

Abstract Algebra Homework 4

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This problem set includes problems from sections 4.4 and 5.3 as well as an extra problem from lecture: in particular, problem 9 from 4.4, 27 and 30 from section 5.3.

9) List every generator of each subgroup of order 8 in \mathbb{Z}_{32} .

Proof: Because \mathbb{Z}_{32} is a finite cyclic group of order 32 generated by 1, by theorem G9 (Cyclic Structure of Finite Groups), there is a *unique* subgroup, let's call it H , of order 8 which is

$$\left\langle \frac{32}{8} \right\rangle = \left\langle 4 \cdot 1 \right\rangle = \left\langle 4 \right\rangle.$$

To find the generators of this subgroup, we want to find all $1 \leq k < 8$ such that $\gcd(8, k) = 1$. As you can see, k can be 1, 3, 5, or 7 (note: this is $U(8)$.) So the generators of H are

$$\{4 \cdot 1, 4 \cdot 3, 4 \cdot 5, 4 \cdot 7\} = \{4, 12, 20, 28\}$$

We can verify the order of each of these elements have order 8:

$$\begin{aligned} |4| &= \frac{32}{\gcd(4, 32)} = \frac{32}{4} = 8 \\ |12| &= \frac{32}{\gcd(12, 32)} = \frac{32}{4} = 8 \\ |20| &= \frac{32}{\gcd(20, 32)} = \frac{32}{4} = 8 \\ |28| &= \frac{32}{\gcd(28, 32)} = \frac{32}{4} = 8 \end{aligned}$$

As we have shown, the generators of H (keeping in mind our binary operator is $+$) are $\{4, 12, 20, 28\}$. \square

27) Let G be a group and define a map $\lambda_g : G \rightarrow G$ by $\lambda_g(a) = ga$. Prove that λ_g is a permutation of G .

Proof: In order to show λ_g is a permutation of G , we need to show $\lambda_g(a)$ is well-defined under multiplication and that λ_g is one-to-one and onto.

First, $\lambda_g(a)$ is well-defined since the multiplication in the group G is well-defined by definition of being a group equipped with operator $*$.

Next, let $a, b \in G$ and $\lambda_g(a) = \lambda_g(b)$. Then we have that $ga = gb$. So, by left-cancellation, we arrive immediately at the fact that $a = b$. Hence, λ_g is one-to-one.

Lastly, to prove λ_g is onto, let $c \in G$. Then $g^{-1}c \in G$ since G is a group (G has inverses). So, $\lambda_g(g^{-1}c) = g(g^{-1}c) = (gg^{-1})c = c$. Hence, λ_g is onto.

Therefore, $\lambda_g : G \rightarrow G$ is a permutation since it is one-to-one and onto. \square

30) Let $\tau = (a_1, a_2, \dots, a_k)$ be a cycle of length k .

a) Prove that if σ is any permutation, then

$$\sigma\tau\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k)) \quad (1)$$

is a cycle of length k .

b) Let μ be a cycle of length k . Prove that there is a permutation σ such that

$$\sigma\tau\sigma^{-1} = \mu.$$

Proof:

a) By right multiplying equation (1) by σ , we see that equation (1) is true precisely if

$$\sigma\tau = (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))\sigma. \quad (2)$$

Now, we want to prove equation (2) is indeed true. Let's consider

$x \notin \{a_1, a_2, \dots, a_k\}$ and $x \in \{a_1, a_2, \dots, a_k\}$ for some x .

If $x \notin \{a_1, a_2, \dots, a_k\}$, then $\sigma\tau(x) = \sigma(x)$ since $\tau(x) = x$. Further, notice that

$$(\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))\sigma(x) = \sigma(x) \quad (3)$$

because the cycle $(\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))$ only acts on the elements $\sigma(a_i)$ for some $1 \leq i \leq k$. As we discussed in recitation, the cycle in (3) fixes everything else, i.e. it does not change to what it maps to. Thus, since we have $x \notin \{a_1, \dots, a_k\}$, the cycle fixes $\sigma(x)$. Therefore, when we consider $x \notin \{a_1, a_2, \dots, a_k\}$, we get the following:

$$\begin{aligned} \sigma\tau(x) &= \sigma(x) \\ &= (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))\sigma(x). \end{aligned}$$

Now, let's consider if $x \in \{a_1, a_2, \dots, a_k\}$. In this case, we have that $x = a_i$ for $1 \leq i \leq k$ as earlier. If $i \neq k$, then $\tau(a_i) = a_{i+1}$. Then, we get that

$$\begin{aligned}\sigma\tau(x) &= \sigma\tau(a_i) \\ &= \sigma(a_{i+1})\end{aligned}$$

So, we effectively get the next element as seen below:

$$(\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))\sigma(a_i) = \sigma(a_{i+1}).$$

If $i = k$, then $\tau(a_k) = a_1$. Hence, $\sigma\tau(a_k) = \sigma(a_1)$, i.e. it forms a k -cycle since it points back to a_1 and

$$(\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))\sigma(a_k) = \sigma(a_1).$$

Therefore $\sigma\tau(x) = (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))\sigma(x)$ for all x and hence equations (1) and (2) are equivalently true. \square

- b) Let $\mu := (b_1, b_2, \dots, b_k)$. Now we just need to have some permutation, σ , in which $\sigma(a_i) = b_i$ and fixes x otherwise, i.e. $\sigma(x) = x$. Then by part a) of this question, we have the following:

$$\begin{aligned}\sigma\tau\sigma^{-1} &= (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k)) \\ &= (b_1, b_2, \dots, b_k) \\ &= \mu\end{aligned}$$

as desired. \square

E1) Use Theorem G9 to dissect the group \mathbb{Z}_{45} . We want to find all subgroups, list all elements of each subgroup, and find all of the generators of each subgroup. In addition, draw a subgroup diagram and show that our work implies that

$$\sum_{\substack{k \geq 1 \\ k|45}} \phi(k) = 45.$$

Solution: To find all of the subgroups of \mathbb{Z}_{45} , we can just look at the positive divisors of 45 per Theorem G9. These are 1, 3, 5, 9, 15, 45. Below we list the elements of each of these subgroups formed. Note that the operator in our group is addition, so the subgroups generated by a are $\{n + a \mid n \in \mathbb{Z}\}$.

$$\begin{aligned}
\langle 0 \rangle &= \{0\} \\
\langle 3 \rangle &= \{3, 6, \dots, 42, 0\} \\
\langle 5 \rangle &= \{5, 10, \dots, 40, 0\} \\
\langle 9 \rangle &= \{9, 18, 27, 36, 0\} \\
\langle 15 \rangle &= \{15, 30, 0\} \\
\langle 45 \rangle &= \{0\}
\end{aligned}$$

The order of each of these subgroups is:

$$\begin{aligned}
|\langle 0 \rangle| &= \frac{45}{1} = 45 \\
|\langle 3 \rangle| &= \frac{45}{3} = 15 \\
|\langle 5 \rangle| &= \frac{45}{5} = 9 \\
|\langle 9 \rangle| &= \frac{45}{9} = 5 \\
|\langle 15 \rangle| &= \frac{45}{15} = 3 \\
|\langle 45 \rangle| &= \frac{45}{45} = 1
\end{aligned}$$

The generators of each of these subgroups are $0\mathbb{Z}$, $3\mathbb{Z}$, $5\mathbb{Z}$, $9\mathbb{Z}$, $15\mathbb{Z}$ and $45\mathbb{Z}$ respectively.

Note: I did not feel up for learning the Tikz package for drawing the subgroup diagrams, so it is hand-drawn and attached.

To show our implies that

$$\sum_{\substack{k \geq 1 \\ k|45}} \phi(k) = 45$$

we can simply calculate $\phi(k)$ for our positive divisors of 45 as seen below.

$$\begin{aligned}
\sum_{\substack{k \geq 1 \\ k|45}} \phi(k) &= \phi(1) + \phi(3) + \phi(5) + \phi(9) + \phi(15) + \phi(45) \\
&= 1 + 2 + 4 + 6 + 8 + 24 \\
&= 45
\end{aligned}$$

□