MA 565 **Midterm** **March 8, 2025**

Name: *MyStevens Username*:

Open book and notes. *Answers must include supporting work.*

Calculators and wolfram alpha can be used for basic computations. *No cooperation.*

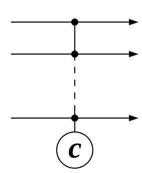
(1) [10 pts] Suppose that $c \in \mathbb{C}$ satisfies $\|c\| = 1$.

(a) [5 pts] Show that an m qubit transformation $|x_1 \dots x_m\rangle \longrightarrow c|x_1 \dots x_m\rangle$ that multiplies a state by c is unitary.

(b) [5 pts] Show that an n qubit transformation

$$|x_1 \dots x_m\rangle \longrightarrow \begin{cases} |x_1 \dots x_m\rangle & \text{if } x_i = 0 \text{ for some } i \\ c|x_1 \dots x_m\rangle & \text{if } x_1 = \dots = x_m = 1. \end{cases}$$

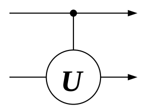
that multiplies the amplitude of the classical state $|1^n\rangle$ by c is unitary. Graphically, the gate for this operation is represented as shown below.



(2) [10 pts] **(Operators with quantum control)** Let U be a unitary transformation of an n-qubit system. Let $U = (u_{ij})$ be the matrix of the operator U. Define a linear transformation CU ("controlled-U") of an $n+1$-qubit system as follows:

$$|0\rangle|\psi\rangle \xrightarrow{CU} |0\rangle|\psi\rangle,$$
$$|1\rangle|\psi\rangle \xrightarrow{CU} |1\rangle U|\psi\rangle.$$

(In the literature you can find different notation for CU, e.g., $C-U$, ${}^{C}U$, or $\Lambda(U)$.) Graphically, the gate for CU is represented as shown below.



(a) [2 pts] What are the dimensions of the matrix for U? What are the dimensions of the matrix for CU?

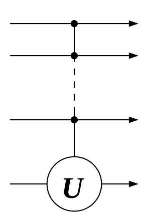
(b) [4 pts] What is structure of the matrix for CU? [Hint. The matrices for U and CU related in some way.]

(c) [4 pts] Explain why CU is a unitary operator.

(d) [+2 pts] In a similar way we can define a version C^mU of U controlled by $x_1, \ldots, x_m$.

$$|x_1 \ldots x_m\rangle|\psi\rangle \xrightarrow{C^mU} \begin{cases} |x_1 \ldots x_m\rangle|\psi\rangle & \text{if } x_i = 0 \text{ for some } i \\ |x_1 \ldots x_m\rangle U|\psi\rangle & \text{if } x_1 = \ldots = x_m = 1 \end{cases}$$

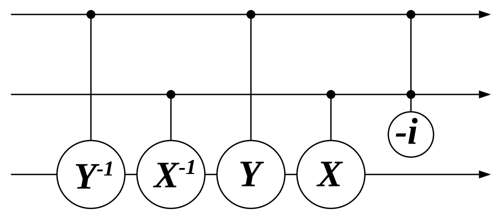
Explain why C^mU is a unitary operator. Graphically, the gate for C^mU is represented as shown below.



(3) [10 pts] **(Realization of the Toffoli (CCNOT) gate using 2-qubit transformations)**. Consider one-qubit unitary operators

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix} \quad \text{and} \quad Y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show that a circuit below computes CCNOT.



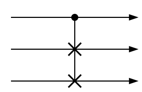
(4) [10 pts] **(Quantum teleportation preserves entanglement)** Suppose Alice has two entangled qubits $|\psi_1\rangle$ and $|\psi_2\rangle$. She applies teleportation to her second qubit and Bob receives $|\psi_2\rangle$. Check that operations applied to the system during teleportation does not change the pair $|\psi_1\rangle|\psi_2\rangle$. As a consequence, we can teleport an entangled n-qubit system by teleporting qubits one by one.

(5) [10 pts] **(Fredkin gate)**. The Fredkin gate (also **CSWAP** gate and conservative logic gate) is a computational gate suitable for reversible computing, invented by E. Fredkin. It can be defined by the 3-bit-to-3-bit map

$$(c, i_1, i_2) \mapsto (c, o_1, o_2) = \begin{cases} (c, i_1, i_2) & \text{if } c = 0, \\ (c, i_2, i_1) & \text{if } c = 1. \end{cases}$$

(a) [6 pts] Prove that this gate is functionally complete (universal) by expressing conjunction, disjunction, and negation using this gate only.

(b) [4 pts] Fredkin transformation defines a map on the classical 3-bit states that can be extended to a transformation of a 3-qubit state. Find the matrix of that transformation and show that it is unitary. Graphically, the gate for cswap is represented as shown below.



(6) [10 pts] **(On quantum parallelism)**.

(a) [5 pts] Give a quantum circuit to create the superposition

$$\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle.$$

(We've seen that this can be done by applying n Hadamard transforms to $|0\ldots0\rangle$ in a special case when $N = 2^n$.) You may assume that j is represented in binary as an n bit string. You may also assume that you have a quantum circuit U_N that on classical inputs produces

$$|j\rangle|0\rangle \longrightarrow \begin{cases} |j\rangle|0\rangle & \text{if } j \geq N, \\ |j\rangle|1\rangle & \text{if } 0 \leq j < N. \end{cases}$$

(I can be more formal and define the operator by $|j\rangle|x\rangle \xrightarrow{U_N} |j\rangle|x \oplus 1_{0\leq j<N}\rangle$.)

(b) [5 pts] Recall that $\{0,1\}^n$ is the set of n-bit strings. Let

$$E = \{\, x_1 \ldots x_n \in \{0,1\}^n \mid x_1 + \ldots + x_n \equiv_2 0 \,\}$$
$$O = \{\, x_1 \ldots x_n \in \{0,1\}^n \mid x_1 + \ldots + x_n \equiv_2 1 \,\}.$$

Construct a quantum circuit U_f that on classical inputs acts as

$$|x\rangle|b\rangle \xrightarrow{U_f} |x\rangle|b \oplus 1_{x\in O}\rangle$$

using CNOT's, CCNOT's, and ancillas if necessary. Then use U_f to construct a quantum circuit that outputs the superposition $\frac{1}{\sqrt{2^{n-1}}} \sum_{x\in E} |x\rangle$ with probability $\frac{1}{2}$ and the superposition $\frac{1}{\sqrt{2^{n-1}}} \sum_{x\in O} |x\rangle$ with probability $\frac{1}{2}$.

(7) [10 pts] Fix $r, N \in \mathbb{N}$, where $r < N$. Consider a vector $v = (v_0, \ldots, v_{N-1})$ defined by

$$v_i = \begin{cases} 1 & \text{if } r \mid i \\ 0 & \text{otherwise.} \end{cases}$$

Let $v' = (v'_0, \ldots, v'_{N-1}) = F_N v$ be the Fourier transform of v.

(a) [2 pts] Write down a formula for the entries of v'.

(b) [8 pts] Assuming r divides N, write down a simple closed form for the formula for the entries v'_i and find indices for which $\|v'_i\|$ are maximal.