Q1. a)

$$X = \begin{cases} \frac{1}{\mu}\log(2U), & U \in (0, \frac{1}{2}] \\ -\frac{1}{\mu}\log(2-2U), & U \in (\frac{1}{2}, 1) \end{cases}, \quad \mu > 0, \mu \in \mathbb{R}, \; U \sim \text{Uniform}(0,1)$$

i.) Let's first find the CDF for X :

- Case 1: $U \in (0, \frac{1}{2}] \Rightarrow X \in (-\infty, 0]$

e^x is monotonically increasing

$$F_X(x) = P(X \leq x) = P\left(\frac{1}{\mu}\log(2U) \leq x\right) \overset{\mu>0}{=} P(\log(2U) \leq \mu x) = P(e^{\log(2U)} \leq e^{\mu x}) =$$

$$= P(2U \leq e^{\mu x}) = P\left(U \leq \frac{e^{\mu x}}{2}\right) = F_U\left(\frac{e^{\mu x}}{2}\right)$$

We have $U \sim \text{Uniform}(0,1) \Rightarrow F_U(u) = u$

$$\Rightarrow F_X(x) = F_U\left(\frac{e^{\mu x}}{2}\right) = \frac{e^{\mu x}}{2}, \text{ when } U \in (0, \tfrac{1}{2}], X \in (-\infty, 0]$$

- Case 2: $U \in (\frac{1}{2}, 1) \Rightarrow X \in (0, +\infty)$

$$F_X(x) = P(X \leq x) = P\left(-\frac{1}{\mu}\log(2-2U) \leq x\right) = P(\log(2-2U) \geq -\mu x) =$$

$$= P(e^{\log(2-2U)} \geq e^{-\mu x}) = P(2-2U \geq e^{-\mu x} - 2) = P\left(U \leq 1 - \frac{e^{-\mu x}}{2}\right) =$$

$$= F_U\left(1 - \frac{e^{-\mu x}}{2}\right) = 1 - \frac{e^{-\mu x}}{2}$$

$$\Rightarrow F_X(x) = \begin{cases} \frac{e^{\mu x}}{2}, & \text{when } U \in (0, \frac{1}{2}], X \in (-\infty, 0] \\ 1 - \frac{e^{-\mu x}}{2}, & \text{when } U \in (\frac{1}{2}, 1), X \in (0, +\infty) \end{cases}$$

Now, let's compute the PDF for X. At values of x where F_X is differentiable,

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x} = \begin{cases} \frac{\mu}{2} \cdot e^{\mu x}, & x \in (-\infty, 0] \\ \frac{\mu}{2} \cdot e^{-\mu x}, & x \in (0, +\infty) \end{cases}$$

Or, $f_X(x) = \frac{\mu}{2} e^{-\mu|x|}, \quad \forall x \in \mathbb{R}$

ii) $M_X(t) = E[e^{tX}] = \int_{x \in S = \mathbb{R}} e^{tx} \cdot f_X(x)dx = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{\mu}{2} \cdot e^{-\mu|x|} dx =$

$$= \underbrace{\frac{\mu}{2} \int_{-\infty}^{0} e^{(t+\mu)x} dx}_{\substack{\text{converges when} \\ t+\mu>0}} + \underbrace{\frac{\mu}{2} \int_{0}^{+\infty} e^{(t-\mu)x} dx}_{\substack{\text{converges when} \\ t-\mu<0}} =$$

$$= \frac{\mu}{(t+\mu)2} \cdot e^{(t+\mu)x}\Big|_{-\infty}^{0} + \frac{\mu}{(t-\mu)2} e^{(t-\mu)x}\Big|_{0}^{+\infty}, \quad |t|<\mu$$

$$= \frac{\mu}{(t+\mu)2}(e^0 - 0) + \frac{\mu}{(t-\mu)2}(0 - e^0) = \frac{\mu}{2}\left(\frac{1}{t+\mu} - \frac{1}{t-\mu}\right) = \frac{\mu^2}{\mu^2 - t^2}, \quad |t|<\mu$$

The characteristic function is given by $\phi_X(t) = E[e^{itX}]$

When the moment generating function exist (we just showed it does exist for $|t|<\mu$), the characteristic function and the moment generating function satisfy the following relationship:

$$\phi_X(t) = M_X(it)$$

$$\Rightarrow \phi_X(t) = \frac{\mu^2}{\mu^2 - i^2t^2} = \frac{\mu^2}{\mu^2 + t^2}, \quad |t|<\mu$$

b.) We are interested in finding the moment generating function for W.

Or, $M_W(t) = E[e^{tW}] = \int_{w \in S} e^{tw} \cdot f_W(w)dw$. The aim is to find $f_W(w)$.

We are given that:

- $Z \sim \text{Exponential}(\lambda), \lambda>0 \Rightarrow f_Z(z) = \lambda \cdot e^{-\lambda z}, z \geq 0, \lambda>0$
- $W|Z=z \sim N(0,z) \Rightarrow f_{W|Z}(w|z) = \frac{1}{\sqrt{2\pi z}} \cdot e^{-\frac{w^2}{2z}}, \quad z>0, w \in \mathbb{R}$

By definition

$$f_{W|Z}(w|z) = \frac{f_{W,Z}(w,z)}{f_Z(z)} \quad \text{or}$$

$$f_{W,Z}(w,z) = f_Z(z) \cdot f_{W|Z}(w|z) = \lambda \cdot e^{-\lambda z} \cdot \frac{1}{\sqrt{2\pi z}} \cdot e^{-\frac{w^2}{2z}} =$$

$$= \frac{\lambda}{\sqrt{2\pi z}} \cdot e^{-\left[\lambda z + \frac{w^2}{2z}\right]} \quad , z>0, w \in \mathbb{R}$$

The marginal density function for W is given by

$$f_W(w) = \int_{-\infty}^{+\infty} f_{W,Z}(w,z)\,dz = \int_0^{+\infty} \frac{\lambda}{\sqrt{2\pi z}} \cdot e^{-\left[\lambda z + \frac{w^2}{2z}\right]} dz$$

$$M_{W|Z}(t) = E_{W|Z}\left[e^{tW} \mid Z=z\right] = \int_{-\infty}^{+\infty} e^{wt} \cdot f_{W|Z}(w,z)\,dw = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi z}} \cdot e^{tw} \cdot e^{-\frac{1}{2}\cdot\frac{w^2}{z}} dw$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{W,Z}(w,z)\,dw = \int_{-\infty}^{+\infty} \frac{\lambda}{\sqrt{2\pi z}}\, e^{-\left[\lambda z + \frac{w^2}{2z}\right]} dw = \lambda \cdot e^{-\lambda z}$$

$$\cancel{e^{-\lambda z}} \cdot \frac{\cancel{\lambda}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{z}} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2z}} = \cancel{\lambda \cdot e^{-\lambda z}} \implies \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2z}} dw = \sqrt{2\pi z}$$

$$M_Z(t) = \int_0^{+\infty} e^{tz} \cdot f_Z(z)\,dz = \int_0^{+\infty} e^{tz} \cdot \lambda \cdot e^{-\lambda z} dz = \lambda \int_0 e^{(t-\lambda)z} dz =$$

$$= \frac{\lambda}{t-\lambda} \cdot e^{(t-\lambda)z} \Big|_0^{+\infty} = \frac{\lambda}{\lambda - t}$$

ii.) $Var(W) = E[Var(W|Z)] + Var[E(W|Z)] =$ $\quad W|Z \sim N(0,Z)$, $Z \sim Exp(\lambda)$

Law of total variance

$= E[Z] + Var[0] =$

$= E[Z] = \frac{1}{\lambda}$

Q2.) a.) $Y_n(x)$ - the discrete random variable defined as the number of realisations that are no greater than x, for fixed $x \in \mathbb{R}$.

$X_1, X_2, \dots, X_n$ - realisations of the first n RVs, $\{X_n\}_{n \geq 1}$, i.i.d. RVs, F_X

Let's define $A_i = \{X_i \leq x\}$, where $i = 1, 2, \dots, n$

Let $I(A_i) = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$

Every X_i, $i = 1, \dots, n$ can either be $> x$ or $\leq x$, or

$$Y_n(x) = \sum_{i=1}^{n} I(A_i) = \sum_{i=1}^{n} I_{\{X_i \leq x\}}$$

It's obvious that the possible values of $Y_n(x)$ are $S = \{0, 1, 2, \dots, n\}$.

This can be interpreted as the set containing the number of successes, where a success is defined as a given $X_i \leq x$.

Let's find the probability mass function of $Y_n(x)$:

$$P_{Y_n}(k) = P_{Y_n}(Y_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{where}$$

n – the number of trials (by problem statement – first n realisations)

k – number of successes

p – success probability for each trial

We notice that this is exactly a Binomial distribution, or

$Y_n(x) \sim \text{Binomial}(n, p)$, which has expectation $= np$, variance $= n.p.(1-p)$

$E[Y_n(x)] = n.p$

$Var[Y_n(x)] = n.p.(1-p)$

b.) $T_n(x) = \frac{Y_n(x)}{n}$

$Y := Y_n(x) \sim \text{Binomial}(n,p)$: $P_{Y_n(x)}(k) = P_{Y_n}(Y_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$

we have $E[Y_i] < \infty$, and

$$\underbrace{\overline{Y_n}}_{RV} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \frac{Y_i}{n} = \sum_{i=1}^{n} T_i$$

LLN: $\overline{Y_n} \xrightarrow{p/a.s.} \mu = E[Y_i] = np$ as $n \to \infty$

$\Rightarrow \overline{T_n} = \frac{1}{n}\sum_{i=1}^{n} T_i = \frac{1}{n} \cdot \overline{Y_n}$ $\Rightarrow \overline{T_n} \longrightarrow \frac{1}{n} \cdot n \cdot p = p$ as $n \to \infty$

Or, also

$$E[T_n(x)] = E\left[\frac{Y_n(x)}{n}\right] = \frac{E[Y_n(x)]}{n} = \frac{np}{p} = p$$

$$Var[T_n(x)] = Var\left[\frac{Y_n(x)}{n}\right] = \frac{Var[Y_n(x)]}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

$\lim_{n\to\infty} E[T_n(x)] \to p$ and $\lim_{n\to\infty} Var[T_n(x)] = 0$

c.) $F_X(x) = \frac{x}{1+x}$, $x > 0$

We derived the pms of $Y_n(x)$ as $P_{Y_n(x)} = \binom{n}{k} p^k (1-p)^{n-k}$ (note that p depends on the fixed $x \in \mathbb{R}$)

For $Y_4(1)$ we have $n=4$, $x=1$

$$F_X(x) = P(X \le x) = \frac{x}{1+x} \Rightarrow F_X(1) = \frac{1}{2}$$ - this gives us the probability

that a random variable among X would be ≤ 1, which is exactly what we defined p to be in a.) in the Binomial distribution.

$\Rightarrow$ The pmf of $Y_4(1)$ is $P_{Y_4(1)} = \binom{4}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{4-k} = \frac{\binom{4}{k}}{16} =$

$= \frac{1}{16}\binom{4}{k}$, $k \in \{0,1,2,3,4\}$