHM: THE TWO-DIMENSIONAL CASE

We note that the shredded pieces may not all be of same length or even not be rectangular in shape and rather have random contours. As the approach discussed above relies on the sparsity of the signals, it can also be modified to handle shredded signal cases where the partitions are uneven or have dissimilar shapes. Among possible extensions of the central ideas presented in this paper, an extension of the methodology to higher dimensional cases is of the most significant



(a) (b) (c)

Fig. 2. Reconstruction results: (a) original images, (b) scrambled shredded strips, (c) reconstructed images.

interest—particularly, the 2D case where shredded images and documents can be restored.

Consider a real-valued 2D signal $\boldsymbol{X}$ as the signal of size $R \times C$ (to be reconstructed), and similarly, the shredded signal $\boldsymbol{Y}$ of the same size, having M vertical shredded strips each of equal size. For the ease of formulation, let $\overline{\boldsymbol{X}}$ and $\overline{\boldsymbol{Y}}$ denote the matrices which are built from the shredded $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. $\overline{\boldsymbol{X}}$ and $\overline{\boldsymbol{Y}}$ both have size $C \times R$. Moreover, $\boldsymbol{\Psi}_C^H \overline{\boldsymbol{V}} \boldsymbol{\Psi}_R^H$ denotes the 2D IDFT representation of the signal $\overline{\boldsymbol{X}}$ where $\boldsymbol{\Psi}_K$ is the normalized DFT matrix of size $K \times K$ where $K \in \{R, C\}$. It can easily be shown that the optimization problem in (3) can be extended to two-dimension in rather similar straightforward manner, such that ,

$$\min_{\boldsymbol{P}, \overline{\boldsymbol{V}}} \quad \left\|(\boldsymbol{P} \otimes \boldsymbol{I}_N)\overline{\boldsymbol{Y}} - \boldsymbol{\Psi}_C^H \overline{\boldsymbol{V}} \boldsymbol{\Psi}_R^H\right\|_2 \tag{12}$$

and can be optimized in a similar way using (6) and (10).

5. EXPERIMENTAL RESULTS

In this section, we present the computational and experimental results including a description of the experimental setups. The proposed non-convex reconstruction approach has been tested on several two-dimensional image signals which are known to be sparse in DFT domain. As benchmark instances we have used several standard texts and gray scale images of size 512×512 for this purpose. The shredded instances are generated by virtually cutting the document pages vertically into 16 shreds producing 512×32 strips and then the strips are indexed by their original order: $0, 1, \cdots, 15$. Example instances of text and image samples and their scrambled versions can be viewed in Fig. 2(a)-(b).

Fig. 2(c) shows the final reconstruction results after the convergence is reached according to the objective minimization function in (12). It can be seen from the images that all the strips matched exactly with their original indices. Based on the order of sparsity for each instance, on average the optimum matching of the 16 shredded strips was achieved after 12-17 iterations and it took only 0.105-0.368 seconds to reach the convergence on a standard PC—a much faster record compared to what one can observe for generic contour and feature based shredded image reconstruction algorithms.

6. CONCLUSIONS AND FUTURE WORK

In this work, we presented a novel non-convex approach to find the best matching of strip-shredded document. This hybrid approach is based on the enhancement of sparsity of the observed signal. The algorithm was tested on several shredded document pages and images and the results obtained suggest that the proposed algorithm demonstrates a great efficiency in reconstructing the shredded signals in terms of the reconstruction rate and computational time. While the numerical results showed that a complete reconstruction was attainable for our specific examples, as a future research avenue, it would be of great interest to use a relatively large number of partitions that may appear in different orientations, as well as, partitions with cross-cut shreds.

7. REFERENCES

[1] C. Faure and N. Vincent, "Document image analysis for active reading," in *Proceedings of the 2007 International Workshop on Semantically Aware Document Processing and Indexing*, New York, NY, USA, 2007, SADPI '07, pp. 7–14, ACM.

[2] L. Likforman-Sulem, A. Zahour, and B. Taconet, "Text line segmentation of historical documents: a survey," *International Journal of Document Analysis and Recognition (IJDAR)*, vol. 9, no. 2, pp. 123–138, 2007.

[3] X Lu, S. Kataria, W. J. Brouwer, J. Z. Wang, P. Mitra, and C. L. Giles, "Automated analysis of images in documents for intelligent document search," *International Journal on Document Analysis and Recognition (IJDAR)*, vol. 12, no. 2, pp. 65–81, 2009.

[4] J. M. Ogier and K. Tombre, "Madonne: Document Image Analysis Techniques for Cultural Heritage Documents," in *International Conference on Digital Cultural Heritage*, Vienna, Austria, Aug. 2006.

[5] A. Ukovich, A. Zacchigna, G. Ramponi, and G. Schoier, "Using clustering for document reconstruction," 2006, vol. 6064, pp. 60640J–60640J–12.

[6] H. Freeman and L. Garder, "Apictorial jigsaw puzzles: The computer solution of a problem in pattern recognition," *IEEE Transactions on Electronic Computers*, vol. EC-13, no. 2, pp. 118–127, April 1964.

[7] D. Goldberg, C. Malon, and M. Bern, "Special issue on the 18th annual symposium on computational geometry - socg2002 a global approach to automatic solution of jigsaw puzzles," *Computational Geometry*, vol. 28, no. 2, pp. 165 – 174, 2004.

[8] M. G. Chung, M. M. Fleck, and D. A. Forsyth, "Jigsaw puzzle solver using shape and color," in *Signal Processing Proceedings, Fourth International Conference on*, 1998, vol. 2, pp. 877–880 vol.2.

[9] F. Toyama, Y. Fujiki, K. Shoji, and J. Miyamichi, "Assembly of puzzles using a genetic algorithm," in *Pattern Recognition Proceedings, 16th International Conference on*, 2002, vol. 4, pp. 389–392 vol.4.

[10] C. Papaodysseus, T. Panagopoulos, M. Exarhos, C. Triantafillou, D. Fragoulis, and C. Doumas, "Contour-shape based reconstruction of fragmented, 1600 BC wall paintings," *IEEE Transactions on Signal Processing*, vol. 50, no. 6, pp. 1277–1288, Jun 2002.

[11] G. Papaioannou, E. A. Karabassi, and T. Theoharis, "Virtual archaeologist: assembling the past," *IEEE Computer Graphics and Applications*, vol. 21, no. 2, pp. 53–59, Mar 2001.

[12] H. C. da Gama Leitao and J. Stolfi, "A multiscale method for the reassembly of two-dimensional fragmented objects," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 24, no. 9, pp. 1239–1251, Sep 2002.

[13] H. Y. Lin and W. C. F. Chiang, "Reconstruction of shredded document based on image feature matching," *Expert Systems with Applications*, vol. 39, no. 3, pp. 3324 – 3332, 2012.

[14] A. Ukovich and G. Ramponi, "Feature extraction and clustering for the computer-aided reconstruction of strip-cut shredded documents," *Journal of Electronic Imaging*, vol. 17, no. 1, pp. 013008–013008–13, 2008.

[15] A. Skeoch, "An investigation into automated shredded document reconstruction using heuristic search algorithms," *Ph. D. Thesis in the University of Bath, UK*, p. 107, 2006.

[16] P. Butler, P. Chakraborty, and N. Ramakrishan, "The deshredder: A visual analytic approach to reconstructing shredded documents," in *Visual Analytics Science and Technology (VAST), IEEE Conference on*, Oct 2012, pp. 113–122.

[17] C. Schauer, M. Prandtstetter, and G. R. Raidl, *A Memetic Algorithm for Reconstructing Cross-Cut Shredded Text Documents*, pp. 103–117, Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[18] M. Prandtstetter and G. R. Raidl, *Combining Forces to Reconstruct Strip Shredded Text Documents*, pp. 175–189, Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[19] M. Prandtstetter and G. R. Raidl, "Meta-heuristics for reconstructing cross cut shredded text documents," in *Proceedings of the 11th Annual Conference on Genetic and Evolutionary Computation*, New York, NY, USA, 2009, GECCO '09, pp. 349–356, ACM.

[20] M. A. O. Marques and C. O. A. Freitas, "Reconstructing strip-shredded documents using color as feature matching," in *Proceedings of the ACM Symposium on Applied Computing*, New York, NY, USA, 2009, SAC '09, pp. 893–894, ACM.

[21] P. Stoica and Y. Selen, "Cyclic minimizers, majorization techniques, and the expectation-maximization algorithm: a refresher," *IEEE Signal Processing Magazine*, vol. 21, no. 1, pp. 112 – 114, Jan. 2004.

[22] R. M. Freund and B. W. Anthony, "Issues in non-convex optimization," `http://goo.gl/idxFSX`, Apr 2004, [Lecture Notes; accessed 31-August-2016].

[23] R. G. Baraniuk, "Compressive sensing [lecture notes]," *IEEE Signal Processing Magazine*, vol. 24, no. 4, pp. 118–121, July 2007.

[24] E. J. Candes and M. B. Wakin, "An introduction to compressive sampling," *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 21–30, March 2008.

[25] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Mathematical Library. Cambridge University Press, 1952.

[26] E. Ruch, R. Schranner, and T. H. Seligman, "Generalization of a theorem by Hardy, Littlewood, and Polya," *Journal of Mathematical Analysis and Applications*, vol. 76, no. 1, pp. 222 – 229, 1980.

[27] I. Roventa, "Hardy–Littlewood–Polya's inequality and a new concept of weak majorization," *Mediterranean Journal of Mathematics*, vol. 13, no. 2, pp. 573–583, 2016.

[28] Harold W. Kuhn, "The Hungarian method for the assignment problem," *Naval Research Logistics Quarterly*, vol. 2, pp. 83–97, 1955.