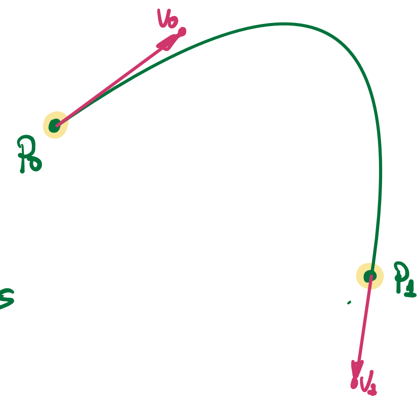
Cubic Hermite Interpolation



Data: P_0, v_0
P_1, V_1

Curve: green

Construct a cubic curve:

- provide two points P_0, P_1 that the curve must pass through
- the tangent vectors v_0, v_1 at these two points (these are essentially the values of the first derivatives at these points, the velocity)
- symmetric way of providing data – each point is treated in the same manner

We look for a curve

$$c(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3,$$

such that

$$c(0) = P_0 \qquad c(1) = P_1$$
$$\text{and}$$
$$c'(0) = v_0 \qquad c'(1) = V_1$$

This leads to a system of equations:

$$\begin{cases} P_0 = a_0 \\ v_0 = a_1 \\ p_1 = a_0 + a_1 + a_2 + a_3 \\ V_1 = a_1 + 2a_2 + 3a_3 \end{cases}$$

which gives us the curve parameters we want in terms of the known/given data

$$\begin{cases} a_0 = p_0 \\ a_1 = v_0 \\ a_2 = 3p_1 - 3p_0 - 2v_0 - V_1 \\ a_3 = -2p_1 + 2p_0 + v_0 + V_1 \end{cases}$$

If we rearrange the terms for $c(t)$ we get

$$c(t) = (1 - 3t^2 + 2t^3)\,p_0 + (t - 2t^2 + t^3)\,v_0 + (-t^2 + t^3)\,V_1 + (3t^2 - 2t^3)\,p_1$$

or in basis function form

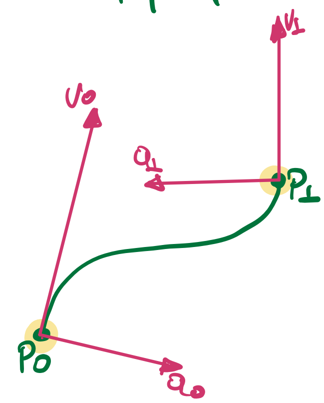
$$c(t) = H_0^3(t)\,p_0 + H_1^3(t)\,v_0 + H_2^3(t)\,V_1 + H_3^3(t)\,p_1,$$

where

$$\begin{cases} H_0^3(t) = 1 - 3t^2 + 2t^3 \\ H_1^3(t) = t - 2t^2 + t^3 \\ H_2^3(t) = -t^2 + t^3 \\ H_3^3(t) = 3t^2 - 2t^3. \end{cases}$$

Another approach without solving a system but rather using desired properties.

Quintic Hermite Interpolation



- direct generalisation of Cubic Hermite interpolation
- Find curve $c(t)$ such that

$$\begin{cases} c(0) = p_0, & c'(0) = v_0, & c''(0) = a_0 \\ c(1) = p_1, & c'(1) = v_1, & c''(1) = a_1 \end{cases} \qquad (1)$$

- simplest solution - quintic (5^{th} order) polynomial curve

We seek a general quintic

$$c(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_5 t^5 \qquad (2)$$

From (1) and (2) it follows

$$p_0 = b_0$$

$$v_0 = b_1$$

$$a_0 = b_2$$

$$p_1 = b_0 + b_1 + b_2 + b_3 + b_4 + b_5$$

$$v_1 = b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5$$

$$a_1 = 2b_2 + 6b_3 + 12b_4 + 20b_5$$

$$c(t) = H_0^5(t)p_0 + H_1^5(t)v_0 + H_2^5(t)a_0 + H_3^5(t)a_1 + H_4^5(t)v_1 + H_5^5(t)p_1, \text{ where}$$

$$H_0^5(t) = 1 - 10t^3 + 15t^4 - 6t^5$$

$$H_1^5(t) = t - 6t^3 + 8t^4 - 3t^5$$

$$H_2^5(t) = \frac{1}{2}t^2 - \frac{3}{2}t^3 + \frac{3}{2}t^4 - \frac{1}{2}t^5$$

$$H_3^5(t) = \frac{1}{2}t^3 - t^4 + \frac{1}{2}t^5$$

$$H_4^5(t) = -4t^3 + 7t^4 - 3t^5$$

$$H_5^5(t) = 10t^3 - 15t^4 + 6t^5$$

- motion planning problems - to control the motion one specifies the position, velocity, and acceleration at several times
- other constraints that could restrict the area the curve may lie - obstacles to avoid
- animation

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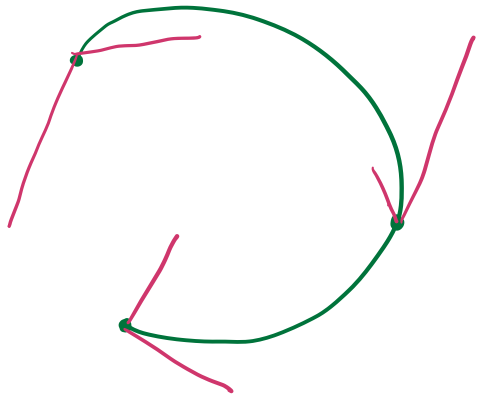
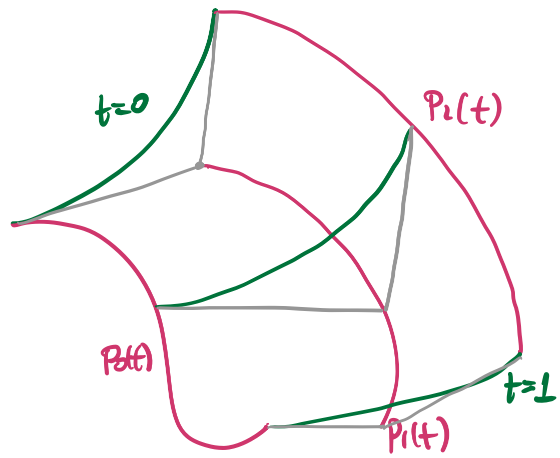


Piecewise Quintic Curve



Animation of parabola

$$c(s,t) = (1-s)^2 P_0(t) + 2s(1-s) P_1(t) + s^2 P_2(t)$$