Homework 1 - Quantum Algorithms

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Problem 1.1

For $\bar{u}, \bar{v} \in \mathbb{C}^n$ show that the following holds:

$$\langle u, v \rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2 - i||u+iv||^2 + i||u-iv||^2)$$

We have:

$$\begin{aligned}
||u+v||^2 &= \langle u+v, u+v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\
||u-v||^2 &= \langle u-v, u-v \rangle = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\
||u+iv||^2 &= \langle u+iv, u+iv \rangle = \langle u, u \rangle + \langle u, iv \rangle + \langle iv, u \rangle + \langle iv, iv \rangle \\
||u-iv||^2 &= \langle u-iv, u-iv \rangle = \langle u, u \rangle - \langle u, iv \rangle - \langle iv, u \rangle + \langle iv, iv \rangle
\end{aligned}$$

Substitute these into the equation, we have right-hand side equal to:

$$\begin{aligned}
&= \frac{1}{4}\left[2\left(\langle u, v \rangle + \langle v, u \rangle\right) - 2i\left(\langle u, iv \rangle + \langle iv, u \rangle\right)\right] \\
&= \frac{1}{4}\left(2\langle u, v \rangle + 2\langle v, u \rangle - 2\underbrace{i \cdot i}_{-1}\langle u, v \rangle - 2\underbrace{i \cdot \bar{i}}_{1}\langle v, u \rangle\right) \\
&= \frac{1}{4}\left(2\langle u, v \rangle + 2\langle v, u \rangle + 2\langle u, v \rangle - 2\langle v, u \rangle\right) \\
&= \frac{1}{4}\left(4\langle u, v \rangle\right) = \langle u, v \rangle
\end{aligned}$$

Problem 1.2

Let $\bar{u} = (i, 1+i)^T$ and $\bar{v} = (2, 1-i)^T$

1. Compute $\langle \bar{u}, \bar{v} \rangle$

$$\langle \bar{u}, \bar{v} \rangle = \langle (i, 1+i)^T, (2, 1-i)^T \rangle$$