Homework 7 - Quantum Algorithms

Son Nguyen

I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1

$$\frac{1}{3}|000\rangle + \frac{2}{3}|010\rangle + \frac{2}{3}|100\rangle$$

The probability of measuring 0 in the first qubit is the sum of the squares of the amplitudes of the states that have 0 in the first qubit. In this case, the probability is

$$P(0_1) = \left|\frac{1}{3}\right|^2 + \left|\frac{2}{3}\right|^2 = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$$

Problem 2

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$HZ = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$HZH = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = X$$

Problem 3

Applying the Hadamard gate to the first qubit in a 2 qubits state $|\psi\rangle$:

$$\begin{aligned}
(H\otimes I)|\psi\rangle &= (H\otimes I)\left(\frac{1}{2}|00\rangle - \frac{i}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)\\
&= \frac{1}{2}(H|0\rangle\otimes|0\rangle) - \frac{i}{\sqrt{2}}(H|0\rangle\otimes|1\rangle) + \frac{1}{\sqrt{2}}(H|1\rangle\otimes|1\rangle)\\
&= \frac{1}{2}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle\right) - \frac{i}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|1\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|1\rangle\right)\\
&= \frac{1}{2\sqrt{2}}(|00\rangle+|10\rangle) - \frac{i}{2}(|01\rangle+|11\rangle) + \frac{1}{2}(|01\rangle-|11\rangle)\\
&= \frac{1}{2\sqrt{2}}|00\rangle - \frac{i}{2}|01\rangle + \frac{1}{2}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle - \frac{i}{2}|11\rangle - \frac{1}{2}|11\rangle\\
&= \frac{1}{2\sqrt{2}}|00\rangle + \frac{1-i}{2}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle - \frac{1+i}{2}|11\rangle
\end{aligned}$$

Verify the coeficients:

$$2\left|\frac{1}{2\sqrt{2}}\right|^2+\left|\frac{1-i}{2}\right|^2+\left|-\frac{1+i}{2}\right|^2=\left|\frac{1}{2}\right|^2+\left|-\frac{i}{\sqrt{2}}\right|^2+\left|\frac{1}{\sqrt{2}}\right|^2=\frac{5}{4}$$

Problem 4

Let assume we have a U that transform state $|\psi\rangle \to |0\rangle$. Where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $\alpha, \beta \in \mathbb{C}$.

$$\begin{aligned} U|\psi\rangle &= U(\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha U|0\rangle + \beta U|1\rangle \\ &= \alpha|0\rangle + \beta|0\rangle \\ &= (\alpha+\beta)|0\rangle \end{aligned}$$

However the Unitary transformation must preserve the innder product. From the above equation, we see that U transfroms $|1\rangle \to |0\rangle$ and $|0\rangle \to |0\rangle$. This is not possible because the inner product of $|0\rangle$ and $|1\rangle$ is, 0 and the inner product of $|0\rangle\, and\, |0\rangle$ is 1. Therefore, there is no such unitary transformation U.

$$\langle 0|1\rangle \neq \langle U0|U1\rangle$$