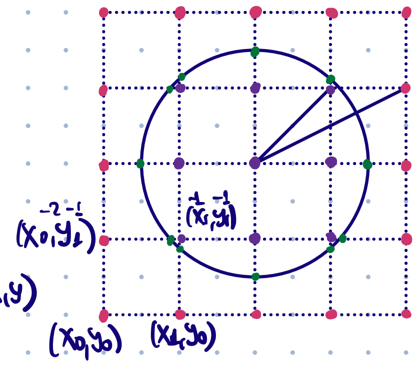
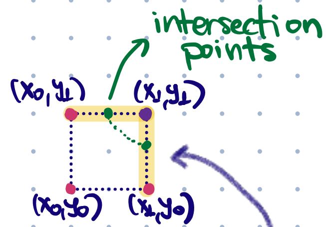
$x^2+y^2-r^2 \le 0$

$r=\sqrt{x^2+y^2}$



$x^2+y^2=r^2=1.5^2$



we do this for each square for the 4 vertice points $(x_0,y_0),(x_1,y_0),(x_0,y_1),(x_1,y_1)$

```
def circle_function(x,y):
    return r - sqrt(x²+y²)
```

Distance from the center (0,0) to the point (x,y) is $\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$.

To determine the sign we have to compare with the radius r.

If r - distance $= r-\sqrt{x^2+y^2} \overset{?}{\ge} 0$ then the point (x,y) is inside and we have a + sign.

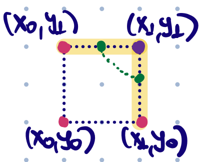
If r - distance $= r-\sqrt{x^2+y^2} < 0$ then the point (x,y) is outside and we have a - sign.

Now, we have to check whether there are sign changes in fore each edge? a square/cube, right?

If there are sign changes in an edge it means there is an intersection with the circle on it!

1.) We should find the intersection points. How can we do this?

Imagine we have this square region. We have collected the edges $((x_0,y_1),(x_1,y_1))$ and $((x_1,y_0),(x_1,y_1))$



In our simple case we can find the exact location. Otherwise, (linear) interpolation or different root-finding techniques could be used. This is zero-crossing problem w.r.t. the surface & edges(grid)

2.) Then we should find the normals to the (circle) at these intersection points.

How is a normal found?

The circle equation is given by $f(x,y)=x^2+y^2-r^2$. Let's find the gradient:

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=(2x,2y)$$

Now, we should normalise it by dividing with $\sqrt{(2x)^2+(2y)^2}=2\sqrt{x^2+y^2}$, so for the normals we get $\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)$. This would point outwards, which in our case is ok due to the quadratic component in the QEF. In general, the normal should point inwards, meaning it should be of the form $\left(\frac{-x}{\sqrt{x^2+y^2}},\frac{-y}{\sqrt{x^2+y^2}}\right)$

Finally, we reached the QEF step!

(1.) $E[x] = \sum_i (\underbrace{n_i}_{\substack{\text{unit normals} \\ n_i = (x_n, y_n)}} \cdot (x_i - \underbrace{p_i}_{\substack{\text{intersection} \\ \text{points}}}))^2$, where p_i, n_i correspond to the intersections (and unit normals) of the contour with the edges of the square.

(1.) can be expressed as the inner product $(Ax-b)^T(Ax-b)$, where

$$A = \begin{pmatrix} -n_0- \\ -n_i- \\ \vdots \end{pmatrix} = \begin{pmatrix} (x_{n_0}, y_{n_0}) \\ \vdots \\ (x_{n_i}, y_{n_i}) \\ \vdots \end{pmatrix} \text{ and } b = n_i . p_i \rightarrow \text{column? hmm}$$

(1.) Becomes $E[x] = x^TA^TAx - 2x^TA^Tb + b^Tb$, where $A^TA \in R^{3\times3}$ symmetric, A^Tb - column vector $\in R^3$, and b^Tb - scalar.

$\hat{x}$ - minimising value of $E[x]$ can be computed by solving $A^TA\hat{x} = A^T.b$ → numerically unstable though (but for our case should be ok)

constructed this Σ outer(n)

→ Improve this by using doubles instead of floats or just use QR decomposition.

$$A = \begin{pmatrix} x_0, y_0 \\ x_1, y_1 \\ x_2, y_2 \end{pmatrix} \Rightarrow A^T = ((x_0, y_0), (x_1, y_1), (x_2, y_2))$$

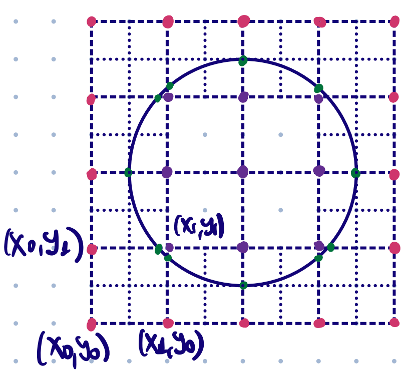
$$\Rightarrow A^TA = [(x_0, y_0), (x_1, y_1), (x_2, y_2)] \begin{pmatrix} (x_0, y_0) \\ (x_1, y_1) \\ (x_2, y_2) \end{pmatrix} = (x_0, y_0)^2 + (x_1, y_1)^2 + (x_2, y_2)^2$$

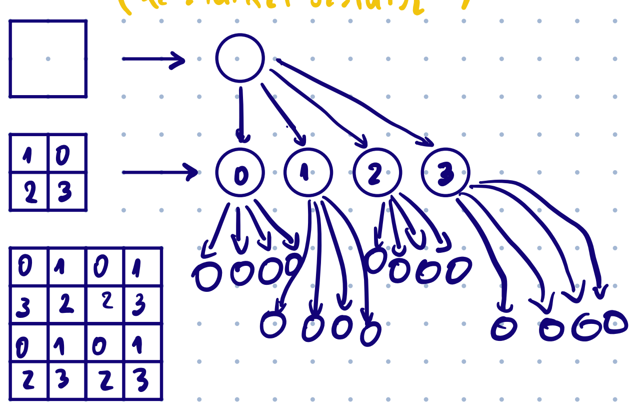
or $\sum$ np.outer($(x_i, y_i), (x_i, y_i)$) !!!

$b = n_i . p_i = $ (normal[0].intx + normal[1].int_y)

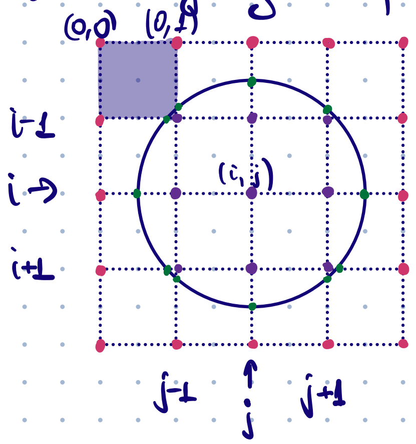
$$A^T = (\underbrace{(x_0, y_0), (x_1, y_1), (x_2, y_2)}_{\text{normals}}) \circ \begin{pmatrix} x_0 . intx_0 + y_0 . int\_y_0 \\ x_1 . intx_1 + y_1 . inty_1 \\ x_2 . intx_2 + y_2 . inty_2 \end{pmatrix}$$

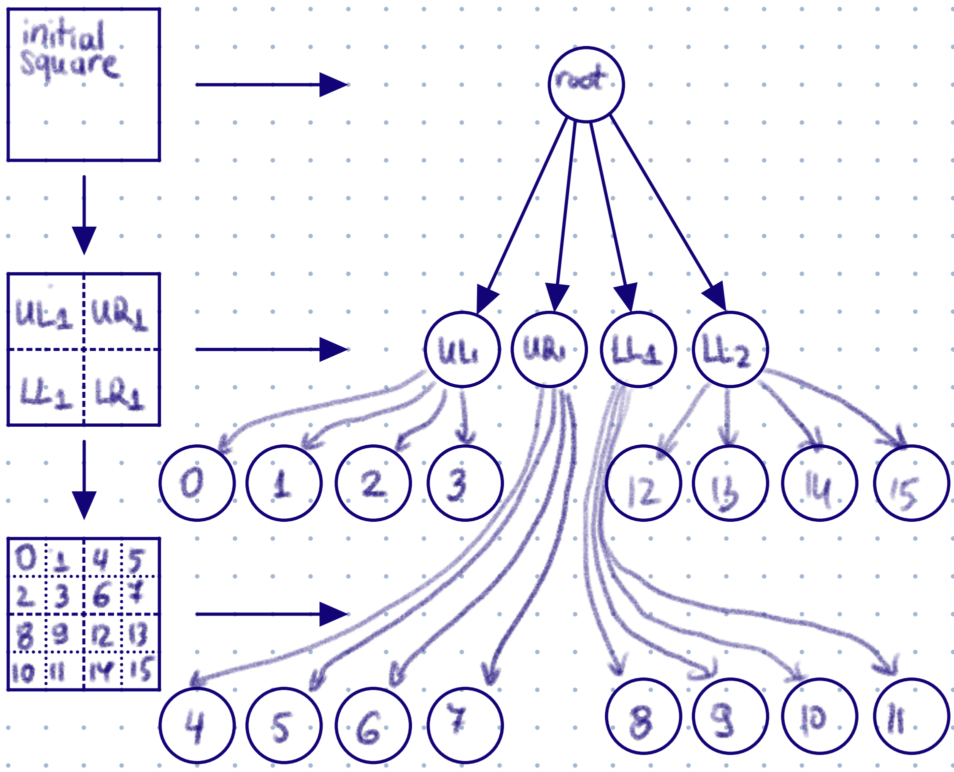
scalar





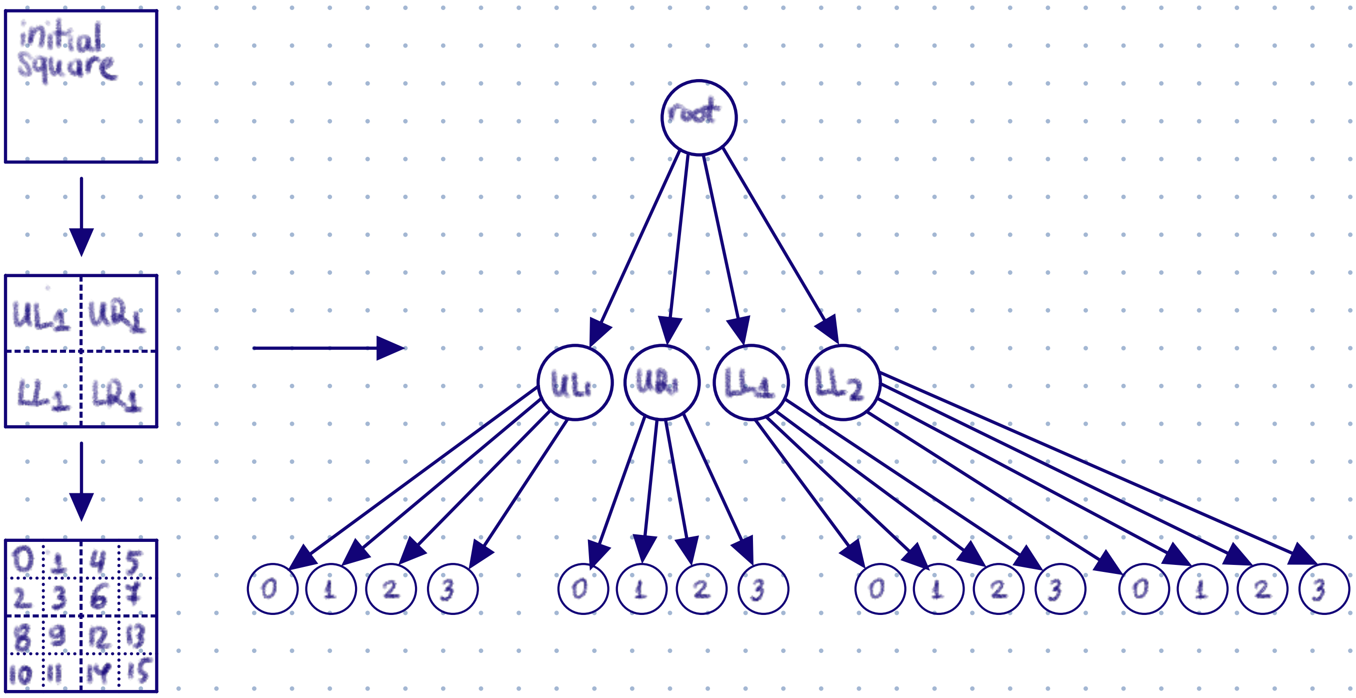
Just for illustration purposes we are going to start with a uniform grid. For the case of formality we are going to show the corresponding quadtree.
Later we are going to expand from uniform grid to an adaptive one with quadtree.

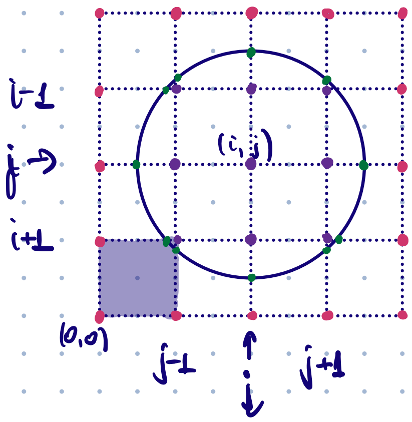




cell by cell approach:

1.) compute at each edge the function value and determine whether it's inside or outside the circle

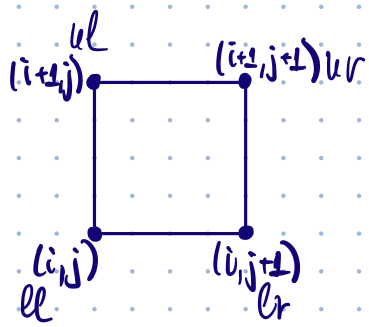




cell by cell approach:

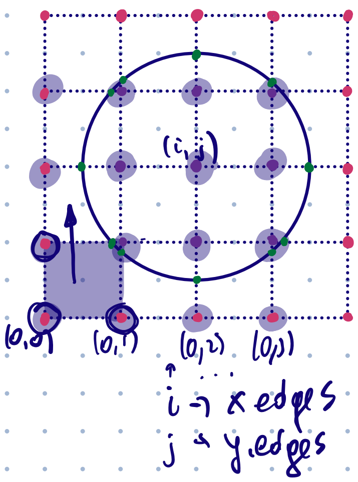
1.) compute at each edge the function value and determine whether it's inside or outside the circle.

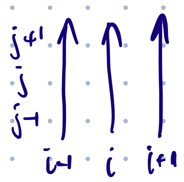
for j in range(yrange): (iterate horizontally)



r-dist>0

(-2,2) F





$\begin{cases} ll == lr & F==F \text{ no change} \\ ll == ul & F==F \text{ no change} \end{cases}$

$\begin{cases} ur == ul & T==F \text{ change} \\ ur == lr & T==F \text{ change} \end{cases}$