Homework 5

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Problem A.9

$$Aa = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \cdot \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} = \begin{pmatrix} 3i \\ 6+2i \\ 6 \end{pmatrix}$$

$$a^\dagger b = \begin{pmatrix} -i & -2i & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} = \begin{pmatrix} -2-4i \end{pmatrix}$$

$$\tilde{a}Bb = \begin{pmatrix} i & 2i & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} = \begin{pmatrix} 8+4i \end{pmatrix}$$

$$ab^\dagger = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1+i & 0 \end{pmatrix} = \begin{pmatrix} 2i & -1+i & 0 \\ 4i & -2+2i & 0 \\ 4 & 2+2i & 0 \end{pmatrix}$$

Problem A.15

From equation (A.42) we have the rotation matrix can be epxpressed as:

$$a' = Ta$$

$$\begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = T \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

Since we are rotation about the x-axis, we have:

$$a'_x = \begin{cases} \hat{i}' = \hat{i} \\ \hat{j}' = \cos(\theta)\hat{j} + \sin(\theta)\hat{k} \\ \hat{k}' = -\sin(\theta)\hat{j} + \cos(\theta)\hat{k} \end{cases} \Rightarrow \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

$$T_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{1}$$

For the rotation about the y-axis, we have:

$$a'_y = \begin{cases} \hat{i}' = \cos(\theta)\hat{i} - \sin(\theta)\hat{k} \\ \hat{j}' = \hat{j} \\ \hat{k}' = \sin(\theta)\hat{i} + \cos(\theta)\hat{k} \end{cases} \Rightarrow \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

The rotation matrix about the y-axis is:

$$T_y = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

From equation (A.63), we have the formula for the change of basis matrix:

$$a^f = Sa^e$$

we have that:

$$\begin{cases} \hat{i}' = \hat{j} \\ \hat{j}' = -\hat{i} \\ \hat{k}' = \hat{k} \end{cases} \Rightarrow \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

Therefore, the matrix S is:

$$S = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we calculate ST_xS^{-1} and ST_yS^{-1}:

$$ST_xS^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & 0\sin(\theta) & \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$ST_yS^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Problem A.26

From equation (A.82) matrix N is normal if:

$$[N^\dagger, N] = 0$$

For the matrix A, we have:

$$A^\dagger A - AA^\dagger = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} = 0$$

$$\Rightarrow A \text{ is diagonalizable}$$

For the matrix B, we have:

$$B^\dagger B - BB^\dagger = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} = 0$$

$$\Rightarrow B \text{ is diagonalizable}$$

To test if they are commutable, we have:

$$AB = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 9 & 0 \\ 9 & -9 & 9 \\ 0 & 9 & 0 \end{pmatrix} = BA$$

$$\Rightarrow A \text{ and } B \text{ commute}$$

To find the eigenvalues and eigenvectors of A, we have:

$$\det(A - \lambda I) = 0 = \begin{vmatrix} 2-\lambda & 2 & -1 \\ 2 & -1-\lambda & 2 \\ -1 & 2 & 2-\lambda \end{vmatrix}$$

$$\Rightarrow \lambda = \{-3, 3\}$$

To find the eigenvector corresponding to $\lambda_1 = -3$, we have:

$$(A - \lambda_1 I)v = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For the eigenvector corresponding to $\lambda_2 = 3$, we have:

$$(A - \lambda_2 I)v = 0$$

$$\Rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Since λ_2 coressponds to 2 eigenvectors, it is degenerate.

We can use the same formula to find the eigenvalues and eigenvectors of B:

$$\det(B - \lambda I) = 0 = \begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 5-\lambda & -1 \\ 2 & -1 & 2-\lambda \end{vmatrix}$$

$$\Rightarrow \lambda = \{0, 3, 6\}$$

Now we check if the eigenvectors of A are also the eigenvector of B:

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \\ 6 \end{pmatrix} = 6v_1$$

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0v_3$$

So v_2 is not a eigenvector of B. For each eigenvalue of B, we have:

For $\lambda = 0$

$$\Rightarrow v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 3$

$$\Rightarrow v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 6$

$$\Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

We can test v_5 on A to see if it is an eigenvector of A:

$$\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3v_5$$

Therefore, v_4 is an eigenvector of A. So the eigenvectors $v1, v3, v4$ of B are also eigenvectors of A.

Problem A.29

From equation (A.93) we have:

$$\det(T) = \lambda_1 \lambda_2 \dots \lambda_n$$
$$\mathrm{Tr}(T) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

$$\det \begin{vmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{vmatrix} = 0 \tag{2}$$

$$\mathrm{Tr} \begin{vmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{vmatrix} = 2 + 2 + 2 = 6 \tag{3}$$

Find the eigenvalues of the matrix, we have that: for $\lambda_1 = 0$, we have $v_1 = A_1 \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}$. For $\lambda_2 = 3$, we have $v_2 = A_2 \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ They are consistent with the value with have in part a.

$$\det(T) = 3 \cdot 3 \cdot 0 = 0$$

$$\text{Tr}(T) = 3 + 3 + 0 = 6$$

To check orthogonality, we have:

$$v_1^\dagger v_2 = \begin{pmatrix} -1 & i & 1 \end{pmatrix} \cdot \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = 0$$

$$v_1^\dagger v_3 = \begin{pmatrix} -1 & i & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$v_2^\dagger v_3 = \begin{pmatrix} -i & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -i \neq 0$$

It means that vectors v_1 is orthogonal to v_2 and v_3, v_2 is not orthogonal to v_3. Now we normalizing the three vectors:

$$A_1^2|-1|^2 + |-i|^2 + |1|^2 = 1$$

$$\Rightarrow A_1 = \pm\frac{1}{\sqrt{3}}$$

$$A_2^2|i|^2 + |1|^2 + |0|^2 = 1$$

$$\Rightarrow A_2 = \pm\frac{1}{\sqrt{2}}$$

$$A_3^2|1|^2 + |0|^2 + |1|^2 = 1$$

$$\Rightarrow A_3 = \pm\frac{1}{\sqrt{2}}$$

The three normalized vectors are:

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}, v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The diagonalized version of T is:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

proof:

$$\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix} = \begin{pmatrix} -1 & i & 1 \\ -i & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} \frac{-1}{3} & \frac{i}{3} & \frac{1}{3} \\ \frac{-i}{3} & \frac{2}{3} & \frac{i}{3} \\ \frac{1}{3} & \frac{-i}{3} & \frac{2}{3} \end{pmatrix}$$

Problem A.30

(a) we have:

$$\begin{aligned}\langle U\alpha|U\beta\rangle &= (U\alpha)^\dagger(U\beta)\\ &= (a^\dagger U^\dagger)(U\beta)\\ &= a^\dagger U^\dagger U\beta\\ &= a^\dagger\beta\\ &= \langle\alpha|\beta\rangle\end{aligned}$$

(b) let λ be the eigenvalue, and $|\alpha\rangle$ be the eigenvector of $\hat{U}$, we have:

$$\hat{U}|\alpha\rangle = \lambda|\alpha\rangle$$

Taking the inner product of $|\alpha\rangle$ with itself, we have:

$$\begin{aligned}\langle\alpha|\alpha\rangle &= \langle\alpha|\hat{U}^\dagger\hat{U}|\alpha\rangle\\ &= \left(\langle\alpha|U^\dagger\right)\left(U|\alpha\rangle\right)\\ &= \left(U|\alpha\rangle\right)^\dagger\left(U|\alpha\rangle\right)\\ &= \left(\lambda|\alpha\rangle\right)^\dagger\left(\lambda|\alpha\rangle\right)\\ &= \left(\lambda^*\langle\alpha|\right)\left(\lambda|\alpha\rangle\right)\\ &= \lambda^*\lambda\langle\alpha|\alpha\rangle\\ 1 &= \lambda^*\lambda = |\lambda|^2\\ \Rightarrow |\lambda| = 1\end{aligned}$$

(c) given λ and μ are the eigenvalues of $\hat{U}$ with their respective eigenvectors $|\alpha\rangle$ and $|\beta\rangle$, we have:

$$\begin{aligned}\langle\alpha|\beta\rangle &= \langle\alpha|\hat{U}^\dagger\hat{U}|\beta\rangle\\ &= \left(\langle\alpha|\hat{U}^\dagger\right)\left(\hat{U}|\beta\rangle\right)\\ &= \left(\hat{U}|\alpha\rangle\right)^\dagger\left(\hat{U}|\beta\rangle\right)\\ &= \left(\lambda|\alpha\rangle\right)^\dagger\left(\mu|\beta\rangle\right)\\ &= \lambda^*\mu\langle\alpha|\beta\rangle\\ 0 &= \lambda^*\mu\langle\alpha|\beta\rangle - \langle\alpha|\beta\rangle\\ &= (\lambda^*\mu - 1)\langle\alpha|\beta\rangle\end{aligned}$$

Now we need to show that $\lambda^*\mu - 1 \neq 0$, multiply both sides by λ:

$$\begin{aligned}\lambda^*\lambda\mu - \lambda &= 0\\ 1\mu - \lambda &= 0\\ \mu &= \lambda\end{aligned}$$

This is contradiction to the assumption that $\mu \neq \lambda$. Therefore, $\lambda^* \mu - 1 \neq 0$. It means that $\langle \alpha | \beta \rangle = 0$, which implies that the eigenvectors of $\hat{U}$ are orthogonal.