PEP553 Midterm Exam
Due Tuesday, October 31, 2023

Formula Page (for reference, you may consult any inanimate source)

Schrödinger's Equation

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \widehat{H}\Psi(x,t) \text{ (time dependent)}$$

$$\widehat{H}\psi(x) = E\psi(x) \text{ (time independent)}$$

$$\Psi(x,t) = \exp\left(-i\frac{E}{\hbar}t\right)\psi(x)$$

Particle in a box problem

$$\widehat{H}\psi_n = E_n\psi_n$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{x}\right)$$

$$\int_0^a \psi_n^* \psi_m dx = \langle \psi_n | \psi_m \rangle = \delta_{nm}$$

$$E_n = \left(\frac{\pi^2\hbar^2}{2ma^2}\right)n^2$$

Quantized Harmonic Oscillator

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + \frac{1}{2}m\omega^2\widehat{x}^2 = \hbar\omega\left(\widehat{a}^\dagger\widehat{a} + \frac{1}{2}\right)$$

$$\widehat{H}\psi_n = E_n\psi_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2...$$

$$\psi_n = \frac{1}{\sqrt{n!}}(\widehat{a}^\dagger)\psi_0$$

$$\int_{-\infty}^{\infty} \psi_m\psi_n dx = \delta_{mn}$$

$$\widehat{a}^\dagger\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$\widehat{a}\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\beta^2 = \frac{m\omega_0}{\hbar}$$

$$\widehat{x} = \frac{\widehat{a} + \widehat{a}^\dagger}{\sqrt{2}\beta}$$

$$\widehat{p} = \frac{m\omega_0}{i}\frac{\widehat{a} - \widehat{a}^\dagger}{\sqrt{2}\beta}$$

Superposition Principle

$$\Psi(x,t) = \sum_n c_n \psi_n(x) \exp\left(-\frac{iE_n}{\hbar}t\right)$$

$$c_n = \int_{-\infty}^{\infty} \psi_n(x)\, \Psi(x,0)\, dx$$

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Instructions: Prepare written solutions to each problem. Use pencil and paper or electronic notepads but whatever method you use, safe as a pdf file and upload to Canvas.

Problem 1: (30 points total) A particle of mass m is in an asymmetric infinite potential well of width a. This is the type of well discussed in class.

The particle is prepared in the state:

$$\Psi = A\left[\psi_1 + 3\psi_3 - \frac{i}{2}\psi_4\right]$$

(a) What value of the constant A will cause this state to be normalized to unity?
(b) What are the possible outcomes of a measurement of the energy made on a system prepared in this state? Your answer should be in terms of the constants *m* and *a* plus Planck's constant.
(c) What average energy (expectation value of the energy) will be measured after many repeated energy measurements on an ensemble of identical systems prepared in this state?

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Problem 2: (40 points total) A particle in a harmonic oscillator potential with spring constant $k = m\omega_0^2$ is known to be in a superposition of two energy eigenstates in which it is two times as likely to be found with energy $E = \frac{9}{2}\hbar\omega_0$ as it is to be found with energy $E = \frac{1}{2}\hbar\omega_0$

(a) Write down a properly normalized state function $\Psi(x,0)$ using harmonic oscillator energy eigenstates that has the above statistical property.

(b) Calculate the energy expectation value $\langle E \rangle$ for this state.

(c) Calculate the expectation values $\langle x \rangle, \langle p \rangle, \langle x^2 \rangle, \langle p^2 \rangle$ for this state.

(d) Use the result of part (c) to find the position-momentum uncertainty product $\Delta x \Delta p$ for this state.

Hint: This problem can best be solved using creation and annihilation operators. If you do that you will not have to do any actual integrations to calculate the expectation values.

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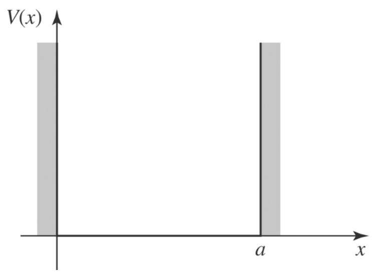
Problem 3: (30 points) A particle of mass m is confined to an infinite potential well of width a as shown below:



At $t = 0$ the particle's wave function is given by

$$\Psi(x,0) = A\left(x - \frac{a}{4}\right) \text{ for } \frac{a}{2} - \frac{a}{4} \le x \le \frac{a}{2} + \frac{a}{4}$$

$$= 0 \text{ for all other values of x}$$

(a) Sketch a graph (hand sketch ok) of the initial wave function and calculate the normalization constant A so that this initial wave function is normalized to unity.

(b) Express $\Psi(x,0)$ as a superposition of the infinite potential well eigenstates and derive an expression for the expansion coefficients. (See for example the equation between Eq. 2.39 and 2.40 in the text, p.35) or the formula sheet provided on this exam) You may use algebraic software to solve any integrals that may arise.

(c) Using the superposition state derived in part (a) derive an expression for the time dependent wave function $\Psi(x,t)$ for this particle.

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Problem 4: (15 total points) For a particle moving in a harmonic oscillator potential, we have the following relevant operators and commutators:

$$\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right) \quad \hat{a}^\dagger = \frac{\beta}{\sqrt{2}}\left(\hat{x} - \frac{i\hat{p}}{m\omega_0}\right)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 = \hbar\omega_0\left(\hat{a}^\dagger a + \frac{1}{2}\right)$$

$$[\hat{x}, \hat{p}] = i\hbar \quad [\hat{a}, \hat{a}^\dagger] = 1$$

(a) (5 points) Show that the commutator exhibits the associative property given by

$$[\widehat{A}\widehat{B}, \widehat{C}] = \widehat{A}[\widehat{B}, \widehat{C}] + [\widehat{A}, \widehat{C}]\widehat{B}$$

(b) (5 points) Establish a simplification for the related expression

$$[\widehat{A}, \widehat{B}\widehat{C}]$$

(c) (5 points each) Using the properties shown in (a) and (b), and the definitions and commutators defined above, calculate the following commutators:

$$[\widehat{H}, \hat{x}]$$
$$[\widehat{H}, \hat{p}]$$
$$[\widehat{H}, \hat{a}]$$
$$[\widehat{H}, \hat{a}^\dagger]$$