Homework 2 - Electromagnetism

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I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1

$$\vec{v} = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$$

- $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

$$(0,0,0) \to (1,0,0) = \int_{(0,0,0)}^{(1,0,0)} \vec{v} \cdot dl = \int_0^1 x^2 dx = \frac{1}{3}$$

$$(1,0,0) \to (1,1,0) = \int_{(1,0,0)}^{(1,1,0)} \vec{v} \cdot dl = \int_0^1 2yzdy = 0 \quad (z=0)$$

$$(1,1,0) \to (1,1,1) = \int_{(1,1,0)}^{(1,1,1)} \vec{v} \cdot dl = \int_0^1 y^2 dz = \int_0^1 dz = 1 \quad (y=1)$$

$$\text{Total} = \frac{1}{3} + 0 + 1 = \frac{4}{3}$$

- $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

$$(0,0,0) \to (0,0,1) = \int_{(0,0,0)}^{(0,0,1)} \vec{v} \cdot dl = \int_0^1 y^2 dz = 0 \quad (y=0)$$

$$(0,0,1) \to (0,1,1) = \int_{(0,0,1)}^{(0,1,1)} \vec{v} \cdot dl = \int_0^1 2yzdy = \int_0^1 2ydy = 1 \quad (z=1)$$

$$(0,1,1) \to (1,1,1) = \int_{(0,1,1)}^{(1,1,1)} \vec{v} \cdot dl = \int_0^1 x^2 dx = \frac{1}{3}$$

$$\text{Total} = 0 + 1 + \frac{1}{3} = \frac{4}{3}$$

- straight line from $(0,0,0)$ to $(1,1,1)$ Since it is a straight line, the function $\vec{v}$ should be evaluated along a specific path parameterization $l(t) = (t,t,t)$, this is because all the variables are dependent on each other.

$$x = t, y = t, z = t, 0 \leq t \leq 1$$

$$dx = dt, dy = dt, dz = dt$$

$$\int_{(0,0,0)}^{(1,1,1)} \vec{v}(l(t)) \cdot dl = \int_0^1 t^2 \cdot 1dt + \int_0^1 2t^2 \cdot 1dt \int_0^1 t^2 \cdot 1dt$$
$$= \int_0^1 t^2 + 2t^2 + t^2 dt = \int_0^1 4t^2 dt = \frac{4}{3}$$

Problem 2

$$T = z^2$$
$$\int_v T dt$$

At first, in the (x, y) coordinate, we have the line go from $(1, 0, 0) \to (0, 1, 0)$. The line can be written as $y = 1 - x$. Therefore the bounds of the integral are:

$$0 \leq x \leq 1, 0 \leq y \leq x - 1 \quad \text{(as } x \text{ goes from } 0 \to 1,\ y \text{ goes from } 0 \to (1 - x))$$

For the plane going bounded by 3 poinnts $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, the equation of the plane is $x + y + z = 1$ or $z = 1 - x - y$. The bound of this integral is:

$$0 \leq z \leq 1 - x - y \quad \text{(z goes from } 0 \to 1 - x - y)$$

Therefore, the integral is:

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{1-x-y} T dt &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z^2 dz dy dx \\
&= \int_0^1 \int_0^{1-x} \left[\frac{z^3}{3} \right]_0^{1-x-y} dy dx \\
&= \int_0^1 \int_0^{1-x} \frac{(1 - x - y)^3}{3} dy dx \\
&= \frac{1}{3} \int_0^1 \int_0^{1-x} (1 - x - y)^3 dy dx
\end{aligned}$$

Substitute $u = 1 - x - y$ and $du = -dy$. The new lower bound is $u = 1 - x$ and the upper bound is $u = 1 - x - (1 - x) = 0$.

$$\begin{aligned}
\frac{-1}{3} \int_0^1 \int_{1-x}^0 u^3 du dx &= \frac{-1}{3} \int_0^1 \left[\frac{u^4}{4} \right]_{1-x}^0 dx \\
&= \frac{-1}{3} \int_0^1 \frac{-(x - 1)^4}{4} dx \\
&= \frac{1}{12} \int_0^1 (x - 1)^4 dx \\
&= \frac{1}{12} \left[\frac{(x - 1)^5}{5} \right]_0^1 \\
&= \frac{1}{60}
\end{aligned}$$

Problem 3

$$\vec{v} = (xy)\hat{x} + (2yz)\hat{y} + (3xz)\hat{z}$$

Divergence theorem states that:

$$\int_v (\nabla \cdot v) d\tau = \int_s v \cdot da$$

$$
\begin{aligned}
\int_v \nabla \cdot \vec{v} d\tau &= \int_0^2 \int_0^2 \int_0^2 \left[\frac{\partial(xy)}{\partial x} + \frac{\partial(2yz)}{\partial y} + \frac{\partial(3xz)}{\partial z} \right] dxdydz \\
&= \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x) dxdydz \\
&= \int_0^2 \int_0^2 \left[yx + 2zx + \frac{3x^2}{2} \right]_0^2 dydz \\
&= \int_0^2 \int_0^2 (2y + 4z + 6) dydz \\
&= \int_0^2 \left[y^2 + 4yz + 6y \right]_0^2 dz \\
&= \int_0^2 (4 + 8z + 12) dz \\
&= \left[4z + 4z^2 + 12z \right]_0^2 \\
&= 8 + 16 + 24 = 48
\end{aligned}
$$

For the surface integral, we have 6 faces. 2 top and bottom faces on the (xy plane with z = 0 and z = 2), 2 faces on the xz plane (y = 0 and y = 2), and 2 faces on the yz plane (x = 0 and x = 2).

$$
\begin{aligned}
\int_s v \cdot da = &\int_0^2 \int_0^2 v_{z=2} dxdy - \int_0^2 \int_0^2 v_{z=0} dxdy + \int_0^2 \int_0^2 v_{x=2} dzdy - \int_0^2 \int_0^2 v_{x=0} dzdy \\
&+ \int_0^2 \int_0^2 v_{y=2} dzdx - \int_0^2 \int_0^2 v_{y=0} dzdx \\
&= \int_0^2 \int_0^2 6x dxdy - \int_0^2 \int_0^2 0 dxdy + \int_0^2 \int_0^2 2y dydz - \int_0^2 \int_0^2 0 dzdy \\
&+ \int_0^2 \int_0^2 4z dxdz - \int_0^2 \int_0^2 0 dxdz \\
&= \int_0^2 \int_0^2 6x dxdy + \int_0^2 \int_0^2 2y dydz + \int_0^2 \int_0^2 4z dxdz \\
&= \int_0^2 \left[3x^2 \right]_0^2 dy + \int_0^2 \left[y^2 \right]_0^2 dz + \int_0^2 \left[2z^2 \right]_0^2 dx \\
&= \int_0^2 12 dy + \int_0^2 4 dz + \int_0^2 8 dx \\
&= [12y]_0^2 + [4z]_0^2 + [8x]_0^2 \\
&= 24 + 8 + 16 = 48
\end{aligned}
$$

Therefore, the two integrals are equal.

Problem 4

$$
\int_S (\nabla \times v) \cdot da = \oint_P v \cdot dl
$$

Since we only include the (yz) plane.

$$\int_S (\nabla \times v) \cdot dydz = \int_0^2 \int_0^{2-z} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} dydz$$
$$= \int_0^2 \int_0^{2-z} \begin{vmatrix} \hat{x} & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} dydz \quad \text{(since we only include the (yz) plane)}$$
$$= \int_0^2 \int_0^{2-z} \left[\frac{\partial(3xz)}{\partial y} - \frac{\partial(2yz)}{\partial z} \right] dydz$$
$$= \int_0^2 \int_0^{2-z} 0 - 2ydydz$$
$$= \int_0^2 \left[-y^2 \right]_0^{2-z} dz$$
$$= \int_0^2 -(2-z)^2 dz$$
$$= -\int_0^2 4 - 4z + z^2 dz$$
$$= -\left[4z - 2z^2 + \frac{z^3}{3} \right]_0^2$$
$$= -8 + 8 - \frac{8}{3} = -\frac{8}{3}$$

Let line $l1$ be the staright like from $(0,0,2) \to (0,0,0)$, line $l2$ be the straight line from $(0,0,0) \to (0,2,0)$, and line $l3$ be the straight line from $(0,2,0) \to (0,0,2)$. We can parameterized the lines as:

$$l1(t) = (0,0,2-t) \quad 0 \le t \le 2$$
$$l2(t) = (0,t,0) \quad 0 \le t \le 2$$
$$l3(t) = (0,2-t,t) \quad 0 \le t \le 2$$

$$\oint_P v \cdot dl = \int_0^2 v(l1(t)) \cdot dl1(t) + \int_0^2 v(l2(t)) \cdot dl2(t) + \int_0^2 v(l3(t)) \cdot dl3(t)$$
$$= \int_0^2 v(0,0,2-t) \cdot (0,0,-1)dt + \int_0^2 v(0,t,0) \cdot (0,1,0)dt + \int_0^2 v(0,2-t,t) \cdot (0,-1,1)dt$$
$$= \int_0^2 -3(2-t) \cdot 0dt \int_0^2 2(t)(0)dt + \int_0^2 2(t)(-1)(2-t) + 3(t)(0)dt$$
$$= 0 + 0 + \int_0^2 -2t(2-t)dt$$
$$= \int_0^2 -4t + 2t^2 dt$$
$$= \left[-2t^2 + \frac{2t^3}{3} \right]_0^2$$
$$= -8 + \frac{16}{3} = -\frac{8}{3}$$

Therefore, the two integrals are equal.

Problem 5

Prove:

- $\int_S f(\nabla \times \vec{A}) \cdot d\vec{a} = \int_S [\vec{A} \times (\nabla f)] \cdot d\vec{a} + \oint_P f\vec{A} \cdot d\vec{l}$

  Left hand side:
  From equation (v) page 21, we have: $\nabla \times (fA) = f(\nabla \times A) - A \times (\nabla f)$

$$\begin{aligned}\int_S f(\nabla \times \vec{A}) \cdot d\vec{a} &= \int_S [\underbrace{\nabla \times (f\vec{A})}_{\text{Stoke's theorem}} + \vec{A} \times (\nabla f)] \cdot d\vec{a} \\ &= \int_S [\vec{A} \times (\nabla f)] \cdot d\vec{a} + \oint_P f\vec{A} \cdot d\vec{l}\end{aligned}$$

- $\int_V \vec{B} \cdot (\nabla \times \vec{A}) d\tau = \int_V \vec{A} \cdot (\nabla \times \vec{B}) d\tau + \oint_S (\vec{A} \times \vec{B}) \cdot d\vec{a}$

  Left hand side:
  From equation (iv) page 21, we have: $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

$$\begin{aligned}\int_V \vec{B} \cdot (\nabla \times \vec{A}) d\tau &= \int_V [\underbrace{\nabla \cdot (\vec{A} \times \vec{B})}_{\text{Divergence Theorem}} + \vec{A} \cdot (\nabla \times \vec{B})] d\tau \\ &= \int_V \vec{A} \cdot (\nabla \times \vec{B}) d\tau + \oint_S (\vec{A} \times \vec{B}) \cdot d\vec{a}\end{aligned}$$