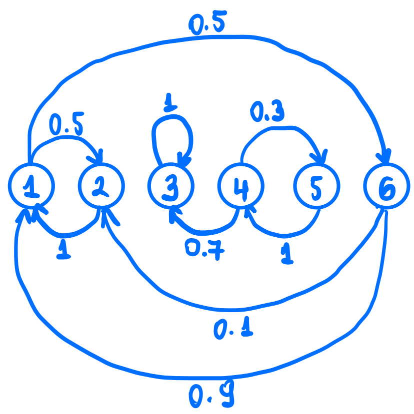
Q1) a.) Transition diagram:



The state space is $S=\{1,2,3,4,5,6\}$

The transition matrix is $P=[p_{ij}]$, $i,j \in S$

Communicating classes:

- $\{1,2,6\}$: $1 \leftrightarrow 2$, $1 \leftrightarrow 6 \Rightarrow 2 \leftrightarrow 6$, and they don't inter-communicate with any other states
- $\{3\}$: $3 \leftrightarrow 3$, only communicates with itself, so is a class itself
- $\{4,5\}$: $4 \leftrightarrow 5$, and they don't inter-communicate with any other states

b)

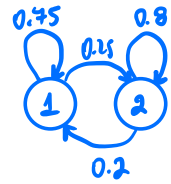
If two states $i,j \in S$ communicate with each other, then they will have the same period. In one closed communicating class all states communicate with each other $\Rightarrow$ it's enough to find the period of only one of the states in one communicating class in order to find the period of the communicating class. We have that $d_i = \gcd\{n : p_{ii}^{(n)} > 0\}$

- $\{1,2,6\}$: $d_1 = \gcd\{n : p_{11}^{(n)} > 0\} = \gcd\{2,2,3\} = 1 = d_2 = d_6 \Rightarrow$ these states are aperiodic and hence the communicating class is aperiodic
- $\{3\}$: $d_3 = \gcd\{n : p_{33}^{(n)} > 0\} = \gcd\{1\} = 1 \Rightarrow \{3\}$ is aperiodic
- $\{4,5\}$: $d_4 = \gcd\{n : p_{44}^{(n)} > 0\} = \gcd\{2\} = 2 = d_5 \Rightarrow \{4,5\}$ has period 2

c) A (Th.4.2.5) Markov chain defined on the discrete state space S has a unique stationary distribution if and only if it is both irreducible and all states are positive recurrent. $C \subseteq S$ is irreducible if all states in C communicate with each other ($i \leftrightarrow j$, $\forall i,j \in C$) and if the entire state space S is irreducible for a given Markov chain, then the chain is irreducible. But our state space S is not irreducible, it consists of three closed communicating classes $\{1,2,6\}, \{3\}, \{4,5\} \overset{\text{th.}}{\Rightarrow}$ the Markov chain doesn't have a unique stationary distribution.

Q2.) **a.)** Let's define our state space to be $S=\{1,2\}$, where 1 corresponds to a sunny day, and 2 corresponds to a sunny day.
Let's denote the transition matrix of the Markov chain: $P=[p_{ij}]=\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$

By problem statement:



transition diagram

1.) $p_{22}=0.8 \Rightarrow p_{21}=1-p_{22}=0.2$

2.) $p_{12}=0.25 \Rightarrow p_{11}=1-p_{12}=0.75$

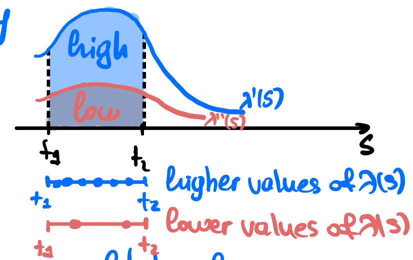
$\Rightarrow$ the transition matrix for this Markov chain is $P=\begin{pmatrix} 0.75 & 0.25 \\ 0.2 & 0.8 \end{pmatrix}$

b.) We can notice that the Markov chain is irreducible (the whole state space is irreducible). Moreover, the state space is finite $\overset{\text{remark}}{\Rightarrow}$ all states in a finite, irreducible state space are positive recurrent $\Rightarrow$ Th.4.2.5 holds and the Markov chain has a unique stationary distribution $\underline{\Pi}$ which is given by the reciprocal of the mean recurrence times for each state: $\pi_i=\mu_i^{-1}$, $\forall i \in S$.
We can approach our problem by finding the unique stationary distribution $\underline{\Pi}$ and from there finding $\mu_1=\pi_1^{-1}$.
In order to find $\underline{\Pi}$ we should solve the system:

$$\begin{cases} \underline{\Pi}=\underline{\Pi}P \\ \pi_1+\pi_2=1 \end{cases} \Rightarrow \begin{cases} \pi_1=\pi_1\,0.75+\pi_2\,0.2 \\ \pi_2=\pi_1\,0.25+\pi_2\,0.8 \\ \pi_1+\pi_2=1 \end{cases} \Rightarrow \pi_1=\frac{4}{9},\ \pi_2=\frac{5}{9},\ \underline{\Pi}=\left(\frac{4}{9},\frac{5}{9}\right)$$

$\Rightarrow \mu_1=\pi_1^{-1}=\frac{9}{4}=2.25$ or if the sun is shining today we have to wait (on average) 2.25 days before another sunny day.

Q3.) a.) We can notice that the expected number of events that occur in the interval $(t_1, t_2]$ is equal to $\int_{t_1}^{t_2} \lambda(s)ds$ (from the Poisson distribution). In other words, the expected number of points falling in our region of interest $(t_1,t_2]$ is equal to the area under the intensity function $\lambda(s)$. For higher values of the intensity function, the area would be bigger ($\lambda(s): [0,\infty) \to [0,\infty)$) and hence the expected number of points in the region would be higher. Analogically, for lower values of the intensity function the expected number of points in the region would be lower.



b.) A Gaussian process has multivariate Gaussian finite-dimensional distributions. Choosing a Gaussian process to model the intensity process would mean that the intensity function $\lambda(t) = E[\Lambda(t)] = E[\overset{\sim N(\mu(t),\sigma(t))}{GP(t)}] = \mu(t) \in \mathbb{R}$ can take negative values which contradicts with the idea behind the intensity function giving the intensity values (average number of points per unit area, which implies intensity ≥ 0) at different points in time, which are non-negative. Choosing a log-Gaussian process ($\Lambda(t) = \exp GP(t)$) means exponentiating the GP which leads to modelling on the logarithmic scale and ensuring the intensity values are positive.

c.) We have the following covariance function: $c(\tau; \sigma, \alpha) = \sigma^2 \exp\{-\alpha\tau\}$, $\forall \tau \geq 0$, where
- σ^2 denotes the variance
- τ denotes the distance between two points t_i, t_j
- α denotes how close two points t_i, t_j should be to influence each other significantly (length-scale). In the stationary case (such as ours), the moment properties of the LGCP process are inherited from $\Lambda(x)$.
  - $m = E[GP(t)] \overset{\text{log-Normal}}{\Rightarrow} \lambda = \exp(\mu + \frac{\sigma^2}{2})$
  - $v^2 = Var[GP(t)] \overset{\text{log-Normal}}{=} [\exp(\sigma^2)-1]\exp(2\mu+\sigma^2) = \lambda^2[\exp(\sigma^2)-1]$
  - covariance function: $c(\tau;\sigma,\alpha) = \sigma^2 \cdot \exp\{-\alpha\tau\} = \sigma^2 \rho(GP(\tau))$

The covariance decreases when α increases (σ, α, τ can't be negative) for fixed σ, τ.
α increasing means two points may be further away from each other in order to

significantly influence one another. This would lead to realisation of the process with distinct areas with more points on average, but with lower density due to the further needed distance between them for significant influence.

The covariance increases when σ increases ($\exp(-\alpha r)$ is always >0).

d.) • LGCP with $\sigma = 10.0$, $\alpha = 0.1$

• LGCP with $\sigma = 0.1$, $\alpha = 10.0$ → low density clusters