Homework 7

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Equation (4.15):

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

Where $R(r)$ is the radial wave function and $Y(\theta,\phi)$ is the angular wave function.

Problem 4.3

(a)

$$\psi(r,\theta,\phi) = Ae^{\frac{-r}{a}}$$

From equation (4.8) we have:

$$\begin{aligned}
E\psi &= -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \\
&= -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + V\psi \\
&\text{Divide both sides by } \psi \\
\Rightarrow E &= -\frac{\hbar^2}{2m}\frac{1}{r^2\psi}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + V \\
\Rightarrow V &= E + \frac{\hbar^2}{2m}\frac{1}{r^2\psi}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)
\end{aligned}$$

Where $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$ is the Laplacian operator.
Plug in $\psi(r,\theta,\phi) = Ae^{\frac{-r}{a}}$ into the equation above, we have:

$$\begin{aligned}
V &= E + \frac{\hbar^2}{2m}\frac{1}{r^2(Ae^{\frac{-r}{a}})}\frac{\partial}{\partial r}\left(r^2\frac{\partial Ae^{\frac{-r}{a}}}{\partial r}\right) \\
&= E + \frac{\hbar^2}{2a^2m} - \frac{\hbar^2}{amr} = E - \frac{\hbar^2}{2ma^2}\left(\frac{2a}{r} - 1\right)
\end{aligned}$$

As $r \to \infty$, $V(r) \to 0$.

$$\begin{aligned}
\lim_{r\to\infty} V(r) &= E - \frac{\hbar^2}{2ma^2}(-1) = 0 \\
\Rightarrow E &= -\frac{\hbar^2}{2ma^2} \\
\Rightarrow V(r) &= -\frac{\hbar^2}{amr}
\end{aligned}$$

(b)

$$\psi(r,\theta,\phi) = Ae^{\frac{-r^2}{a^2}}$$

$$\begin{aligned}
V(r) &= E + \frac{\hbar^2}{2m}\frac{1}{r^2\psi}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \\
&= E + \frac{\hbar^2}{2m}\frac{1}{r^2(Ae^{\frac{-r^2}{a^2}})}\frac{\partial}{\partial r}\left(r^2\frac{\partial Ae^{\frac{-r^2}{a^2}}}{\partial r}\right) \\
&= E - \frac{3\hbar^2}{a^2m} + \frac{2\hbar^2r^2}{a^4m}
\end{aligned}$$

As $V(0) = 0$

$$\begin{aligned}
\lim_{r\to\infty} V(r) &= E - \frac{3\hbar^2}{a^2m} + \frac{2\hbar^2r^2}{a^4m} = 0 \\
\Rightarrow E &= \frac{3\hbar^2}{a^2m} \\
\Rightarrow V(r) &= \frac{2\hbar^2r^2}{a^4m}
\end{aligned}$$

Problem 4.4

Equation(4.27):

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\left(\frac{d}{dx}\right)^m P_l(x)$$

Where $P_l(x)$ is the l^{th} Lengendre polynomial. Equation (4.28)

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

The normalized angular wave functions is called spherical harmonics. Equation (4.32)

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta)$$

Constructing Y_0^0, Y_2^1

$$\begin{aligned}
Y_0^0 &= \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} \\
Y_2^1 &= -\left(\frac{15}{8\pi}\right)^{\frac{1}{2}}\sin(\theta)\cos(\theta)e^{i\phi}
\end{aligned}$$

To check orthogonality, we need to integrate the product of the two spherical harmonics over the unit sphere.

$$\begin{aligned}
\int_0^\pi\int_0^{2\pi} Y_0^0 Y_2^1 \sin(\theta)d\theta d\phi &= \int_0^\pi\int_0^{2\pi}\left[\left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\right]^*\left[-\left(\frac{15}{8\pi}\right)^{\frac{1}{2}}\sin(\theta)\cos(\theta)e^{i\phi}\right]\sin(\theta)d\theta d\phi \\
&= 0 \quad \text{(Wolfram Alpha)} \\
&\Rightarrow \text{Orthogonal}
\end{aligned}$$

To normalize the spherical harmonics for Y_0^0, Y_2^1, we have:

$$\int_0^\pi \int_0^{2\pi} |Y_0^0|^2 \sin(\theta) d\theta d\phi = \int_0^\pi \int_0^{2\pi} \left[\left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\right]^2 \sin(\theta) d\phi d\theta = 1$$
$$\int_0^\pi \int_0^{2\pi} |Y_2^1|^2 \sin(\theta) d\theta d\phi = \int_0^\pi \int_0^{2\pi} \left[-\left(\frac{15}{8\pi}\right)^{\frac{1}{2}} \sin(\theta)\cos(\theta)e^{i\phi}\right]^2 \sin(\theta) d\phi d\theta = 1$$

*Calculated on Wolfram

Problem 4.7

Find $Y_l^l(\theta, \phi)$, and $Y_3^2(\theta, \phi)$, we have:

$$P_3^2 = 15\sin^2(\theta)\cos(\theta)$$

Plug equation (4.28) into equation (4.27), we have:

$$\begin{aligned} P_l^m &= (-1)^m (1-x^2)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^m P_l(x) \\ &= (-1)^m (1-x^2)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^m \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2-1)^l \\ &= \frac{(-1)^m}{2^l l!} (1-x^2)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^{l+m} (x^2-1)^l \end{aligned}$$

For $Y_3^2(\theta, \phi)$:

$$\begin{aligned} Y_3^2(\theta, \phi) &= \sqrt{\frac{(2\cdot 3)+1}{4\pi} \frac{1!}{5!}} e^{2i\phi} P_3^2(\cos\theta) \\ &= \sqrt{\frac{7}{4\pi} \frac{1}{120}} e^{2i\phi} P_3^2(\cos\theta) \\ &= \sqrt{\frac{7}{480\pi}} e^{2i\phi} \left[\frac{1}{2^3 \cdot 3!}(1-x^2)\left(\frac{d}{dx}\right)^5 (x^2-1)^3\right]_{x=\cos\theta} \\ &= \sqrt{\frac{7}{480\pi}} e^{2i\phi} \left[\frac{1}{48}(1-x^2)720x\right]_{x=\cos\theta} \\ &= \sqrt{\frac{7}{480\pi}} e^{2i\phi} \left[15(1-\cos^2(\theta))\cos(\theta)\right] \end{aligned}$$

Equation (4.18):

$$\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2(\theta) Y$$

Plug in $Y_3^2(\theta,\phi)$ into the left hand side of the equation above, we have:

$$\sin(\theta)\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial}{\partial\theta}\left(\sqrt{\frac{7}{480\pi}}e^{2i\phi}\left[15(1-\cos^2(\theta))\cos(\theta)\right]\right)\right]+\frac{\partial^2}{\partial\phi^2}\left(\sqrt{\frac{7}{480\pi}}e^{2i\phi}\left[15(1-\cos^2(\theta))\cos(\theta)\right]\right)$$
$$=-3e^{2i\phi}\sqrt{\frac{105}{2\pi}}\cos(\theta)\sin^4(\theta)\quad\text{(Wolfram Alpha)}$$

For the right hand side of the equation above, we have:

$$\begin{aligned}&-3(3+1)\sin^2(\theta)\sqrt{\frac{7}{480\pi}}e^{2i\phi}\left[15(1-\cos^2(\theta))\cos(\theta)\right]\\&=-3e^{2i\phi}\sqrt{\frac{105}{2\pi}}\cos(\theta)\sin^4(\theta)\quad\text{(Wolfram Alpha)}\end{aligned}$$

For $Y_l^l(\theta,\phi)$:

$$\begin{aligned}Y_l^l(\theta,\phi)&=\sqrt{\frac{2l+1}{4\pi}\frac{(l-l)!}{(l+l)!}}e^{il\phi}P_l^l(\cos\theta)\\&=\sqrt{\frac{2l+1}{4\pi(2l)!}}e^{il\phi}\left[\frac{(-1)^l}{2^l l!}(1-x^2)^{\frac{l}{2}}\underbrace{\left(\frac{d}{dx}\right)^{2l}(x^2-1)^l}_{(2l)!}\right]_{x=\cos\theta}\\&=\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l(\theta)\end{aligned}$$

Check satisfication of equation (4.18) for $Y_l^l(\theta,\phi)$, plug in $Y_l^l(\theta,\phi)$ into the left hand side of the equation above, we have:

$$\begin{aligned}&\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\left[\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l(\theta)\right]\right)+\frac{\partial^2}{\partial\phi^2}\left[\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l(\theta)\right]\\&=\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\left[\left(\sin^2(\theta)\frac{\partial^2}{\partial\theta^2}e^{il\phi}\sin^l(\theta)\right)+\left(\frac{\partial^2}{\partial\phi^2}e^{il\phi}\sin^l(\theta)\right)\right]\\&=\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\left[\left(l^2e^{il\phi}\sin^l(\theta)\cos^2(\theta)-le^{il\phi}\sin^{(l+2)}(\theta)\right)+\left(-e^{il\phi}l^2\sin^l(\theta)\right)\right]\\&=\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}(-l\sin^2(\theta)-\sin^2(\theta))le^{il\phi}\sin^l(\theta)\\&=-l(l+1)\sin^2(\theta)\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l(\theta)\\&=-l(l+1)\sin^2(\theta)Y_l^l(\theta,\phi)\end{aligned}$$

Problem 4.8

For $l \neq l'$

$$\begin{aligned}\int_{-1}^{1} P_l(x)P_{l'}(x)dx &= \int_{-1}^{1} \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \frac{1}{2^{l'} l'!}\left(\frac{d}{dx}\right)^{l'} (x^2-1)^{l'} dx \\ &= 0\end{aligned}$$

For $l = l'$

$$\begin{aligned}\int_{-1}^{1} P_l(x)P_{l'}(x)dx &= \int_{-1}^{1} \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l dx \\ &= \frac{2}{2l+1}\end{aligned}$$

$$\Rightarrow \int_{-1}^{1} P_l(x)P_{l'}(x)dx = \left(\frac{2}{2l+1}\right)\delta_{ll'}$$