Midterm Exam PEP 553 fall 2024

Son Nguyen, I pledge my honor that I have abided by the Stevens Honor System.

$$\Psi = A\left[\psi_1 + 3\psi_3 - \frac{i}{2}\psi_4\right]$$

mass = m

width = a

a) $\int_0^a |\Psi(x)|^2 dx = 1$

$$\Leftrightarrow \int_0^a A^2\left[\psi_1 + 3\psi_3 - \frac{i}{2}\psi_4\right]^2 dx = 1$$

$$\Leftrightarrow A^2 \int_0^a \left(\psi_1 + 3\psi_3 - \frac{i}{2}\psi_4\right)^* \left(\psi_1 + 3\psi_3 - \frac{i}{2}\psi_4\right) dx = 1$$

$$\Leftrightarrow A^2 \int_0^a \underline{|\psi_1|^2} + 3\psi_1^*\psi_3 - \frac{i}{2}\psi_1^*\psi_4 + \underline{9|\psi_3|^2} + 3\psi_3^*\psi_1$$

$$- \frac{3i}{2}\psi_3^*\psi_4 - \frac{i}{2}\psi_4^*\psi_1 - \frac{3i}{2}\psi_4^*\psi_3 - \underline{\frac{1}{4}|\psi_4|^2}\, dx = 1$$

$$\Leftrightarrow A^2\left[1 + 9 - \frac{1}{4}\right] = 1 \Leftrightarrow \frac{39}{4}(A^2) = 1$$

$$\Leftrightarrow A^2 = \frac{4}{39}$$

$$\Leftrightarrow A = \frac{2}{\sqrt{39}}$$

b) possible outcomes

$$\Psi = \frac{2}{\sqrt{39}}\left[\psi_1 + 3\psi_3 - \frac{i}{2}\psi_4\right]$$

$$\Leftrightarrow \frac{2}{\sqrt{39}}\psi_1 + \frac{6}{\sqrt{39}}\psi_3 - \frac{i}{\sqrt{39}}\psi_4 .$$

$$E_n = \left(\frac{\pi^2 \hbar^2}{2ma^2}\right) n^2$$

for ψ_1, $n=1$

$$E_1 = \left(\frac{\pi^2 \hbar^2}{2ma^2}\right).$$

for ψ_3, $n=3$

$$E_3 = \left(\frac{\pi^2 \hbar^2}{2ma^2}\right) 3^2 = \frac{9\pi^2\hbar^2}{2ma^2}$$

for ψ_4, $n=4$

$$E_4 = \left(\frac{\pi^2 \hbar^2}{2ma^2}\right) 4^2 = \frac{16\pi^2\hbar^2}{2ma^2}$$

c) $\langle E \rangle = \sum_n |c_n|^2 E_n .$

$$= \frac{4}{39}\frac{\pi^2 \hbar^2}{2ma^2} + \frac{12}{13} \cdot \frac{9\pi \hbar^2}{2ma^2} + \left(-\frac{i}{\sqrt{39}}\right)\frac{16\pi^2\hbar^2}{2ma^2}$$

$$= 8 \frac{\pi^2 \hbar^2}{2ma^2} = \frac{4\pi^2\hbar^2}{2ma^2}$$

Problem 2:

$k = m\omega_0^2$ $\quad E_1 = \frac{9}{2}\hbar\omega_0 \quad E_2 = \frac{1}{2}\hbar\omega_0$

a) $\Psi(x,0) = C_1\Psi_1 + C_2\Psi_2$

We have

$$|c_1|^2 = 2|C_2|^2$$

$$2|C_2|^2 + |C_2|^2 = 1 \Leftrightarrow$$

$$3|C_2|^2 = 1$$

$$\Leftrightarrow C_2 = \sqrt{\frac{1}{3}}$$

$$\Rightarrow C_1 = \sqrt{\frac{2}{3}}$$

$$\Leftrightarrow \Psi(x,0) = \sqrt{\frac{2}{3}}\Psi_1 + \sqrt{\frac{1}{3}}\Psi_2.$$

b) $\langle E \rangle$ $\quad E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$

$$E = \frac{9}{2}\hbar\omega_0 = \frac{8}{2} + \frac{1}{2}\hbar\omega \Rightarrow n = 4$$

$$E = \frac{1}{2}\hbar\omega_0 = 0 + \frac{1}{2}\hbar\omega_0 \Rightarrow n = 0.$$

$$\langle E \rangle = |C_0|^2 E_0 + |C_4|^2 E_4 = \frac{1}{3}\cdot\frac{1}{2}\hbar\omega_0 + \frac{2}{3}\cdot\frac{9}{2}\hbar\omega_0$$

$$\Rightarrow \langle E \rangle = \frac{19}{6} \hbar w_0$$

c) calculate the $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,0)|^2 \, dx$$

$$= \int_{-\infty}^{\infty} x \left| \sqrt{\frac{2}{3}} \psi_4 + \sqrt{\frac{1}{3}} \psi_0 \right|^2 dx$$

$$= \int_{-\infty}^{\infty} \frac{2}{3} \underbrace{x |\psi_4|^2 dx}_{0} + \int_{-\infty}^{\infty} \frac{1}{3} \underbrace{x |\psi_0|^2 dx}_{0}$$

$$\Rightarrow \langle x \rangle = 0$$

$$\Rightarrow \langle p \rangle = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi(x,0)|^2 \, dx.$$

$$\Leftrightarrow \frac{2}{3} \underbrace{\int_{-\infty}^{\infty} x^2 |\psi_4|^2 dx}_{\langle x^2 \rangle_4} + \frac{1}{3} \underbrace{\int_{-\infty}^{\infty} x^2 |\psi_0|^2 dx.}_{\langle x^2 \rangle_0}$$

$$\langle x^2 \rangle_n = \left(\frac{1}{2} + n \right) \frac{\hbar}{m w}$$

$$\Rightarrow \langle x^2 \rangle_4 = \left(\frac{1}{2} + 4 \right) \frac{\hbar}{m w} = \frac{9 \hbar}{2 m w}$$

$$\Rightarrow \langle x^2 \rangle_0 = \frac{\hbar}{2m\omega}$$

$$\Rightarrow \langle x^2 \rangle = \frac{2}{3} \cdot \frac{9}{2} \frac{\hbar}{m\omega} + \frac{1}{3} \cdot \frac{\hbar}{2m\omega}$$

$$= \frac{18}{6} \frac{\hbar}{m\omega} + \frac{1}{6} \frac{\hbar}{m\omega} = \frac{19}{6} \frac{\hbar}{m\omega}$$

$$\hat{p} = \frac{m w_0}{i} \frac{\hat{a} - \hat{a}^+}{\sqrt{2}\beta} \qquad \beta = \sqrt{\frac{m w_0}{h}}.$$

$$\Rightarrow \frac{m W_0}{i} \frac{\hat{a} - \hat{a}^+}{\sqrt{2}\sqrt{\frac{m w_0}{\hbar}}} = \frac{m W_0}{i} \cdot \frac{\sqrt{\hbar}}{\sqrt{2 m W_0}} (\hat{a} - \hat{a}^+).$$

$$= \frac{\sqrt{m W_0 \hbar}}{i\sqrt{2}} (\hat{a} - \hat{a}^+)$$

$$\langle p^2 \rangle_n = \int_{-\infty}^{\infty} \psi_n^* \hat{p} \psi_n \, dx \qquad = -i \frac{\sqrt{m w_0 \hbar}}{\sqrt{2}} (\hat{a} - \hat{a}^+)$$

$$= \int_{-\infty}^{\infty} \psi_n^* \left(-i \sqrt{\frac{m W_0 \hbar}{2}} (\hat{a} - \hat{a}^+) \right)^2 \psi_n \, dx.$$

$$= \frac{\hbar m w}{2} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}^2 - 2\hat{a}\hat{a}^+ + \hat{a}^{+2}) \psi_n \, dx.$$

$$= \frac{\hbar m \omega}{2} \int_{-\infty}^{\infty} n |\psi_n|^2 + |n+1| |\psi_n|^2 \, dx$$

$$= \hbar \frac{m\omega}{2} (2n+1).$$

$$\Rightarrow \langle p^2 \rangle_4 = \frac{9\hbar m\omega}{2}$$

$$\Rightarrow \langle p^2 \rangle_0 = \frac{\hbar m\omega}{2}$$

$$\Rightarrow \langle p^2 \rangle = \frac{9\hbar m\omega}{2} \cdot \frac{2}{3} + \frac{m\hbar\omega}{2} \cdot \frac{1}{3} = \frac{19\hbar m\omega}{6}$$

$$\Delta x = \langle x^2 \rangle - \langle x \rangle = \frac{19\hbar}{6m\omega}$$

$$\Delta p = \langle p^2 \rangle - \langle p \rangle = \frac{19\hbar m\omega}{6}$$

$$\Delta x \Delta p = \frac{19\hbar^2 \cancel{m\omega}}{6^2 \cancel{m\omega}} = \frac{19\hbar^2}{36}$$

problem 3:

$$\Psi(x,0) = A\left(x - \frac{a}{4}\right). \text{ for } \frac{a}{2} - \frac{a}{4} \leq x \leq \frac{a}{2} + \frac{a}{4}.$$

$$= 0 \quad \text{otherwise}$$

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx = 1$$

$$\Leftrightarrow \int_{\frac{a}{2}-\frac{a}{4}}^{\frac{a}{2}+\frac{a}{4}} \left[A\left(x - \frac{a}{4}\right)\right]^2 dx = 1 \Leftrightarrow A^2 \int_{\frac{a}{4}}^{\frac{3a}{4}} \left(x - \frac{a}{4}\right)^2 dx = 1$$

$$\Leftrightarrow A^2 \frac{a^3}{24} = 1$$

(a) $A = \sqrt{\frac{24}{a^3}}$

$$\Rightarrow \psi(x,0) = \sqrt{\frac{24}{a^3}}\left(x - \frac{a}{4}\right)$$

b) $$\Psi(x,t) = \sum_n c_n \psi_n(x) \exp\left(-\frac{iE_n t}{\hbar}\right).$$

$$c_n = \int_{-\infty}^{\infty} \psi_n(x)\, \Psi(x,0)\, dx.$$

$$= \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$

$$= \sqrt{\frac{2}{a}} \int_{\frac{a}{4}}^{\frac{3a}{4}} \sin\left(\frac{n\pi x}{a}\right) \sqrt{\frac{24}{a^3}}\left(x - \frac{a}{4}\right) dx$$

$$= \sqrt{\frac{48}{a^4}} \int_{\frac{a}{4}}^{\frac{3a}{4}} \sin\left(\frac{n\pi x}{a}\right)\left(x - \frac{a}{4}\right) dx = c_n$$

$$c_n = \frac{\sqrt{3}\sqrt{\frac{1}{a^4}}\, a^3 \left(-8\cos\left(\frac{n\pi}{4}\right) + (8 - n^2\pi^2)\cos\left(\frac{3n\pi}{4}\right) + 4n\pi \sin\left(\frac{3n\pi}{4}\right)\right)}{n^3\pi^3}$$

$$\psi(x,0) = \sum_n c_n \psi_n(x)$$

$$\psi(x,t) = \sum_n c_n \psi_n(x) \exp\left(-\frac{iE_n t}{\hbar}\right)$$

Sketch : $\psi(x)$