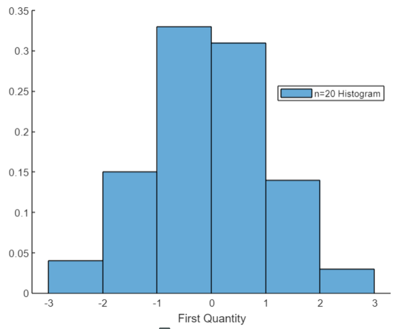
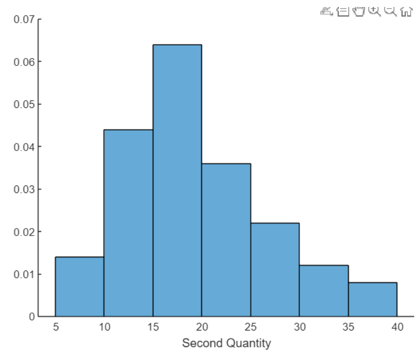
1)

(1) $n = 20$

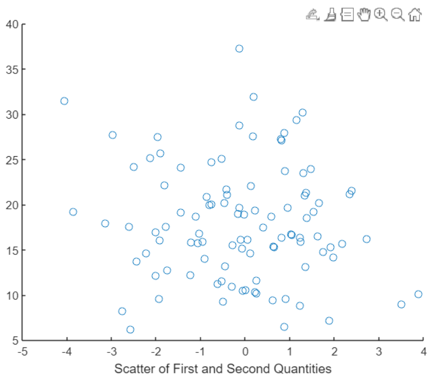
sample quantities = 100



$\frac{\bar{x}-2}{\sqrt{3^2/n}}$ histogram



$\frac{n-1\ s}{3^2}$ histogram



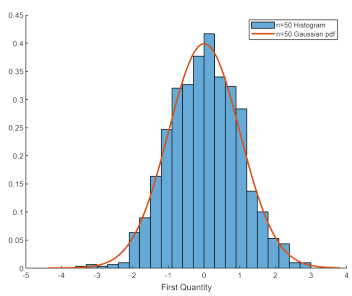
Scatterplot of both quantities

(2) Based on the scatterplot of the $\frac{\bar{x}-2}{\sqrt{s^2/n}}$ and $\frac{(n-1)s^2}{3^2}$ random variables, my claim is that they are independent.

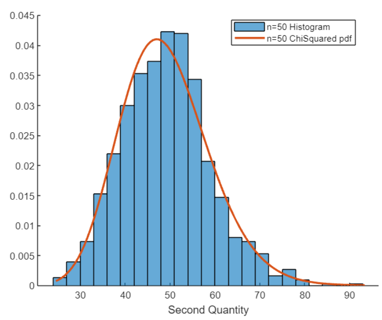
(3) $n = 50$

sample quantities = 1000

*Here we can see the overlay of the theoretical Gaussian pdf.

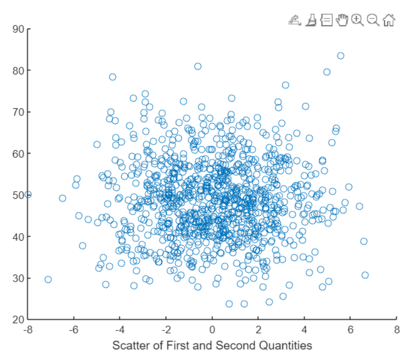


$\frac{\bar{x}-2}{\sqrt{3^2/n}}$ histogram



$\frac{(n-1)s^2}{3^2}$ histogram

*Here we can see the overlay of the theoretical chi-squared pdf.



Scatterplot of both quantities

2)

a) $W \sim N(2,4)$
$X \sim N(-1,3)$
$Y \sim N(1,2)$
$\Big\}$ Independent

(1) Consider: $A = W + 3X - 2Y$

$$M_A(t) = M_W(t) \cdot M_X(3t) \cdot M_Y(-2t)$$
$$= e^{2t+8t^2} \cdot e^{-3t+\frac{9}{2}\cdot 9t^2} \cdot e^{-2t+2\cdot 4t^2}$$
$$= e^{2t+8t^2} \cdot e^{-3t+\frac{81}{2}t^2} \cdot e^{-2t+8t^2}$$
$$= e^{(2t+8t^2)+(-3t+\frac{81}{2}t^2)+(-2t+8t^2)}$$
$$= e^{-3t+\frac{113}{2}t^2}$$

$\therefore A \sim N(-3, \sqrt{113})$

(2) Consider $B = \left(\frac{(W-2)^2}{16}\right) + \left(\frac{(X+1)^2}{9}\right) + \left(\frac{(Y-1)^2}{4}\right)$

*Recall $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 \sim \chi^2(n)$, where X_i are iid Gaussians

$\therefore B = \chi^2(1) + \chi^2(1) + \chi^2(1)$

$$M_B(t) = (1-2t)^{-\frac{1}{2}} \cdot (1-2t)^{-\frac{1}{2}} \cdot (1-2t)^{-\frac{1}{2}}$$
$$= (1-2t)^{-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}}$$
$$= (1-2t)^{-\frac{3}{2}}$$

$\therefore B \sim \chi^2(3)$

(3) Consider: $C = \dfrac{\frac{W-2}{4}}{\sqrt{\left(\frac{(X+1)^2}{9}+\frac{(Y-1)^2}{4}\right)/2}}$

*Recall $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 \sim \chi^2(n)$, where X_i are iid Gaussians

$\therefore \left(\frac{(X+1)^2}{9}+\frac{(Y-1)^2}{4}\right) \sim \chi^2(1) + \chi^2(1)$

$\sim \chi^2(2)$, let's denote this as K

$\therefore C = \dfrac{\frac{W-2}{4}}{\sqrt{K/2}}$, where $K \sim \chi^2(2)$

* $\frac{W-2}{4} \sim N(0,1)$, let's denote this as L

$C = \dfrac{L}{\sqrt{K/2}}$, where $L \sim N(0,1)$ and $K \sim \chi^2(2)$

*Recall $\frac{X}{\sqrt{Y/n}}$, when $X \sim N(0,1)$ and $Y \sim \chi^2_n$ is a T distribution with n degrees of freedom.

$\therefore C \sim T(2)$

(4) Consider: $D = \dfrac{\frac{(X^2+1)}{9}}{\frac{(Y-1)^2}{4}}$

*Recall $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 \sim \chi^2(n)$, where X_i are iid Gaussians

$\therefore \frac{(X+1)^2}{9} \sim \chi^2(1)$ and $\frac{(Y-1)^2}{4} \sim \chi^2(1)$

*Recall that a ratio of chi-square variables is an F-distribution.

$\therefore D \sim F(1,1)$

b) Let $X \sim N(-1, 9)$, $Y \sim \chi^2(12)$, $T \sim T(10)$, and $F \sim F(8, 9)$

(1)

(1) $\mathbb{P}(0 < X < 1) \approx 0.04369$

In[1]:= `CDF[NormalDistribution[-1, 9], 1] - CDF[NormalDistribution[-1, 9], 0]`

Out[1]= $-\frac{1}{2}\mathrm{Erfc}\left[-\frac{1}{9\sqrt{2}}\right] + \frac{1}{2}\mathrm{Erfc}\left[-\frac{\sqrt{2}}{9}\right]$

In[2]:= $N\left[-\frac{1}{2}\mathrm{Erfc}\left[-\frac{1}{9\sqrt{2}}\right] + \frac{1}{2}\mathrm{Erfc}\left[-\frac{\sqrt{2}}{9}\right]\right]$

Out[2]= 0.0436937

(2) $\mathbb{P}(3 < Y < 14) \approx 0.6948$

In[3]:= `CDF[ChiSquareDistribution[12], 14] - CDF[ChiSquareDistribution[12], 3]`

Out[3]= $-\mathrm{GammaRegularized}\left[6, 0, \frac{3}{2}\right] + \mathrm{GammaRegularized}[6, 0, 7]$

In[4]:= $N\left[-\mathrm{GammaRegularized}\left[6, 0, \frac{3}{2}\right] + \mathrm{GammaRegularized}[6, 0, 7]\right]$

Out[4]= 0.694836

(3) $\mathbb{P}(0 < T < 1) \approx 0.3296$

In[5]:= `CDF[StudentTDistribution[10], 1] - CDF[StudentTDistribution[10], 0]`

Out[5]= $\frac{256\,043}{234\,256\sqrt{11}}$

In[6]:= $N\left[\frac{256\,043}{234\,256\sqrt{11}}\right]$

Out[6]= 0.329553

(4) $\mathbb{P}(0 < F < 1) \approx 0.5055$

In[7]:= `CDF[FRatioDistribution[8, 9], 1] - CDF[FRatioDistribution[8, 9], 0]`

Out[7]= $\mathrm{BetaRegularized}\left[\frac{8}{17}, 4, \frac{9}{2}\right]$

In[8]:= $N\left[\mathrm{BetaRegularized}\left[\frac{8}{17}, 4, \frac{9}{2}\right]\right]$

Out[8]= 0.505456

(2) Given $\alpha = 0.05$, we must calculate .025 and .975 quantiles of $X, Y, T,$ and F:

X: .025 quantile is −18.6397
.975 quantile is 16.6397

In[13]:= `Quantile[NormalDistribution[-1, 9], 0.025]`

Out[13]= -18.6397

In[14]:= `Quantile[NormalDistribution[-1, 9], 0.975]`

Out[14]= 16.6397

T: .025 quantile is −2.2281
.975 quantile is 2.2281

In[15]:= `Quantile[StudentTDistribution[10], 0.025]`

Out[15]= -2.22814

In[16]:= `Quantile[StudentTDistribution[10], 0.975]`

Out[16]= 2.22814

Y: .025 quantile is 4.4038
.975 quantile is 23.3367

In[11]:= `Quantile[ChiSquareDistribution[12], 0.025]`

Out[11]= 4.40379

In[12]:= `Quantile[ChiSquareDistribution[12], 0.975]`

Out[12]= 23.3367

F: .025 quantile is 0.2295
.975 quantile is 4.1020

In[17]:= `Quantile[FRatioDistribution[8, 9], 0.025]`

Out[17]= 0.229503

In[18]:= `Quantile[FRatioDistribution[8, 9], 0.975]`

Out[18]= 4.10196

3)

a) We're given that (X,Y) are jointly distributed as Gaussian with means μ_X, μ_Y; variances σ_X^2, σ_Y^2; and ρ parameter.

$$\therefore f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right)\right]}$$

Calculating $f_X(x)$: $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \Rightarrow$

μ_X μ_Y σ_X σ_Y

```
In[18]:= Integrate[PDF[BinormalDistribution[{μ1, μ2}, {σ1, σ2}, ρ], {x, y}], {y, -Infinity, Infinity}]
```

Out[18]= $\dfrac{e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}}{\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma_1\sqrt{-\frac{1}{(-1+\rho^2)\sigma_2^2}}\,\sigma_2}$ if $\mathrm{Re}\left[\frac{1}{(-1+\rho^2)\sigma_2^2}\right] \le 0$

*The integral was evaluated using mathematica.

$$\Rightarrow \frac{1}{\sigma_X\sqrt{2\pi}\sqrt{1-\rho^2}\sqrt{-\frac{1}{(-1+\rho^2)\sigma_Y^2}}\,\sigma_Y} e^{\left[-\frac{1}{2}\left(\frac{(x-\mu_X}{\sigma_X}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_X\sqrt{2\pi}\sqrt{\frac{1\cancel{(1-\rho^2)}}{\cancel{(1-\rho^2)}\sigma_Y^2}}\,\sigma_Y} e^{\left[-\frac{1}{2}\left(\frac{(x-\mu_X}{\sigma_X}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_X\sqrt{2\pi}\sqrt{\frac{1}{\sigma_Y^2}}\,\sigma_Y} e^{\left[-\frac{1}{2}\left(\frac{(x-\mu_X}{\sigma_X}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_X\sqrt{2\pi}\,\frac{1}{\cancel{\sigma_Y}}\,\cancel{\sigma_Y}} e^{\left[-\frac{1}{2}\left(\frac{(x-\mu_X}{\sigma_X}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_X\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{(x-\mu_X}{\sigma_X}\right)^2\right]}$$

As we can see, this is the pdf of a Gaussian distributin with mean $= \mu_X$ and variance $= \sigma_X^2$ (i.e., std dev $= \sigma_X$).

Calculating $f_Y(y)$: $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \Rightarrow$

μ_X μ_Y σ_X σ_Y

```
In[17]:= Integrate[PDF[BinormalDistribution[{μ1, μ2}, {σ1, σ2}, ρ], {x, y}], {x, -Infinity, Infinity}]
```

Out[17]= $\dfrac{e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}}{\sqrt{2\pi}\sqrt{1-\rho^2}\sqrt{-\frac{1}{(-1+\rho^2)\sigma_1^2}}\,\sigma_1\,\sigma_2}$ if $\mathrm{Re}\left[\frac{1}{(-1+\rho^2)\sigma_1^2}\right] \le 0$

*The integral was evaluated using mathematica.

$$\Rightarrow \frac{1}{\sigma_Y\sqrt{2\pi}\sqrt{1-\rho^2}\sqrt{-\frac{1}{(-1+\rho^2)\sigma_X^2}}\,\sigma_X} e^{\left[-\frac{1}{2}\left(\frac{(y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_Y\sqrt{2\pi}\sqrt{\frac{1\cancel{(1-\rho^2)}}{\cancel{(1-\rho^2)}\sigma_X^2}}\,\sigma_X} e^{\left[-\frac{1}{2}\left(\frac{(y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_Y\sqrt{2\pi}\sqrt{\frac{1}{\sigma_X^2}}\,\sigma_X} e^{\left[-\frac{1}{2}\left(\frac{(y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_Y\sqrt{2\pi}\,\frac{1}{\cancel{\sigma_X}}\,\cancel{\sigma_X}} e^{\left[-\frac{1}{2}\left(\frac{(y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

$$\Rightarrow \frac{1}{\sigma_Y\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{(y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

As we can see, this is the pdf of a Gaussian distributin with mean $= \mu_Y$ and variance $= \sigma_Y^2$ (i.e., std dev $= \sigma_Y$).

b)

From part (a), we have shown $X \sim N(\mu_x, \overset{\nearrow \text{variance}}{\sigma_x^2})$ and $Y \sim (\mu_Y, \overset{\nearrow \text{variance}}{\sigma_Y^2})$ ...

Let $U = aX + bY$

Let $W = aX$

$$\Longrightarrow \begin{array}{r} U = aX + bY \\ -\ W = aX \\ \hline U - W = bY \end{array} \qquad \therefore \quad X = \frac{W}{a}, \quad Y = \frac{U-W}{b}$$

$$\mathbb{J} = \begin{bmatrix} \frac{\partial}{\partial U}\frac{W}{a} & \frac{\partial}{\partial W}\frac{W}{a} \\ \frac{\partial}{\partial U}\frac{U-W}{b} & \frac{\partial}{\partial W}\frac{U-W}{b} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{a} \\ \frac{1}{b} & \frac{-1}{b} \end{bmatrix}$$

$\therefore \det(\mathbb{J}) = -\frac{1}{ab}$

$$f_{UW}(u,w) = \left|-\frac{1}{ab}\right| f_{XY}\left(\frac{w}{a}, \frac{u-w}{b}\right)$$

$$= \frac{1}{ab} f_{XY}\left(\frac{w}{a}, \frac{u-w}{b}\right)$$

$$= \frac{1}{ab} \cdot \frac{1}{2\pi\sigma_x\sigma_Y\sqrt{1-\rho^2}} e^{\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{\frac{w}{a}-\mu_x}{\sigma_x}\right)^2 + \left(\frac{\frac{u-w}{b}-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(\frac{w}{a}-\mu_x)(\frac{u-w}{b}-\mu_Y)}{\sigma_x\sigma_Y}\right)\right]}$$

Now to find the pdf of U i.e., $aX+bY$:

$$f_U(u) = \int_{-\infty}^{\infty} f_{UW}(u,w)\,dw \Longrightarrow$$

μ_x μ_Y σ_x σ_Y

```
In[3]:= Integrate[1/(ab) * PDF[BinormalDistribution[{μ1, μ2}, {σ1, σ2}, ρ], {w/a, (u-w)/b}], {w, -Infinity, Infinity}]
```

Out[3]= $\dfrac{e^{-\frac{(-u+a\mu_1+b\mu_2)^2}{2(a^2\sigma_1^2+2ab\rho\sigma_1\sigma_2+b^2\sigma_2^2)}}}{ab\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma_1\sigma_2\sqrt{-\frac{a^2\sigma_1^2+2ab\rho\sigma_1\sigma_2+b^2\sigma_2^2}{a^2b^2(-1+\rho^2)\sigma_1^2\sigma_2^2}}}$ if condition

* The integral was evaluated using mathematica.

$$\Longrightarrow \frac{1}{ab\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma_x\sigma_Y\sqrt{-\frac{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}{a^2b^2(-1+\rho^2)\sigma_x^2\sigma_Y^2}}} e^{\frac{-1}{2}\left[\left(\frac{-u+a\mu_x+b\mu_Y}{\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}}\right)^2\right]}$$

$$\Longrightarrow \frac{1}{ab\sqrt{2\pi}\sqrt{1-\rho^2}\sqrt{-\frac{(a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2)\cancel{(\sigma_x^2\sigma_Y^2)}}{a^2b^2(-1+\rho^2)\cancel{\sigma_x^2\sigma_Y^2}}}} e^{\frac{-1}{2}\left[\left(\frac{-u+a\mu_x+b\mu_Y}{\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}}\right)^2\right]}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sqrt{-\frac{(a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2)\cancel{(a^2b^2)}}{\cancel{a^2b^2}(-1+\rho^2)}}} e^{\frac{-1}{2}\left[\left(\frac{-u+a\mu_x+b\mu_Y}{\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}}\right)^2\right]}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sqrt{\frac{(a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2)}{(1-\rho^2)}}} e^{\frac{-1}{2}\left[\left(\frac{-u+a\mu_x+b\mu_Y}{\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}}\right)^2\right]}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}\sqrt{\frac{(a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2)\cancel{(1-\rho^2)}}{\cancel{(1-\rho^2)}}}} e^{\frac{-1}{2}\left[\left(\frac{-u+a\mu_x+b\mu_Y}{\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}}\right)^2\right]}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}} e^{\frac{-1}{2}\left[\left(\frac{-u+a\mu_x+b\mu_Y}{\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}}\right)^2\right]}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}} e^{\frac{-1}{2}\left[\left(\frac{-(u-(a\mu_x+b\mu_Y))}{\sqrt{a^2\sigma_x^2+2ab\rho\sigma_x\sigma_Y+b^2\sigma_Y^2}}\right)^2\right]}$$

As we can see, this is a pdf of a Gaussian distribution with mean $= a\mu_x + b\mu_Y$ and std dev $= \sqrt{a^2\sigma_x^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_x\sigma_Y}$.

4. $f_{XYZ}(x, y, z) = \begin{cases} K(x + 2y + 3z), & 0 \le x, y, z \le 1 \\ 0, & \text{otherwize} \end{cases}$

a)
$$\int_0^1 \int_0^1 \int_0^1 K(x + 2y + 3z)\,dx\,dy\,dz = 1$$
$$K\int_0^1 \int_0^1 \int_0^1 x + 2y + 3z\,dx\,dy\,dz = 1$$
$$K\int_0^1 \int_0^1 \tfrac{1}{2}x^2 + 2yx + 3zx \Big|_0^1 dy\,dz = 1$$
$$K\int_0^1 \int_0^1 \tfrac{1}{2} + 2y + 3z\,dy\,dz = 1$$
$$K\int_0^1 \tfrac{1}{2}y + y^2 + 3zy \Big|_0^1 dz = 1$$
$$K\int_0^1 \tfrac{1}{2} + 1 + 3z\,dz = 1$$
$$K\int_0^1 \tfrac{3}{2} + 3z\,dz = 1$$
$$K\left(\tfrac{3}{2}z + \tfrac{3}{2}z^2 \Big|_0^1\right) = 1$$
$$3K = 1$$

$\therefore K = \frac{1}{3}$ is the constant that makes the function a joint pdf.

b) $f_X(x) = ?$

$$f_{XY}(x, y) = \int_0^1 \tfrac{1}{3}(x + 2y + 3z)\,dz$$
$$= \tfrac{1}{3}\int_0^1 x + 2y + 3z\,dz$$
$$= \tfrac{1}{3}\left(xz + 2yz + \tfrac{3}{2}z^2 \Big|_0^1\right)$$
$$= \tfrac{1}{3}\left(x + 2y + \tfrac{3}{2}\right)$$

$$f_X(x) = \int_0^1 f_{XY}(x,y)\,dy$$
$$= \int_0^1 \tfrac{1}{3}\left(x + 2y + \tfrac{3}{2}\right)dy$$
$$= \tfrac{1}{3}\int_0^1 x + 2y + \tfrac{3}{2}\,dy$$
$$= \tfrac{1}{3}\left(xy + y^2 + \tfrac{3}{2}y \Big|_0^1\right)$$
$$= \tfrac{1}{3}\left(x + 1 + \tfrac{3}{2}\right)$$
$$= \tfrac{1}{3}\left(x + \tfrac{5}{2}\right)$$
$$= \tfrac{1}{3}x + \tfrac{5}{6}$$

$$f_X(x) = \begin{cases} \tfrac{1}{3}x + \tfrac{5}{6}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

5.

$M_X(t)$ exists for $t \in [-\alpha, \alpha]$, where $\alpha > 0$...

Consider $Y = aX + b$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Recall: $M_X(t) = E[e^{tX}]$

$$\therefore M_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}]$$
$$= E[e^{taX + tb}]$$
$$= E[e^{taX} \cdot e^{tb}]$$
$$= E[e^{taX}] \cdot E[e^{tb}]$$

*We can separate these because e^{tb} has no rv associated with it.

*We were given that $M_X(t)$ exists (is finite) when $t \in [-\alpha, \alpha]$, where $\alpha > 0$. In the case of Y, we can see that e^{tb} will remain finite as long as t is a finite number (we already know b is because of the given info) and $E[e^{taX}]$ will remain finite as long as $t \in [\frac{-\alpha}{|a|}, \frac{\alpha}{|a|}]$:

Consider $t = \frac{-\alpha}{|a|}$

$a < 0$: $$M_Y\left(\frac{-\alpha}{|a|}\right) = E\left[e^{\frac{-\alpha}{\cancel{|a|}}\cancel{a}X}\right] \cdot E\left[e^{\left(\frac{-\alpha}{|a|}b\right)}\right]$$
$$= E[e^{\alpha X}] \cdot e^{\left(\frac{-\alpha}{|a|}b\right)}$$

We know this exists because of the given info regarding $M_X(t)$.

$a > 0$: $$M_Y\left(\frac{-\alpha}{|a|}\right) = E\left[e^{\frac{-\alpha}{\cancel{|a|}}\cancel{a}X}\right] \cdot E\left[e^{\left(\frac{-\alpha}{|a|}b\right)}\right]$$
$$= E[e^{-\alpha X}] \cdot e^{\left(\frac{-\alpha}{|a|}b\right)}$$

We know this exists because of the given info regarding $M_X(t)$.

Consider $t = \frac{\alpha}{|a|}$

$a < 0$: $$M_Y\left(\frac{\alpha}{|a|}\right) = E\left[e^{\frac{\alpha}{\cancel{|a|}}\cancel{a}X}\right] \cdot E\left[e^{\left(\frac{\alpha}{|a|}b\right)}\right]$$
$$= E[e^{-\alpha X}] \cdot e^{\left(\frac{\alpha}{|a|}b\right)}$$

We know this exists because of the given info regarding $M_X(t)$.

$a > 0$: $$M_Y\left(\frac{\alpha}{|a|}\right) = E\left[e^{\frac{\alpha}{\cancel{|a|}}\cancel{a}X}\right] \cdot E\left[e^{\left(\frac{\alpha}{|a|}b\right)}\right]$$
$$= E[e^{\alpha X}] \cdot e^{\left(\frac{\alpha}{|a|}b\right)}$$

We know this exists because of the given info regarding $M_X(t)$.

Notice though that t isn't defined when $a = 0$ because division by zero...

$$\therefore M_Y(t) = E[e^{taX}] \cdot E[e^{tb}] = E[e^{taX}] \cdot e^{tb}$$

As we can see for $M_Y(t)$, if $t \notin [\frac{-\alpha}{|a|}, \frac{\alpha}{|a|}]$, then we can conclude that $M_Y(t)$ will not exist because $E[e^{taX}]$ will not exist based on the information provided for $M_X(t)$.

6.

Let $X_1, X_2, \ldots, X_n$ be n independent rvs, where $X_i \sim Poi(\lambda_i)$.

$\therefore M_{X_i}(t) = e^{\lambda_i(e^t - 1)}$ *Based on mgf table in slides...

Let $Y = X_1 + X_2 + \ldots + X_n$

$$\begin{aligned}
\therefore M_Y(t) &= E[e^{tY}] \\
&= E[e^{t(X_1+X_2+\ldots+X_n)}] \\
&= E[e^{tX_1+tX_2+\ldots+tX_n}] \\
&= E[e^{tX_1} \cdot e^{tX_2} \cdot \ldots \cdot e^{tX_n}] \\
&= E[e^{tX_1}] \cdot E[e^{tX_2}] \cdot \ldots \cdot E[e^{tX_n}] \\
&= M_{X_1}(t) \cdot M_{X_2}(t) \cdot \ldots \cdot M_{X_n}(t)
\end{aligned}$$

$$\begin{aligned}
&= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \cdot \ldots \cdot e^{\lambda_n(e^t-1)} \\
&= e^{\lambda_1(e^t-1) + \lambda_2(e^t-1) + \ldots + \lambda_n(e^t-1)} \\
&= e^{(e^t-1)[\lambda_1+\lambda_2+\ldots\lambda_n]} \\
&= e^{(\lambda_1+\lambda_2+\ldots+\lambda_n)(e^t-1)}
\end{aligned}$$

*Notice that this is the mgf for the Poisson distribution, where $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

$\therefore Y \sim Poi(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$

7)

$$M_X(t) = \frac{1}{4} + \frac{1}{2}e^{t} + \frac{1}{4}e^{2t}$$

$E[X] = 1$

$Var[X] = \frac{1}{2}$

$$M_X'(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{2t}$$

$$M_X'(0) = E[X] = \frac{1}{2}(1) + \frac{1}{2}(1)$$

$$= 1$$

$$M_X''(t) = \frac{1}{2}e^{t} + e^{2t}$$

$$M_X''(0) = E[X^2] = \frac{1}{2} + 1$$

$$= \frac{3}{2}$$

$$\therefore Var[X] = E[X^2] - (E[X])^2$$

$$= \frac{3}{2} - 1$$

$$= \frac{1}{2}$$