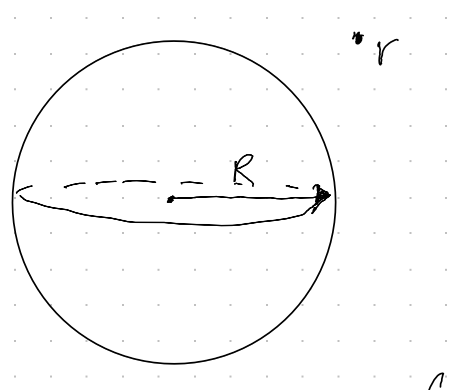
Homework 4.

1) Uniformly charged solid sphere.

R, q

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{|r-r'|}\,dv$$



$$\rho = \frac{q}{\frac{4}{3}\pi R^3}$$

$(r \geq R)$

$$V(r) = -\int_{r_0}^{r} E \cdot dl$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr$$

$$= -\frac{1}{4\pi\varepsilon_0} q \left[-\frac{1}{r}\right]_{\infty}^{r}$$

$$= -\frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right) = \frac{q}{4\pi\varepsilon_0 r} = V_{out}(r)$$

$(r < R)$

$$V(r) = -\int_{r_0}^{r} E\cdot dl = -\int_{\infty}^{R}\frac{q}{4\pi\varepsilon_0 r^2}\,dr - \int_{R}^{r}\frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\,r\,dr$$

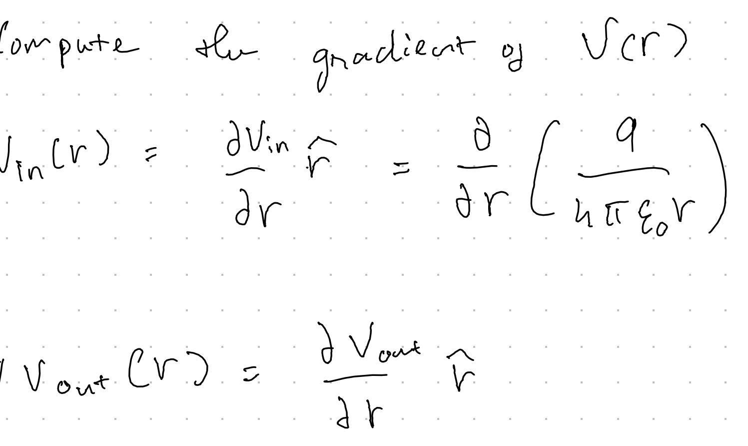
$$= -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{R}\frac{1}{r^2}\,dr - \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R^3}\int r\,dr.$$

$$= \frac{-q}{4\pi\varepsilon_0}\left(-\frac{1}{R}\right) - \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\left(\frac{r^2}{2}-\frac{R^2}{2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2}{2}-\frac{R^2}{2}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{r^2}{2R^3} + \frac{R^2}{2R^3}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{3}{2R} - \frac{r^2}{2R^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$



Compute the gradient of $V(r)$

$$\nabla V_{in}(r) = \frac{\partial V_{in}}{\partial r}\hat{r} = \frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon_0 r}\right)\hat{r} = \boxed{-\frac{q}{4\pi\varepsilon_0 r^2}\hat{r}}$$

$$\nabla V_{out}(r) = \frac{\partial V_{out}}{\partial r}\hat{r}$$

$$= \frac{\partial}{\partial r}\left[\frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{2R}\cdot\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\cancel{2}R}\left[-\left(\frac{\cancel{2}r}{R^2}\right)\right]\hat{r} = \boxed{-\frac{1}{4\pi\varepsilon_0}\cdot\frac{qr}{R^3}\hat{r}}$$

2) Find energy stored in uniformly charged solid sphere R, q

$$W = \frac{1}{2}\int \rho V\,d\tau$$

$$\rho = \frac{3q}{4\pi R^3}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

$$W = \frac{\rho}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3-\frac{r^2}{R^2}\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{\rho}{\cancel{2}}\frac{q}{\cancel{2}R}\frac{1}{4\cancel{\pi}\varepsilon_0}(2\cancel{\pi})(\cancel{2})\int_0^R\left(3r^2 - \frac{r^4}{R^2}\right)dr.$$

$$= \frac{\rho q}{4\varepsilon_0 R}\left[\frac{3r^3}{3} - \frac{r^5}{5R^2}\right]_0^R$$

$$= \frac{3q}{4\pi R^3}\cdot\frac{q}{4\varepsilon_0 R}\left(R^3 - \frac{R^3}{5}\right)$$

$$= \frac{3q}{\cancel{4}\pi \cancel{R^3}}\cdot\frac{\cancel{4}}{5}\cancel{R^3}\cdot\frac{q}{4\varepsilon_0 R} = \frac{3}{5}\frac{q^2}{4\pi\varepsilon_0 R}$$

$$W = \frac{\varepsilon_0}{2}\int E^2\,d\tau$$

$r \leq R$

$$-E = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{qr}{R^3}\hat{r} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}$$

$r \geq R$

$$-E = -\frac{q}{4\pi\varepsilon_0 r^2} \Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}$$

$$W = \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}E^2 r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{\varepsilon_0}{2}(2\pi)(2)\int_0^{\infty}r^2\cdot E^2\,dr.$$

$$= 2\pi\varepsilon_0 \underbrace{\int_0^R r^2E^2\,dr}_{\text{Inside}} + \underbrace{\int_R^{\infty}r^2E^2\,dr}_{\text{outside}}$$

$$= 2\pi\varepsilon_0\left[\int_0^R r^2\left(\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\right)^2 dr + \int_R^{\infty}\left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 r^2\,dr\right]$$

$$= 2\pi\varepsilon_0\left(\frac{q}{4\pi\varepsilon_0}\right)^2\left[\int_0^R\frac{r^4}{R^6}\,dr + \int_R^{\infty}\frac{1}{r^2}\,dr\right]$$

$$= \frac{2\pi q^2}{16\pi^2\varepsilon_0}\left[\frac{r^5}{5R^6}\Big|_0^R + -\frac{1}{r}\Big|_R^{\infty}\right]$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{5R} + \left(-\underbrace{\frac{1}{\infty}}_{0} - \frac{1}{R}\right)\right]$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{5R} + \frac{1}{R}\right) = \frac{q^2}{8\pi\varepsilon_0}\frac{6}{5R} = \frac{q^2}{4\pi\varepsilon_0 R}\frac{3}{5}$$

$$W = \frac{\varepsilon_0}{2}\left(\int_V E^2\,d\tau + \oint_S VE\cdot da\right).$$