MA331 - Probability and Statistics review - Sample solution

To solve this test you only need knowledge acquired in MA222 or equivalent. This assignment is important to self-assess your knowledge about prerequisites. **Collaboration is not allowed.**

To get full credit you need to:

- Clearly state the correct answer.
- Justify and explain each step in your solution and the final answer.
- Make very clear that you understand the logic behind the steps.
- Using a symbolic algebra system to perform computations does not exonerate you to explain and motivate every step as mentioned above. If you do any computation using a symbolic algebra system, you must state the command used.
- Correctly interpreting the statement of each problem is part of this test.

Very important: Honor code applies fully. You are required to sign the pledge. You must submit your own work only. **It is prohibited to post the following problems or ask anything on any website/forum/tutoring system or any other virtual means as for example Chegg.**

Problem

A coin with diameter d is thrown onto a floor made of square tiles with side length $l > d$. What is the probability that the coin lands completely within one of the tiles? (that is, it does not cross any of the sides of a square)?

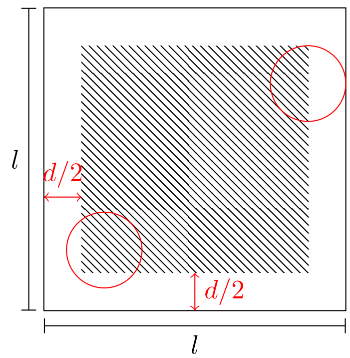
Solution:

Let X and Y be the coordinates of the center of the coin, measured from the lower left corner of a square on the floor.

For the coin to not touch the edge, its center must fall at a distance greater than $d/2$ from the sides (hatched area).

The probability is given by:

$$\left(\frac{l-d}{l}\right)^2 = \left(1 - \frac{d}{l}\right)^2 .$$



Problem

There are six romantic couples in a tango class [1] Men and women are paired at random to start the class. What is the probability that the dance partners are the same as the romantic couples?

Solution:

Let

$$M_1, M_2, M_3, M_4, M_5, M_6$$

represent six men, and

$$W_1, W_2, W_3, W_4, W_5, W_6$$

represent six women.

It is sufficient to randomly arrange the women. The total number of arrangements is $6! = 720$.

$$\mathbb{P}\left(M_i \text{ dances with } W_j\right) = \frac{1}{6!} = \frac{1}{720} \approx 1.4 \times 10^{-3} \quad \text{for } i = 1, \ldots, 6\,.$$

Problem

Imagine three persons each choose a random real number from the interval $[0, 1]$. Calculate the probability that the sum of the squares of their chosen numbers does not exceed 1.

Solution:

Let the real numbers in question be random variables X_1, X_2, and X_3 with distribution $\mathrm{Unif}(0, 1)$. These three random variables, when allowed to vary freely between 0 and 1, trace out a cube of side length 1 in $\mathbb{R}^3$. The region of interest is the set of all points (X_1, X_2, X_3) such that

$$X_1^2 + X_2^2 + X_3^2 \leq 1$$

[1] A Tango couple is 1 man and 1 woman.

This inequality defines the interior of the unit sphere in $\mathbb{R}^3$. Because the support of each of these random variables is limited to $[0,1]$, we only consider the section of this sphere when the coordinates x, y, and z are all greater than or equal to 0. The probability of a point, chosen at random according to the distributions of these random variables, falls within the sphere, is simply the ratio of the volume of the section of the sphere to the volume of the cube. Therefore,

$$\begin{aligned}\mathbb{P}\left(X_1^2 + X_2^2 + X_3^2 \leq 1\right) &= \frac{\frac{4}{3}\pi \cdot (1)^3}{(1)^3} \\ &= \frac{\pi}{6}\end{aligned}$$

Problem

Two dice are thrown. Consider the events

- $A =$ "the sum is equal to 3"
- $B =$ "the sum is equal to 7"
- $C =$ "at least one of the dice shows a 1"

Calculate $\mathbb{P}(A \mid C)$ and $\mathbb{P}(B \mid C)$. Are A and C independent? What about B and C?

Solution:

Let

- $A = \{\text{the sum is } 3\}$
- $B = \{\text{the sum is } 7\}$
- $C = \{\text{at least one is a } 1\}$

We have:

$$\mathbb{P}(A \mid C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A)}{\mathbb{P}(C)} = \frac{2/36}{11/36} = \frac{2}{11}$$

since $A \subseteq C$.

A and C are not independent.

$$\mathbb{P}(B \mid C) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} = \frac{2/36}{11/36} = \frac{2}{11} \neq \mathbb{P}(B) = \frac{1}{6}$$

B and C are not independent.

Problem

Let X, Y be r.v's with support $\{1,2,3,4\}$. The next formula gives the joint proba function:

$$\mathbb{P}(X{=}i,\, Y{=}j) = \frac{i+j}{80}\,.$$

Compute $\mathbb{P}(X{=}Y)$, $\mathbb{P}(XY{=}6)$ and $\mathbb{P}(1{\leq}X{\leq}2,\, 2{<}Y{\leq}4)$.

Solution:

Let X and Y be discrete random variables such that $\mathcal{S}_X = \mathcal{S}_Y = \{1,2,3,4\}$. Also,

$$\mathbb{P}(X=i, Y=j) = \frac{i+j}{80}$$

1. $\mathbb{P}(X=Y) = \sum_{i=1}^{4} \mathbb{P}(X=i, Y=i) = \sum_{i=1}^{4} \frac{2i}{80} = \frac{1}{40}\sum_{i=1}^{4} i = \frac{1}{4}$
2. $\mathbb{P}(XY=6) = \mathbb{P}(X=2, Y=3) + \mathbb{P}(X=3, Y=2)$

$$= 2\cdot\frac{2+3}{80} = \frac{1}{8}$$

3. $\mathbb{P}(1 \leq X < 2, 2 < Y \leq 4)$

$$= \mathbb{P}(X=1, Y=3) + \mathbb{P}(X=1, Y=4) + \mathbb{P}(X=2, Y=3) + \mathbb{P}(X=2, Y=4)$$

$$= \frac{1+3}{80} + \frac{1+4}{80} + \frac{2+3}{80} + \frac{2+4}{80} = \frac{20}{80} = \frac{1}{4}.$$

Problem

The random variable X takes values -1, 0, 1 with probabilities $\frac{1}{8}, \frac{2}{8}, \frac{5}{8}$ respectively.

1. Calculate $\mathbb{E}(X)$.
2. Find the probability function of $Y = X^2$ and use it to calculate $\mathbb{E}(Y)$.
3. Calculate $\mathbb{E}(X^2)$ using the formula for the expected value of a function of a random variable.
4. Calculate $\mathrm{Var}(X)$.

Solution:

1.

x	-1	0	1
$p(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{5}{8}$
$xp(x)$	$-\frac{1}{8}$	0	$\frac{5}{8}$

$$\mathbb{E}(X) = \frac{1}{2}\,.$$

2.

x	-1	0	1
$y = x^2$	1	0	1
$p(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{5}{8}$

$$\Rightarrow \begin{array}{c|cc} y & 0 & 1 \\ \hline p(y) & \frac{2}{8} & \frac{6}{8} \end{array}$$

$$\mathbb{E}(Y) = \frac{6}{8}.$$

3. $$\begin{array}{c|ccc} x & -1 & 0 & 1 \\ x^2 & 1 & 0 & 1 \\ \hline x^2p(x) & \frac{1}{8} & 0 & \frac{5}{8} \end{array}$$

$$\mathbb{E}(X^2) = \frac{6}{8}.$$

4. $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{6}{8} - \left(\frac{1}{2}\right)^2 = \frac{6}{8} - \frac{2}{8} = \frac{4}{8} = \frac{1}{2}$.

Problem

Two players, A and B, alternately and independently flip a coin and the first player to obtain a head wins. Assume player A flips first.

1. If the coin is fair, what is the probability that A wins?
2. Suppose that $\mathbb{P}\,(\text{head}) = p$, not necessarily $\frac{1}{2}$. What is the probability that A wins?

Solution:

1.

$$\mathbb{P}\,(A \text{ wins}) = \sum_{i=1}^{\infty} \mathbb{P}\,(A \text{ wins on } i\text{th toss}) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{1}{2}\right)^4 \frac{1}{2} + \ldots = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{2i+1} = \frac{2}{3}.$$

2.

$$\mathbb{P}\,(A \text{ wins}) = p + (1-p)p^2 + (1-p)^2p^4 + \ldots = \sum_{i=0}^{\infty} p(1-p)^{2i} = \frac{p}{1-(1-p)^2}.$$

Problem

Let $f(x) = x + ax^2$ on $[0, 1]$.

1. Find a that makes f a density function.
2. Find the cumulative distribution function (c.d.f.) of X with density f.
3. Calculate $\mathbb{P}\left(\frac{1}{2} < X < 1\right)$.
4. Calculate $\mathbb{E}\,[X]$ and $\text{Var}\,[X]$.

Solution:

Given $f(x) = x + ax^2$ on $[0, 1]$.

1. $\int_0^1 x + ax^2\,dx = \left[\frac{x^2}{2} + \frac{ax^3}{3}\right]_0^1 = \frac{1}{2} + \frac{a}{3} = 1 \Rightarrow a = \frac{3}{2}$.

2. $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{2} + \frac{x^3}{2} & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$.

3. $\mathbb{P}\left(\frac{1}{2} < X \leq 1\right) = F(1) - F\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{4} + \frac{1}{8}\right) = 1 - \frac{3}{16} = \frac{13}{16}$.

4. $\mathbb{E}[X] = \int_0^1 \frac{x^2}{2} + \frac{x^3}{2}\,dx = \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{3} + \frac{3}{8} = \frac{8+9}{24} = \frac{17}{24}$.

$\mathbb{E}\left[X^2\right] = \int_0^1 x^3 + \frac{3}{2}x^4\,dx = \left[\frac{x^4}{4} + \frac{3x^5}{10}\right]_0^1 = \frac{1}{4} + \frac{3}{10} = \frac{22}{40} = \frac{11}{20}$.

$\mathrm{Var}[X] = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = \frac{11}{20} - \left(\frac{17}{24}\right)^2 = \frac{11}{20} - \frac{289}{576} = \frac{316-289}{576} = \frac{27}{576} \approx 0.0469$.

Problem

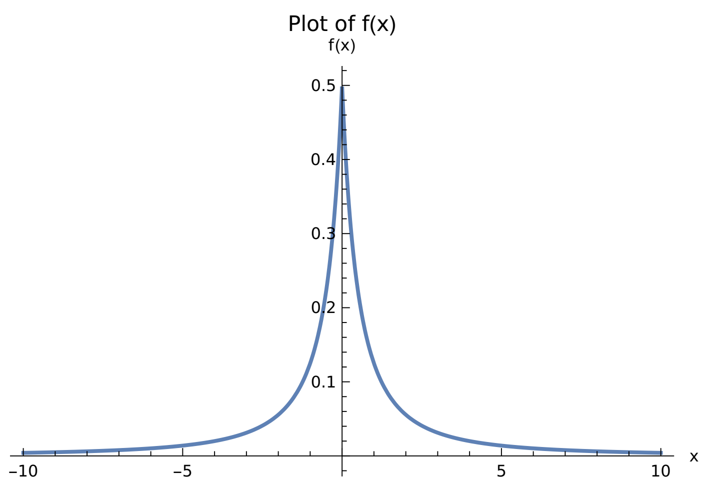
Let X be a variable with density $f(x) = \frac{1}{2(1+|x|)^2}$ with $x \in \mathbb{R}$.

1. Sketch the graph of $f(x)$.
2. Calculate $\mathbb{P}(-1 < X < 2)$.
3. Calculate $\mathbb{P}(|X| > 1)$.
4. Calculate $\mathbb{E}[X]$

Solution:

Given $f(x) = \frac{1}{2(1+|x|)^2}, \quad x \in \mathbb{R}$.

1.



2.

$$\begin{aligned}
\mathbb{P}(-1 < X < 2) &= \int_{-1}^{2} f(x)\,dx \\
&= \int_{-1}^{0} \frac{1}{2(1-x)^2}\,dx + \int_{0}^{2} \frac{1}{2(1+x)^2}\,dx \\
&= \frac{1}{2}\left[-\frac{1}{1-x}\right]_{-1}^{0} + \frac{1}{2}\left[-\frac{1}{1+x}\right]_{0}^{2} \\
&= \frac{1}{2}\left[\left(-1+\frac{1}{2}\right) + \left(-\frac{1}{3}+1\right)\right] \\
&= \frac{1}{4} + \frac{1}{3} = \frac{7}{12}.
\end{aligned}$$

3. $\mathbb{P}(|X| > 1) = 2\int_{1}^{\infty} \frac{1}{2(1+x)^2}\,dx = 2\left[-\frac{1}{1+x}\right]_{1}^{\infty} = 2\left(0 - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}$.
4. No, because $\int_{-\infty}^{\infty} \frac{x}{2(1+|x|)^2}\,dx$ does not converge.

Problem

Let $X \sim \text{Bin}(100, 1/3)$. An "exact" calculation on a computer gives $\mathbb{P}(X \leq 30) = 0.2765539$. Use the CLT to give an approximation of $\mathbb{P}(X \leq 30)$.

Solution:

Let $X \sim \text{Bin}(100, \frac{1}{3})$. We have $\mu = \frac{100}{3}$ and $\sigma^2 = \frac{100 \cdot \frac{2}{3}}{3}$.

$$\mathbb{P}(X \leq 30) = \mathbb{P}\left(\frac{X - 100/3}{\sqrt{100 \cdot (2/3)^2/3}} \leq \frac{30 - 100/3}{\sqrt{100 \cdot (2/3)^2/3}}\right)$$

$$\approx \Phi(-0.707) = 0.2398,$$