Dimensional Equations

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Assume:

* $\rho_l = \rho_s = \rho$ = constant
* ρ_g constant

$$\frac{D\phi_g}{Dt} = \nabla \cdot \underline{u}_l$$

$H_s = \rho C_p (T - T_i) - \rho L$

$$\frac{DH}{Dt} + \nabla \cdot (H_l \underline{u}_l) = \nabla \cdot (\bar{k} \nabla T)$$

$H = \phi_s H_s + \phi_l H_l$, $H_l = \rho C_p (T - T_i)$

$$\frac{DS}{Dt} + \nabla \cdot (S_l \underline{u}_l) = \nabla \cdot (\phi_l D_s \nabla S_l)$$

$S = \phi_s S_s + \phi_l S_l$ as mass ratio

bulk salinity

$$\frac{DG}{Dt} + \nabla \cdot (\rho_l \xi_l \underline{u}_l) + \nabla \cdot (\rho_g \underline{u}_g) = \nabla \cdot (\rho_l \phi_l D_\xi \nabla \xi_l)$$

$G = \rho_l \phi_l \xi_l + \rho_g \phi_g$

bulk gas mass per unit volume

$$W_g = \phi_g \left(\frac{V_{stokes}}{K_1(\lambda)} + \frac{2G(\lambda)}{\phi_l} W_l \right)$$

vertical only

$\underline{u}_l = W_l \hat{\underline{z}}$, $u_g = W_g \hat{\underline{z}}$

$\lambda = R_B / R_T$, $R_T \sim R_0 \phi_l^{q}$

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Conservation of Water

Liquid phase contains : H_2O, Salt (aq), Gas(aq)

mass ratio $S_\ell = \dfrac{\text{kg of salt}}{\text{kg of liquid}}$

density of liquid phase ρ_ℓ
(in principle a function of composition)

mass ratios : $\underset{\text{wat.}}{W_\ell} + \underset{\text{salt}}{S_\ell} + \underset{\text{gas}}{\xi_\ell} = 1$

By analogy solid phase contains : H_2O (s) , Salt (s)

$W_s + S_s = 1$

$W_s = \dfrac{\text{kg ice}}{\text{kg of solid}}$, $S_s = \dfrac{\text{kg of salt}}{\text{kg of solid}}$

Total mass of H_2O in system per volume $= \rho_\ell \phi_\ell W_\ell + \rho_s \phi_s W_s$

Can be advected in liquid phase

$$\Rightarrow \frac{D}{Dt}\left(\rho_\ell \phi_\ell W_\ell + \rho_s \phi_s W_s\right) + \nabla \cdot \left(\rho_\ell W_\ell \underline{u}_\ell\right) = 0$$

liq Darcy vel

Assumptions :

* $W_\ell \approx 1$, $W_s \approx 1$
* $\rho_\ell = \rho_s = \rho =$ constant

$\Downarrow$

$$\frac{D\phi_g}{Dt} = \nabla \cdot \underline{u}_\ell$$

* Assume gas volume change doesn't drive much flow (i.e $\phi_g << 1$)

$$\Rightarrow \nabla \cdot \underline{u}_\ell = 0$$

Conservation of Heat

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Enthalpy per unit volume H

specific heat capacity C_p s.t. $\left.\frac{\partial H}{\partial T}\right|_{\text{fixed phase}} = \rho C_p$.

$H_s = \rho_s C_{p,s}(T - T_i) - \rho_s L$ → latent heat of fusion

$H_\ell = \rho_\ell C_{p,\ell}(T - T_i)$

at $T = T_i$ $\frac{H_s}{\rho_s} - \frac{H_\ell}{\rho_\ell} = -L$

T_i is reference temperature

* Neglect enthalpy in gas phase as $\rho_g \ll \rho_\ell, \rho_s$

$\Rightarrow$ total enthalpy per unit vol $= H = \phi_s H_s + \phi_\ell H_\ell = (\rho_s \phi_s C_{p,s} + \rho_\ell \phi_\ell C_{p,\ell})(T - T_i) - \rho_s \phi_s L$.

$(+H_0)$ set enthalpy of liquid at $T = T_i$ to $H_0 = 0$.

Enthalpy can be advected in liquid or conducted:

$$\frac{DH}{Dt} + \nabla \cdot (H_\ell \underline{u}_\ell) = \nabla \cdot (\bar{k} \nabla T)$$

thermal conductivity: $\bar{k} = \phi_\ell k_\ell + \phi_s k_s + \phi_g k_g$ (typically $k_g \ll k_\ell, k_s$).

can use $\frac{D\phi_g}{Dt} = \nabla \cdot \underline{u}_\ell$

& $C_{p,g} = k_g = 0$, $C_{p,s} = C_{p,\ell} = C_p$, $\rho_s = \rho_\ell = \rho$

$\Rightarrow$

$$\rho C_p (1 - \phi_g) \frac{DT}{Dt} + \rho C_p \underline{u}_\ell \cdot \nabla T = \rho L \frac{D\phi_s}{Dt} + \nabla \cdot (k(1-\phi_g)\nabla T)$$

Can allow insulating effect of air even if we approximate $\phi_s + \phi_\ell \approx 1$ everywhere else as $k_g = 0$

$$\frac{DH}{Dt} + \nabla \cdot (H_\ell \underline{u}_\ell) = \nabla \cdot (k(1-\phi_g)\nabla T)$$

Assume: $C_{p,s} = C_{p,\ell} = C_p$ $\qquad \kappa = \frac{k}{\rho C_p}$

$\rho_s = \rho_\ell = \rho$

$k_s = k_\ell = k$

$\phi_s + \phi_\ell = 1$ (i.e. $\phi_g \ll 1$)

$$\rho C_p \frac{DT}{Dt} + \rho C_p \underline{u}_\ell \cdot \nabla T = \rho L \frac{D\phi_s}{Dt} + k \nabla^2 T$$

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Conservation of Salt

$$\frac{D}{Dt}\left(\rho_l \phi_l S_l + \rho_s \phi_s S_s\right) + \nabla\cdot\left(\rho_l S_l \underline{u}_l\right) = \nabla\cdot\left(\rho_l \phi_l D_s \nabla S_l\right)$$

$\rho_l \phi_l S_l + \rho_s \phi_s S_s$: total mass of salt per unit volume

$\nabla\cdot(\rho_l S_l \underline{u}_l)$: advection in liquid

$\nabla\cdot(\rho_l \phi_l D_s \nabla S_l)$: molecular diffusion in liquid

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Conservation of Gas

$$\frac{D}{Dt}\left(\rho_l \phi_l \xi_l + \rho_g \phi_g\right) + \nabla\cdot\left(\rho_l \xi_l \underline{u}_l\right) + \nabla\cdot\left(\rho_g \underline{u}_g\right) = \nabla\cdot\left(\rho_l \phi_l D_g \nabla \xi_l\right)$$

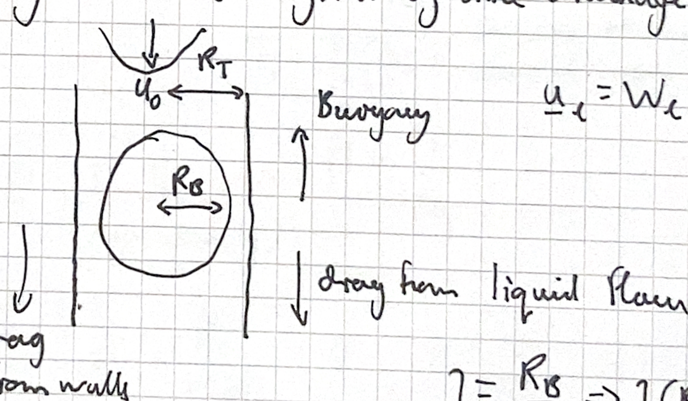
$\underline{u}_g$: gas Darcy vel

$\nabla\cdot(\rho_g \underline{u}_g)$: advection by bubbles

Gas Darcy Velocity

* Assume Background Liquid Darcy flow $\underline{u}_l$ is given by brine drainage or imposed
* Vertical direction $\underline{u}_g = W_g \hat{\underline{z}}$ $\qquad$ $\underline{u}_l = W_l \hat{\underline{z}}$

Force balance bubble in tube



$$\Rightarrow V_g = \frac{V_{stokes}}{K_1(\lambda)} + G(\lambda)\, U_0$$

V_g: interstitial bubble velocity.

U_0: maximum interstitial liquid velocity in Poiseuille profile

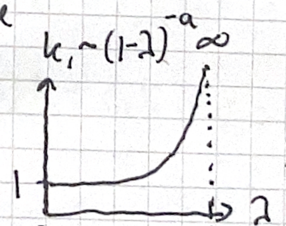
$$V_{stokes} = \frac{\Delta\rho g R_B^2}{3\mu_l} \qquad U_0 = \frac{2W_l}{\phi_l}$$

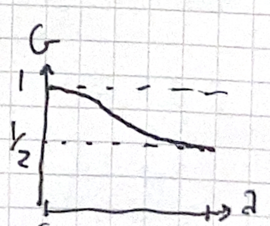
$$\lambda = \frac{R_B}{R_T} \Rightarrow \lambda(R_B, \phi_l)$$

$$R_T \sim R_0 \phi_l^{q} \quad \text{(Experimental)}$$

$\Delta\rho \simeq \rho_l \qquad W_g = V_g \phi_g$

$$W_g = \phi_g\left(\frac{\rho_l g R_B^2}{3\mu K_1(\lambda)} + \frac{2G(\lambda)}{\phi_l} W_l\right)$$





Scales in 1D Problem

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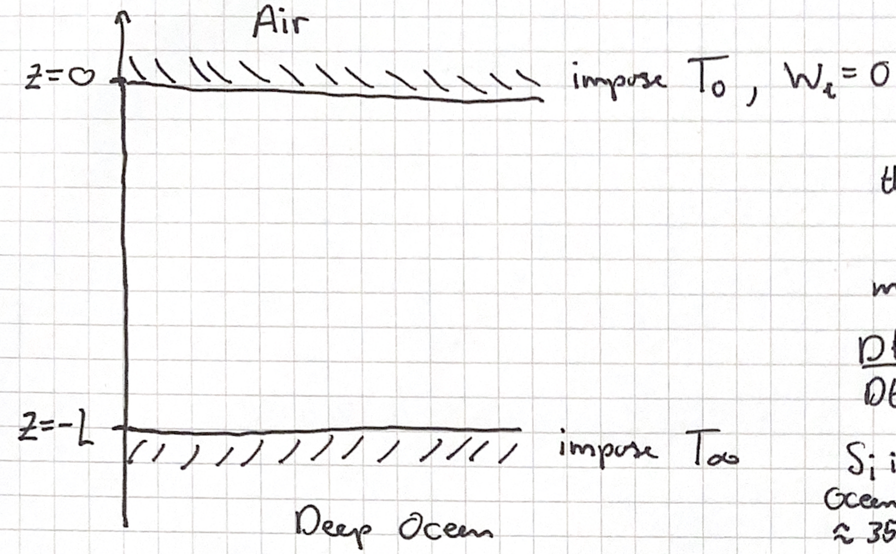

conductivity

thermal diffusivity $\kappa = \dfrac{K}{\rho c_p}$

spec heat capacity

moving frame V

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} - V\frac{\partial f}{\partial z}.$$

S_i is Ocean salinity $\approx 35\,g/kg$

scale of thermal diffusive growth

domain depth L

Lengths $z \sim L$ **Time** $t \sim L^2/\kappa$ (thermal diffusion) **Velocity** $V, W_\ell, W_g \sim \kappa/L$

Salt differences $\Theta = \dfrac{S - S_i}{\Delta S}$ $\Delta S = S_E - S_i$ ($230\,g/kg$)

(relative to ocean water scaled by typical variation).

$\Theta = 0$ ocean water $\mathcal{C} = \dfrac{S_i}{\Delta S}$.

$\Theta = -\mathcal{C}$ fresh

Temperature differences $\theta = \dfrac{T - T_i}{T_i - T_E}$ $\Delta T = T_i - T_E$

(relative to ocean freezing temperature).

$T_i = T_L(S_i)$ liquidus freezing temp of ocean water.

Dissolved gas $\omega = \dfrac{\chi_\ell}{\chi_{sat}}$ (relative to saturation concentration).

Bulk gas $\Gamma = \dfrac{G}{\rho_g}$ (gas density)

Enthalpy $H \sim \rho c_p \Delta T$ (from temp scale)

$$\Rightarrow \mathcal{H}_\ell = \frac{H_\ell}{\rho c_p \Delta T} = \theta \qquad \mathcal{H}_s = \frac{H_s}{\rho c_p \Delta T} = \theta - \underbrace{\frac{L}{\rho c_p}}_{\text{Stefan number}}$$

Bulk enthalpy: $\mathcal{H} = \dfrac{H}{\rho c_p \Delta T} = \phi_s \mathcal{H}_s + \phi_\ell \mathcal{H}_\ell = \phi_s(\theta - St) + \phi_\ell \theta = (1-\phi_g)\theta - \phi_s St$

Bulk salt: $\Theta = \dfrac{S - S_i}{\Delta S} = \dfrac{\phi_s(S_s - S_i)}{\Delta S} + \dfrac{\phi_\ell(S_\ell - S_i)}{\Delta S} - \dfrac{\phi_g S_i}{\Delta S} = \phi_s\Theta_s + \phi_\ell\Theta_\ell - \phi_g\mathcal{C}$

neglect $\phi_g \ll 1$ for reduced

Bulk gas: $\Gamma = \dfrac{G}{\rho_g} = \left(\dfrac{\rho_\ell \chi_{sat}}{\rho_g}\right)\omega\phi_\ell + \phi_g = \chi\omega\phi_\ell + \phi_g$

1D Non Dimensional Equations

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$$\frac{D\phi_g}{Dt} = \frac{\partial W_\ell}{\partial z}$$

$$\lambda = \frac{R_B}{R_T} = \frac{R_B}{R_0 \phi_\ell^q} = \frac{\Lambda}{\phi_\ell^q}$$

$$W_g = \phi_g \left(\frac{\mathcal{B}}{K_1(\lambda)} + \frac{2G(\lambda)}{\phi_\ell} W_\ell \right)$$

Liner liquidus $T_L(S_\ell) = T_i + \Gamma_L(S_i - S_\ell)$

s.t $\Delta T = \Gamma_L \Delta S$

$\Rightarrow \theta_L(\Theta_\ell) = -\Theta_\ell$

Solidus (no segregation coeff)

$\theta_s = -1$

$k_s = k_\ell = k_g$

$$\frac{DH}{Dt} + \frac{\partial}{\partial z}(H_\ell W_\ell) = \frac{\partial^2 \theta}{\partial z^2}$$

$\to$ if $k_s = k_\ell$, $k_g = 0$ then $\frac{\partial}{\partial z}\left((1-\phi_g)\frac{\partial \theta}{\partial z}\right)$

$H = (1-\phi_g)\theta - \phi_s St$ $\quad H_\ell = \theta$

$$\frac{D\Theta}{Dt} + \frac{\partial}{\partial z}\left((\Theta_\ell + C_e) W_\ell\right) = \frac{1}{Le_s} \frac{\partial}{\partial z}\left(\phi_\ell \frac{\partial \Theta_\ell}{\partial z}\right)$$

$\Theta = \phi_s \Theta_s + \phi_\ell \Theta_\ell - \phi_g C_e$

$$\frac{D\Gamma}{Dt} + \frac{\partial}{\partial z}(\chi \omega W_\ell) + \frac{\partial W_g}{\partial z} = \frac{\chi}{Le_g} \frac{\partial}{\partial z}\left(\phi_\ell \frac{\partial \omega}{\partial z}\right)$$

$\Gamma = \chi \omega \phi_\ell + \phi_g$

Non-Dimensional Numbers

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$$St = \frac{L}{C_p \Delta T} \qquad C = \frac{S_i}{\Delta S} \qquad Le_S = \frac{D_S}{\kappa} \qquad Le_\xi = \frac{D_\xi}{\kappa}$$

$$\chi = \frac{\rho_\ell \xi_{sat}}{\rho_g} \qquad \mathcal{B} = \frac{V_{st}}{V} = \frac{\rho_\ell g R_B^2 L}{3 \mu \kappa} \qquad \lambda = \frac{R_B}{R_0}$$

We also have non-dimensional BCs:

$z=0$ (top):

cold temp: $\theta_0(t)$

No salt flux or fixed salinity Θ_0

Fixed gas e.g $\Gamma_0 = \Gamma_{sat}(H_0, \theta_0)$

$z=-1$ (ocean):

$\theta(z=-1) = \theta_\infty$

Ocean salinity $\Theta(z=-1)=0$

$\Gamma(z=-1) = \chi$

Reduced Model (makes enthalpy method much more tractable).

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* When gas doesn't move ϕ_g is controlled by freezing, for saturated liquid χ is the maximum volume of gas that can be exsolved

Approximation $\chi \ll 1 \Rightarrow$ neglect terms of $O(\phi_g)$ except in gas equation

$$\mathcal{H} = \Theta - \phi_s St \qquad \text{and } \phi_s + \phi_l = 1.$$

$$\Theta = \phi_s \Theta_s + \phi_l \Theta_l \qquad \therefore \frac{\partial W_l}{\partial z} \simeq 0 \text{ (no flow driven by gas)}$$

$$\Gamma = \chi \phi_l \omega + \phi_g$$

other conservation equations stay the same.

* This can breakdown if large gas volume accumulates below impermeable layer or gas velocity becomes much larger than liquid velocity.

* For consistency impose extra condition that ϕ_g is no larger than void fraction

Approximate scheme: (No gas fraction)

In reality

Solid : $\phi_s = 1$, $\phi_l = 0$

$\phi_s = 1 - \phi_g$, $\phi_l = 0$ (the same)

other phases (mush/liquid) : $\phi_s + \phi_l = 1$

ϕ_s the same

$\phi_l = 1 - \phi_s - \phi_g$

$\therefore \phi_g \leq \text{void frac} = 1 - \phi_s$

(so that $\phi_l \geq 0$)

This stops arbitrary gas accumulating in one cell and gives Boundary Layer.

* This approximation lets us evaluate Γ given $\mathcal{H}, \Theta, W_l$ as input.