Problem Set 2

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Optimal Path

The infinitely lived representative agent maximises the following utility function subject to constraints,

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right\}$$

$u\left(c_t\right)=\ln c_t$, subject to:

$$\begin{gathered}c_t+i_t=y_t\\ y_t=k_t^{1-\theta}\left(zh_t\right)^{\theta}\\ i_t=k_{t+1}-(1-\delta)k_t\end{gathered}$$

We can plug the constraints into the objective function to simplify the problem.

$$\begin{aligned}c_t&=y_t-i_t\\ &=k_t^{1-\theta}\left(zh_t\right)^{\theta}+(1-\delta)k_t-k_{t+1}\end{aligned}$$

Since we are solving the social planners problem there is no aggregate uncertainty. Therefore the social planner maximises a deterministic objective function, this holds throughout the subsequent sections of this question.

$$\max_{k_{t+1}}\quad E_0\left\{\sum_{t=0}^{\infty}\beta^t ln\left(k_t^{1-\theta}\left(zh_t\right)^{\theta}+(1-\delta)k_t-k_{t+1}\right)\right\}$$

Which gives the first order conditions:

$$\beta^t\frac{1}{k_t^{1-\theta}\left(zh_t\right)^{\theta}+(1-\delta)k_t-k_{t+1}}=\beta^{t+1}\frac{k_{t+1}^{1-\theta}\left(zh_t\right)^{\theta}+(1-\delta)k_{t+1}}{k_{t+1}^{1-\theta}\left(zh_{t+1}\right)^{\theta}+(1-\delta)k_{t+1}-k_{t+2}}$$

In the steady state $k_t=k_{t+1}=k^*$

$$\frac{1}{k^{*1-\theta}\left(zh_t\right)^{\theta}+(1-\delta)k^*-k^*}=\beta\frac{k^{*1-\theta}\left(zh_t\right)^{\theta}+(1-\delta)k^*}{k^{*1-\theta}\left(zh_t\right)^{\theta}+(1-\delta)k^*-k^*}$$

So the numerators cancel out and we get:

$$1 = \beta\big(k^{*1-\theta}\,(zh_t)^{\theta} + (1-\delta)k^*\big)$$

From here we get the steady state for capital,

$$k^* = zh\left[\frac{\beta(1-\theta)}{1-\beta(1-\delta)}\right]^{\frac{1}{\theta}}$$

The problem states to set the labour share $\theta = 0.67$ and $h_t = 0.31 \forall t$. Our steady state is a function of labour productivity, the time discount factor β and the depreciation rate δ. Given a target capital to output ratio we can solve for z in the production function.

$$y_t = k_t^{1-\theta}(zh_t)^{\theta}$$
$$z_t = \left(\frac{y_t}{k_t^{1-\theta}h_t^{\theta}}\right)^{\frac{1}{\theta}}$$

If $\frac{y^*}{k^*} = 4$, we can normalise $y = 1$ such that we impose an initial steady state at k=4. Such that by plugging in the exogenous and steady state values $z = 1.63$. furthermore, given the restriction that the investment output ratio must equal o.25 we know that,

$$i^* = k^* - (1-\delta)k^*$$
$$i^* = \delta k^*$$
$$\frac{i^*}{y^*} = \frac{1}{4}, \text{ given y = 1}$$
$$\frac{1}{4} = 4\delta \rightarrow \delta = \frac{1}{16}$$

Finally, solving for β in the steady state expression for capital,

$$\beta = \frac{\left[\frac{k^*}{zh}^{\theta}\frac{1}{1-\theta}\right]}{1+(1-\delta)\left[\frac{k^*}{zh}^{\theta}\frac{1}{1-\theta}\right]}$$
$$\beta = 0.98$$

Therefore our using the constraints in the problem the initial steady state is,

$$\begin{bmatrix} k^* & y^* & i^* & c^* & \delta & \beta & \theta & z \end{bmatrix} = \begin{bmatrix} 4 & 1 & \frac{1}{4} & \frac{3}{4} & \frac{1}{16} & 0.98 & 0.67 & 1.63 \end{bmatrix}$$

If we were to exogenously shock the economy and double the parameter z, the economy would jump to a new steady state. This steady state is found from plugging in the new labour productivity parameter to the closed form expression for the capital steady.

$$\begin{bmatrix} \hat{k}^* & \hat{y}^* & \hat{i}^* & \hat{c}^* & \delta & \beta & \theta & z \end{bmatrix} = \begin{bmatrix} 8 & 2 & \frac{1}{2} & \frac{3}{2} & \frac{1}{16} & 0.98 & 0.67 & 1.63 \end{bmatrix}$$

Transition Paths

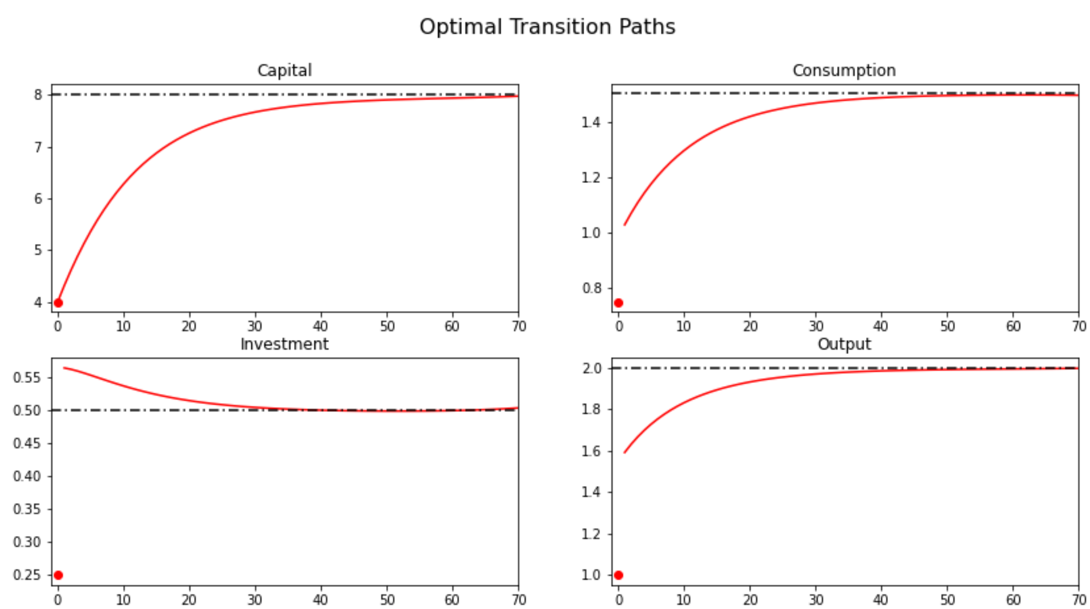
In period $t = 0$, the economy is in a steady state. Following the exogenous shock and doubling of labour productivity the economy is shifted out of the steady state. Such a shift may come from the development of labour enhancing technology. Capital increases with a decreasing rate towards the new steady state level of 8. Since all the constraints hold with equality and we have defined all variables in terms of capital, consumption, investment and output are also shifted to their new state.

This problem can be set up as the following,

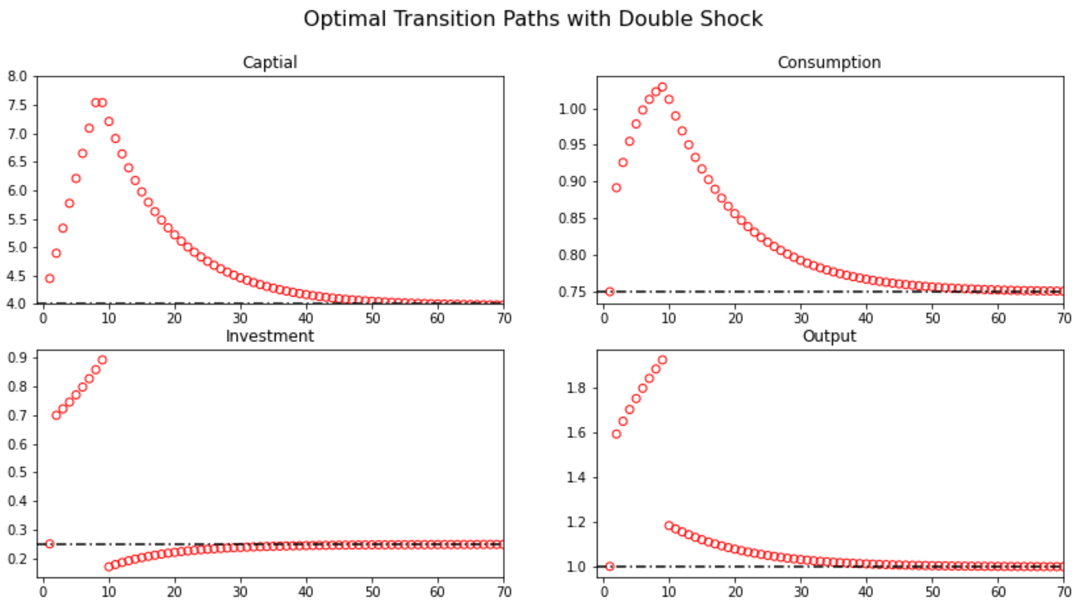
$$\begin{aligned} f(k_0, k_1, k_2) &= 0 \\ f(k_1, k_2, k_3) &= 0 \\ f(k_2, k_3, k_4) &= 0 \\ &\vdots \\ f(k_{n-2}, k_{n-1}, k_n) &= 0 \end{aligned}$$

where $f(k_{n-2}, k_{n-1}, k_n)$ is the Euler equation. Given that we know both the first and final steady state, they may be plugged in for k_0 and k_n, this produces an identified system of equations to solve for (n-2) unknowns. The roots to the above system of equations will map the transition from the initial steady state to the final one. Care has to be taken with the final steady state since k_{t+2} is known. The precise methods are seen in the code file.

At first, investment increases significantly from 0.25 to around 0.56. This is above it's steady state value and proceeds to converge downwards to the second steady state value of 0.5. The initial increase in production capabilities therefore are in the first period directed to investment. Consumption increases steadily from which the agent will take value from a higher consumption level and a smooth increase. Convergence to the new steady state appears to take around 50 to 60 time periods.



To test the model, labour productivity is shocked again. A temporary increase in labour productivity may be attributed to changing international mobility laws and the skill set of incoming immigrant workers. In this case, in period $t = 0$ the economy is in the initial steady state and the labour productivity is exogenously doubled. Consumers adjust to the new change and as can be seen from figure two, begin to converge on the second steady state through the same mechanism as discussed above. However, in $t = 10$, the labour productivity is halved back to it's original value.



In the figure above, given the second shock in $t = 10$, the economy converges back to the initial steady state values. Again, investment swings down relatively more than consumption which converges steadily back to the initial steady state value of 0.75.

Optimal Covid-19 Lockdown

As shown in the previous problem set, Covid-19 has brought our ability to telework to the forefront of public debate. Consider the following toy one-sector economy which captures a trade off choice faced by social planner. Two key parameters define the economy, a human contact (HC) and telework (TW) score. Risk only occurs at the workplace and therefore we define two rates of infection. Firstly the unconditional infection rate,

$$i = \beta(HC) m\left(H_f\right) \quad \text{with} \quad m\left(H_f\right) = \frac{i_0 H_f}{N} \tag{1}$$

Where $m(H_f)$ represents a meeting probability. $\beta(HC) \in [0, 1]$ is the conditional infection rate which depends on how much human contact working in this sector entails. The production function of this economy is a CES production function defined as,

$$Y = \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_f H_{nf}^{\frac{\rho-1}{0}} \right)^{\frac{\rho}{\rho-1}} \tag{2}$$

The social planner's problem is defined as,

$$\max_{\{H_j \in [0,N]\}_{j=\{f,nf\}}} Y\left(H_f, H_{nf}\right) - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega D$$

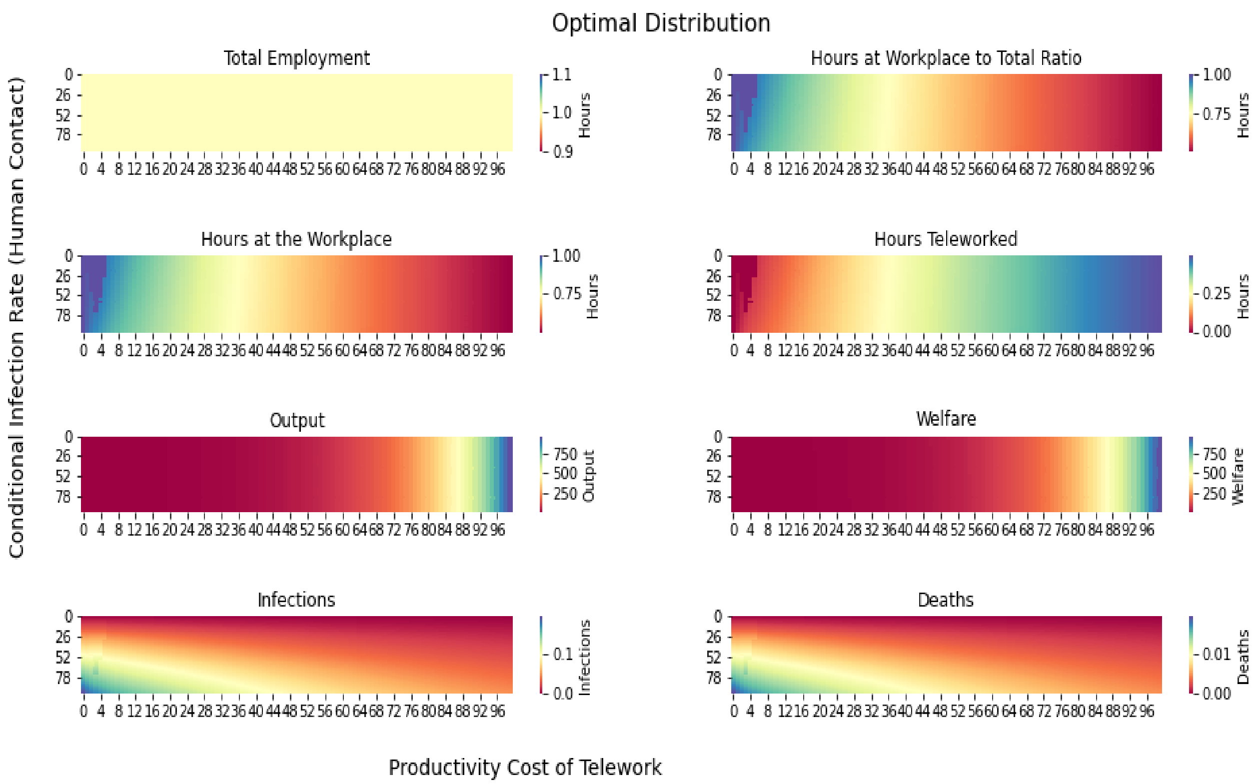
subject to

$$H = H_f + H_{nf} \leq N$$

where for the parameters of this problem we define $C(TW) \in [0, 1]$. This implicitly assumes that people working at home are at best as productive as those at the workplace. This may or may not be appropriate and would be an interesting extension of the project.

To solve the problem, I created a (100×100) grid of all values and combinations of both $\beta(HC)$ and $C(TW)$. I then solved the maximisation problem, iterating over each combination of values for each parameter to find the optimal H_f and H_{nf}. This returns two (100×100) matrices of optimal H_f and H_{nf} values. These are plotted as heat maps to show how each value changes as the relative values of the key parameters change. $\beta(HC)$ is recorded on the y axis and $C(TW)$ on the x axis. The productivity cost of telework enters the production function multiplicatively and as such since it is bounded between 0 and 1, the higher the score the better.

The first case under the parameters defined in the question is presented below.



As can be seen from the top right graph, the constraint on employment is binding for all levels of $\beta(HC)$ and $C(TW)$. From the second row, we see both the hours worked at the workplace and those worked at home. They fit together almost like a puzzle as one increases, the other decreases. When the risk of being infected at work is 0 the social planner sends all the workforce to the workplace. This is to be expected due to the cost of working from home. As the risk of infection increases, the social planner sends more people to work from home since they want to reduce the welfare cost of deaths. Furthermore, as the cost of telework decreases, moving to the right of the figure. The proportion of people working at home increases also. Understandably, as the conditional infection rate increases so does the number of infections and deaths.

Both output and welfare remain relatively constant around low infection rates and a high productivity cost of telework, only when the cost begins to decrease and the social planner can productively send people home does it increase.

This case is best taken as a base line study to run comparative statistics on as the parameters such as ρ and ω change.

In the first case, ρ is set to 1.1 such that working at home and at the work place are relatively easy to substitute for the social planner. This explains why the hours worked at the workplace and at home are mirror images of each other and changes in whether people work at home or the work place is driven by the productivity cost margin as this differentiates the two options for the social planner. Consider the case presented below where ρ increases to 10,

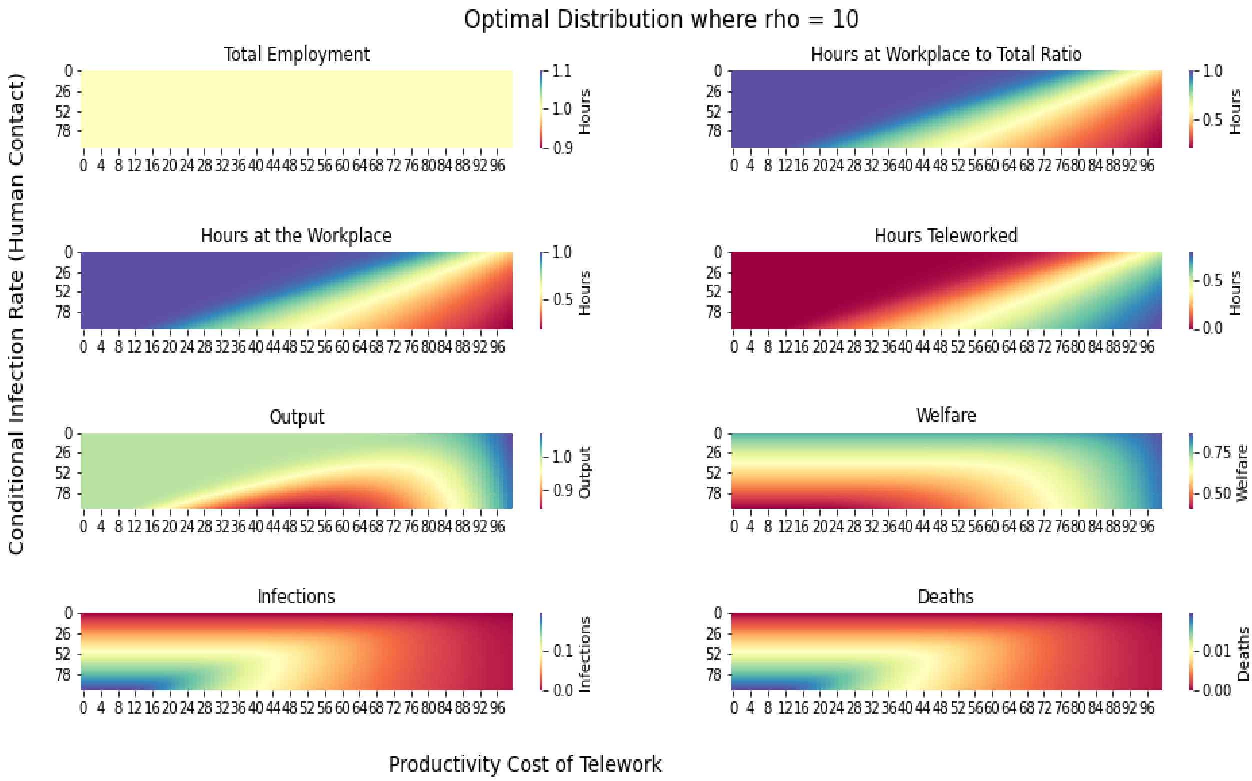


Figure 1: Caption

ρ increasing to 10 reflects a decrease in the complementarity between working at home and at the workplace. Interestingly, the constraint on the number of hours worked is still binding, however the other figures show a large shift in the optimal solution. As the conditional infection rate increases, the social planner continues to keep people working at the workplace even as the cost of telework is high due to decrease in substitutability. This has significant consequences for the economy and population as a whole. The number of infections and consequently deaths increases due to more people having to go to work and be exposed to the virus.

The absolute value of output has decreased severely which reflects a more constrained economy. Output now begins to decrease at a lower level than under a higher rate of substitutability. When the rate of infection is high, and the cost of telework is around .5 output decreases since the social planner is forced to send people to work from home despite the relative cost both in terms of substitution and productivity.

This is an important finding for policy considerations. In industries and their surrounding communities in which working from home and in the workplace are not easy substitutes, a greater number of deaths can be expected as a result. Due to the increased number of deaths welfare decreases for the social planner since $\omega > 0$ represents the cost to the social planner from the death of the agents. For comparison, consider the case in which the social planner didn't care about the deaths of agents and $\omega = 0$ and $\rho = 10$.

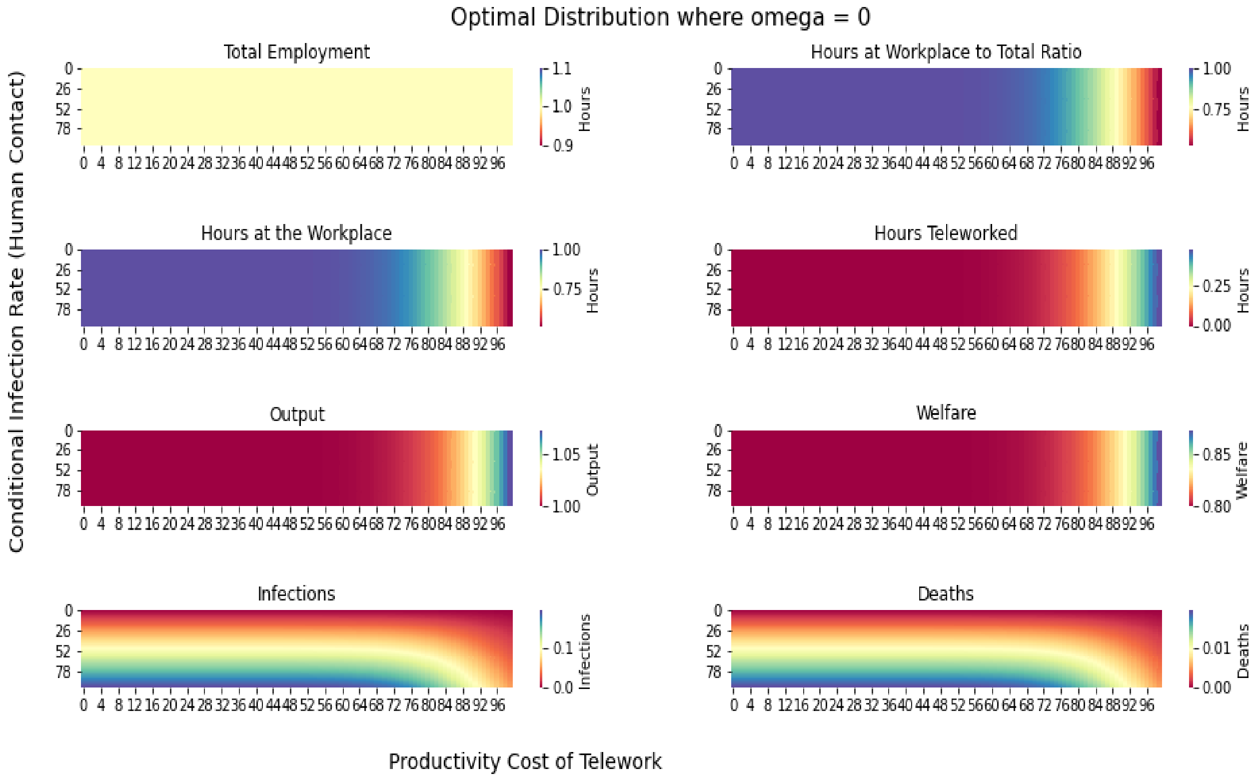


Figure 2: Caption

The social planner faces a high low degree of substitution and doesn't suffer welfare losses from increased deaths. As to be expected, the social planner chooses to keep people working at the workplace until the productivity of working from home and the work place are approximately equal. This is regardless of the conditional infection rate which leads to large increases in the number of infections and deaths.

Welfare however remains low overall due to the low rate of substitutability between working at home and the workplace as seen previously.

Finally, as a thought experiment I was interested to see when the social planner would allow for slack in the constraint on the total number of hours worked. ρ is returned back to the original 1.1 and from increasing ω to 100 such that the social planner cares a lot about the deaths of agents, we see that at high conditional infection rates and a high cost of telework the social planner leaves a section of the work force as unemployed. The number of infected and deaths decreases significantly and the majority of those who do work, work from home. Output decreases and approaches zero, reflecting a total failure of the sector as the rate of infection and cost of telework increase.

This is interesting and relevant to consider as part of the debate around Covid-19 policy. Are policy makers taking excessive risks with human lives in order to save the economy? What is the value of a human life when traded off against output levels?

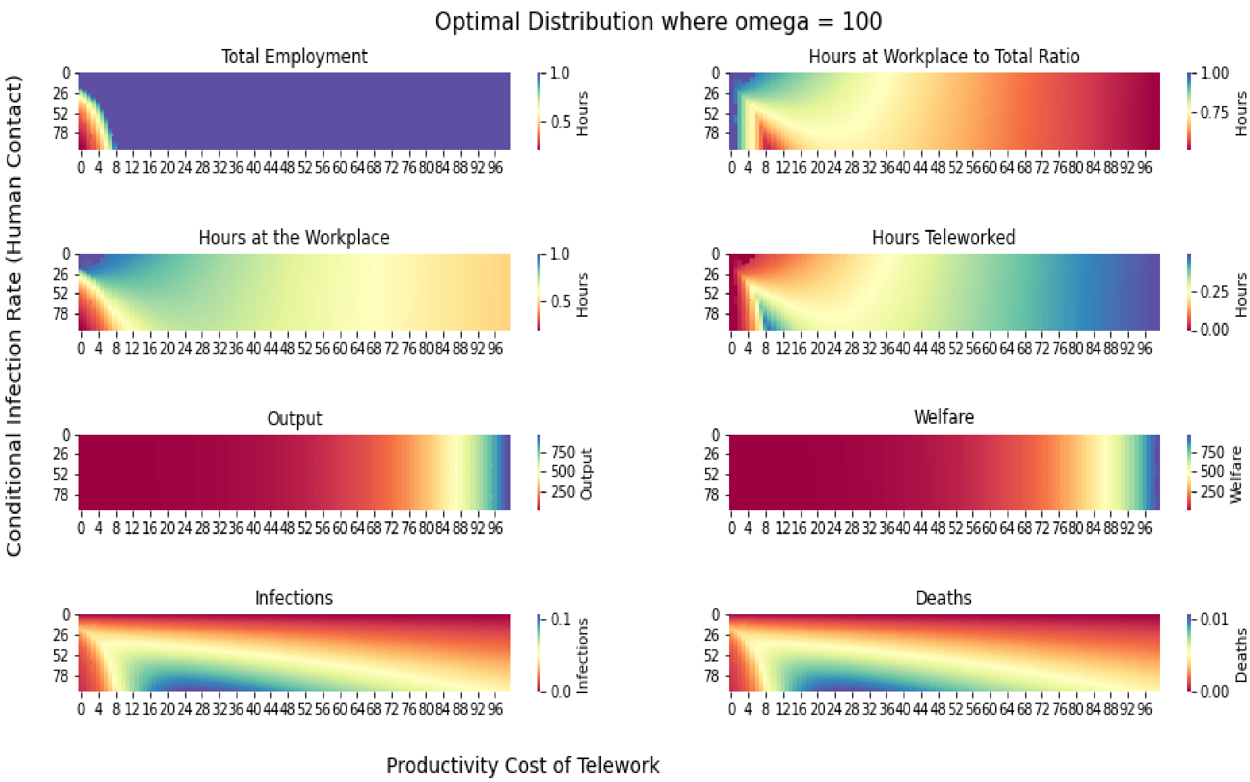


Figure 3: Caption

Overall there are a few main conclusions. For the social planner, when working at home and at the work place can be substituted with easily they freely move people around in response to rising rates of infection in line with productivity changes. This ability is reduced however when that substitutabiltiy decreases. Furthermore, the value we place on a life is key to the findings as this will drive the social planners welfare loss calculations. These results have also backed up previous findings that overall, a greater ability to work from home increases the agility of the economy to deal with negative shocks.