Commutator Discussion

This is a supplement to the lecture notes on the quantum harmonic oscillator notes. I will complete some of the derivations relating to commutators.

Definition: The commutator between two operators is defined as follows:

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A} \tag{1}$$

Three useful properties of commutators:

"Sign" property:

$$\left[\hat{A},\hat{B}\right]=-\left[\hat{B},\hat{A}\right] \tag{2}$$

Distributive Property:

$$\begin{aligned}\left[\hat{A}+\hat{B},\hat{C}\right]&=\left(\hat{A}+\hat{B}\right)\hat{C}-\hat{C}\left(\hat{B}+\hat{A}\right)\\&=\hat{A}\hat{C}-\hat{C}\hat{A}+\hat{B}\hat{C}-\hat{C}\hat{B}\\&=\left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right]\end{aligned} \tag{3}$$

Associative Property (from left):

$$\begin{aligned}\left[\hat{A}\hat{B},\hat{C}\right]&=\hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B}\\&=\hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}\\&=\hat{A}\left(\hat{B}\hat{C}-\hat{C}\hat{B}\right)+\left(\hat{A}\hat{C}-\hat{C}\hat{A}\right)\hat{B}\\&=\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}\end{aligned} \tag{4}$$

There is a similar associative property from the right, can you see what it is?

The canonical commutator is between the position and momentum operators:

$$\left[\hat{x},\hat{p}\right]=i\hbar \tag{5}$$

This may be taken as a postulate or it can be shown to be consistent with the coordinate representation of the two operators:

$$\hat{\mathrm{p}}\Rightarrow\frac{\hbar}{i}\frac{d}{dx}\qquad\hat{x}\Rightarrow x \tag{6}$$

To see how the definition of Eq. (5) is consistent with Eq. (6) we introduce the concept of a test function, $f(x)$. The operators act on this function to produce a resultant function. We operate on the test function with the commutator and carry out the operations as far as possible using the chain rule for differentiation.

$$\begin{aligned}
\left[\hat{x},\hat{p}\right] f(x) &= (\hat{x}\hat{p}-\hat{p}\hat{x}) f(x) \\
&= \hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x) \\
&= x\left(\frac{\hbar}{i}\frac{d}{dx}\right) f(x) - \left(\frac{\hbar}{i}\frac{d}{dx}\right) x f(x) \\
&= \left(\frac{\hbar}{i}\right)\left[x\frac{df(x)}{dx} - \frac{d}{dx}\big(xf(x)\big)\right] \\
&= \left(\frac{\hbar}{i}\right)\left[x\frac{df(x)}{dx} - \frac{dx}{dx}f(x) - x\frac{df(x)}{dx}\right] \\
&= \left(\frac{\hbar}{i}\right)\left[-f(x)\right] \\
&= i\hbar f(x) \Rightarrow \left[\hat{x},\hat{p}\right] = i\hbar
\end{aligned} \tag{7}$$

As pointed out in Griffiths the commutator may be taken as a postulate and then the form for $\hat{p}$ given in Eq. (6) can be shown to follow.

Application to the Quantum Harmonic Oscillator

In discussions of the harmonic oscillator it is useful to define two new operators, called the creation and annihilation operators, formed as linear combinations of the position and momentum operators.

$$\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right) \tag{8}$$

$$\hat{a}^{\dagger} = \frac{\beta}{\sqrt{2}}\left(\hat{x} - \frac{i\hat{p}}{m\omega_0}\right) \tag{9}$$

Here we have defined the constant β to be

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}} \tag{10}$$

We wish to compute the commutator $\left[\hat{a},\hat{a}^{\dagger}\right]$. This may be accomplished by using the definitions in Eq's (8) an (9), the commutator Eq. (5) and the distributive property, Eq. (3).

Commutator Discussion

$$\begin{aligned}
\left[\hat{a},\hat{a}^{\dagger}\right] &= \left[\left(\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i\hat{p}}{m\omega_0}\right)\right),\left(\frac{\beta}{\sqrt{2}}\left(\hat{x}-\frac{i\hat{p}}{m\omega_0}\right)\right)\right] \\
&= \left(\frac{\beta}{\sqrt{2}}\right)^2\left[\left(\left(\hat{x}+\frac{i\hat{p}}{m\omega_0}\right)\right),\left(\hat{x}-\frac{i\hat{p}}{m\omega_0}\right)\right] \\
&= \left(\frac{\beta^2}{2}\right)\left[\left[\hat{x},\left(\hat{x}-\frac{i\hat{p}}{m\omega_0}\right)\right]+\left[\frac{i\hat{p}}{m\omega_0},\left(\hat{x}-\frac{i\hat{p}}{m\omega_0}\right)\right]\right] \\
&= \left(\frac{\beta^2}{2}\right)\left[[\hat{x},\hat{x}]-\left[\hat{x},\frac{i\hat{p}}{m\omega_0}\right]+\left[\frac{i\hat{p}}{m\omega_0},\hat{x}\right]-\left[\frac{i\hat{p}}{m\omega_0},\frac{i\hat{p}}{m\omega_0}\right]\right]
\end{aligned} \tag{11}$$

This may be simplified by recognizing that for any operator $\hat{A}$, $\left[\hat{A},\hat{A}\right]=0$. Then we have

$$\begin{aligned}
\left[\hat{a},\hat{a}^{\dagger}\right] &= \left(\frac{\beta^2}{2}\right)\left[[\hat{x},\hat{x}]-\left[\hat{x},\frac{i\hat{p}}{m\omega_0}\right]+\left[\frac{i\hat{p}}{m\omega_0},\hat{x}\right]-\left[\frac{i\hat{p}}{m\omega_0},\frac{i\hat{p}}{m\omega_0}\right]\right] \\
&= \left(\frac{\beta^2}{2}\right)\left[-\left[\hat{x},\frac{i\hat{p}}{m\omega_0}\right]+\left[\frac{i\hat{p}}{m\omega_0},\hat{x}\right]\right] \\
&= -2\left(\frac{\beta^2}{2}\right)\left(\frac{i}{m\omega_0}\right)[\hat{x},\hat{p}] \\
&= -i\left(\frac{\beta^2}{m\omega_0}\right)(i\hbar) \\
&= -\left(\frac{m\omega_0}{\hbar}\frac{\beta^2}{m\omega_0}\right)(i)^2\hbar \\
&= 1
\end{aligned} \tag{12}$$

Make sure you follow all of the steps.