Homework 6

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Problem 3.5

(a)

$$\begin{aligned}
\langle f|xf\rangle &= \int_{-\infty}^{\infty} f^* x f dx \\
&= \int_{-\infty}^{\infty} x f^* f dx \quad \text{(since } x \text{ is real: } x = x^*\text{)} \\
&= \int_{-\infty}^{\infty} (xf)^* f dx \\
&= \langle xf|f\rangle \Rightarrow x^\dagger = x
\end{aligned}$$

$$\begin{aligned}
\langle f|if\rangle &= \int_{-\infty}^{\infty} f^* i f dx \\
&= \int_{-\infty}^{\infty} i f^* f dx \\
&= \int_{-\infty}^{\infty} (-if)^* f dx \\
&= \langle -if|f\rangle \Rightarrow i^\dagger = -i
\end{aligned}$$

$$\begin{aligned}
\langle f|\frac{d}{dx}f\rangle &= \int_{-\infty}^{\infty} f^* \frac{df}{dx} dx \\
&= f^* f\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df^*}{dx} f dx \\
&= -\int_{-\infty}^{\infty} \frac{df^*}{dx} f dx = \langle -\frac{d}{dx}f|f\rangle \Rightarrow -\frac{d}{dx} = \frac{d}{dx}^\dagger
\end{aligned}$$

(b)

$$\begin{aligned}
\langle f|\hat{Q}\hat{R}f\rangle &= \int_{-\infty}^{\infty} f^*\hat{Q}\hat{R}f dx \\
&= \int_{-\infty}^{\infty} f^*\hat{Q}\underbrace{\left(\hat{R}f\right)}_{g} dx \\
&= \langle f|\hat{Q}g\rangle = \langle \hat{Q}^\dagger f|g\rangle \\
&= \int_{-\infty}^{\infty} \underbrace{(\hat{Q}^\dagger f)^*}_{h}(\hat{R}f)dx \\
&= \langle \hat{R}^\dagger h|f\rangle = \langle \hat{R}^\dagger\hat{Q}^\dagger f|f\rangle \Rightarrow (\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger
\end{aligned}$$

Next, we need to prove $(\hat{Q}+\hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger$

$$\begin{aligned}
\langle f(\hat{Q}+\hat{R})|f\rangle &= \int_{-\infty}^{\infty} f^*(\hat{Q}+\hat{R})f dx \\
&= \int_{-\infty}^{\infty} f^*\left(\hat{Q}f+\hat{R}f\right) dx \\
&= \langle f|\hat{Q}f\rangle + \langle f|\hat{R}f\rangle \\
&= \langle \hat{Q}^\dagger f|f\rangle + \langle \hat{R}^\dagger f|f\rangle \\
&= \int_{-\infty}^{\infty} (\hat{Q}^\dagger f)^* f + (\hat{R}^\dagger f)^* f dx \\
&= \int_{-\infty}^{\infty} \left(\hat{Q}^\dagger f + \hat{R}^\dagger f\right)^* f dx \\
&= \int_{-\infty}^{\infty} \left[\left(\hat{Q}^\dagger + \hat{R}^\dagger\right) f\right]^* f dx \\
&= \langle\left(\hat{Q}^\dagger + \hat{R}^\dagger\right) f|f\rangle \Rightarrow (\hat{Q}+\hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger
\end{aligned}$$

$$\begin{aligned}
\langle f|c\hat{Q}f\rangle &= \int_{-\infty}^{\infty} f^* c\hat{Q} f dx \\
&= c\langle f|\hat{Q}f\rangle = c\langle \hat{Q}^\dagger f|f\rangle \\
&= \int_{-\infty}^{\infty} c(\hat{Q}^\dagger f)^* f dx \\
&= \int_{-\infty}^{\infty} (c^*\hat{Q}^\dagger f)^* f dx \\
&= \langle c^*\hat{Q}^\dagger f|f\rangle \Rightarrow (c\hat{Q})^\dagger = c^*\hat{Q}^\dagger
\end{aligned}$$

(c) Equation(2.48):

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x)$$

$$
\begin{aligned}
\hat{a}_+^\dagger &= \left[\frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)\right]^\dagger \\
&= \left(\frac{1}{\sqrt{2\hbar m\omega}}\right)^* \left[(-i\hat{p})^\dagger + (m\omega x)^\dagger\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \left[i\hat{p}^\dagger + m\omega x\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \left[i\left(-i\hbar\frac{d}{dx}\right)^\dagger + m\omega x\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \left[i\underbrace{\left(i\hbar\left(-\frac{d}{dx}\right)\right)}_{\hat{p}} + m\omega x\right] = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega x) = \hat{a}_- \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \left[\hbar\frac{d}{dx} + m\omega x\right]
\end{aligned}
$$

Problem 3.14

(a) Prove equation (3.64): $[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$

$$
\begin{aligned}
[\hat{A} + \hat{B}, \hat{C}]f(x) &= [(\hat{A} + \hat{B})\hat{C} - \hat{C}(\hat{A} + \hat{B})]f(x) \\
&= \hat{A}\hat{C}f(x) + \hat{B}\hat{C}f(x) - \hat{C}\hat{A}f(x) - \hat{C}\hat{B}f(x) \\
&= \hat{A}\hat{C}f(x) - \hat{C}\hat{A}f(x) + \hat{B}\hat{C}f(x) - \hat{C}\hat{B}f(x) \\
&= [\hat{A}, \hat{C}]f(x) + [\hat{B}, \hat{C}]f(x) \\
&\Rightarrow [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]
\end{aligned}
$$

Prove equation (3.65): $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

$$
\begin{aligned}
[\hat{A}\hat{B}, \hat{C}]f(x) &= \hat{A}\hat{B}\hat{C}f(x) - \hat{C}\hat{A}\hat{B}f(x) \\
[\hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}]f(x) &= \hat{A}\hat{B}\hat{C}f(x) \underbrace{-\hat{A}\hat{C}\hat{B}f(x) + \hat{A}\hat{C}\hat{B}f(x)}_{=0} - \hat{C}\hat{A}\hat{B}f(x) \\
&= [\hat{A}\hat{B}, \hat{C}]f(x) \Rightarrow [\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}
\end{aligned}
$$

(b) Prove $[x^n, \hat{p}] = i\hbar n x^{n-1}$

$$
\begin{aligned}
[x^n, \hat{p}]f(x) &= x^n \hat{p} f(x) - \hat{p} x^n f(x) \\
&= x^n \left(-i\hbar \frac{d}{dx}\right) f(x) - \left(-i\hbar \frac{d}{dx}\right) x^n f(x) \\
&= -x^n i\hbar \frac{d}{dx} f(x) + i\hbar \frac{d}{dx}(x^n f(x)) \\
&= -x^n i\hbar \frac{df}{dx} + i\hbar(n x^{n-1} f(x) + x^n \frac{df}{dx}) \\
&= -x^n i\hbar \frac{df}{dx} + i\hbar n x^{n-1} f(x) + i\hbar x^n \frac{df}{dx} \\
&= i\hbar n x^{n-1} f(x)
\end{aligned}
$$

(c)

$$
\begin{aligned}
[f(x), \hat{p}]F &= f(x)\hat{p}F - \hat{p} f(x) F \\
&= f(x)\left(-i\hbar \frac{d}{dx}\right) F - \left(-i\hbar \frac{d}{dx}\right) f(x) F \\
&= f(x)(-i\hbar \frac{dF}{dx}) + i\hbar \frac{df}{dx} F + i\hbar f(x) \frac{dF}{dx} \\
&= i\hbar \frac{df}{dx} F \Rightarrow [f(x), \hat{p}] = i\hbar \frac{df}{dx}
\end{aligned}
$$

(d) Equation (2.54): $\hat{H} = \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2})$

$$
\begin{aligned}
[\hat{H}, \hat{a}_\pm] &= \hat{H}\hat{a}_\pm - \hat{a}_\pm \hat{H} \\
&= \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2})\hat{a}_\pm - \hat{a}_\pm \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2}) \\
&= \hbar\omega(\hat{a}_-\hat{a}_+\hat{a}_\pm - \frac{1}{2}\hat{a}_\pm) - \hbar\omega(\hat{a}_\pm\hat{a}_-\hat{a}_+ - \hat{a}_\pm \frac{1}{2}) \\
&= \hbar\omega(\hat{a}_-\hat{a}_+\hat{a}_\pm - \frac{1}{2}\hat{a}_\pm - \hat{a}_\pm\hat{a}_-\hat{a}_+ + \hat{a}_\pm \frac{1}{2}) \\
&= \hbar\omega(\hat{a}_-\hat{a}_+\hat{a}_\pm - \hat{a}_\pm\hat{a}_-\hat{a}_+) \\
&= \hbar\omega(\hat{a}_-\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_-\hat{a}_+) \quad \text{or} \quad \hbar\omega(\hat{a}_-\hat{a}_+\hat{a}_+ - \hat{a}_+\hat{a}_-\hat{a}_+)
\end{aligned}
$$

We know that $[\hat{a}_-, \hat{a}_+] = 1$ and $[\hat{a}_+, \hat{a}_-] = -1$

$$
\begin{aligned}
[\hat{H}, \hat{a}_-] &= \hbar\omega(\hat{a}_-\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_-\hat{a}_+) \\
&= \hbar\omega\hat{a}_-[\hat{a}_+, \hat{a}_-] = -\hbar\omega\hat{a}_- \\
[\hat{H}, \hat{a}_+] &= \hbar\omega(\hat{a}_-\hat{a}_+\hat{a}_+ - \hat{a}_+\hat{a}_-\hat{a}_+) \\
&= \hbar\omega[\hat{a}_-, \hat{a}_+]\hat{a}_+ = \hbar\omega\hat{a}_+
\end{aligned}
$$

$$
\Rightarrow [\hat{H}, \hat{a}_\pm] = \pm\hbar\omega\hat{a}_\pm
$$

Problem 3.20

The energy-time uncertainty principle is given by:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

From equation (3.75) we have $\Delta E = \sigma_H$ and $\Delta_t = \frac{\sigma_x}{\left|\frac{d\langle x\rangle}{dt}\right|}$

$$\Rightarrow \sigma_H \frac{\sigma_x}{\left|\frac{d\langle x\rangle}{dt}\right|} \geq \frac{\hbar}{2}$$

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)]$$

For this equation, we have:

$$A = \frac{1}{\sqrt{2}}$$

$$\langle H\rangle = \frac{1}{2}E_1 + \frac{1}{2}E_2 = \frac{5\pi^2\hbar^2}{4ma^2}$$

$$\langle H^2\rangle = \frac{1}{2}E_1^2 + \frac{1}{2}E_2^2 = \frac{17\hbar^4\pi^4}{8m^2a^4}$$

$$\langle x\rangle = \frac{a}{2} - \frac{16a}{9\pi^2}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)$$

$$\langle x^2\rangle = \frac{a^2}{3} - \frac{5a^2}{16\pi^2} - \frac{16a^2}{9\pi^2}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)$$

$$\frac{d\langle x\rangle}{dt} = \frac{8\hbar}{3ma}\sin\left(\frac{3\hbar\pi^2}{2ma^2}t\right)$$

$$\Delta E \Delta t = \sqrt{\left(\frac{17\hbar^4\pi^4}{8m^2a^4}\right) - \left(\frac{5\pi^2\hbar^2}{4ma^2}\right)^2} \frac{\sqrt{\left(\frac{a^2}{3} - \frac{5a^2}{16\pi^2} - \frac{16a^2}{9\pi^2}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)\right) - \left(\frac{a}{2} - \frac{16a}{9\pi^2}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)\right)^2}}{\left|\frac{8\hbar}{3ma}\sin\left(\frac{3\hbar\pi^2}{2ma^2}t\right)\right|} \geq \frac{\hbar}{2}$$

Problem 3.24

Let operator $\hat{Q}$ has the othonormal basis $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$, the matrix representing the operator can be written as:

$$\begin{aligned}
\hat{Q}|v_1\rangle &= \hat{Q}_{11}|v_1\rangle + \hat{Q}_{21}|v_2\rangle + \cdots + \hat{Q}_{n1}|v_n\rangle \\
\hat{Q}|v_2\rangle &= \hat{Q}_{12}|v_1\rangle + \hat{Q}_{22}|v_2\rangle + \cdots + \hat{Q}_{n2}|v_n\rangle \\
&\vdots \\
\hat{Q}|v_n\rangle &= \hat{Q}_{1n}|v_1\rangle + \hat{Q}_{2n}|v_2\rangle + \cdots + \hat{Q}_{nn}|v_n\rangle
\end{aligned}$$

or

$$\hat{Q}|v_i\rangle = \sum_{j=1}^{n} \hat{Q}_{ji}|v_j\rangle$$

The goal is to find the matrix element $\hat{Q}_{ji}$, and we can do that by taking the inner product of both sides:

$$\langle v_j|\hat{Q}|v_i\rangle = \hat{Q}_{ji}$$

Take the complex of both sides:

$$\langle v_j|\hat{Q}|v_i\rangle^* = \hat{Q}^*_{ji} = \langle v_i|\underbrace{\hat{Q}^\dagger}_{\hat{Q}}|v_j\rangle = \hat{Q}_{ij}$$

Problem 3.25

Since $|1\rangle, |2\rangle$ are orthonormal basis, we have:

$$\begin{aligned}
\hat{H}|1\rangle &= H_{11}|1\rangle + H_{21}|2\rangle \\
&= \epsilon(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)|1\rangle \\
&= \epsilon(|1\rangle + |2\rangle) = \underbrace{\epsilon}_{H_{11}}|1\rangle + \underbrace{\epsilon}_{H_{21}}|2\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{H}|2\rangle &= H_{12}|1\rangle + H_{22}|2\rangle \\
&= \epsilon(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)|2\rangle \\
&= \epsilon(-|2\rangle + |1\rangle) = \underbrace{\epsilon}_{H_{12}}|1\rangle + \underbrace{-\epsilon}_{H_{22}}|2\rangle
\end{aligned}$$

So the matrix representation of $\hat{H}$ is:

$$\hat{H} = \begin{pmatrix} \epsilon & \epsilon \\ \epsilon & -\epsilon \end{pmatrix}$$

To find the eigenvalues and eigenvectors of $\hat{H}$, we need to solve the equation:

$$\begin{aligned} H|\psi\rangle &= E|\psi\rangle \\ H|\psi\rangle - EI|\psi\rangle &= 0 \\ (H - EI)|\psi\rangle &= 0 \\ \Rightarrow \det(H - EI) &= 0 \Rightarrow E = \pm\epsilon\sqrt{2} \end{aligned}$$

For $E_1 = \epsilon\sqrt{2}$, its eigenvector is $\begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}$

For $E_2 = -\epsilon\sqrt{2}$, its eigenvector is $\begin{pmatrix} 1 \\ -\sqrt{2} - 1 \end{pmatrix}$