Homework 4

Son [Joe] Nguyen

Problem 2.19

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n\pi x}{a}\right) \right]$$

We have:

$$e^{ix} = \cos x + i \sin x$$
$$\Rightarrow e^{i\left(\frac{n\pi x}{a}\right)} = \cos\left(\frac{n\pi x}{a}\right) + i \sin\left(\frac{n\pi x}{a}\right)$$
$$\Rightarrow e^{-i\left(\frac{n\pi x}{a}\right)} = \cos\left(\frac{n\pi x}{a}\right) - i \sin\left(\frac{n\pi x}{a}\right)$$

Adding and subtracting the two equations above, we have:

$$e^{i\left(\frac{n\pi x}{a}\right)} + e^{-i\left(\frac{n\pi x}{a}\right)} = 2\cos\left(\frac{n\pi x}{a}\right)$$
$$e^{i\left(\frac{n\pi x}{a}\right)} - e^{-i\left(\frac{n\pi x}{a}\right)} = 2i\sin\left(\frac{n\pi x}{a}\right)$$

$$\begin{aligned} f(x) &= b_o + \sum_{n=1}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n\pi x}{a}\right) \right] \\ &= b_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{i\left(\frac{n\pi x}{a}\right)} - e^{-i\frac{n\pi x}{a}}}{2i} \right) + b_n \left(\frac{e^{i\left(\frac{n\pi x}{a}\right)} + e^{-i\frac{n\pi x}{a}}}{2} \right) \\ &= b_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2i} + \frac{b_n}{2} \right) e^{i\left(\frac{n\pi x}{a}\right)} + \left(\frac{b_n}{2} - \frac{a_n}{2i} \right) e^{-i\left(\frac{n\pi x}{a}\right)} \end{aligned}$$

when $n = 0$, we have $c_0 = b_0$, when $n > 0$, we have $c_n = \frac{-ia_n + b_n}{2}$, and when $n < 0$, we have $c_n = \frac{ia_{-n} + b_{-n}}{2}$:

$$\begin{aligned} f(x) &= c_0 + \sum_{n=1}^{\infty} c_n e^{i\left(\frac{n\pi x}{a}\right)} + \sum_{n=-1}^{-\infty} c_n e^{i\left(\frac{n\pi x}{a}\right)} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{a}\right)} \end{aligned}$$

Using the equation above, we have:

$$f(x)e^{\frac{-in\pi x}{a}} = \sum_{k=-\infty}^{\infty} c_k e^{\frac{i(n-k)\pi x}{a}}$$

multiply both sides by $e^{\frac{-in\pi x}{a}}$

Now we integrate both sides from $-a$ to a:

$$\int_{-a}^{a} f(x)e^{\frac{-in\pi x}{a}}dx = \int_{-a}^{a} \sum_{k=-\infty}^{\infty} c_k e^{\frac{i(n-k)\pi x}{a}}dx \quad \text{now we simplify the right hand side of the equation}$$
$$= \sum_{k=-\infty}^{\infty} c_k \int_{-a}^{a} e^{\frac{i(n-k)\pi x}{a}}dx$$

For the intergration on the right hand side, when $k \neq n$, we have:

$$\begin{aligned}
\int_{-a}^{a} e^{\frac{i(n-k)\pi x}{a}}dx &= \frac{a}{i(n-k)\pi}\left[e^{\frac{i(n-k)\pi x}{a}}\right]_{-a}^{a} \\
&= \frac{a}{i(n-k)\pi}\left[e^{\frac{i(n-k)\pi a}{a}} - e^{\frac{-i(n-k)\pi a}{a}}\right] \\
&= \frac{a}{i(n-k)\pi}\left[e^{i(n-k)\pi} - e^{-i(n-k)\pi}\right] \\
&= \frac{a}{i(n-k)\pi}\left(\cos((n-k)\pi) + i\sin((n-k)\pi) - (\cos((k-n)\pi) - i\sin((k-n)\pi))\right) \\
&= 0
\end{aligned}$$

When $k = n$, we have:

$$\begin{aligned}
\int_{-a}^{a} e^{\frac{i(n-k)\pi x}{a}}dx &= \int_{-a}^{a} e^{\frac{i(n-n)\pi x}{a}}dx \\
&= \int_{-a}^{a} e^0 dx \\
&= \int_{-a}^{a} 1dx \\
&= 2a
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\int_{-a}^{a} f(x)e^{\frac{-in\pi x}{a}}dx &= \sum_{k=-\infty}^{\infty} c_k \int_{-a}^{a} e^{\frac{i(n-k)\pi x}{a}}dx \\
&= c_n \int_{-a}^{a} e^{\frac{i(n-n)\pi x}{a}}dx \\
&= c_n 2a
\end{aligned}$$

$$\Rightarrow c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{\frac{-in\pi x}{a}}dx$$

Substitute $k = \frac{n\pi}{a}$ into $f(x)$:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (\sqrt{2\pi} c_n) e^{ik_n x} \end{aligned}$$

Now we just need to prove that $\sqrt{2\pi} c_n = F(k)\Delta k$:

$$\begin{aligned} \sqrt{2\pi} c_n &= \sqrt{2\pi} \frac{1}{2a} \int_{-a}^{a} f(x) e^{\frac{-in\pi x}{a}} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{2\pi}{2a} \int_{-a}^{a} f(x) e^{\frac{-in\pi x}{a}} dx \\ &= \left[\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{\frac{-in\pi x}{a}} dx \right] \frac{\pi}{a} \\ &= F(k_n)\Delta k \end{aligned}$$

From the previous part, we have:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k_n) e^{ik_n x} \Delta k \quad , k_n = \frac{n\pi}{a}$$

$$F(k_n) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ik_n x} dx$$

For the function $f(x)$, when $a \to \infty$:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk \\ F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \end{aligned}$$

Problem 2.20

$$\Psi(x, 0) = A e^{-a|x|}$$

(a) Normalize the wave function:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx \\
&= \int_{-\infty}^{\infty} |Ae^{-a|x|}|^2 dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2a|x|} dx \\
&= A^2 2 \int_{0}^{\infty} e^{-2ax} dx \\
&= A^2 2 \frac{1}{2a} \\
\Rightarrow A &= \sqrt{a} \\
\Rightarrow \Psi(x,0) &= \sqrt{a} e^{-a|x|}
\end{aligned}$$

(b) we have:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega t)} dk$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx$$

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{a} e^{-a|x|} e^{-ikx} dx \\
&= \frac{\sqrt{a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx} dx \\
&= \frac{\sqrt{a}}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-a|x|-ikx} dx + \frac{\sqrt{a}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-a|x|-ikx} dx \\
&= \frac{\sqrt{a}}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-a(-x)-ikx} dx + \int_{0}^{\infty} e^{-ax-ikx} dx \right) \\
&= \frac{\sqrt{a}}{\sqrt{2\pi}} \left(\frac{1}{a-ik} + \frac{1}{a+ik} \right) \\
&= \sqrt{\frac{a}{2\pi}} \left(\frac{2a}{a^2+k^2} \right)
\end{aligned}$$

(c) Construct $\Psi(x,t)$ from equation (2.101):

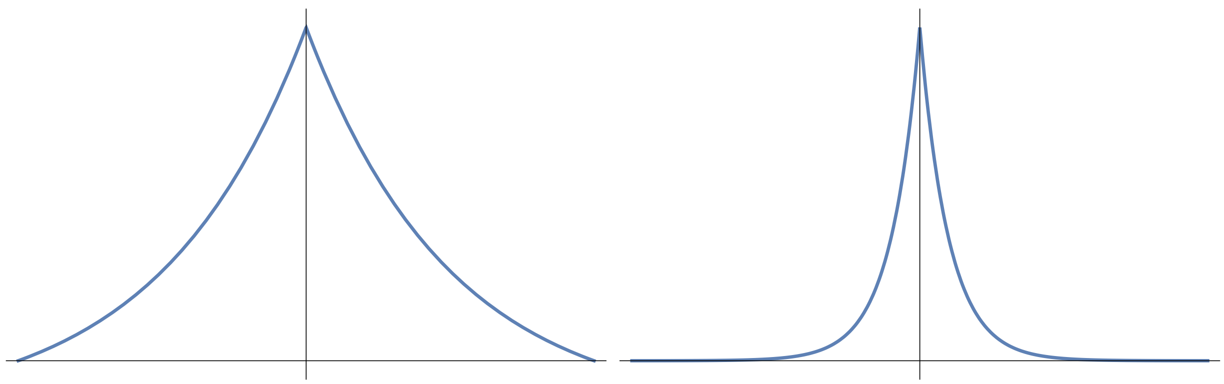
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{hk^2}{2m} t)} dk$$

Substitute $\phi(k)$ into the equation above:

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{a}{2\pi}} \left(\frac{2a}{a^2+k^2} \right) e^{i(kx - \frac{hk^2}{2m}t)} dk \\
&= \frac{a^{\frac{3}{2}}}{\pi} \int_{-\infty}^{\infty} \frac{1}{k^2+a^2} e^{i(kx - \frac{hk^2}{2m}t)} dk
\end{aligned}$$

(d) Discussing the limiting cases:

- When a is very large, the wave function is getting closer and closer to 0. And for $\phi(k)$, when a get larger, the function will approach $\sqrt{\frac{2}{\pi}}$



(a) (b)

Figure 1: $\Psi(x, 0)$

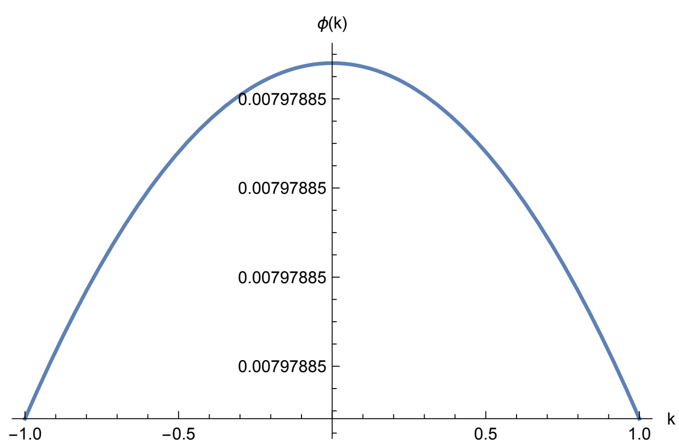


Figure 2: $\phi(k)$ when a = 10000

- when a is very small, the graphs will be reverse from what we have when a is large.

Problem 2.21

$$\Psi(x,0) = Ae^{-ax^2}$$

(a) Normalize the wave function:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx \\
&= \int_{-\infty}^{\infty} |Ae^{-ax^2}|^2 dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = A^2 \sqrt{\frac{\pi}{2a}} \\
\Rightarrow A &= \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}
\end{aligned}$$

(b) Find the wave function:

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-ax^2-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-ax^2-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-\frac{k^2}{4a}} \sqrt{\frac{\pi}{a}} = \frac{1}{(2a\pi)^{\frac{1}{4}}} e^{-\frac{k^2}{4a}} \\
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(2a\pi)^{\frac{1}{4}}} e^{-\frac{k^2}{4a}} e^{i(kx - \frac{\hbar k^2}{2m} t)} dk \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(2a\pi)^{\frac{1}{4}}} \int_{-\infty}^{\infty} e^{-\left(\frac{1}{4a} + \frac{i\hbar}{2m} t\right) k^2 + ikx} dk \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(2a\pi)^{\frac{1}{4}}} e^{-\frac{x^2}{4\left(\frac{1}{4a} + \frac{i\hbar}{2m} t\right)}} \sqrt{\frac{\pi}{\frac{1}{4a} + \frac{i\hbar}{2m} t}} \\
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{1}{\gamma} e^{\frac{-ax^2}{\gamma^2}}, \quad \gamma = \sqrt{1 + \frac{2i\hbar at}{m}}
\end{aligned}$$

(c) Find $|\Psi(x,t)|^2$:

$$\omega = \sqrt{\frac{a}{\left[1 + \left(\frac{2\hbar at}{m}\right)^2\right]}}$$

$$
\begin{aligned}
|\Psi(x,t)|^2 &= \Psi^*\Psi \\
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{\left(e^{\frac{-ax^2}{\gamma^2}}\right)^*}{\gamma^*} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{e^{\frac{-ax^2}{\gamma^2}}}{\gamma}
\end{aligned}
$$

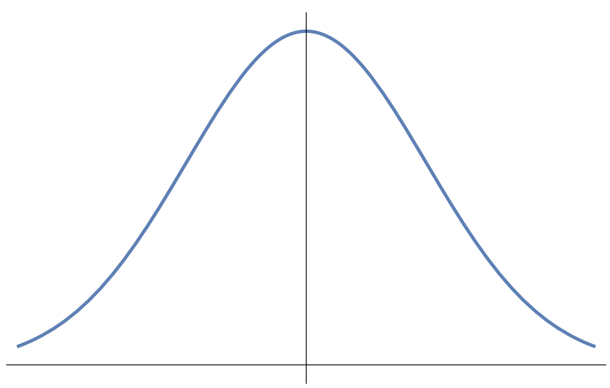
instead of setting $\gamma = \sqrt{1 + \frac{2i\hbar at}{m}}$, we set $\gamma = \frac{2\hbar at}{m}$, then we have:

$$
\begin{aligned}
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{\left(e^{-\frac{ax^2}{1+i\gamma}}\right)^*}{\left(\sqrt{1+i\gamma}\right)^*} \frac{\left(e^{-\frac{ax^2}{1+i\gamma}}\right)}{\left(\sqrt{1+i\gamma}\right)} \\
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{\left(e^{-\frac{ax^2}{1-i\gamma}}\right)}{\left(\sqrt{1-i\gamma}\right)} \frac{\left(e^{-\frac{ax^2}{1+i\gamma}}\right)}{\left(\sqrt{1+i\gamma}\right)} \\
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{ax^2}{1-i\gamma}} e^{-\frac{ax^2}{1+i\gamma}}}{\sqrt{1-i\gamma}\sqrt{1+i\gamma}} \\
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{2ax^2}{1+\gamma^2}}}{\sqrt{1+\gamma^2}}
\end{aligned}
$$

We can rewrite $\omega = \sqrt{\frac{a}{(1+\gamma^2)}}$, then $|\Psi(x,t)|^2$ can be written as:

$$
\sqrt{\frac{2}{\pi}}\omega e^{-2\omega^2 x^2}
$$

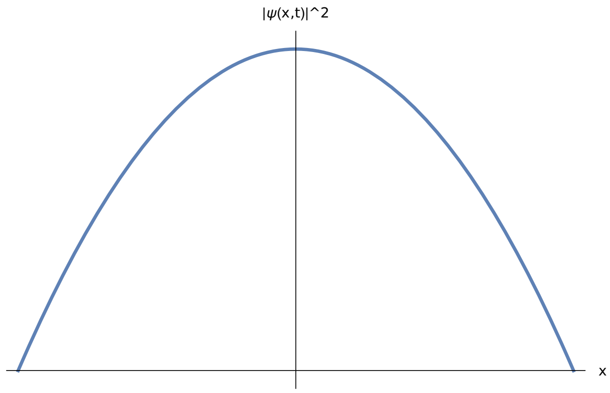
For t = 0, we have:



(a)

Figure 3: $|\Psi(x,0)|^2$

For t very large we have:



(a)

Figure 4: $|\Psi(x,0)|^2, \quad t \to \infty$

(d) Find $\langle x \rangle, \langle p \rangle, \langle x^2 \rangle, \sigma_x, \sigma_p$:

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 dx \\
&= \int_{-\infty}^{\infty} x \sqrt{\frac{2}{\pi}} \omega e^{-2\omega^2 x^2} dx \\
&= \sqrt{\frac{2}{\pi}} \omega \int_{-\infty}^{\infty} x e^{-2\omega^2 x^2} dx = 0 \quad \text{because } x e^{-2\omega^2 x^2} \text{is an odd function} \\
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} = 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 |\Psi(x,t)|^2 dx \\
&= \sqrt{\frac{2}{\pi}} \omega \int_{-\infty}^{\infty} x^2 e^{-2\omega^2 x^2} dx \\
&= \sqrt{\frac{2}{\pi}} \omega \frac{\sqrt{\pi}}{2(2\omega^2)\sqrt{2\omega^2}} = \frac{1}{4\omega^2} \\
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{2\omega} \\
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{a}\hbar
\end{aligned}
$$

(e) The uncertainty principle $\sigma_x \sigma_p \geq \frac{\hbar}{2}$:

$$
\sigma_x \sigma_p = \frac{1}{2\omega} \sqrt{a}\hbar = \frac{1}{2} \sqrt{\frac{1+\gamma^2}{a}} \hbar \sqrt{a} = \frac{\sqrt{1+\gamma^2}}{2} \hbar \geq \frac{\hbar}{2}
$$

Problem 2.22

$$\int_{-3}^{1}(x^3-3x^2+2x-1)\delta(x+2)dx$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

$$\int_{-3}^{1}(x^3-3x^2+2x-1)\delta(x+2)dx = (-2)^3-3(-2)^2+2(-2)-1 = -8-12-4-1 = -25$$

$$\int_{0}^{\infty} [\cos(3x)+2]\,\delta(x-\pi)dx = \cos(3\pi)+2 = -1$$

$$\int_{-1}^{1} \exp(|x|+3)\delta(x-2)dx = 0$$

Problem 2.23

(a) $\delta(cx) = \frac{1}{|c|}\delta(x)$

$$\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \int_{-\infty}^{\infty} f(x)\frac{1}{|c|}\delta(x)dx = \frac{1}{|c|}f(0), \quad c \in \mathbb{R}$$

(b) $\frac{\partial\theta}{dx} = \delta(x)$

$$\int_{-\infty}^{\infty} f(x)\frac{\partial\theta}{dx}dx = \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

Problem 2.26

$$\delta(x) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}$$

Substitute back to the first equation, we have:

$$\delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk$$