MATH 70095 - Applicable Maths
Autumn 2022 - Assessed Coursework 1

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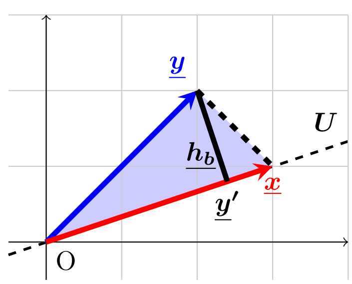
Problem Q1

From the problem statement we have that $\underline{x}, \underline{y} \in \mathbb{R}^2$ are Euclidean vectors, and we want to find the area of the triangle (the shaded area in the figure below) formed in the Euclidean plane by connecting the points represented by $\underline{x}, \underline{y}$, and the origin (labelled as O in the below figure).



The triangle is in the Euclidean plane, therefore we are equipped with the Euclidean norm which gives the length of an Euclidean vector. Or for $\underline{x} \in \mathbb{R}^n$ we have

$$||\underline{x}|| = \sqrt{x_1^2 + ... + x_n^2} = \sqrt{\underline{x}^T \underline{x}}.$$

We can give the area (A) of a triangle by $A = \frac{b.h_b}{2}$ where b is the base of the triangle and h_b is the height to this base. In our case b is equal to the length of the base of the triangle which is given by $\underline{x}$ (problem statement), or $b = ||\underline{x}||$.

Now, the aim is to find the height h_b. The height starts from the point represented by $\underline{y}$ and is orthogonal to the base $\underline{x}$. Let U be the line passing through the origin O and the point represented by $\underline{x}$. We can interpret our current aim as first finding the projection of $\underline{y}$ onto U which is a one-dimensional subspace of $\mathbb{R}^2$ and its basis is $\underline{x}$, finding an expression for $\underline{y}'$ in terms of the vectors $\underline{x}$

and $\underline{y}$. Then, denoting the vector corresponding to the height h_b, connecting the datapoint $\underline{y}$ with $\underline{y}' \in U$, as $\underline{h_b} = \underline{y}' - \underline{y}$. Finally, using Pythagorean theorem in the triangle formed by the origin O, the point represented by $\underline{y}$, and the point represented by $\underline{y}'$, in order to find the length of the height:

$$||\underline{y}||^2 = ||\underline{h_b}||^2 + ||\underline{y}'||^2$$

Since $\underline{y}' \in U$ it can be written $\underline{y}' = \xi \underline{x}$ for some $\xi \in \mathbb{R}$. We can use this along with the definition of orthogonality ($\underline{x} \perp \underline{h_b}$) to deduce:

$$\langle \underline{x}, \underline{h_b} \rangle = \langle \underline{x}, \underline{y}' - \underline{y} \rangle = \underline{x}^T(\underline{y}' - \underline{y}) = \xi \underline{x}^T \underline{x} - \underline{x}^T \underline{y} = 0$$

Therefore

$$\xi = \frac{\underline{x}^T \underline{y}}{\underline{x}^T \underline{x}} = \frac{\underline{x}^T \underline{y}}{||\underline{x}||^2}, \underline{y}' = \xi \underline{x} = \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \underline{y}$$

For the squared norm of $\underline{y}'$ we obatin:

$$||\underline{y}'||^2 = \underline{y}'^T \underline{y}' = \left(\frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \underline{y} \right)^T \cdot \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \underline{y} = \underline{y}^T \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \cdot \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \underline{y}$$

A note on the above calculation:

$$(xx^T y)^T = (x(x^T y))^T = (x^T y)^T x^T = y^T x x^T$$

Calculating the length of the height using Pythagorean theorem:

$$||\underline{h_b}||^2 = ||\underline{y}||^2 - ||\underline{y}'||^2 = \underline{y}^T \underline{y} - \underline{y}^T \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \cdot \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \underline{y}$$

Now, let's first find the squared area of the shaded triangle for the sake of simplified notation:

$$A^2 = \frac{1}{4} ||\underline{x}||^2 ||\underline{h_b}||^2 = \frac{1}{4} ||\underline{x}||^2 \left(\underline{y}^T \underline{y} - \underline{y}^T \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \cdot \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \underline{y} \right) =$$

$$= \frac{1}{4} \left(\left(\underline{y}^T \underline{y}\right) ||\underline{x}||^2 - \underline{y}^T \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \cdot \frac{\underline{x}\underline{x}^T}{||\underline{x}||^2} \underline{y} ||\underline{x}||^2 \right) =$$

$$= \frac{1}{4} \left(\left(\underline{y}^T \underline{y}\right) \left(\underline{x}^T \underline{x}\right) - \underline{y}^T \frac{\underline{x} \left(\underline{x}^T \underline{x}\right) \underline{x}^T}{\left(\underline{x}^T \underline{x}\right)} \underline{y} \right) = \frac{1}{4} \left(\left(\underline{y}^T \underline{y}\right) \left(\underline{x}^T \underline{x}\right) - \underline{y}^T \underline{x}\underline{x}^T \underline{y} \right) =$$

$$= \frac{1}{4} \left(\left(\underline{y}^T \underline{y}\right) \left(\underline{x}^T \underline{x}\right) - \left(\underline{x}^T \underline{y}\right) \left(\underline{x}^T \underline{y}\right) \right) = \frac{1}{4} \left(\left(\underline{y}^T \underline{y}\right) \left(\underline{x}^T \underline{x}\right) - \left(\underline{x}^T \underline{y}\right)^2 \right)$$

A note on the above calculation:

$$\left(\underline{y}^T \underline{x}\right) \in \mathbb{R} \Longrightarrow \left(\underline{y}^T \underline{x}\right) = \left(\underline{y}^T \underline{x}\right)^T = \left(\underline{x}^T \underline{y}\right)$$

Or for A we get

$$A = \frac{1}{2}\sqrt{\left(\underline{y}^T\underline{y}\right)\left(\underline{x}^T\underline{x}\right) - \left(\underline{x}^T\underline{y}\right)^2}$$

which is what we wanted.

Problem Q2

$$M = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & \pi \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{\pi}{2} \end{bmatrix}$$

Part a) + Part b)

Let's compute the characteristic polynomial (using the rule of Sarrus for the determinant):

$$|M-\lambda I_3| = \begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\lambda & 0 \\ 0 & 0 & \frac{\pi}{2}-\lambda \end{bmatrix} = \left(\frac{1}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right)\left(\frac{\pi}{2}-\lambda\right) - \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\left(\frac{\pi}{2}-\lambda\right) =$$

$$= \left(\frac{\pi}{2}-\lambda\right)(\lambda-1)(\lambda+1)$$

The eigenvalaues are $\lambda_1 = \frac{\pi}{2}$, $\lambda_2 = 1$, and $\lambda_3 = -1$. They all appear only once as roots of the characteristic polynomial, so all of them have algebraic multiplicity equal to 1, or $\mu_M(\lambda_i) = 1$, $i = 1, 2, 3$.

Let's compute the eigenvectors corresponding to each eigenvalue by solving $(M - \lambda_i I_3)\,\underline{x} = 0, i = 1, 2, 3$:

$\underline{\lambda_1 = \frac{\pi}{2}}$:

$$\begin{bmatrix} \frac{1}{2}-\frac{\pi}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\frac{\pi}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{x} = 0 \Longleftrightarrow \begin{bmatrix} 1-\pi & \sqrt{3} & 0 \\ \sqrt{3} & -1-\pi & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{x} = 0$$

We obtain:

$$\begin{cases} (1-\pi)x_1 + \sqrt{3}x_2 = 0 \\ \sqrt{3}x_1 - (1+\pi)x_2 = 0 \\ 0x_3 = 0 \end{cases}$$

$$\begin{cases} (1-\pi)\sqrt{3}x_1 + 3x_2 = 0 \\ (1-\pi)\sqrt{3}x_1 - (1-\pi)(1+\pi)x_2 = 0 \\ 0x_3 = 0 \end{cases}$$

$$\begin{cases}(4-\pi^2)x_2 = 0\\ (1-\pi)\sqrt{3}x_1 - (1-\pi)(1+\pi)x_2 = 0\\ 0x_3 = 0\end{cases}$$

The solutions are $x_1 = 0, x_2 = 0, x_3 = p$, for every $p \in \mathbb{R}$. The eigenvector should be nonzero so lets chose $x_3 = 1$, and the eigenvector corresponding to this eigenvalue is $\underline{v_1} = (0,0,1)^T$.

$\underline{\lambda_2 = 1}$:

$$\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ \frac{\sqrt{3}}{2} & -\frac{3}{2} & 0\\ 0 & 0 & \frac{\pi}{2}-1\end{bmatrix}\underline{x} = 0 \Longleftrightarrow \begin{bmatrix}-1 & \sqrt{3} & 0\\ \sqrt{3} & -3 & 0\\ 0 & 0 & \pi-2\end{bmatrix}\underline{x} = 0$$

Solving the system by Gaussian elimination:

$$\left[\begin{array}{ccc|c}-1 & \sqrt{3} & 0 & 0\\ \sqrt{3} & -3 & 0 & 0\\ 0 & 0 & \pi-2 & 0\end{array}\right] \xrightarrow{\text{row 2 } + \sqrt{3} \text{ row 1}} \left[\begin{array}{ccc|c}-1 & \sqrt{3} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \pi-2 & 0\end{array}\right]$$

We obtain

$$\begin{cases}x_1 = \sqrt{3}x_2\\ x_3 = 0\end{cases}$$

So the corresponding nonzero eigenvector is $\underline{v_2} = (\sqrt{3},1,0)^T$

$\underline{\lambda_3 = -1}$:

$$\begin{bmatrix}\frac{3}{2} & \frac{\sqrt{3}}{2} & 0\\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ 0 & 0 & \frac{\pi}{2}+1\end{bmatrix}\underline{x} = 0 \Longleftrightarrow \begin{bmatrix}3 & \sqrt{3} & 0\\ \sqrt{3} & 1 & 0\\ 0 & 0 & \pi+2\end{bmatrix}\underline{x} = 0$$

Solving the system by Gaussian elimination:

$$\left[\begin{array}{ccc|c}3 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 1 & 0 & 0\\ 0 & 0 & \pi+2 & 0\end{array}\right] \xrightarrow{\text{row 1 - } \sqrt{3} \text{ row 2}} \left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ \sqrt{3} & 1 & 0 & 0\\ 0 & 0 & \pi+2 & 0\end{array}\right]$$

We obtain

$$\begin{cases}\sqrt{3}x_1 = -x_2\\ x_3 = 0\end{cases}$$

So the corresponding nonzero eigenvector is $\underline{v_3} = (-\frac{1}{\sqrt{3}},1,0)^T$

Thus all eigenvalues have a geometric multiplicity equal to 1, or $\gamma_M(\lambda_i) = 1$, $i = 1,2,3$.

We obtain $\gamma_M(\lambda_i) = \mu_M(\lambda_i)$, $i = 1, 2, 3$. Therefore, $M \in \mathbb{R}^{3\times3}$ will admit 3 linearly independent eigenvectors, and these eigenvectors form an eigenbasis. That means we can diagonalise M. The diagonalisation of M is given by $M = P\Lambda P^{-1}$, where:

$$P = \begin{bmatrix}\underline{v_1}, & \underline{v_2}, & \underline{v_3}\end{bmatrix} = \begin{bmatrix}0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 1 \\ 1 & 0 & 0\end{bmatrix},$$

$$\Lambda = \begin{bmatrix}\lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3\end{bmatrix} = \begin{bmatrix}\frac{\pi}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{bmatrix}$$

For P^{-1} we have (P is constructed using the eigenbasis, hence it is full rank and invertible):

$$P^{-1} = \begin{bmatrix}0 & 0 & 1 \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0\end{bmatrix}$$

Part c)

We showed that $M = P\Lambda P^{-1}$ so let's try to make a geometric interpretation of the linear transformation(s) represented by M by using the matrices from this eigendecomposition.

Let's note that the eigenbasis we obtained is orthogonal. So, the matrix P^{-1} changes the standard basis to an orthogonal one. (Of course, because the eigenvectors are defined up to a multiplicative constant we can scale the eigenvectors and even obtain an orthonormal basis). The geometric interpretation of that is transforming the coordinate system but at the same time preserving the orthogonal properties.

The matrix Λ is just scaling the axes in the new coordinate system (within Euclidean space) by values equal to the eigenvalues it contains in its diagonal.

And P is reverting the image back to the original basis.

Simply said, the linear transformation represented by M is scaling in mutually orthogonal directions.

Problem Q3

Ann and Ben are taking turns until one of them wins, so there are two possible scenarios of the players tunrs given who is baking in the first round.

Ann starts first:

$$Ann, Ben, Ann, Ben, ..., \text{someone wins}$$

Ben starts first:

$$Ben, Ann, Ben, Ann, ..., \text{someone wins}$$

The event of ending the competition after the first round means that the person baking in this first round got a score of at least 9 and won the competition. Therefore, there are two scenarios depending on the baker in the first round:

$\longrightarrow$ Ann starts first

$$Ann$$

which is equivalent to Ann wins.

$\longrightarrow$ Ben starts first

$$Ben$$

which is equivalent to Ben wins.

Let's translate this to the language of conditional probabilities. We are interested in the event of Ann winning which we'll denote as

$$E = \{\text{Ann wins}\}$$

and the two possible conditional scenarios:

Scenario 1:

$$F_1 = \{\text{The competition ends after the first round when Ann starts}\}$$

$$P(E|F_1) = \frac{P(E \cap F_1)}{P(F_1)} = \frac{\alpha}{\alpha} = 1$$

The probability of the competition ending after the first round when Ann starts is the same as the probability that Ann receives a score of 9 or above, and this is given by the problem statement and is equal to α, or $P(F_1) = \alpha$.

The probability that Ann wins if the competition ends after the first round when Ann is baking is the same as the probability that Ann receives a score of 9 or above, or $P(E \cap F_1) = \alpha$.

Scenario 2:

$$F_2 = \{\text{The competition ends after the first round when Ben starts}\}$$

$$P(E|F_2) = \frac{P(E \cap F_2)}{P(F_2)} = \frac{0}{\beta} = 0$$

The probability of the competition ending after the first round when Ben starts is the same as the probability that Ben receives a score of 9 or above, and this is given by the problem statement and it is equal to β, or $P(F_2) = \beta$.

The probability that Ann wins if the competition ends after the first round when Ben starts is 0, because in the general case if Ann starts second she could win only on even turns and winning on an odd turn doesn't exist as a possibility.