Enthalpy Method With $\phi_s + \phi_l + \phi_g = 1$.

Bulk temperature T, Salinity S

$S = \phi_s S_s + \phi_l S_l$ as $S_g = 0$.

Linearised phase boundaries

$$T_L(S) = T_E - \Gamma(S_l - S_E) \quad (1)$$

$$T_S(S) = \max\left(T_E, T_E - \frac{\Gamma}{d_c}(S_s - d_c S_E)\right) \quad (2)$$

d_c distribution coeff

in mushy phase approximate $S_s = d_c S_l$.

eutectic point (T_E, S_E) s.t. $T_L(S_E) = T_S(S_E) = T_E$

fresh water freezes at $0°C$ $T_L(0) = 0 \Rightarrow T_E + \Gamma S_E = 0$

ϕ_g reduces maximum bulk salinity by occupying volume

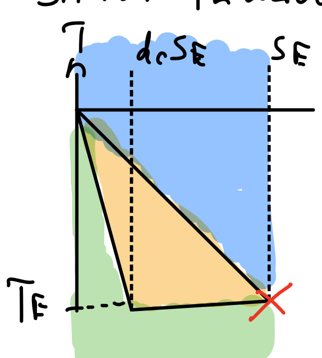
* Phase space for sub-eutectic Mushy Layer changes with ϕ_g

consider liquid phase $S = \phi_l S_l$ $\quad 0 \le S_l \le S_E$ for sub eutectic mushy layer.

$\phi_l = 1 - \phi_g$ as $\phi_s = 0$ $\Rightarrow$ $S = (1-\phi_g) S_l$

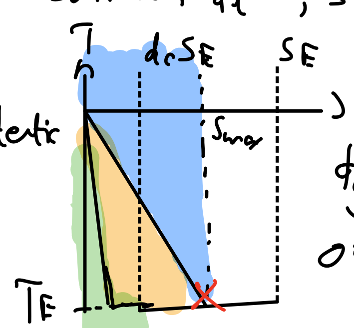
$\therefore$ $0 \le S \le (1-\phi_g) S_E$ so as ϕ_g increases eutectic point shifts inwards (on liquidus $\phi_s = 0$, $S = (1-\phi_g)S_l$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ sub into

... solidus $\phi_l = 0$, $S = (1-\phi_g) S_s$ $\Big\}$ (1) & (2))



$\phi_g = 0$

× = eutectic point moves as $\phi_g \uparrow$



$\phi_g = 1/2$

$0 \le S \le S_{max} = (1-\phi_g) S_E$

Define Bulk Quantities

(ξ_{sat} = constant otherwise going to get non linear equations for ϕ_g everywhere).

Enthalpy : $H = \phi_s H_s + \phi_l H_l + \phi_g H_g$

Salt : $S = \phi_s S_s + \phi_l S_l$

Gas : $G = \rho_l \xi_l \phi_l + \rho_g \phi_g$ $\quad G = \rho_g \Gamma$

take $\rho_l = \rho_s = \rho = const$ $\quad \varepsilon = \rho_g / \rho << 1.$

$\rho_g = const$

Neglect H_g term as $O(\varepsilon)$.

Take $c_{p,l} = c_{p,s}$

$H_s = \rho c_p (T - T_i) - \rho L$

$H_L = \rho c_p (T - T_i)$

$H_g = 0.$

$H = \rho c_p (T - T_i)(\phi_s + \phi_l) - \phi_s \rho L$

Note only freezing / melting changes latent heat term

gas acts as a void carrying no enthalpy

so have effective heat capacity $H = \rho c_{eff} (T - T_i) - \rho \phi_s L$

Breaks down when $\phi_g = 1$ $\quad c_{eff} = c_p (1 - \phi_g).$

as no energy can be retained in system.

ND.

$$\mathcal{H} = \rho c_p \Delta T \mathcal{H} \qquad \theta = \frac{T - T_i}{\Delta T} \qquad \Delta T = T_i - T_E \qquad \Delta S = S_E - S_i$$

$$\Theta = \frac{S - S_i}{\Delta S} \qquad \Theta_l = \frac{S_l - S_i}{\Delta S}, \qquad \Theta_s = \frac{S_s - S_i}{\Delta S}.$$

$$\mathcal{C}_e = \frac{S_i - d_c S_i}{\Delta S} = \frac{\text{difference in liquid solid salinity}}{\text{salinity variation.}}$$

$$\mathcal{C}_e = \frac{(1-d_c)S_i}{\Delta S} \qquad \underset{S_i \to 0}{0} \le \mathcal{C}_e \le \underset{S_i \to S_E}{\infty} \qquad St = \frac{L}{c_p \Delta T}$$

$$\xi_l = \xi_{sat}\,\omega \quad \text{so} \quad \omega_{sat} = 1. \qquad \chi = \frac{\rho \xi_{sat}}{\rho_g} \qquad \phi_g = O(\chi)$$

$$\frac{H}{\rho c_p \Delta T} = \theta(1-\phi_g) - \phi_s St = \mathcal{H} \qquad \frac{S_i}{\Delta S} = \frac{\mathcal{C}_e}{1-d_c}$$

$$\boxed{\begin{aligned} \mathcal{H}_s &= \theta - St \\ \mathcal{H}_l &= \theta. \end{aligned}}$$

$$\Theta = \frac{S - S_i}{\Delta S} = \frac{\phi_s S_s + \phi_l S_l + \phi_g \times 0 - \phi_s S_i - \phi_l S_i - \phi_g S_i}{\Delta S}$$

$$\Theta = \phi_s \Theta_s + \phi_l \Theta_l - \phi_g \frac{S_i}{\Delta S} = \phi_s \Theta_s + \phi_l \Theta_l - \frac{\phi_g \mathcal{C}_e}{(1-d_c)}$$

So: $\dfrac{-\mathcal{C}_e}{(1-d_c)} \leq \Theta_l \leq 1$

fresh   eutectic

if $\phi_s = 0$ $\Theta = (1-\phi_g)\Theta_l - \dfrac{\phi_g \mathcal{C}_e}{(1-d_c)}$

$\dfrac{-\mathcal{C}_e}{(1-d_c)}(1-\phi_g) - \dfrac{\phi_g \mathcal{C}_e}{(1-d_c)} = \dfrac{-\mathcal{C}_e}{(1-d_c)}$ $\Big|$ $\Theta_l = 1$ $1 - \phi_g\left(1 + \dfrac{\mathcal{C}_e}{1-d_c}\right)$

So $\dfrac{-\mathcal{C}_e}{(1-d_c)} \leq \Theta \leq \Theta_{max} = 1 - \phi_g\left(1 + \dfrac{\mathcal{C}_e}{1-d_c}\right)$

bulk fresh

the reduced eutectic point for a given ϕ_g $\Theta_{max}(\phi_g)$

s.t. $\Theta_{max}(0) = 1$.

note $S_{max} = (1-\phi_g)S_E$ from earlier

$\Theta_{max} = \dfrac{S_{max} - S_i}{\Delta S} = \dfrac{(1-\phi_g)S_E - S_i}{\Delta S}$

$= \dfrac{S_E - S_i}{\Delta S} - \phi_g \dfrac{S_E}{\Delta S} = 1 - \phi_g\left(1 + \dfrac{\mathcal{C}_e}{1-d_c}\right).$

$\dfrac{S_E}{\Delta S} - \dfrac{S_i}{\Delta S} = 1$ $\dfrac{S_E}{\Delta S} = 1 + \dfrac{\mathcal{C}_e}{1-d_c}$

$$\Gamma = \frac{G}{\rho_g} = \frac{\rho \omega \xi_{sat} \phi_l}{\rho_g} + \phi_g = \chi \omega \phi_l + \phi_g$$

SO :

$$\phi_s + \phi_l + \phi_g = 1$$

$$\mathcal{H} = \theta(1-\phi_g) - \phi_s St$$

$$\Theta = \phi_s \Theta_s + \phi_l \Theta_l - \frac{\phi_g C_e}{1-d_c}$$

eutectic point.

$$-\frac{C_e}{1-d_c} \le \Theta \le \Theta_{max}(\phi_g)$$

$$\Gamma = \chi \omega \phi_l + \phi_g$$

saturation when $\omega = 1$, $\phi_g = 0$

$\Gamma_{sat} = \chi \phi_l$ ∴

$\Gamma \ge \Gamma_{sat}(\mathcal{H}, \Theta)$: $\omega = 1$

$\Gamma < \Gamma_{sat}(\mathcal{H}, \Theta)$: $\phi_g = 0$.

ω, ϕ_g
ϕ_s, ϕ_l
$\theta, \Theta_l, \Theta_s$
from $\mathcal{H}, \Theta, \Gamma$.

Phases :	liquid (+ gas)	solid + liquid (+ gas) mushy	eutectic	solid (+ gas)
	$\mathcal{H} \ge \mathcal{H}_L$	$\mathcal{H}_E \le \mathcal{H} < \mathcal{H}_L$	$\mathcal{H}_S \le \mathcal{H} < \mathcal{H}_E$	$\mathcal{H} < \mathcal{H}_S$
	$\phi_s = 0$	$\theta = \theta_L(\Theta_l)$	$\theta = \theta_S(\Theta_s)$	$\phi_l = 0$
	Θ_s N/A	$\Theta_s = d_c \Theta_l - C_e$	$\Theta_l = 1$	Θ_l N/A.

Working out ND $\left\{ \frac{S_S}{S_\ell} = d_c << 1 \right\}$

$S_S = d_c S_\ell$

$$\frac{\Delta S \Theta_S + S_i}{\Delta S \Theta_\ell + S_i} = d_c = \frac{\Theta_S + \frac{\mathcal{C}_e}{1-d_c}}{\Theta_\ell + \frac{\mathcal{C}_e}{1-d_c}}$$

$$d_c\left((1-d_c)\Theta_\ell + \mathcal{C}_e\right) = \Theta_S(1-d_c) + \mathcal{C}_e .$$

$$d_c(1-d_c)\Theta_\ell + d_c \mathcal{C}_e - \mathcal{C}_e = \Theta_S(1-d_c) .$$

$$d_c(1-d_c)\Theta_\ell - \mathcal{C}_e(1-d_c) = \Theta_S(1-d_c)$$

$$d_c \Theta_\ell - \mathcal{C}_e = \Theta_S .$$

in Mushy phase $\mathcal{H}_E \leq \mathcal{H} < \mathcal{H}_L$

$$\Theta_S = d_c \Theta_\ell - \mathcal{C}_e .$$

(as $d_c \to 0$ $\Theta_S \to -\mathcal{C}_e$).

ND Liquidus & Solidus: so $\Delta T = \Gamma \Delta S$.

$T_L = T_E - \Gamma(S_\ell - S_E)$ $\quad$ $T_i = T_E - \Gamma(S_i - S_E)$

$$\Theta_L = \frac{T - T_i}{\Delta T} = \frac{T_E - T_i}{\Delta T} - \frac{\Gamma(\Delta S \Theta_\ell + S_i - S_E)}{\Delta T}$$

$$\Theta_L = -1 - \frac{(\Delta S \Theta_\ell + S_i - S_E)}{\Delta S} = -1 - (\Theta_\ell - 1)$$
$$= -1 - \Theta_\ell + 1 = -\Theta_\ell$$

$$\boxed{\theta_L(\Theta_L) = -\Theta_e}$$

$$T_S = \max\left(T_E, T_E - \frac{\Gamma}{d_c}\left(S_S - d_c S_E\right)\right)$$

$$\theta_S = \frac{T_S - T_i}{\Delta T} = \max\left(-1, -1 - \frac{1}{d_c \Delta S}\left(\Delta S \Theta_S + S_i - d_c S_E\right)\right)$$

$$= \max\left(-1, -1 - \left(\frac{\Theta_S}{d_c} + \frac{e_e}{d_c(1-d_c)} - \frac{S_E}{\Delta S}\right)\right)$$

$$\frac{S_E - S_i}{\Delta S} = \frac{S_E}{\Delta S} - \frac{e_e}{1-d_c} = 1.$$

$$\theta_S = \max\left(-1, -1 - \left(\frac{\Theta_S}{d_c} + \frac{e_e}{d_c(1-d_c)} - 1 - \frac{e_e}{1-d_c}\right)\right).$$

$$\theta_S = \max\left(-1, -\frac{\Theta_S}{d_c} + \frac{e_e}{1-d_c} - \frac{e_e}{d_c(1-d_c)}\right)$$

$$\theta_S = \max\left(-1, -\frac{\Theta_S}{d_c} + \frac{e_e}{(1-d_c)d_c}(-1 + d_c)\right)$$

$$\theta_S = \max\left(-1, -\frac{\Theta_S}{d_c} - \frac{e_e}{d_c}\right)$$

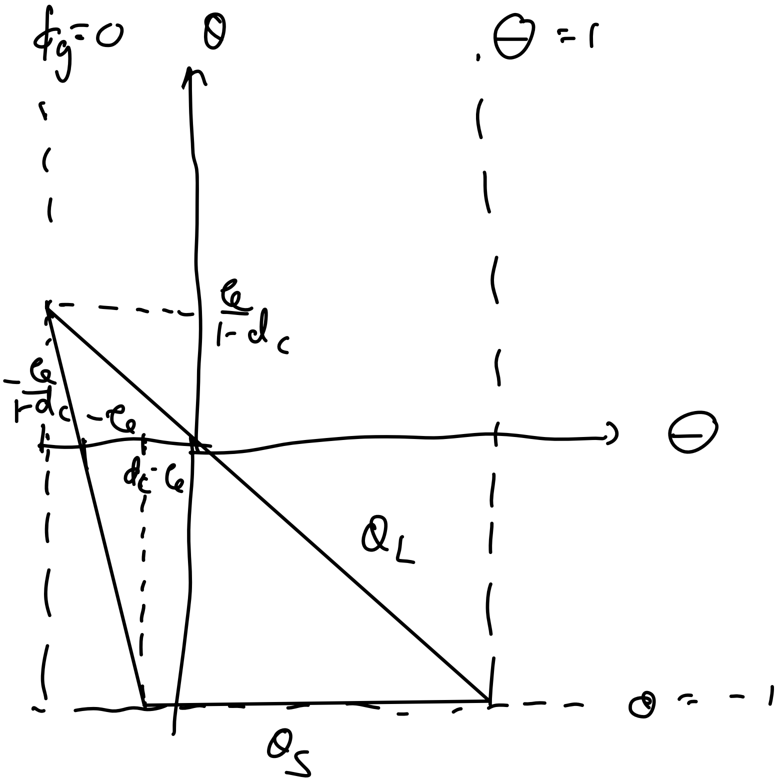
$$\boxed{\theta_S = \max\left(-1, -1/d_c\,(\Theta_S + e_e)\right).}$$

$\Theta_S = -\frac{e_e}{1-d_c}$ $\theta_S = \frac{e_e}{1-d_e}$

$\Theta_S = -e_e$ $\theta_S = 0.$

$\Theta_S = d_c - e_e,$ $\theta_S = -1$

$$\boxed{\begin{array}{l} \Theta_S + e_e = d_c \Theta_L \text{ in mush} \\ \Theta_S + e_e = d_c \Rightarrow \Theta_L = 1 \text{ in mush} \end{array}}$$



Non Dimensional (T, S) phase plane at $\phi_g=0$.

Now we calculate phase relations for $\Gamma < \Gamma_{sat}$.

so $\phi_g=0$ and $\omega = \Gamma / \chi \phi_l(\mathcal{H}, \Theta)$

Liquid Θ:

$\phi_s=0$, Θ_s N/A, $\phi_g=0 \Rightarrow \phi_l=1$, $\omega = \Gamma/\chi$.

Need to find θ & Θ_l.

$\mathcal{H}=\theta$

$\Theta=\Theta_l$

Mushy phase. $\ominus$

$\theta = \theta_L = -\Theta_e \qquad \phi_g = 0 \qquad \phi_s = 1 - \phi_e$

$\Theta_s = d_c \Theta_e - C_e . \quad \omega = \Gamma / \chi \phi_e$

Need to find θ, ϕ_e

$\mathcal{H} = \theta - (1-\phi_e) St \quad \Rightarrow \quad 1 - \phi_e = \dfrac{\theta - \mathcal{H}}{St} \Rightarrow \phi_e = 1 - \dfrac{\theta - \mathcal{H}}{St}.$

$\Theta = (1-\phi_e)(-d_c \theta - C_e) - \phi_e \theta$

$$\Theta = \frac{(\theta - \mathcal{H})}{St}(-d_c\theta - C_e) - \left(1 - \frac{\theta - \mathcal{H}}{St}\right)\theta$$

$St\,\Theta = (\theta - \mathcal{H})(-d_c\theta - C_e) - (St - \theta + \mathcal{H})\theta.$

$St\,\Theta = -d_c\theta^2 + d_c\theta\mathcal{H} - \theta C_e + C_e\mathcal{H} - (St + \mathcal{H})\theta + \theta^2$

$St\,\Theta = \theta^2(1-d_c) + \theta(d_c\mathcal{H} - C_e - St - \mathcal{H}) + C_e\mathcal{H}.$

$A\theta^2 - B\theta + C = 0 \quad B \quad C$

$\theta^2(1-d_c) - \theta((1-d_c)\mathcal{H} + C_e + St) + C_e\mathcal{H} - St\,\Theta = 0$

$$\theta = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$B^2 - 4AC = \left[(1-d_c)\mathcal{H} + \mathcal{C}_e + St\right]^2 - 4(1-d_c)(\mathcal{C}_e\mathcal{H} - St\,\Theta)$$

this is always +ve.

to get a -ve temperature must take -ve root

so $\theta = \dfrac{B - \sqrt{B^2-4AC}}{2A}$. $\quad \Theta_e = -\theta$

$\Theta_s = -d_c\theta - \mathcal{C}_e$

$\phi_e = 1 - \dfrac{\theta - \mathcal{H}}{St}$ $\quad \phi_g = 0, \ \phi_s = 1 - \phi_e, \ \omega = \dfrac{\Gamma}{\chi \phi_e}$.

$\mathcal{H}_L$ Boundary: $\phi_g = 0 \quad \phi_s = 0 \quad \phi_e = 1$

$\ominus \quad \theta = -\Theta_e \quad \Theta_s = -d_c\theta - \mathcal{C}_e$.

$\mathcal{H}_L = \theta$ so $\mathcal{H}_L^-(\Theta) = -\Theta$.

$\Theta = -\theta$.

Eutectic Region $\ominus$ $\theta = \theta_S(\Theta_S)$, $\Theta_L = 1$, $\phi_g = 0$

$\omega = \Gamma / \chi \phi_L$ $\quad \phi_S = 1 - \phi_L$.

need Θ_S & ϕ_L

$\mathcal{H} = \theta - (1-\phi_L) St$.

$\Theta = (1-\phi_L)\Theta_S + \phi_L$

we actually always have that $\Theta_S \leq \theta_c - \theta_e$

so $\theta = -1$

otherwise we intersect the solidus.

so $\mathcal{H} = -1 - (1-\phi_L) St$

$\Theta = (1-\phi_L)\Theta_S + \phi_L$

so $\frac{\mathcal{H}+1}{St} = \phi_L - 1$ so $\phi_L = \frac{\mathcal{H}+1}{St} + 1$

and $\phi_S = -\frac{(\mathcal{H}+1)}{St}$.

$\Theta_S = \frac{\Theta - \phi_L}{1 - \phi_L}$, $\theta = -1$.

$\mathcal{H}_E$ $\ominus$ Bdry: $\theta = -1$, $\theta_l = 1$, $\phi_g = 0$, $\Theta_S = d_c - e_e$.

$\omega = \Gamma / \chi\phi_l$ $\phi_S = 1 - \phi_l$.

$\mathcal{H}_E = -1 - (1-\phi_l) St$

$\Theta = \phi_l + (1-\phi_l)(d_c - e_e)$.

$\phi_l^E (1 - d_c + e_e) + (d_c - e_e) = \Theta$

$$\phi_l^E = \frac{\Theta + e_e - d_c}{1 - d_c + e_e}$$

$\mathcal{H}_E^-(\Theta) = -1 - St(1 - \phi_l^E(\Theta))$.

intersect solidus at $\Theta = d_c - e_e$.

intersect liquidus at $\Theta = 1$.

$\mathcal{H}_S$ $\ominus$ Bdry: $\theta = \theta_S(\Theta_S)$, $\phi_l = 0$, $\phi_g = 0$

$\Theta_l = 1$, $\omega = \Gamma / \chi\phi_l$, $\phi_S = 1$.

$\mathcal{H}_S = \theta_S - St$

$\Theta = \Theta_S$ so $\mathcal{H}_S^- = \max(-1 - St, \frac{-1}{d_c}(\Theta + e_e) - St)$.

Solid Region $\phi_s = 1$, $\phi_g = 0$, $\phi_l = 0$, $\Theta_c = 1$

$w = 1$ (continuity).

$H = \Theta - St$

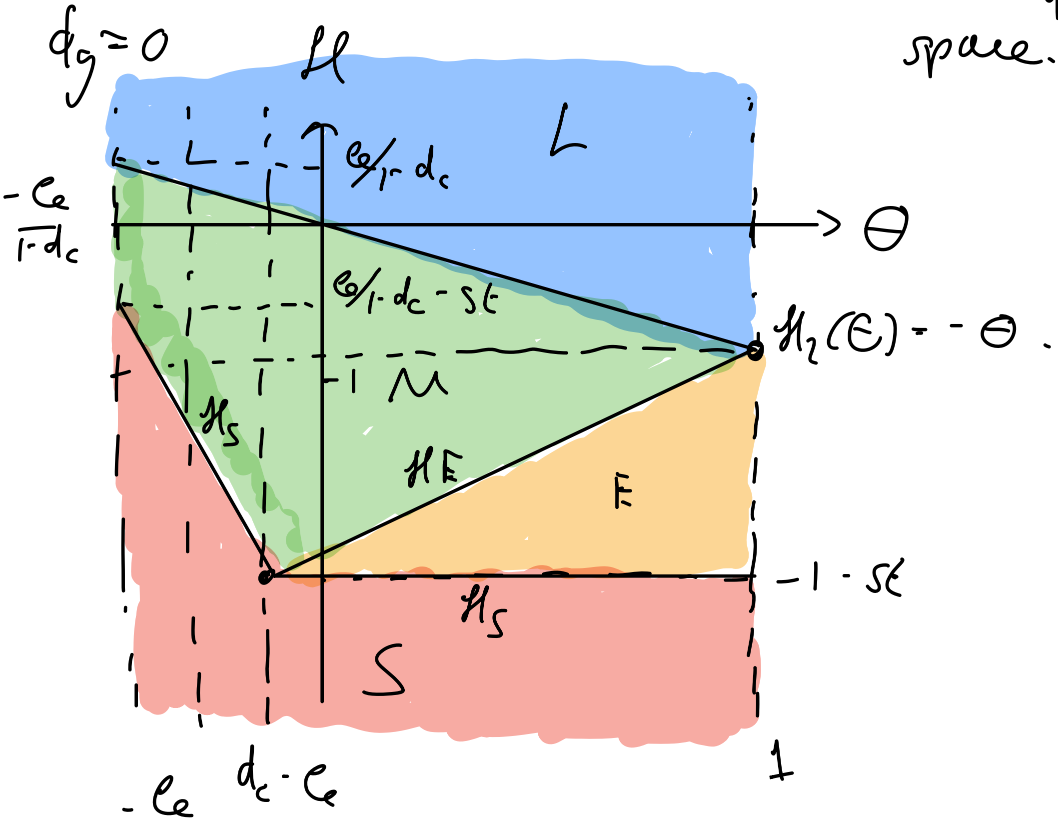
$\Theta = \Theta_s$

$\Theta = H + St$, $\Theta_s = \Theta$.

$\Gamma < \Gamma_{sat} = \chi \phi_e(H, \Theta)$

phase diagram in enthalpy space.

$\phi_g = 0$



Now do same calculations with

$\Gamma > \Gamma_{sat}$

so $\omega = 1$ & $\phi_g > 0$

These should match up as $\phi_g \to 0$

Liquid Region: $\phi_s = 0$, $\theta_s = d_c\theta_e - c_e$, $\omega = 1$

$\phi_l = 1 - \phi_g$ $\oplus$

$\mathcal{H} = \theta(1 - \phi_g)$

$\theta = (1-\phi_g)\theta_e - \dfrac{\phi_g c_e}{1-d_c}$

$\Gamma = \chi(1-\phi_g) + \phi_g$ $\quad \phi_g(1-\chi) = \Gamma - \chi$

so $\phi_g = \dfrac{\Gamma - \chi}{1-\chi}$ (Note $\chi \le \Gamma \le 1$)

we kn $\phi_s, \theta_s, \phi_g, \omega$

$\Rightarrow \phi_l = 1 - \phi_g$ known

need θ_1 & θ

$\theta = \mathcal{H}/_{1-\phi_g}$ $\quad \theta_e = \dfrac{\theta + \dfrac{\phi_g c_e}{1-d_c}}{1-\phi_g}$

Mushy Regin $\oplus$: $\omega = 1$, $\theta = -\Theta_e$, $\Theta_s = d_c\Theta_e - C_e$.

$\Gamma = \chi\phi_e + \phi_g$ so $\phi_g = \Gamma - \chi\phi_l$. $\phi_s = 1 - \phi_e - \phi_g$

need ϕ_l & θ

$$H = \theta(1-\phi_g) - (1-\phi_l-\phi_g)St.$$

$$\Theta = -\phi_e\theta + (1-\phi_e-\phi_g)(-d_c\theta - C_e) - \frac{\phi_g C_e}{1-d_c}.$$

$$\Theta = -\phi_l\theta + \phi_s(-d_c\theta - C_e) - \frac{\phi_g C_e}{1-d_c}$$

$$H = \theta(1-\phi_g) - \phi_s St$$

so $$\boxed{\frac{H + \phi_s St}{(1-\phi_g)} = \theta}$$

Find ϕ_l in terms of θ assuming $\underline{\underline{d_c = 0}}$

$$H = \theta(1-\Gamma+\chi\phi_e) - (1-\phi_l-\Gamma+\chi\phi_l)St$$

$$\Theta = -\phi_l\theta - (1-\phi_l-\Gamma+\chi\phi_l)C_e - C_e(\Gamma-\chi\phi_l)$$

$$\Theta = -\phi_l\theta - C_e(1-\phi_l)$$

so $\Theta = \phi_l(-\theta + C_e) - C_e$ so $$\phi_l = \frac{\Theta + C_e}{C_e - \theta}$$

Mathematica finds 2 solutions for θ.

$$\theta_1 = -\frac{1}{2A}\left(B - \sqrt{4AC + E^2}\right)$$

solution of

$$\theta_2 = -\frac{1}{2A}\left(B + \sqrt{4AC + E^2}\right)$$

$$A\theta^2 - B\theta - C = 0$$

First show $B = E$

$$B = \mathcal{C} + \mathcal{H} + \mathcal{S}t - \mathcal{C}\Gamma - \mathcal{S}t\Gamma + \mathcal{C}\chi + \mathcal{I}_i\chi - \mathcal{C}d_c$$
$$-2\mathcal{H}d_c - \mathcal{S}t\,d_c + \mathcal{C}\Gamma d_c + \mathcal{S}t\Gamma d_c + \mathcal{H}\chi d_c$$
$$-\mathcal{I}_i\chi d_c + \mathcal{H}d_c^2 - \mathcal{H}\chi d_c^2$$

factorise by d_c

$$B = \mathcal{C} + \mathcal{H} + \mathcal{S}t - \mathcal{C}\Gamma - \mathcal{S}t\Gamma + \mathcal{C}\chi + \mathcal{I}_i\chi$$
$$+ d_c(-\mathcal{C} - 2\mathcal{H} - \mathcal{S}t + \mathcal{C}\Gamma + \mathcal{S}t\Gamma + \mathcal{H}\chi$$
$$-\mathcal{I}_i\chi)$$
$$+ d_c^2(\mathcal{H} - \mathcal{H}\chi)$$

$B = E$ confirmed group by the number

$$B = G(1 - \Gamma + \chi - d_c + \Gamma d_c)$$
$$+ H(1 - 2d_c + \chi d_c + d_c^2 - \chi d_c^2)$$
$$+ C_i(\chi)(1 - d_c)$$
$$+ St(1 - \Gamma - d_c + \Gamma d_c)$$

$$B = G(1 - d_c - \Gamma(1 - d_c) + \chi)$$
$$+ H((1 - d_c)^2 + \chi d_c(1 - d_c))$$
$$+ C_i \chi(1 - d_c)$$
$$+ St(1 - d_c - \Gamma(1 - d_c))$$

$$B = G((1 - d_c)(1 - \Gamma) + \chi)$$
$$+ H(1 - d_c)(1 - d_c + \chi d_c)$$
$$+ C_i \chi(1 - d_c)$$
$$+ St(1 - d_c)(1 - \Gamma)$$

$$B = (1-d_c)\left[(C_e + St)(1-\Gamma) + \mathcal{H}(1-d_c+d_c\chi) + L_i\chi\right] + C_e\chi$$

$$\theta_1 = -\frac{1}{2A}\left(B - \sqrt{4AC+B^2}\right)$$

$$\theta_2 = -\frac{1}{2A}\left(B + \sqrt{4AC+B^2}\right)$$

$$A = (\Gamma-1)(d_c-1)^2$$

$$C = St\, L_i(\chi-1) + C_e(\mathcal{H} - St\,\Gamma + St\,\chi) + (C_e\mathcal{H} - St\, L_i)d_c(\chi-1)$$

$$= St\, L_i\chi - St\, L_i + C_e\mathcal{H} - St\, C_e\Gamma + C_e St\,\chi + C_e\mathcal{H}d_c(\chi-1) - St\, L_i d_c(\chi-1)$$

$$C = St\, L_i\big((\chi-1) - d_c(\chi-1)\big) + C_e\mathcal{H}\big(1 + d_c(\chi-1)\big) + St\, C_e(\chi-\Gamma)$$

$$C = St\Gamma_1(1-d_c)(\chi-1)$$
$$+ C_e\mathcal{H}((1-d_c)+d_c\chi)$$
$$+ St\, C_e(\chi-\Gamma)$$

$$C = (1-d_c)\left[(\chi-1)St\Gamma_1 + C_e\mathcal{H}\right] + d_c C_e\mathcal{H}\chi$$
$$+ St\, C_e(\chi-\Gamma).$$

$$C = (1-d_c)\left[C_e\mathcal{H} - (1-\chi)St\Gamma_1\right] + C_e\left[d_c\mathcal{H}\chi - St(\Gamma-\chi)\right]$$

in the limit $\Gamma = \chi = 0$

$$C = (1-d_c)\left[C_e\mathcal{H} - St\Gamma_1\right]$$
$$A = -(1-d_c)^2$$
$$B = (1-d_c)\left[C_e + St + \mathcal{H}(1-d_c)\right]$$

Agrees with previous work

Nale

Note $\theta_1 = -\frac{1}{2A}\left(B - \sqrt{B^2+4AC}\right)$

$\theta_2 = -\frac{1}{2A}\left(B + \sqrt{B^2+4AC}\right)$

in limit $\Gamma = \chi = 0$

$d_c = 0$

renamed

$\Theta = \Sigma_1^T$

$C = e_e\mathcal{H} - \delta t\,\Sigma_1^T$

$A = -1 < 0$

$B = e_e + \delta t + \mathcal{H}$

in Mush $\mathcal{H} < -\Sigma_1^T$

$-e_e < \Sigma_1^T < 1.$ $\quad e_e > -\Sigma_1^T > -1$

$-1 < -\Sigma_1^T < e_e.$

so most -ve $\mathcal{H}$ is $\mathcal{H} = -1$

lowest value of B is $e_e + \delta t - 1 > 0.$

so $B > 0.$

to get a negative temp Take θ_1!

$\theta_1 = \frac{1}{2}\left(B - \sqrt{\cdots}\right)$ $\quad \theta_2 = \frac{1}{2}\left(B + \sqrt{\cdots}\right)$

TL DR :

$$\theta = -\frac{1}{2A}\left(B - \sqrt{B^2 + 4AC}\right)$$

$$A = (\Gamma - 1)(1 - d_c)^2$$

$$B = (1-d_c)\left[(C_e + St)(1-\Gamma) + \mathcal{H}(1 - d_c + d_c \chi) + \Gamma_1 \chi\right] + C_e \chi$$

$$C = (1-d_c)\left[C_e \mathcal{H} - (1-\chi) St\, \Gamma_1\right] + C_e\left[d_c \mathcal{H} \chi - St(\Gamma - \chi)\right]$$

in Supersaturated + Mushy Region.

Liquidus Body: $\omega = 1$, $\phi_S = 0$, $\Theta_S = d_c \Theta_e - C_e$.

$$\theta = -\Theta_e.$$

$$\phi_e = 1 - \phi_g$$

$$\mathcal{H}_L^+ = \theta(1-\phi_g)$$

$$\Theta = (1-\phi_g)(-\theta) - \frac{\phi_g C_e}{1-d_c}.$$

$$\Gamma = \phi_g + \chi(1-\phi_g) \qquad \phi_g = \frac{\Gamma - \chi}{1-\chi}.$$

ϕ_g is known

$$\frac{\Theta + \frac{\phi_g C_e}{1-d_c}}{1-\phi_g} = -\theta.$$

so $\mathcal{H}_L^+(\Theta) = -\left(\Theta + \frac{\phi_g C_e}{1-d_c}\right).$

Eutectic Region (+): $\omega = 1$ $\theta = -1$, $\theta_e = 1$,

$$\mathcal{H} = -1(1-\phi_g) - (1-\phi_l-\phi_g)St. \quad \phi_s = 1-\phi_l-\phi_g$$

$$\Theta = \phi_l + (1-\phi_l-\phi_g)\Theta_S - \frac{\phi_g C_e}{1-d_c}$$

$$\Gamma = \chi\phi_l + \phi_g. \quad \phi_g = \Gamma - \chi\phi_l$$

solve for ϕ_l, ϕ_g, Θ_S

$$\Theta_S = \frac{\Theta + \frac{\phi_g C_e}{1-d_c} - \phi_l}{1-\phi_l-\phi_g}$$

$$\mathcal{H} = \phi_g - 1 - (1-\phi_l-\phi_g)St$$

$$\mathcal{H} = \Gamma - \chi\phi_l - 1 - (1-\phi_l-\Gamma+\chi\phi_l)St.$$

$$\mathcal{H} = \phi_l(-\chi + St - \chi St) + \Gamma - 1 - St + \Gamma St.$$

$$\mathcal{H} = \phi_l(St(1-\chi) - \chi) - (1-\Gamma)(1+St)$$

$$\phi_1 = \frac{(1-\Gamma)(1+St) + \mathcal{H}}{St(1-\chi) - \chi}$$

$$\phi_g = \Gamma - \chi\phi_1 .$$

Eutectic Boundary$^+$: $\omega = 1, \quad \theta = -1, \quad \theta_l = 1$

$$\theta_s = d_c - c_e$$

$$\phi_s + \phi_l + \phi_g = 1$$

$$\mathcal{H}_E^+ = -1(1-\phi_g) - \phi_s St$$

$$\theta = \phi_l + \phi_s(d_c - c_e) - \frac{\phi_g c_e}{1-d_c}$$

$$\Gamma = \phi_e\chi + \phi_g$$

$$d_c = 0$$

$$\phi_g = \Gamma - \phi_e\chi$$

$$\phi_s = 1 - \phi_g - \phi_l = 1 - \Gamma + \phi_l\chi - \phi_l$$

$$\theta = \phi_l - c_e(1 - \Gamma + \phi_l\chi - \phi_l) - c_e(\Gamma - \phi_e\chi)$$

$$\Theta = \phi_l(1 - C_e\chi + C_e + C_e\chi)$$

$$+ C_e\Gamma - C_e - C_e\Gamma$$

$$\Theta = \phi_l(1 + C_e) - C_e$$

$$\phi_l = \frac{\Theta + C_e}{1 + C_e}$$ same as before

but now $\phi_g = \Gamma - \dfrac{\chi(\Theta + C_e)}{1 + C_e}$

$$H_E^+(\Theta, \Gamma) = -1(1 - \phi_g) - \phi_s St$$

at $\Theta = -C_e$ $\phi_l = 0$ $\phi_g = \Gamma$, $\phi_s = 1 - \Gamma$ ✓

$$H_E^+ = -1(1 - \Gamma + \chi\phi_l^E) - (1 - \phi_l^E - \Gamma + \chi\phi_l^E)St$$

$$H_E^+ = -1(1 - \Gamma + \chi\phi_l^E) - St(1 - \Gamma + \phi_l^E(\chi - 1)).$$

Solidus Boundary: $\phi_l = 0$, $\omega = 1$, $\Theta = \Theta_S$, C_l N/A

$$H_S^+ = (1 - \phi_g)\Theta_S - (1 - \phi_g)St. = (1 - \phi_g)(\Theta_S - St)$$

$$\Theta = (1 - \phi_g)\Theta_S - \frac{\phi_g C_e}{1 - \phi_c} \qquad \Gamma = \phi_g$$

$$\Theta_s = \frac{\Theta + \frac{r e_e}{1-d_c}}{1-r}$$

$$\theta_s(\Theta_s) = \max\left(-1, -\tfrac{1}{d_c}(\Theta_s + e_e)\right)$$

$$\theta_s(\Theta_s) = \begin{cases} -1, & \Theta_s > d_c - e_e \\ -\frac{1}{d_c}(\Theta_s + e_e), & \Theta_s \leq d_c - e_e \end{cases}$$

$$\theta_s = \max\left(-1, \frac{-1}{d_c}\left(\frac{\Theta + \frac{r e_e}{1-d_c}}{1-r} + e_e\right)\right).$$

so $\mathcal{H}_s^+(\Theta) = (1-r)\max\left(-1-St, \frac{-1}{d_c}\left(\frac{\Theta + \frac{r e_e}{1-d_c}}{1-r} + e_e\right) - St\right)$.

Solid Region ⊕: $\phi_l = 0$, θ_l N/A, $w = 1$

$$\mathcal{H} = \theta(1-\phi_g) - (1-\phi_g)St = (1-\phi_g)(\theta - St)$$

$$\Theta = \phi_s \theta_s - \frac{\phi_g e_e}{1-d_c} = (1-r)\theta_s - \frac{r e_e}{1-d_c}$$

$$r = \phi_g$$

$$\sigma_s = \frac{\sigma + \frac{r u}{1-d_c}}{1-n}$$

$$\theta = \frac{u}{1-n} + St$$

$$d_g = n$$

$$d_s = 1-n$$

Calculating Phase Boundaries.

Need total phase boundaries

$\mathcal{H}_{L,E,S}(\Theta, \Gamma)$ & $\Gamma_{sat}(\mathcal{H}, \Theta)$.

Note that for $\Gamma \leq \Gamma_{sat}$ phase boundaries are independent of Γ

$\therefore \Gamma_{sat}(\mathcal{H}, \Theta) = \chi \phi_l(\mathcal{H}, \Theta)$.

where $\phi_l = \phi_l$ subsaturated

$$\Gamma_{sat}(\mathcal{H}, \Theta) = \begin{cases} \chi & \mathcal{H} \geq \mathcal{H}_L^-(\Theta) \\ \chi\left(1 - \left(\frac{\theta - \mathcal{H}}{St}\right)\right), & \mathcal{H}_E^-(\Theta) \leq \mathcal{H} < \mathcal{H}_L^-(\Theta) \\ \chi\left(1 + \left(\frac{\mathcal{H}+1}{St}\right)\right), & \mathcal{H}_S^- < \mathcal{H} \leq \mathcal{H}_E^-(\Theta) \\ 0 & , \mathcal{H} \leq \mathcal{H}_S^-(\Theta) \end{cases}$$

where

$$\theta = \frac{B - \sqrt{B^2 - 4AC}}{2A}.$$

$A = 1 - d_c$

$B = (1 - d_c)\mathcal{H} + C_e + St$

$C = C_e \mathcal{H} - St\,\Theta$

$\mathcal{H}_L^-(\Theta) = -\Theta$

$$d_1^E = \frac{\Theta + C_e - d_c}{1 - d_c + C_e}$$

$\mathcal{H}_E^-(\Theta) = -1 - St(1 - \phi_l^F(\Theta))$.

$\mathcal{H}_S^- = \max\left(-1 - St, \frac{-1}{d_c}(\Theta + C_e) - St\right)$.

To calculate
phase boundaries we need intersects with Γ_{sat}

occurs at $\Gamma_L^{sat} = \Gamma_{sat}(\mathcal{H}_L^-(\theta), \theta)$

$$\Gamma_E^{sat} = \Gamma_{sat}(\mathcal{H}_E^-(\theta), \theta)$$

$$\Gamma_S^{sat} = \Gamma_{sat}(\mathcal{H}_S^-(\theta), \theta)$$

then $\mathcal{H}_L(\theta, \Gamma) = \begin{cases} \mathcal{H}_L^-(\theta) & , \Gamma \leq \Gamma_L^{sat}(\theta) \\ \mathcal{H}_L^+(\theta, \Gamma) & , \Gamma > \Gamma_L^{sat}(\theta) \end{cases}$

same for E & S.

when $\mathcal{H} = \mathcal{H}_L^-(\theta)$ $\Gamma_L^{sat} = \chi$

when $\mathcal{H} = \mathcal{H}_S^-(\theta)$ $\Gamma_S^{sat} = 0$

$\mathcal{H} = \mathcal{H}_E^-(\theta)$ $\Gamma_E^{sat} = \chi\left(\dfrac{\theta + c_e - d_c}{1 - d_c + c_e}\right)$

$$\mathcal{H}_E^- = -1 - St(1 - \phi_1^E(\theta))$$

$$\Gamma_E^{sat} = \chi\left(1 - (1 - \phi_1^E(\theta))\right)$$

$$= \chi \phi_1^E(\theta) = \chi\left(\frac{\theta + c_e - d_c}{1 - d_c + c_e}\right)$$

$$H_L(\theta,\Gamma) = \begin{cases} -\theta, & \Gamma \le \chi \\ -\left(\theta + \frac{C_e}{1-d_c}\left[\frac{\Gamma-\chi}{1-\chi}\right]\right), & \Gamma > \chi \end{cases}$$

$$H_E(\theta,\Gamma) = \begin{cases} -1 - St(1-\phi_l^E(\theta)), & \Gamma \le \chi\phi_l^E(\theta) \\ -1(1-\phi_g) - \phi_s St, & \Gamma > \chi\phi_l^E(\theta) \end{cases}$$

$\phi_l^E = \frac{\theta + C_e}{1 + C_e}$ same as before $(d_c = 0)$

$(\Gamma - \chi\phi_l^E)$

$\phi_s = 1 - \phi_l^E - \phi_g \qquad \phi_g = \Gamma - \frac{\chi(\theta + C_e)}{1 + C_e}$

$$H_S(\theta,\Gamma) = \begin{cases} \max\left(-1-St, \frac{-1}{d_c}(\theta + C_e) - St\right), & \Gamma = 0 \\ (1-\Gamma)\max\left(-1-St, \frac{-1}{d_c}\left(\frac{\theta + \frac{\Gamma C_e}{1-d_c}}{1-\Gamma} + C_e\right) - St\right), & \Gamma > 0 \end{cases}$$

Bounds: H can be anything. $\quad 0 \le \Gamma \le 1$.

$-\frac{C_e}{(1-d_c)} \le \theta \le \theta_{max} = 1 - \phi_g\left(1 + \frac{C_e}{1-d_c}\right)$

bulk fresh

the reduced eutectic point for a given ϕ_g $\theta_{max}(\phi_g)$ s.t. $\theta_{max}(0) = 1$.

To plot $\Theta_{max}(r)$

we can use defined in liquid $f_g = \frac{r-\chi}{1-\chi}$.

$r \geq \chi$?

$r < \chi$ $f_g = 0$

$$\Theta_{max} = \begin{cases} 1, & r \leq \chi \\ 1 - \left(\frac{r-\chi}{1-\chi}\right)\left(1 + \frac{c_e}{1-d_c}\right), & r > \chi. \end{cases}$$