Homework 8 - Quantum Algorithms

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I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1

For the first circuit, let state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. After measurment, the state collapse to either $p(|0\rangle) = |\alpha|^2$ or $p(|1\rangle) = |\beta|^2$. Applying U after the measurment give us either: $p(U|0\rangle) = |\alpha|^2$ or $p(U|1\rangle) = |\beta|^2$.

For the second circuit, we have $|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle$. Applying CNOT_{01} give us:

$$\begin{aligned}\text{CNOT}_{01}[(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle] &= \text{CNOT}_{01}(\alpha|00\rangle + \beta|10\rangle) \\ &= \alpha|00\rangle + \beta|11\rangle\end{aligned}$$

CNOT gate creates a second identical state on the second second qubit. Now we applying unitary U to the first qubit give us.

$$(U \otimes I)(\alpha|00\rangle + \beta|11\rangle) = \alpha(U|0\rangle \otimes |0\rangle) + \beta(U|1\rangle \otimes |1\rangle)$$

Now measuring the second qubit. If the second qubit is measured to be $|0\rangle$, the state of the first qubit is $U|0\rangle$ with probability $|\alpha|^2$. If the second qubit is measured to be $|1\rangle$, the state of the first qubit is $U|1\rangle$ with probability $|\beta|^2$.

Problem 2

In the first circuit, after the measurment, we have state $|0\rangle$. Applying U give us $U|0\rangle$. For the second circuit, starting with state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Measurment will result in either $|0\rangle$ or $|1\rangle$. Applying SWAP gate to the state $|0\rangle$ give us:

$$\begin{aligned}\text{SWAP}(|0\rangle \otimes |0\rangle) &= |0\rangle \otimes |0\rangle \\ &= |0\rangle \otimes |0\rangle = |00\rangle \\ \text{SWAP}(|0\rangle \otimes |1\rangle) &= |1\rangle \otimes |0\rangle \\ &= |0\rangle \otimes |1\rangle = |01\rangle\end{aligned}$$

Now applying U to the first qubit gives us:

$$\begin{aligned}(U \otimes I)|0\rangle \otimes |0\rangle = U|0\rangle \otimes |0\rangle \\ (U \otimes I)|0\rangle \otimes |1\rangle = U|0\rangle \otimes |1\rangle\end{aligned}$$

No matter the coeficient α and β, we always have $U|0\rangle$ as the state of the first qubit.

Problem 3

For the first circuit:
Starting with $|\psi\rangle = \alpha|0\rangle + \beta|0\rangle$ state, and $|\phi\rangle$ state, after measurment on $|\psi\rangle$ we have:

$$p(|0\rangle) = |\alpha|^2$$
$$p(|1\rangle) = |\beta|^2$$

Applying Control-$U_{\psi,\phi}$

$$p(CU_{\psi,\theta}(|0\rangle \otimes |\phi\rangle) = |0\rangle \otimes |\phi\rangle) = |\alpha|^2$$
$$p(CU_{\psi,\theta}(|1\rangle \otimes |\phi\rangle) = |1\rangle \otimes U|\phi\rangle) = |\beta|^2$$

For the second circuit:

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes |\phi\rangle \\ &= \alpha(|0\rangle \otimes |\phi\rangle) + \beta(|1\rangle \otimes |\phi\rangle) \end{aligned}$$

Applying Control-$U_{\psi,\phi}$:

$$CU_{\psi,\phi}[\alpha(|0\rangle \otimes |\phi\rangle) + \beta(|1\rangle \otimes |\phi\rangle)]$$
$$= \alpha(|0\rangle \otimes |\phi\rangle) + \beta(|1\rangle \otimes U|\phi\rangle)$$

After measurment:

$$p(|0\rangle \otimes |\phi\rangle) = |\alpha|^2$$
$$p(|1\rangle \otimes U|\phi\rangle) = |\beta|^2$$