1. Let $X \sim Unif(0,1)$. Show that $X^2 \sim Beta(\frac{1}{2}, 1)$. $\quad f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$

First, note that $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ according to Wolfram Alpha.

Now, find pdf of Beta$(\frac{1}{2}, 1)$. $\Rightarrow x^{\frac{1}{2}-1}(1-x)^{1-1} \cdot \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(1/2)\cdot\Gamma(1)} = x^{-1/2} \cdot \frac{\Gamma(3/2)}{\sqrt{\pi}\cdot 1} = x^{-1/2} \cdot \frac{\sqrt{\pi}}{2\sqrt{\pi}} = \frac{1}{2\sqrt{x}}$.

Let $Y = g(X) = X^2$. $S_X = [0,1]$ where g is increasing and one-to-one. We find that $h(Y) = g^{-1}(Y) = \sqrt{Y} = X$. Now we can use the formula to find pdf of Y.

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| = f_X(\sqrt{y}) \cdot \left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2\sqrt{y}} = \text{pdf of Beta}(\tfrac{1}{2}, 1) \checkmark$$

note: $\sqrt{y} > 0 \ \forall y$.

2. Let $U \sim Unif(0,1)$ and let $\lambda > 0$. Find the pdf and cdf of $X = -\frac{1}{\lambda}\ln U$.

Let $g(U) = X$. $S_U = [0,1]$ on which g is one-to-one and decreasing. $h(X) = g^{-1}(X) = U = e^{-\lambda X}$. Use the fomulato find pdf of X. $\quad$ note: $f_U(u) = \begin{cases} 1 & u \in [0,1] \\ 0 & u \notin [0,1] \end{cases}$

$$f_X(x) = f_U(h(x)) \cdot |h'(x)| = f_U(e^{-\lambda x}) \cdot |-\lambda e^{-\lambda x}| = 1 \cdot \lambda e^{-\lambda x}, \ x \in [0, \infty). \quad (\text{b.c } \lambda > 0)$$

Integrate to find cdf: $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \lambda e^{-\lambda t}\,dt = (-e^{-\lambda t})\Big|_0^x$ $= -e^{-\lambda x} - (-1) = 1 - e^{-\lambda x}$. $\Rightarrow \boxed{f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \text{ and } F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}}$

3. Let $A = (0,1)^2$. Find B determined by the transformation $y_1 = u(x_1, x_2) = x_1 + x_2$ and $y_2 = v(x_1, x_2) = x_1 - x_2$. Plot the regions A and B and compute the jacobian.

$y_1 + y_2 = 2x_1 \Rightarrow x_1 = \frac{y_1 + y_2}{2}$. $\quad y_1 - y_2 = 2x_2 \Rightarrow x_2 = \frac{y_1 - y_2}{2}$. $\quad J = \begin{vmatrix} D_1 x_1 & D_2 x_1 \\ D_1 x_2 & D_2 x_2 \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$.

$0 < x_1 < 1 \Rightarrow 0 < \frac{y_1 + y_2}{2} < 1 \Rightarrow 0 < y_1 + y_2 < 2 \Rightarrow -y_1 < y_2 < 2 - y_1$

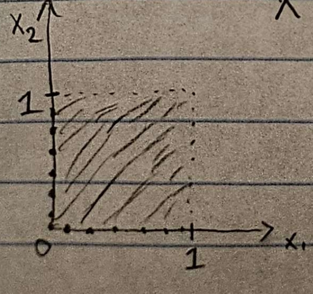
$0 < x_2 < 1 \Rightarrow 0 < \frac{y_1 - y_2}{2} < 1 \Rightarrow 0 < y_1 - y_2 < 2 \Rightarrow y_2 < y_1 < 2 + y_2$.

$-y_1 > -2 - y_2$ and $-y_1 < -y_2 \Rightarrow 2 - y_1 < 2 - y_2 \Rightarrow -2 - y_2 < y_2 < 2 - y_2 \Rightarrow -2 < 2y_2 < 2$

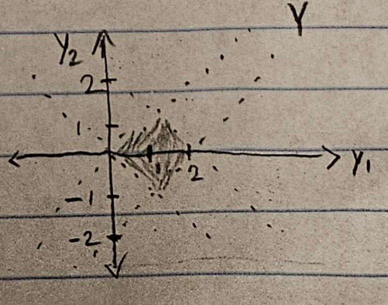
$\Rightarrow -1 < y_2 < 1$.

$-y_1 < y_2 < 2 - y$ and $y_2 < y_1 < 2 - y_1$

X



Y



det $J = -\frac{1}{2}$

$J = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$

4. Let X_1, X_2 be independent r.v. $Y_1 = u(X_1)$ and $Y_2 = v(X_2)$. $S_{X_1,X_2} = S_{Y_1,Y_2}$

a. Compute the jacobian. Let $w_1(Y_1) = X_1$ and $w_2(Y_2) = X_2$.

$$J = \begin{vmatrix} D_1 w_1 & D_2 w_1 \\ D_1 w_2 & D_2 w_2 \end{vmatrix} = \begin{vmatrix} w_1' & 0 \\ 0 & w_2' \end{vmatrix} = \boxed{|w_1'(y_1) \cdot w_2'(y_2)|}.$$

b. Apply the formula. Note that $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$ by independence.

$$\Rightarrow f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(w_1(y_1), w_2(y_2)) \cdot |w_1'(y_1) \cdot w_2'(y_2)|$$
$$= f_{X_1}(w_1(y_1)) \cdot f_{X_2}(w_2(y_2)) \cdot |w_1'(y_1)| \cdot |w_2'(y_2)| = \boxed{f_{X_1}(w_1(y_1)) \cdot |w_1'(y_1)| \cdot f_{X_2}(w_2(y_2)) \cdot |w_2'(y_2)|}$$

c. Apply formula to find f_{Y_1} and f_{Y_2}: $f_{Y_1}(y_1) = f_{X_1}(w_1(y_1)) \cdot |w_1'(y_1)|$.

$f_{Y_2}(y_2) = f_{X_2}(w_2(y_2)) \cdot |w_2'(y_2)|$.

Now, $f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = f_{X_1}(w_1(y_1)) \cdot |w_1'(y_1)| \cdot f_{X_2}(w_2(y_2)) \cdot |w_2'(y_2)| = f_{Y_1,Y_2}(y_1,y_2)$.

Therefore, Y_1, Y_2 are independent.

5. Let X_1 and X_2 be two independent r.v. with distribution $N(\mu, \sigma^2)$.

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

a. I disagree

b. Find f_{Y_1,Y_2}. First, $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2\right)$

$\cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2\right) = \frac{1}{\sigma^2 \cdot 2\pi} \cdot \exp\left(-\frac{1}{2}\left(\left(\frac{x_1-\mu}{\sigma}\right)^2 + \left(\frac{x_2-\mu}{\sigma}\right)^2\right)\right)$.

Now find inverses: $Y_1 + Y_2 = 2X_1 \Rightarrow X_1 = \frac{Y_1+Y_2}{2}$ and $Y_1 - Y_2 = 2X_2 \Rightarrow X_2 = \frac{Y_1 - Y_2}{2}$,

Apply the formula: $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right) \cdot \frac{1}{2}$ ← J derived in Prob. 3.

$$-\frac{1}{2}\left(\left(\frac{y_1+y_2-2\mu}{2\sigma}\right)^2 + \left(\frac{y_1-y_2-2\mu}{2\sigma}\right)^2\right) = -\frac{1}{8\sigma^2}(2y_1^2 + 2y_2^2 + 8\mu^2 - 8\mu y_1) = -\frac{.25y_1^2 + .25y_2^2 + \mu^2 - \mu y_1}{\sigma^2}.$$

$$\Rightarrow \boxed{f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{4\pi\sigma^2} \cdot \exp\left(-\frac{.25y_1^2 + .25y_2^2 + \mu^2 - \mu y_1}{\sigma^2}\right)}$$

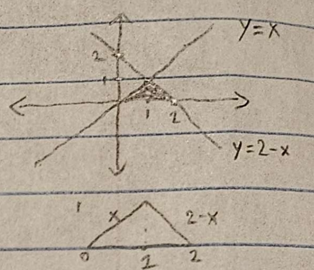
c. $f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{4\pi\sigma^2} \cdot \exp\left(-\left(\frac{\mu}{\sigma}\right)^2\right) \cdot \exp\left(\frac{-.25y_1^2 + \mu y_1}{\sigma^2}\right) \cdot \exp\left(\frac{-.25y_2^2}{\sigma^2}\right)$

$= C \cdot g(y_1) \cdot h(y_2)$ for a constant C, and functions g, h of y_1, y_2 respectively.

Because the joint pdf is seperable in this manner (and the domain lacks dependence)

I believe Y_1, Y_2 are independent.

6. Let $f_{X,Y}(x,y) = \begin{cases} 3xy & 0 \le y \le x \quad 0 \le y \le 2-x \\ 0 & \text{otherwise} \end{cases}$



a. $P(X \le 1/2, Y \le 1) = P(X \le \frac{1}{2}, Y \le X) =$

$\int_0^{1/2}\int_0^x 3xy\,dy\,dx = \int_0^{1/2} \frac{3xy^2}{2}\Big|_{y=0}^{y=x} dx = \int_0^{1/2} \frac{3x^3}{2}dx$

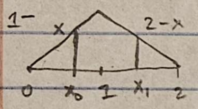
$= \frac{3x^4}{8}\Big|_0^{1/2} = \frac{3}{8}\left(\frac{1}{16}-0\right) = \boxed{\frac{3}{128} \approx .0234}$

$F(\frac{1}{2},\frac{1}{4}) = P(X \le \frac{1}{2}, Y \le \frac{1}{4}) = P(X \le \frac{1}{4}, Y \le X) + P(\frac{1}{4} \le X \le \frac{1}{2}, Y \le \frac{1}{4})$

$= \int_0^{1/4}\int_0^x 3xy\,dy\,dx + \int_{1/4}^{1/2}\int_0^{1/4} 3xy\,dy\,dx = \int_0^{1/4} \frac{3xy^2}{2}\Big|_{y=0}^{y=x} dx + \int_{1/4}^{1/2} \frac{3xy^2}{2}\Big|_{y=0}^{y=1/4} dx$

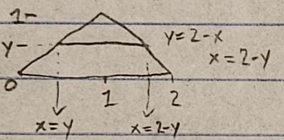
$= \int_0^{1/4} \frac{3x^3}{2}dx + \int_{1/4}^{1/2} \frac{3x}{32}dx = \frac{3x^4}{8}\Big|_0^{1/4} + \frac{3x^2}{64}\Big|_{1/4}^{1/2} = \frac{3}{2048} - 0 + \frac{3}{256} - \frac{3}{1024} = \boxed{\frac{21}{2048} \approx .0103}$

b. Find f_X and f_Y. $\quad f_X(x) = \begin{cases} \int_0^x 3xy\,dy & \text{if } x<1 \\ \int_0^{2-x} 3xy\,dy & \text{if } x>1 \end{cases} \Rightarrow \boxed{f_X(x) = \begin{cases} .5(3x^3) & \text{if } 0 \le x \le 1 \\ .5(3x^3-12x^2+12x) & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}}$



$\Rightarrow \int_0^x 3xy\,dy = \frac{3xy^2}{2}\Big|_{y=0}^{y=x} = \frac{3x^3}{2} \qquad \int_0^{2-x} 3xy\,dy = \frac{3x(2-x)^2}{2} = \frac{3x^3-12x^2+12x}{2}$

f_Y:



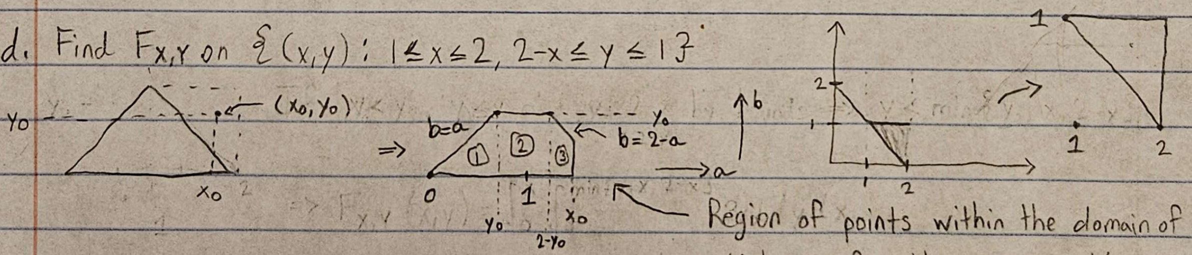
$\Rightarrow \int_y^{2-y} 3xy\,dx = \frac{3x^2y}{2}\Big|_{x=y}^{x=2-y} = \frac{3(2-y)^2y}{2} - \frac{3y^3}{2} = \frac{3}{2}(4y-4y^2) = 6(y-y^2)$

$\Rightarrow \boxed{f_Y(y) = \begin{cases} 6y-6y^2 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$

c. $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \frac{3}{2}\int_0^1 x^4 dx + \frac{3}{2}\int_1^2 (x^4-4x^3+4x^2)dx = \frac{3}{2}\left(\frac{x^5}{5}\Big|_0^1 + \left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_1^2\right)$

$= \frac{3}{2}\left(\frac{1}{5} - 0 + \frac{32}{5} - 16 + \frac{32}{3} - \frac{1}{5} + 1 - \frac{4}{3}\right) = \frac{3}{2}\left(\frac{28}{3} - 15 + \frac{32}{5}\right) = \frac{3}{2}\cdot\frac{11}{15} = \boxed{\frac{11}{10} = 1.1}$

d. Find $F_{X,Y}$ on $\{(x,y): 1 \le x \le 2, 2-x \le y \le 1\}$



Region of points within the domain of the pdf which satisfy $X \le x_0$ and $Y \le y_0$.

Integrate over that region (split into 3 parts)

$\underbrace{\int_0^y\int_0^a 3ab\,db\,da}_{1} + \underbrace{\int_y^{2-y}\int_0^y 3ab\,db\,da}_{2} + \underbrace{\int_{2-y}^x\int_0^{2-a} 3ab\,db\,da}_{3} = \underbrace{\int_0^y \frac{3ab^2}{2}\Big|_{b=0}^{b=a} da}_{1} + \underbrace{\int_y^{2-y} \frac{3ab^2}{2}\Big|_{b=0}^{b=y} da}_{2}$

$+ \underbrace{\int_{2-y}^x \frac{3ab^2}{2}\Big|_{b=0}^{b=2-a} da}_{3} = \underbrace{\int_0^y \frac{3a^3}{2}da}_{1} + \underbrace{\int_y^{2-y} \frac{3ay^2}{2}da}_{2} + \underbrace{\int_{2-y}^x \frac{3}{2}a^3 - 6a^2 + 6a\,da}_{3} = \underbrace{\frac{3y^4}{8}}_{1} - \underbrace{3(y-1)y^2}_{2} - \underbrace{\frac{3y^4 - 8y^3 - 3x^4 + 16x^3 - 24x^2 + 16}{8}}_{3}$

$\Rightarrow -3y^3 + 3y^2 + y^3 + \frac{3}{8}x^4 - 2x^3 + 3x^2 - 2 = \frac{3}{8}x^4 - 2x^3 + 3x^2 - 2 + 3y^2 - 2y^3.$

$\Rightarrow \boxed{F_{X,Y}(x,y) = \frac{3}{8}x^4 - 2(x^3+y^3) + 3(x^2+y^2) - 2 \quad \text{for } (x,y): 1 \le x \le 2 \text{ and } 2-x \le y \le 1.}$

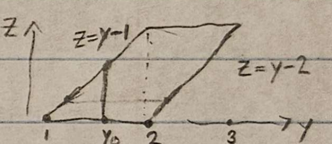
7. $f_{X_1,X_2}(x_1,x_2) = \begin{cases} x_1 - x_2 & \text{if } x_1 \in [1,2],\ x_2 \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

Let $Y = X_1 + X_2$. Find F_Y. Let $Z = X_2$. $\Rightarrow$ $X_1 = Y - X_2 = Y - Z = u(Y,Z)$ and $X_2 = v(Y,Z) = Z$.

$J = \begin{pmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. $|\det J| = |1 - 0| = 1$. $0 \le X_2 \le 1 \Rightarrow 0 \le Z \le 1$.

$1 \le X_1 \le 2 \Rightarrow 1 \le Y - Z \le 2 \Rightarrow 1 + Z \le Y \le 2 + Z$.

Now $f_{Y,Z}(y,z) = f_{X_1,X_2}(y-z, z) \cdot 1 = y - 2z$ for $0 \le z \le 1$ and $1+z \le y \le 2+z$.



Now $f_{(X_1+X_2)} = f_Y(y) = \int_0^{y-1} y - 2z\, dz$ if $1 \le y \le 2$

$\int_{y-2}^{1} y - 2z\, dz$ if $2 < y \le 3$

$\Rightarrow (yz - z^2)\Big|_0^{y-1} = y^2 - y - y^2 + 2y - 1 - 0 - 0 = y - 1$

$\Rightarrow (yz - z^2)\Big|_{y-2}^{1} = y - 1 - y^2 + 2y + y^2 - 4y + 4 = 3 - y$

$\boxed{\Rightarrow f_Y(y) = \begin{cases} y - 1 & \text{if } 1 \le y \le 2 \\ 3 - y & \text{if } 2 < y \le 3 \\ 0 & \text{otherwise} \end{cases}}$

b. $f_{X_1} = \int_0^1 x_1 - x_2\, dx_2 = x_1 - \frac{1}{2}$ $\qquad f_{X_2} = \int_1^2 x_1 - x_2\, dx_1 = \frac{3}{2} - x_2$

$\Rightarrow f_{X_1} \cdot f_{X_2} = (x_1 - \frac{1}{2})(\frac{3}{2} - x_2) \ne x_1 - x_2 = f_{X_1,X_2}$ $\Rightarrow$ $\boxed{X_1, X_2 \text{ are not independent.}}$