MATH 70095 - Applicable Maths
Autumn 2022 - Unassessed Coursework

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October 17, 2022

Problem Q1

Part a)

Let's apply the quotient rule:

$$f'(x) = \frac{x'(x+e^{-x}) - x(x+e^{-x})'}{(x+e^{-x})^2} = \frac{(x+e^{-x}) - x(1-e^{-x})}{(x+e^{-x})^2}$$

Here comes the first difference, it is in the signs of the two terms in the numerator. The quotient rule wasn't applied correctly, the subtraction terms were swapped. Finally, the correct result is:

$$f'(x) = \frac{e^{-x}(x+1)}{(x+e^{-x})^2}$$

The difference in the sign comes from the previous mistake.

Part b)

Let's begin with integration by parts:

$$\int_0^{\frac{\pi}{4}} x^2 \sin 2x \, dx = \int_0^{\frac{\pi}{4}} x^2 (-\frac{1}{2}) \, d\cos 2x =$$

$$= \left[x^2(-\frac{1}{2}\cos 2x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\frac{1}{2}\cos 2x \, dx^2 = \left[x^2(-\frac{1}{2}\cos 2x)\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} x\cos 2x \, dx$$

The first difference is caused by a mistake in the integration by parts method. The derivative of x^2 was calculated when it shouldn't have been. Only the derivative of $sin 2x$ should be calculated.

Now, let's apply the integration by parts method to the newly derived integral:

$$\int_0^{\frac{\pi}{4}} x^2 \sin 2x \, dx = \left[x^2(-\frac{1}{2}\cos 2x)\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} x(\frac{1}{2}) \, d\sin 2x =$$
$$= \left[-\frac{1}{2}x^2 \cos 2x\right]_0^{\frac{\pi}{4}} + \left[x(\frac{1}{2}\sin 2x)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

The difference in the first term comes from the previous mistake. The difference in the third term is from wrongly applying the integration by parts method by mistaking the sign in front of the integral.

Let's calculate the final answer:

$$\int_0^{\frac{\pi}{4}} x^2 \sin 2x \, dx = \left[-\frac{1}{2}x^2 \cos 2x\right]_0^{\frac{\pi}{4}} + \left[x(\frac{1}{2}\sin 2x)\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{4}} =$$
$$= 0 - 0 + \frac{\pi}{8} - 0 + 0 - \frac{1}{4} = \frac{\pi}{8} - \frac{1}{4}$$

The difference in the last sign comes from the previous mistake in the third term.

Problem Q2

Part a)

We can notice that $f(x, y, z) = \log g(x, y) + \log h(y, z)$ is a sum of two logarithms applied to functions with two arguments. The only differences between the two terms are the function names and arguments. Having that in mind, we can ease the work of calculating the partial derivatives of f by alternating the function names and arguments of the partial derivatives of a function of the following type: $\log F(x_1, x_2)$.

The gradient is given by

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}.$$

Let's calculate the partial derivatives. The derivative is a linear operator so we can calculate the derivative of each term separately by also applying the chain rule to the terms:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\log g(x, y) + \frac{\partial}{\partial x}\log h(y, z) = \frac{\partial}{\partial x}\log g(x, y) = \frac{1}{g(x, y)}\frac{\partial g(x, y)}{\partial x}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} \log g(x,y) + \frac{\partial}{\partial z} \log h(y,z) = \frac{\partial}{\partial z} \log h(y,z) = \frac{1}{h(y,z)} \frac{\partial h(y,z)}{\partial z}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \log g(x,y) + \frac{\partial}{\partial y} \log h(y,z) = \frac{1}{g(x,y)} \frac{\partial g(x,y)}{\partial y} + \frac{1}{h(y,z)} \frac{\partial h(y,z)}{\partial y}$$

We have:

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{1}{g(x,y)} \frac{\partial g(x,y)}{\partial x} \\ \frac{1}{g(x,y)} \frac{\partial g(x,y)}{\partial y} + \frac{1}{h(y,z)} \frac{\partial h(y,z)}{\partial y} \\ \frac{1}{h(y,z)} \frac{\partial h(y,z)}{\partial z} \end{bmatrix}$$

Part b)

The Hessian is given by

$$H_f = \nabla^2 f(x,y,z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}.$$

Calculating the derivatives using the results from $Part a)$ and by applying the product rule, chain rule and making use of the linearity of the derivative operator.

We already have:

$$\frac{\partial f}{\partial x} = \frac{1}{g(x,y)} \frac{\partial g(x,y)}{\partial x}$$

Therefore:

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial}{\partial x} \frac{1}{g(x,y)} \right) \frac{\partial g(x,y)}{\partial x} + \frac{1}{g(x,y)} \frac{\partial^2 g(x,y)}{\partial x^2} =$$

$$= -\frac{1}{g^2(x,y)} \left(\frac{\partial g(x,y)}{\partial x} \right)^2 + \frac{1}{g(x,y)} \frac{\partial^2 g(x,y)}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \left(\frac{\partial}{\partial y} \frac{1}{g(x,y)} \right) \frac{\partial g(x,y)}{\partial x} + \frac{1}{g(x,y)} \frac{\partial^2 g(x,y)}{\partial x \partial y} =$$

$$= -\frac{1}{g^2(x,y)} \frac{\partial g(x,y)}{\partial y} \frac{\partial g(x,y)}{\partial x} + \frac{1}{g(x,y)} \frac{\partial^2 g(x,y)}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial x \partial z} = 0$$

We already have calculated:

$$\frac{\partial f}{\partial z} = \frac{1}{h(y,z)} \frac{\partial h(y,z)}{\partial z}$$

Using the above results we easily get:

$$\frac{\partial^2 f}{\partial z \partial x} = 0$$

$$\frac{\partial^2 f}{\partial z \partial y} = -\frac{1}{h^2(y,z)} \frac{\partial h(y,z)}{\partial y} \frac{\partial h(y,z)}{\partial z} + \frac{1}{h(y,z)} \frac{\partial^2 h(y,z)}{\partial z \partial y}$$

$$\frac{\partial^2 f}{\partial z^2} = -\frac{1}{h^2(y,z)} \left(\frac{\partial h(y,z)}{\partial z} \right)^2 + \frac{1}{h(y,z)} \frac{\partial^2 h(y,z)}{\partial z^2}$$

We already have calculated:

$$\frac{\partial f}{\partial y} = \frac{1}{g(x,y)} \frac{\partial g(x,y)}{\partial y} + \frac{1}{h(y,z)} \frac{\partial h(y,z)}{\partial y}$$

Using the above results we easily get:

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{1}{g^2(x,y)} \frac{\partial g(x,y)}{\partial x} \frac{\partial g(x,y)}{\partial y} + \frac{1}{g(x,y)} \frac{\partial^2 g(x,y)}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial y \partial z} = -\frac{1}{h^2(y,z)} \frac{\partial h(y,z)}{\partial z} \frac{\partial h(y,z)}{\partial y} + \frac{1}{h(y,z)} \frac{\partial^2 h(y,z)}{\partial y \partial z}$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{g^2(x,y)} \left(\frac{\partial g(x,y)}{\partial y} \right)^2 + \frac{1}{g(x,y)} \frac{\partial^2 g(x,y)}{\partial y^2} - \frac{1}{h^2(y,z)} \left(\frac{\partial h(y,z)}{\partial y} \right)^2 + \frac{1}{h(y,z)} \frac{\partial^2 h(y,z)}{\partial y^2}$$

Problem Q3

Part a)

First let's prove the vectors are linearly independent. This is equivalent to showing that the following system has only a trivial solution $x = y = z = 0$:

x + 2y = 0
y + 2z = 0
2x + y + z = 0
x + 3z = 0

Solving the system by Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right] \xrightarrow{\text{row 3 - 2*row 1}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right] \xrightarrow{\text{row 3 + 3*row 2}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 7 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{\text{row 3 / 7}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right] \xrightarrow{\text{row 4 - row 1}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right] \xrightarrow{\text{row 4 + 2*row 2}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

$$\xrightarrow{\text{row 4 - 7*row 3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{row 2 - 2*row 3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{row 1 - 2*row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution of the system is x = y = z = 0 and it is unique. So, the system has only a trivial solution. Therefore, the vectors are linearly independent. We also know (problem statement) that they span U. We can conclude that these vectors form a basis of the vector space U (U is a subspace of the vector space R^4 hence it's a vector space).

We can construct the matrix B (containing the basis vectors in its columns) with which we are going to calculate the matrix $P = B(B^T B)^{-1} B^T$.

$$B = \begin{bmatrix} \underline{b_1} & \underline{b_2} & \underline{b_3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$$B^T B = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 5 \\ 4 & 6 & 3 \\ 5 & 3 & 14 \end{bmatrix}$$

For calculating the inverse matrix we'll use the formula

$$A^{-1} = \frac{1}{|A|} C^T$$

where C is the matrix of cofactors.

First we should calculate the determinant (using the Rule of Sarrus):

$$|B^T B| = 6 * 6 * 14 + 4 * 3 * 5 * 2 - 5 * 6 * 5 - 6 * 3 * 3 - 4 * 4 * 14 = 196$$

Now, let's construct C (which is a symmetric matrix because $B^T B$ is symmetric):

$$C = C^T = \begin{bmatrix} 75 & -41 & -18 \\ -41 & 59 & 2 \\ -18 & 2 & 20 \end{bmatrix}$$

We get

$$(B^T B)^{-1} = \frac{1}{196}\begin{bmatrix} 75 & -41 & -18 \\ -41 & 59 & 2 \\ -18 & 2 & 20 \end{bmatrix} = \begin{bmatrix} \frac{75}{196} & -\frac{41}{196} & -\frac{18}{196} \\ -\frac{41}{196} & \frac{59}{196} & \frac{2}{196} \\ -\frac{18}{196} & \frac{2}{196} & \frac{20}{196} \end{bmatrix}$$

For P we have:

$$P = B(B^T B)^{-1} B^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{75}{196} & -\frac{41}{196} & -\frac{18}{196} \\ -\frac{41}{196} & \frac{59}{196} & \frac{2}{196} \\ -\frac{18}{196} & \frac{2}{196} & \frac{20}{196} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} =$$

$$= \frac{1}{28}\begin{bmatrix} -1 & 11 & -2 \\ -11 & 9 & 6 \\ 13 & -3 & -2 \\ 3 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}$$

Finally, let's calculate the orthogonal projection of $\underline{v} = [1, 2, 1, 3]^T$:

$$P\underline{v} = \frac{1}{4}\begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}, & \frac{9}{4}, & \frac{5}{4}, & \frac{11}{4} \end{bmatrix}^T$$

Part b)

Let's construct the matrix such that the given euclidean vectors are its columns:

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix}$$

We are going to calculate the Gram matrix G using the Euclidean product (because we are in the 4-dimensional Euclidean vector space equipped with the usual Euclidean inner product):

$$
\begin{aligned}
g_{11} &= 1^2 + 0^2 + 2^2 + 1^2 = 6 \\
g_{12} = g_{21} &= 1 * 2 + 0 * 1 + 2 * 1 + 1 * 0 = 4 \\
g_{13} = g_{31} &= 1 * 0 + 0 * 2 + 2 * 1 + 1 * 3 = 5 \\
g_{14} = g_{41} &= 1 * 2 + 0 * 1 + 2 * 1 + 1 * 2 = 6 \\
\\
g_{22} &= 2^2 + 1^2 + 1^2 + 0^2 = 6 \\
g_{23} = g_{32} &= 2 * 0 + 1 * 2 + 1 * 1 + 0 * 3 = 3 \\
g_{24} = g_{42} &= 2 * 2 + 1 * 1 + 1 * 1 + 0 * 2 = 6 \\
\\
g_{33} &= 0^2 + 2^2 + 1^2 + 3^2 = 14 \\
g_{34} = g_{43} &= 0 * 2 + 2 * 1 + 1 * 1 + 3 * 2 = 9 \\
\\
g_{44} &= 2^2 + 1^2 + 1^2 + 2^2 = 10
\end{aligned}
$$

We get:

$$
G = \begin{bmatrix} 6 & 4 & 5 & 6 \\ 4 & 6 & 3 & 6 \\ 5 & 3 & 14 & 9 \\ 6 & 6 & 9 & 10 \end{bmatrix}
$$

The columns vectors of U are linearly independent if and only if the determinant of the corresponding Gram matrix G is nonzero. Indeed, $|G| = 196$. Therefore, the given vectors are linearly independent.