Abstract Algebra Homework 7

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March 18, 2016

This problem set includes problems 10.3 numbers $4d)$, 11.3 numbers $7, 16, 17$ and 11.4 number 5.

4) Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$.

4d) Show that T/U is abelian.

<u>Proof</u>: Note that we have already showed that U is normal in T in part 4c). To show that T/U is abelian, we need to show that $(AU)(BU) = (BU)(AU)$ for all $A, B \in T$.

Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and let $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$. Then we have that

$$AB = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}$$

and

$$BA = \begin{pmatrix} a'a & a'b + b'c \\ 0 & c'c \end{pmatrix}.$$

This shows that $AB \neq BA$ in general. However, we want to show that $(AU)(BU) = (BU)(AU)$. Note that $(AU)(BU) = ABU$ and $(BU)(AU) = BAU$ since U is normal. Let $C = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in U$ where $z \in \mathbb{R}$. Then we have that

$$ABU = \begin{pmatrix} aa' & z(ab' + bc') \\ 0 & cc' \end{pmatrix}$$

and

$$BAU = \begin{pmatrix} a'a & z(a'b + b'c) \\ 0 & c'c \end{pmatrix}.$$

Notice that $aa' = a'a$ and $cc' = c'c$ since $a, a', c, c' \in \mathbb{R}$. So we see that ABU and BAU only differ in the upper right entry. This does not matter though since U is matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$. Notice that both $z(ab' + bc')$ and $z(a'b + b'c) \in \mathbb{R}$. Thus, AB and BA define the same coset in U, meaning that $ABU = BAU$. Thus T/U is abelian. $\square$

7) In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.

a) List the elements in $H+N$ and $H \cap N$.

Solution: We have that

$$\begin{aligned} H+N &= \{h+n \mid h \in H \text{and} \, n \in N\} \\ &= \{h+n \mid h \in \langle 4 \rangle \text{and} \, n \in \langle 6 \rangle\} \\ &= \{0,2,4,6,8,10,12,14,16,18,20,22\} \\ &= \langle 2 \rangle \end{aligned}$$

We also see that $H \cap N = \{0, 12\}$.

Remarks: Note that $H+N = \Big\langle \gcd(4,6) \Big\rangle$ in $\mathbb{Z}_{24}$ and also that $H \cap N = \Big\langle \operatorname{lcm}(4,6) \Big\rangle \mathbb{Z}$ in $\mathbb{Z}_{24}$. □

b) List the cosets in HN/N, showing the elements in each coset.

Solution: We know that

$$(H+N)/N := \{g+n | g \in H+N\}.$$

So we see that the cosets that partition $H+N$ are the following:

$$\begin{aligned} 0+N &= \{0,6,12,18\} \\ 2+N &= \{2,8,14,20\} \\ 4+N &= \{4,10,16,22\} \end{aligned}$$

c) List the costs in $H/(H \cap N)$, showing the elements in each coset.

Solution: We know that

$$H/(H \cap N) := \{aH \cap N | a \in H\}.$$

So we see that the cosets that partition the group H are the following:

$$\begin{aligned} 0+H \cap N &= \{0,12\} \\ 4+H \cap N &= \{4,16\} \\ 8+H \cap N &= \{8,20\} \end{aligned}$$

d) Give the correspondence between $()H+N)/N$ and $H/(H \cap N)$ described in the proof of the Second Isomorphism Theorem.

Solution: Recall that all subgroups of an abelian group are normal. Since $\mathbb{Z}_{24}$ is an abelian group, we have that H and N are normal in $\mathbb{Z}_{24}$. So we can apply the Second Isomorphism Theorem which tells us

$$H/(H \cap N) \cong (H+N)/N.$$

□

16) If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

Proof: To show that G is isomorphic to a subgroup of $G/H \times G/K$, we need to define a function ϕ and show it is a group homomorphism. Then we will show that $\ker \phi = \{e\}$ and use the First Isomorphism Theorem.

Let $\phi : G \mapsto G/H \times G/K$ be defined as $\phi(g) = (gH, gK)$. Clearly the function ϕ is well-defined. We now need to show this is indeed a group homomorphism. Let $a, b \in G$. Then

$$\begin{aligned} \phi(ab) &= (abH, abK) \\ &= (aH, aK)(bH, bK) \\ &= \phi(a)\phi(b). \end{aligned}$$

So ϕ is a homomorphism. From the First Isomorphism Theorem, we know that

$$G/\ker\phi \cong \phi(G).$$

Since $\phi(G)$ is a subgroup of $G/H \times G/K$ and we want to show $G \cong \phi(G)$, it suffices for us to show that $\ker\phi = \{e\}$, i.e. $\ker\phi = H \cap K$.

Let $g \in \ker\phi$. Then $\phi(g) = (gH, gK) = (H, K)$. That is, $gH = H$ and $gK = K$. Thus $g \in H \cap K$ and we have that $\ker\phi \subset H \cap K$. Contrarily, if $g \in H \cap K$ then we clearly see that $\phi(g) = (gH, gK) = (H, K)$ so that $H \cap K \subset \ker\phi$. Thus $\ker\phi = H \cap K$.

Since we have shown that $\ker\phi = H \cap K$, i.e. ϕ is one-to-one, we have proven that

$$G \cong \phi(G)$$

which is a subgroup of $G/H \times G/K$. □

17) Let $\phi : G_1 \mapsto G_2$ be a surjective group homomorphism. Let H_1 be a normal subgroup of G_1 and suppose that $\phi(H_1) = H_2$. Prove or disprove that $G_1/H_1 \cong G_2/H_2$.

Proof: We will disprove that $G_1/H_1 \cong G_2/H_2$ by giving a counterexample. □

Give counter example.

5) Let G be a group and let i_g be an inner automorphism of G and define a map $G \mapsto Aut(G)$ by $g \mapsto i_g$. Prove that this map is a homomorphism with image $Inn(G)$ and kernel $Z(G)$. Use this result to conclude that

$$G/Z(G) \cong Inn(G)$$