Homework 1 - Electromagnetism

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I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1

Prove the identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Left hand-side:

$$
\begin{aligned}
\vec{B} \times \vec{C} &= (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \times (C_x\hat{x} + C_y\hat{y} + C_z\hat{z}) \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\
&= (B_yC_z - B_zC_y)\hat{x} + (B_zC_x - B_xC_z)\hat{y} + (B_xC_y - B_yC_x)\hat{z}
\end{aligned}
$$

$$
\begin{aligned}
\vec{A} \times (\vec{B} \times \vec{C}) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix} \\
&= [A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)]\hat{x} - \\
&\quad [A_x(B_xC_y - B_yC_x) - A_z(B_yC_z - B_zC_y)]\hat{y} + \\
&\quad [A_x(B_xC_z - B_zC_x) - A_y(B_yC_z - B_zC_y)]\hat{z}
\end{aligned}
$$

Right hand-side:

$$\vec{A} \cdot \vec{C} = A_xC_x + A_yC_y + A_zC_z$$

$$
\begin{aligned}
\vec{B}(\vec{A} \cdot \vec{C}) &= (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})\underbrace{(A_xC_x + A_yC_y + A_zC_z)}_{\text{scalar}} \\
&= (B_xA_xC_x + B_xA_yC_y + B_xA_zC_z)\hat{x} + \\
&= (B_yA_xC_x + B_yA_yC_y + B_yA_zC_z)\hat{y} + \\
&= (B_zA_xC_x + B_zA_yC_y + B_zA_zC_z)\hat{z}
\end{aligned}
$$

$$
\begin{aligned}
\vec{C}(\vec{A}\cdot\vec{B}) &= (C_x\hat{x} + C_y\hat{y} + C_z\hat{z})(A_xB_x + A_yB_y + A_zB_z) \\
&= (C_xA_xB_x + C_xA_yB_y + C_xA_zB_z)\hat{x} + \\
&\quad (C_yA_xB_x + C_yA_yB_y + C_yA_zB_z)\hat{y} + \\
&\quad (C_zA_xB_x + C_zA_yB_y + C_zA_zB_z)\hat{z}
\end{aligned}
$$

$$
\begin{aligned}
\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) &= \underbrace{(B_xA_yC_y + B_xA_zC_z - C_xA_yB_y - C_xA_zB_z)}_{\text{x-component}}\hat{x} + (\dots)\hat{y} + (\dots)\hat{z} \\
\vec{A}\times(\vec{B}\times\vec{C}) &= [A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)]\hat{x} + (\dots)\hat{y} + (\dots)\hat{z} \\
&= \underbrace{(A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z)}_{\text{x-component}}\hat{x} + (\dots)\hat{y} + (\dots)\hat{z}
\end{aligned}
$$

Problem 2

1. Prove $\nabla\cdot(\nabla\times\vec{A}) = 0$

$$
\begin{aligned}
\nabla\times\vec{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}
\end{aligned}
$$

$$
\begin{aligned}
&\nabla\cdot(\nabla\times\vec{A}) \\
&= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}\right] \\
&= \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_y}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial y\partial x} + \frac{\partial^2 A_x}{\partial y\partial z} + \frac{\partial^2 A_y}{\partial z\partial x} - \frac{\partial^2 A_x}{\partial z\partial y} = 0
\end{aligned}
$$

2. Prove $\nabla\times\nabla\psi = 0$

$$
\nabla\psi = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial y}\hat{y} + \frac{\partial\psi}{\partial z}\hat{z}
$$

$$
\begin{aligned}
\nabla\times\nabla\psi &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2\psi}{\partial y\partial z} - \frac{\partial^2}{\partial z\partial y}\right)\hat{x} - \left(\frac{\partial^2\psi}{\partial x\partial z} - \frac{\partial^2\psi}{\partial z\partial x}\right)\hat{y} + \left(\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x}\right)\hat{z} = 0
\end{aligned}
$$

Problem 3

1. $f(x,y,z) = x^2y^3 + z^4$

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \\
&= 2xy^3\hat{x} + 3x^2y^2\hat{y} + 4z^3\hat{z}
\end{aligned}
$$

2. $\vec{v} = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$

$$\begin{aligned}
\nabla \cdot \vec{v} &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot (y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}) \\
&= \frac{\partial y^2}{\partial x} + \frac{\partial(2xy + z^2)}{\partial y} + \frac{\partial 2yz}{\partial z} \\
&= 0 + 2x + 2y = 2(x + y)
\end{aligned}$$

3. $\vec{v} = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

$$\begin{aligned}
\nabla \times \vec{v} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \left(\frac{\partial(-2xz)}{\partial y} - \frac{\partial(3xz^2)}{\partial z}\right)\hat{x} - \left(\frac{\partial(-2xz)}{\partial x} - \frac{\partial x^2}{\partial z}\right)\hat{y} + \left(\frac{\partial 3xz^2}{\partial x} - \frac{\partial x^2}{\partial y}\right)\hat{z} \\
&= -6xz\hat{x} + 2x\hat{y} + 3z^2\hat{z}
\end{aligned}$$

Problem 4

1. Prove $\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)$

$$\begin{aligned}
\nabla \cdot (f\vec{A}) &= \nabla \cdot (fA_x\hat{x} + fA_y\hat{y} + fA_z\hat{z}) \\
&= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) \\
&= \frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x} + \frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y} + \frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z} \\
&= f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z} \\
&= f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)
\end{aligned}$$

2. Prove $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

$$
\begin{aligned}
\nabla \cdot (\vec{A} \times \vec{B}) &= \nabla \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= \nabla \cdot [(A_y B_z - A_z B_y)\hat{x} - (A_x B_z - A_z B_x)\hat{y} + (A_x B_y - A_y B_x)\hat{z}] \\
&= \frac{\partial}{\partial x}(A_y B_z - A_z B_y) - \frac{\partial}{\partial y}(A_x B_z - A_z B_x) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x) \\
&= \frac{\partial A_y}{\partial x} B_z + A_y \frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x} B_y - A_z \frac{\partial B_y}{\partial x} - \\
&\quad \frac{\partial A_x}{\partial y} B_z - A_x \frac{\partial B_z}{\partial y} + \frac{\partial A_z}{\partial y} B_x + A_z \frac{\partial B_x}{\partial y} + \\
&\quad \frac{\partial A_x}{\partial z} B_y + A_x \frac{\partial B_y}{\partial z} - \frac{\partial A_y}{\partial z} B_x - A_y \frac{\partial B_x}{\partial z} \\
&= \underbrace{B_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_y}{\partial x} \right) + B_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + B_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)}_{\vec{B} \cdot (\nabla \times \vec{A})} - \\
&\quad \underbrace{\left[A_x \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + A_y \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + A_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \right]}_{\vec{A} \cdot (\nabla \times \vec{B})}
\end{aligned}
$$

3. Prove $\nabla \times (\vec{A}f) = f(\nabla \times \vec{A}) - \vec{A} \times \nabla f$

$$
\begin{aligned}
\nabla \times (\vec{A}f) &= \nabla \times (A_x f\hat{x} + A_y f\hat{y} + A_z f\hat{z}) \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x f & A_y f & A_z f \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(A_z f) - \frac{\partial}{\partial z}(A_y f)\right]\hat{x} - \left[\frac{\partial}{\partial x}(A_z f) - \frac{\partial}{\partial z}(A_x f)\right]\hat{y} + \left[\frac{\partial}{\partial x}(A_y f) - \frac{\partial}{\partial y}(A_x f)\right]\hat{z} \\
&= \left[\frac{\partial A_z}{\partial y}f + A_z\frac{\partial f}{\partial y} - \frac{\partial A_y}{\partial z}f - A_y\frac{\partial f}{\partial z}\right]\hat{x} - \\
&\quad \left[\frac{\partial A_z}{\partial x}f + A_z\frac{\partial f}{\partial x} - \frac{\partial A_x}{\partial z}f - A_x\frac{\partial f}{\partial z}\right]\hat{y} + \\
&\quad \left[\frac{\partial A_y}{\partial x}f + A_y\frac{\partial f}{\partial x} - \frac{\partial A_x}{\partial y}f - A_x\frac{\partial f}{\partial y}\right]\hat{z} \\
&= f\hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{x}\left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial y}\right) + \\
&\quad - f\hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{y}\left(A_z\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial z}\right) + \\
&\quad f\hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \hat{z}\left(A_x\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial x}\right) \\
&= f\underbrace{\left[\hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]}_{\nabla\times\vec{A}} - \\
&\quad \underbrace{\left[\hat{x}\left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial y}\right) - \hat{y}\left(A_z\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial z}\right) + \hat{z}\left(A_x\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial x}\right)\right]}_{\vec{A}\times\nabla f} \\
&= f(\nabla \times \vec{A}) - \vec{A} \times \nabla f
\end{aligned}
$$