Homework 3

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Problem 2.10

From equation 2.60 in the text book, we have:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

From equation 2.48, we have:

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{p} + m\omega x\right)$$

And the formula for $\psi_n(x)$ is:

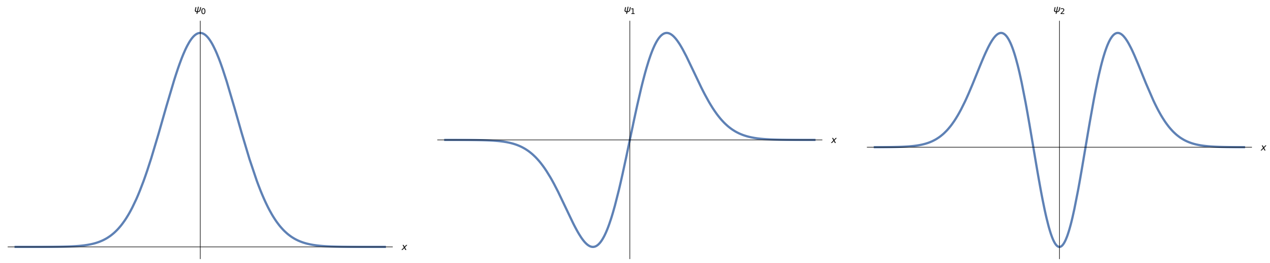
$$\psi_n = \frac{1}{\sqrt{n!}}\left(\hat{a}_+\right)^n \psi_0$$

Therefore we have $\psi_2(x) = \frac{1}{\sqrt{2!}}(\hat{a}_+)^2\psi_0$

We can calculate $(\hat{a}_+)^2\psi_0$ as follows:

$$\begin{aligned}
\hat{a}_+\psi_0 &= \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \quad \text{where } \hat{p} = -i\hbar\frac{d}{dx} \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[\hbar\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x\left(e^{-\frac{m\omega}{2\hbar}x^2}\right)\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[2m\omega x\left(e^{-\frac{m\omega}{2\hbar}x^2}\right)\right] \\
\left(\hat{a}_+\right)^2\psi_0 &= \frac{2m\omega}{2\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(xe^{-\frac{m\omega}{2\hbar}x^2}\right) \\
&= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(-\hbar\frac{d}{dx}xe^{-\frac{m\omega}{2\hbar}x^2} + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right) \\
&= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(-\hbar e^{-\frac{m\omega}{2\hbar}x^2} + 2m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right)
\end{aligned}$$

So $\psi_2(x) = \frac{1}{\sqrt{2}}\frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(-\hbar e^{-\frac{m\omega}{2\hbar}x^2} + 2m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right)$



(a) Plot of $\psi_0(x)$ (b) Plot of $\psi_1(x)$ (c) Plot of $\psi_2(x)$

Figure 1: Plots of $\psi_n(x)$ for $n = 0, 1, 2$

To check the orthogonality of the wave functions, we can calculate the integral of the product of the wave functions over the range of $[-\infty, \infty]$. If the integral is zero, then the wave functions are orthogonal. As we can see from the sketch. We have $\psi_0(x)$ is an even function, $\psi_1(x)$ is an odd function, and $\psi_2(x)$ is an even function. Therefore, the product of $\psi_0(x)$ and $\psi_1(x)$ is an odd function, and the product of $\psi_1(x)$ and $\psi_2(x)$ is also an odd function. So the integral of the product of $\psi_0(x)$ and $\psi_1(x)$ over the range of $[-\infty, \infty]$ is zero, and the integral of the product of $\psi_1(x)$ and $\psi_2(x)$ over the range of $[-\infty, \infty]$ is also zero. Therefore, the wave functions are orthogonal.

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0(x)\psi_2(x)dx &= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \frac{1}{\sqrt{2}} \frac{1}{\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left(-\hbar e^{-\frac{m\omega}{2\hbar}x^2} + 2m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right) dx \\ &= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left(\frac{2m\omega x^2}{\hbar} - 1\right) e^{-\frac{m\omega}{2\hbar}x^2} dx = 0\end{aligned}$$

Problem 2.11

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

Set constant $A = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$, we have:

$$\psi_0(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} x|\psi_0(x)|^2 dx \\ &= \int_{-\infty}^{\infty} x(Ae^{-\frac{m\omega}{2\hbar}x^2})^2 dx = 0\end{aligned}$$

We have $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = 0$, because $\langle x\rangle = 0$. Therefore, the expectation value of the momentum is zero.

$$
\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 |\psi_0(x)|^2 dx \\
&= \int_{-\infty}^{\infty} x^2 (Ae^{-\frac{m\omega}{2\hbar}x^2})^2 dx = \frac{\hbar}{2m\omega}
\end{aligned}
$$

$$
\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi(x,t)^* \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \Psi(x,t) dx
$$

$$
\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_0(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi_0(x) dx \\
&= \int_{-\infty}^{\infty} (Ae^{-\frac{m\omega}{2\hbar}x^2}) \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 (Ae^{-\frac{m\omega}{2\hbar}x^2}) dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} Ae^{-\frac{m\omega}{2\hbar}x^2} \frac{d^2}{dx^2}(Ae^{-\frac{m\omega}{2\hbar}x^2}) dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} Ae^{-\frac{m\omega}{2\hbar}x^2} A \left(-\frac{e^{\frac{-m\omega x^2}{2\hbar}}}{h} + \frac{e^{\frac{-m\omega x^2}{2\hbar}} m^2\omega^2 x^2}{\hbar^2}\right) dx \\
&= \frac{m\omega\hbar}{2}
\end{aligned}
$$

We have the equation of $\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}} x e^{\frac{m\omega}{2\hbar}x^2}$

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x |\psi_1(x)|^2 dx \\
&= \int_{-\infty}^{\infty} x \left(\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}} x e^{\frac{m\omega}{2\hbar}x^2}\right)^2 dx = 0
\end{aligned}
$$

Therefore, $\langle p \rangle = 0$

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 |\psi_1(x)|^2 dx \quad \text{set } \alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \text{ and } \beta = \sqrt{\frac{m\omega}{\hbar}} x \\
&= \int_{-\infty}^{\infty} x^2 \left(\alpha\sqrt{2}\beta e^{\frac{-y^2}{2}}\right)^2 dx \\
&= \int_{-\infty}^{\infty} \frac{\hbar}{m\omega}\beta^2 (2\alpha^2\beta^2 e^{-y^2}) \sqrt{\frac{\hbar}{m\omega}} d\beta \\
&= \frac{3\alpha^2\hbar^2\sqrt{\pi}}{2m^2\sqrt{\frac{\hbar\omega}{m}}} \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_1(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi_1(x) dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \psi_1(x) \frac{d^2\psi_1(x)}{dx^2} dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \left(\alpha\sqrt{2}\beta e^{\frac{-y^2}{2}}\right) \frac{d^2}{dx^2} \left(\alpha\sqrt{2}\beta e^{\frac{-y^2}{2}}\right) \sqrt{\frac{\hbar}{m\omega}} d\beta \\
&= -\hbar^2 \int_{-\infty}^{\infty} \sqrt{2}\alpha\beta e^{\frac{-y^2}{2}} \left(\sqrt{2}\alpha\frac{m\omega}{\hbar}(-3y + y^3) e^{\frac{-y^2}{2}}\right) \sqrt{\frac{\hbar}{m\omega}} d\beta \\
&= \frac{3m\omega\hbar}{2}
\end{aligned}$$

We need to prove the uncertainty principle that $\sigma_x \sigma_p \geq \frac{\hbar}{2}$ from equation (1.40)
For $\psi_0(x)$:

$$\begin{aligned}
\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 &= \frac{\hbar}{2m\omega} \\
\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 &= \frac{m\omega\hbar}{2} \\
\sigma_x \sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}} &= \frac{\hbar}{2}
\end{aligned}$$

For $\psi_1(x)$:

$$\begin{aligned}
\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 &= \frac{3\alpha^2\hbar^2\sqrt{\pi}}{2m^2\sqrt{\frac{\hbar\omega}{m}}} \\
\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 &= \frac{3m\omega\hbar}{2} \\
\sigma_x \sigma_p = \sqrt{\frac{3\alpha^2\hbar^2\sqrt{\pi}}{2m^2\sqrt{\frac{\hbar\omega}{m}}}}\sqrt{\frac{3m\omega\hbar}{2}} &= \frac{3\hbar}{2}
\end{aligned}$$

The expected value of $\langle T\rangle$ and $\langle V\rangle$ for ψ_0 are:

$$\begin{aligned}\langle T\rangle &= \frac{\langle p^2\rangle}{2m} = \frac{\omega\hbar}{4}\\ \langle V\rangle &= \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{\hbar\omega}{4}\end{aligned}$$

The expected value of $\langle T\rangle$ and $\langle V\rangle$ for ψ_1 are:

$$\begin{aligned}\langle T\rangle &= \frac{\langle p^2\rangle}{2m} = \frac{3\omega\hbar}{4}\\ \langle V\rangle &= \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{3\hbar\omega}{4}\end{aligned}$$

Problem 2.12

From equation 2.5:

$$\begin{aligned}x &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)\\ \hat{p} &= i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)\end{aligned}$$

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty}\psi_n^* x\psi_n dx = \int_{-\infty}^{\infty}\psi_n^*\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)\psi_n dx = 0\\ \langle p\rangle &= m\frac{d\langle x\rangle}{dt} = 0\end{aligned}$$

From example 2.5, we have:

$$\begin{aligned}\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle &= \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)\\ \Rightarrow \langle x^2\rangle &= (\frac{1}{2}+n)\frac{\hbar}{m\omega}\end{aligned}$$

Now we find $\langle p^2\rangle$:

$$\begin{aligned}\langle p^2\rangle &= \int_{-\infty}^{\infty}\psi_n^*\hat{p}^2\psi_n dx\\ &= \int_{-\infty}^{\infty}\psi_n^*\left(i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)\right)^2\psi_n dx\\ &= \int_{-\infty}^{\infty}\psi_n^*\left(\frac{-\hbar m\omega}{2}\right)\left(\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right)\psi_n dx\\ &= \left(\frac{-\hbar m\omega}{2}\right)\int_{-\infty}^{\infty}\psi_n^*\left(\hat{a}_+\hat{a}_-\psi_n + \hat{a}_-\hat{a}_+\psi_n\right)dx\\ &= \left(\frac{-\hbar m\omega}{2}\right)\int_{-\infty}^{\infty}n|\psi_n|^2 + (n+1)|\psi_n|^2 dx\\ &= \left(\frac{-\hbar m\omega}{2}\right)(2n+1)\end{aligned}$$

Problem 2.13

$$\Psi(x,0) = A[3\psi_0(x) + 4\psi_1(x)]$$

Normalize the wave function:

$$\begin{aligned}
|\Psi(x,0)|^2 &= |A|^2[9|\psi_0(x)|^2 + 24|\psi_0(x)\psi_1(x)| + 16|\psi_1(x)|^2] = 1 \\
&= |A|^2(9 + 16 + 24 \cdot 0) = 1 \\
&= |A|^2 = \frac{1}{25} \\
&= A = \frac{1}{5} \\
\Rightarrow \Psi(x,0) &= \frac{1}{5}[3\psi_0(x) + 4\psi_1(x)]
\end{aligned}$$

We have $\Psi(x,t) = \sum_{n=0}^{\infty} c_n\psi_n(x)e^{\frac{-iE_nt}{\hbar}}$ and $\Psi(x,0) = \frac{1}{5}[3\psi_0(x) + 4\psi_1(x)]$:

$$\begin{aligned}
\Rightarrow \Psi(x,0) &= \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x) = \sum_{n=0}^{\infty} c_n\psi_n(x) \quad \text{plug in } t = 0 \\
\Rightarrow c_0 &= \frac{3}{5} \quad \text{and} \quad c_1 = \frac{4}{5} \quad \text{and} \quad c_n = 0 \quad \text{for } n \neq 0, 1 \\
\Rightarrow \Psi(x,t) &= \frac{3}{5}\psi_0(x)e^{\frac{-iE_0t}{\hbar}} + \frac{4}{5}\psi_1(x)e^{\frac{-iE_1t}{\hbar}} \\
\Rightarrow \Psi(x,t) &= \frac{3}{5}\psi_0(x)e^{\frac{-i\omega t}{2}} + \frac{4}{5}\psi_1(x)e^{\frac{-3i\omega t}{2}} \quad \text{where } E_n = (n + \frac{1}{2})\hbar\omega
\end{aligned}$$

$$\begin{aligned}
|\Psi(x,t)|^2 &= \Psi(x,t)^* \cdot \Psi(x,t) \\
&= \left(\frac{3}{5}\psi_0(x)e^{\frac{i\omega t}{2}} + \frac{4}{5}\psi_1(x)e^{\frac{3i\omega t}{2}}\right) \cdot \left(\frac{3}{5}\psi_0(x)e^{\frac{-i\omega t}{2}} + \frac{4}{5}\psi_1(x)e^{\frac{-3i\omega t}{2}}\right) \\
&= \frac{1}{25}\left(9\psi_0(x)^2 + 16\psi_1(x)^2 + 12\psi_0(x)\psi_1(x)e^{-i\omega t} + 12\psi_0(x)\psi_1(x)e^{i\omega t}\right) \\
&= \frac{1}{25}\left(9\psi_0(x)^2 + 16\psi_1(x)^2 + 24\psi_0(x)\psi_1(x)\cos(\omega t)\right)
\end{aligned}$$

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} x\frac{1}{25}\left(9\psi_0(x)^2 + 16\psi_1(x)^2 + 24\psi_0(x)\psi_1(x)\cos(\omega t)\right)dx \\
&= \frac{1}{25}\int_{-\infty}^{\infty}(9\langle x\rangle_0 + 16\langle x\rangle_1 + 24\cos(\omega t)x\psi_0(x)\psi_1(x)) \quad \text{where } \langle x\rangle_n \text{ is the expected value of } x \text{ for } \psi_n \\
&= \frac{24}{25}\cos(\omega t)\int_{-\infty}^{\infty} x\psi_0(x)\psi_1(x)dx \quad \text{plug in } x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
&= \frac{24}{25}\cos(\omega t)\int_{-\infty}^{\infty}\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\psi_0(x)\psi_1(x)dx
\end{aligned}$$

$$= \frac{24}{25}\cos(\omega t)\int_{-\infty}^{\infty}\sqrt{\frac{\hbar}{2m\omega}}(a_+\psi_0 + a_-\psi_0)\psi_1 dx \quad \text{where } (a_-\psi_0) = 0 \text{ and } a_+\psi_n = \sqrt{n+1}\psi_n$$

$$= \frac{24}{25}\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty}\psi_1^2 dx = \frac{24}{25}\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -\frac{24}{25}\sin(\omega t)\sqrt{\frac{\hbar m\omega}{2}}$$

Ehnrenfest's theorem states that $\frac{d\langle p\rangle}{dt} = \langle -\frac{\partial V}{\partial x}\rangle$:

$$\frac{d\langle p\rangle}{dt} = -\frac{24}{25}\cos(\omega t)\omega\sqrt{\frac{\hbar m\omega}{2}}$$

$$\langle -\frac{\partial V}{\partial x}\rangle = \langle -\frac{\partial}{\partial V}\left(\frac{1}{2}m\omega^2 x^2\right)\rangle = -m\omega^2\langle x\rangle = -\frac{24}{25}\cos(\omega t)\omega\sqrt{\frac{\hbar m\omega}{2}}$$

To measure the energy of a wavefunction. The probabilty is the constant $|c_n|^2$. So for the wavefunction $\Psi(x,t)$, the probability of measuring the energy E_0 is $|c_0|^2 = \frac{9}{25}$ and the probability of measuring the energy E_1 is $|c_1|^2 = \frac{16}{25}$.