Homework 1 - Quantum Algorithms

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I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1.1

For $\bar{u}, \bar{v} \in \mathbb{C}^n$ show that the following holds:

$$\langle u, v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2 - i||u+iv||^2 + i||u-iv||^2)$$

We have:

$$\begin{aligned}
||u+v||^2 &= \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
||u-v||^2 &= \langle u-v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle \\
||u+iv||^2 &= \langle u+iv, u+iv\rangle = \langle u,u\rangle + \langle u,iv\rangle + \langle iv,u\rangle + \langle iv,iv\rangle \\
||u-iv||^2 &= \langle u-iv, u-iv\rangle = \langle u,u\rangle - \langle u,iv\rangle - \langle iv,u\rangle + \langle iv,iv\rangle
\end{aligned}$$

Substitute these into the equation, we have right-hand side equal to:

$$\begin{aligned}
&= \frac{1}{4}\left[2\left(\langle u,v\rangle + \langle v,u\rangle\right) - 2i\left(\langle u,iv\rangle + \langle iv,u\rangle\right)\right] \\
&= \frac{1}{4}\left(2\langle u,v\rangle + 2\langle v,u\rangle - 2\underbrace{i\cdot i}_{-1}\langle u,v\rangle - 2\underbrace{i\cdot \bar{i}}_{1}\langle v,u\rangle\right) \\
&= \frac{1}{4}\left(2\langle u,v\rangle + 2\langle v,u\rangle + 2\langle u,v\rangle - 2\langle v,u\rangle\right) \\
&= \frac{1}{4}\left(4\langle u,v\rangle\right) = \langle u,v\rangle
\end{aligned}$$

Problem 1.2

Let $\bar{u} = (i, 1+i)^T$ and $\bar{v} = (2, 1-i)^T$

1. Compute $\langle \bar{u}, \bar{v}\rangle$ We have the vector:

$$\bar{u} = \begin{bmatrix} i \\ 1+i \end{bmatrix}, \bar{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

$$\begin{aligned}
\langle \bar{u}, \bar{v}\rangle &= \langle (i, 1+i)^T, (2, 1-i)^T\rangle \\
\bar{u}^\dagger \cdot \bar{v} &= \begin{bmatrix} -i & 1-i \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1-i \end{bmatrix} \\
&= -2i + 1^2 - 2i + ii = -4i
\end{aligned}$$

2. Compute $||\bar{u}||$

$$\begin{aligned}
||\bar{u}|| &= \sqrt{\langle \bar{u}, \bar{u} \rangle} \\
&= \sqrt{\bar{u}^\dagger \cdot \bar{u}} \\
&= \sqrt{\begin{bmatrix} -i & 1-i \end{bmatrix} \cdot \begin{bmatrix} i \\ 1+i \end{bmatrix}} \\
&= \sqrt{-i \cdot i + (1-i)(1+i)} \\
&= \sqrt{1+2} = \sqrt{3}
\end{aligned}$$

3. Express $\frac{1}{2-i}$ in the form $a + bi$ with $a, b \in \mathbb{R}$

$$\begin{aligned}
\frac{1}{2-i} &= \frac{1}{2-i} \cdot \frac{2+i}{2+i} \\
&= \frac{2+i}{5} = \frac{2}{5} + \frac{1}{5}i
\end{aligned}$$

Problem 1.3

Prove that a composition of linear transformations $\phi, \psi : V \to V$ is linear.
Let the transformation ϕ be defined as $\phi, \psi : V \to V$. Let $v, u \in V$ and $\alpha, \beta \in \mathbb{C}$, the composition of ϕ and ψ is defined as $\psi \circ \phi : V \to U$:

$$(\psi \circ \phi)(\alpha v + \beta u) = \psi(\phi(\alpha v + \beta u))$$

Since ϕ is linear, we have:

$$\phi(\alpha v + \beta u) = \phi(\alpha u) + \phi(\beta v) = \alpha\phi(u) + \beta\phi(u)$$

Plug that back into the equation, we have:

$$\psi(\phi(\alpha v + \beta u)) = \psi(\alpha\phi(u) + \beta\phi(u))$$

Since ψ is linear, we have:

$$\psi(\alpha\phi(u) + \beta\phi(u)) = \psi(\alpha\phi(u)) + \psi(\beta\phi(v)) = \alpha(\psi \circ \phi)(u) + \beta(\psi \circ \phi)(v)$$

Therefore, the composition of linear transformations ϕ and ψ is linear.

$$(\psi \circ \phi)(\alpha v + \beta u) = \alpha(\psi \circ \phi)(u) + \beta(\psi \circ \phi)(v)$$

Problem 1.4

$$\begin{aligned}
||1 - e^{2ix}||^2 &= (1 - e^{2ix}) \cdot (\overline{1 - e^{2ix}}) \\
&= [1 - (\cos(2x) + i\sin(2x))] \cdot [\overline{1 - (\cos(2x) + i\sin(2x))}] \\
&= [1 - \cos(2x) - i\sin(2x)] \cdot [1 - \cos(2x) + i\sin(2x)] \\
&= 1 - \cos(2x) + i\sin(2x) - \cos(2x) + \cos^2(2x) - i\sin(2x)\cos(2x) \\
&\quad - i\sin(2x) + i\sin(2x)\cos(2x) + \sin^2(2x) \\
&= 1 - 2\cos(2x) + \underbrace{\cos^2(2x) + \sin^2(2x)}_{1} \\
&= 2 - 2\cos(2x) \\
&= 2(1 - \cos(2x)) = 2 \cdot 2\sin^2(x) = 4\sin^2(x)
\end{aligned}$$