

Abstract Algebra Homework 6

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This problem set includes problems from sections 11.3, 9.3, and 10.3.

8) If G is an abelian group and $n \in \mathbb{N}$, show that $\phi : G \rightarrow G$ defined by $g \mapsto g^n$ is a group homomorphism.

Proof: Let $a, b \in G$. It is easy to see that $\phi : G \rightarrow G$ is well-defined for all $g \in G$. To show ϕ is a group homomorphism we need to show that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$. We have that

$$\begin{aligned}\phi(ab) &= (ab)^n \\ &= b^n a^n \\ &= a^n b^n \because G \text{ is abelian} \\ &= \phi(a)\phi(b)\end{aligned}$$

Hence, $\phi : G \mapsto G$ is indeed a group homomorphism. □

10) If $\phi : G \rightarrow H$ is a group homomorphism and G is cyclic, prove that $\phi(G)$ is also cyclic.

Proof: Let $G = \langle g \rangle$. We want to show there exist $x \in \phi(G)$ such that $\phi(G) = \langle x \rangle$. It is natural to think $\phi(g)$ generates $\phi(G)$. Then we have that

$$\phi(g^n) = \phi(\underbrace{g \cdot g \cdot g \cdots g}_{n \text{ times}}) = \underbrace{\phi(g)\phi(g)\cdots\phi(g)}_{n \text{ times}} = (\phi(g))^n.$$

Since every element of G is of the form g^n , then this shows that every element $\phi(G)$ is of the form $(\phi(g))^n$. Thus $\phi(G)$ is also cyclic and $\phi(G) = \langle \phi(g) \rangle$. □

8) Prove that \mathbb{Q} is not isomorphic to \mathbb{Z} .

Proof: Without loss of generality, assume the group binary operation is $+$ for both groups. We claim that $\mathbb{Q} \not\cong \mathbb{Z}$ since \mathbb{Z} is cyclic ($\mathbb{Z} = \langle 1 \rangle$) while \mathbb{Q} is not. If \mathbb{Q} were isomorphic to \mathbb{Z} then \mathbb{Q} must be cyclic. We will show that \mathbb{Q} is not cyclic.

Clearly \mathbb{Q} is not generated by 0 since $\langle 0 \rangle = \{0\}$. So we will consider any cyclic subgroup of \mathbb{Q} generated by a non-zero element $a \in \mathbb{Q}$. Then, by definition of being a cyclic group, we have that

$$Q = \langle a \rangle = \{na \mid n \in \mathbb{Z}\}.$$

Notice that this subgroup in particular does not contain the element $\frac{a}{2}$ for example. If it did, then we would have that $\frac{a}{2} = \frac{1}{2}a = na$ for some $n \in \mathbb{Z}$. But, earlier we showed that $a \neq 0$ since \mathbb{Q} is not generated by 0. Thus, n must be $\frac{1}{2}$. However, $\frac{1}{2} \notin \mathbb{Z}$. Therefore, \mathbb{Q} is not cyclic.

Since we have shown that \mathbb{Q} is not cyclic while \mathbb{Z} is, it follows that $\mathbb{Q} \not\cong \mathbb{Z}$ by the Structure Theorem for groups presented in recitation. \square

47) If $G \cong \overline{G}$ and $H \cong \overline{H}$, show that $G \times H \cong \overline{G} \times \overline{H}$.

Proof: To show two groups are isomorphic, we need to define a bijective group homomorphism between the two groups. Since $G \cong \overline{G}$, there exist a bijective group homomorphism $\phi : G \mapsto \overline{G}$. Similarly, since $H \cong \overline{H}$, there exist a bijective group homomorphism $\psi : H \mapsto \overline{H}$. So define

$$\phi \times \psi : G \times H \mapsto \overline{G} \times \overline{H}$$

by $(\phi \times \psi)(g, h) = (\phi(g), \psi(h))$ where $G \times H := \{(g, h) \mid g \in G, h \in H\}$. We now need to check that $\phi \times \psi$ is indeed a bijective group homomorphism.

i) To be explicit, we should verify that $\phi \times \psi$ is well-defined. It is well-defined if $(g, h) = (g', h') \implies (\phi \times \psi)(g, h) = (\phi \times \psi)(g', h')$. By definition of $\phi \times \psi$ to the left hand side, we have

$$(\phi(g), \psi(h)) = (\phi(g'), \psi(h'))$$

which is equivalent to the right hand side, so $\phi \times \psi$ is well-defined.

ii) We now show $\phi \times \psi$ is one-to-one. Let $(g_1, h_1), (g_2, h_2) \in G \times H$ such that

$$(\phi \times \psi)(g_1, h_1) = (\phi \times \psi)(g_2, h_2).$$

Then we have that $(\phi(g_1), \psi(h_1)) = (\phi(g_2), \psi(h_2))$ which is true if and only if $\phi(g_1) = \phi(g_2)$ and $\psi(h_1) = \psi(h_2)$. Since both ϕ and ψ are bijective, we have that $g_1 = g_2$ and $h_1 = h_2$ respectively. Thus $(g_1, h_1) = (g_2, h_2)$ and $\phi \times \psi$ is one-to-one.

iii) We now show that $\phi \times \psi$ is onto and thus we will have shown it is bijective. Let $(\overline{g}, \overline{h}) \in \overline{G} \times \overline{H}$. We want to show there exist (g, h) such that

$$(\phi \times \psi)(g, h) = (\overline{g}, \overline{h}).$$

We have that

$$\begin{aligned} (\phi \times \psi)(g, h) &= (\phi(g), \psi(h)) \\ &= (\overline{g}, \overline{h}). \end{aligned}$$

So $\phi \times \psi$ is onto and we then have that it is bijective.

iv) Lastly we need to show $\phi \times \psi$ is a homomorphism. Let $(g_1, h_1), (g_2, h_2) \in G \times H$. We want to show

$$(\phi \times \psi)((g_1, h_1)(g_2, h_2)) = (\phi \times \psi)(g_1, h_1)(\phi \times \psi)(g_2, h_2) \quad (1)$$

By definition, we have that $(g_1 h_1)(g_2 h_2) = (g_1 g_2, h_1 h_2)$. Working with the left hand side of equation (1), we have that

$$\begin{aligned} (\phi \times \psi)((g_1, h_1)(g_2, h_2)) &= (\phi \times \psi)(g_1 g_2, h_1 h_2) \\ &= (\phi(g_1 g_2), \psi(h_1 h_2)) \\ &= (\phi(g_1)\phi(g_2), \psi(h_1)\psi(h_2)) \because \phi, \psi \text{ homomorphisms} \end{aligned}$$

Working with the right hand side of equation (1), we also see that

$$\begin{aligned}(\phi \times \psi)(g_1, h_1)(\phi \times \psi)(g_2, h_2) &= (\phi(g_1), \psi(h_1))(\phi(g_2), \psi(h_2)) \\ &= (\phi(g_1)\phi(g_2), \psi(h_1)\psi(h_2))\end{aligned}$$

Thus, $\phi \times \psi$ is indeed a group homomorphism. Since the function $\phi \times \psi$ is bijective and a group homomorphism, we have that

$$G \times H \cong \overline{G} \times \overline{H}.$$

□

4) Let T be the group of nonsingular upper triangular 2×2 matrices with entries in \mathbb{R} . Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$.

4a) Show that U is a subgroup of T .

Proof: To show $U < T$ we will use the Subgroup Test.

i) We need to show the identity of T , I_2 is an element of U . Simply let $x = 0$ and we get the identity matrix. So the identity $I_2 \in U$.

ii) If $h_1, h_2 \in U$, we need to show $h_1 h_2 \in U$. Let $h_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $h_2 = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ for $x, y \in \mathbb{R}$. Then we have that

$$h_1 h_2 = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix}$$

which is an element of U since $y+x \in \mathbb{R}$.

iii) Lastly, we need to show that if $h \in U$ then $h^{-1} \in U$. Let $h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then $h^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$. Since $x \in \mathbb{R}$, $-x \in \mathbb{R}$. Thus $h^{-1} \in U$. □

4b) Prove that U is abelian.

Proof: To show U is abelian, we will simply calculate $h_1 h_2$ and compare it to $h_2 h_1$ for $h_1, h_2 \in U$. If $h_1 h_2 = h_2 h_1$ then U is abelian.

Let $h_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $h_2 = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$. Then we have that

$$h_1 h_2 = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix}$$

and also that

$$h_2 h_1 = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}.$$

Since $x, y \in \mathbb{R}$, $x+y = y+x$. Thus, $h_1 h_2 = h_2 h_1$ and so U is abelian. □

4c) Prove that U is normal in T .

Proof: To show U is normal in T , we need to show that for all $t \in T$, $tUt^{-1} \subset U$.

Let $t = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for $a, b, c \in \mathbb{R}$. Then

$$\begin{aligned}t^{-1} &= \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix}\end{aligned}$$

So we then have that

$$\begin{aligned}
 tUt^{-1} &= \begin{pmatrix} a & c \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \\
 &= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & -\frac{b}{c} + \frac{ax+b}{c} \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & \frac{ax}{c} \\ 0 & 1 \end{pmatrix}
 \end{aligned}$$

Notice that $\frac{ax}{c} \in \mathbb{R}$ since $a, x, c \in \mathbb{R}$ and note that since $ac \neq 0$, then $c \neq 0$ so we are not dividing by 0. Thus, $tUt^{-1} \subset U$. and hence U is normal in T . \square

4e) Is T normal in $GL_2(\mathbb{R})$?

Proof: No, T is *not* normal in $GL_2(\mathbb{R})$. Consider the following matrices:

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in T \text{ and } g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{R}).$$

Then we have that

$$\begin{aligned} gtg^{-1} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

Hence we see that $gtg^{-1} \notin T$. Then $gTg^{-1} \not\subset T$. Therefore, T is not a normal subgroup of $GL_2(\mathbb{R})$. □