

Abstract Algebra Homework 8

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This problem set includes problems 2, 24, 28, 34, and 38 from section 16.6.

2) Let R be the ring of 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

where $a, b \in \mathbb{R}$. Show that although R is a ring that has no identity, we can find a subring S of R with an identity.

Proof: We first show that R has no identity.

Show this

We claim that $S = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ where $a \in \mathbb{R}$ is a subring of R . We will use the Subring Test to show that this is indeed a subring.

S is clearly nonempty since a is any real number.

We now show that for all $r, s \in S, rs \in S$. Let $r = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ where $a, b \in \mathbb{R}$. Then we have that

$$\begin{aligned} rs &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix} \\ &\in S \end{aligned}$$

since $ab \in \mathbb{R}$ because a and b are both in \mathbb{R} .

Next we show that for all $r, s \in S, r - s \in S$. Let r and s be as above. Then

$$\begin{aligned} r - s &= \begin{pmatrix} a - b & 0 \\ 0 & 0 \end{pmatrix} \\ &\in S \end{aligned}$$

since $a - b \in \mathbb{R}$ because both a and b are in \mathbb{R} .

Thus we have shown that $S = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is indeed a subring of R . □

24) Let R be a ring with a collection of subrings $\{R_\alpha\}$. Prove that $\bigcap R_\alpha$ is a subring of R . Give an example to show that the union of two subrings need not be a subring.

Proof: Let S be the intersection of a collection of subrings of the ring R . That is, $S = \bigcap_{i \in I} S_i$ where I is an indexed set and each S_i is a subring of R . We will use the Subring Test to show that S is indeed a subring of R .

We first begin with a claim and its proof to use it later on.

Claim: If S is a subring of a ring R then $0 \in S$.

Proof of Claim: If S is a subring of R then S is nonempty. Let $x \in S$. Then since S is a ring and has closure under additive inverses and addition, we have that $x + (-x) \in S$. By definition of additive inverses, $x + (-x) = 0$. Thus $0 \in S$.

We now check the conditions of the Subring Test hold.

i) To show that S is nonempty, just apply the result from the claim. Since $0 \in S_i$ for each $i \in I$ we have that $0 \in \bigcap_{i \in I} S_i$; i.e. $0 \in S$.

ii) Next we show for all $a, b \in S, a - b \in S$. Let $a, b \in S$. By definition of S we see $a, b \in S_i$ for each $i \in I$. By assumption that each S_i is a subring (and so S_i is a ring), we have that $a - b \in S_i$ for each $i \in I$. Then by definition of intersection, this means that $a - b \in \bigcap_{i \in I} S_i$ for each $i \in I$. That is, $a - b \in S$.

iii) Lastly we show that for all $a, b \in S, ab \in S$. Let $a, b \in S$. By definition of S we see that $a, b \in S_i$ for each $i \in I$. By assumption that each S_i is a subring (and so S_i is a ring), we have that $ab \in S_i$ for each $i \in I$. Then by definition of intersection, this means that $ab \in \bigcap_{i \in I} S_i$ for each $i \in I$. That is, $ab \in S$.

Thus S is a subring by the Subring Test. □

To give an example to show that the union of two subrings need not be a subring, consider the following:

$$R = \mathbb{Z} \quad S = \{2n \mid n \in \mathbb{Z}\} \quad T = \{3n \mid n \in \mathbb{Z}\}.$$

Note that R is a ring and S and T are subrings of R (one can easily verify this). We will show that $S \cup T$ is not a subring of R . Consider two elements: $2 \in S, 3 \in T$. Clearly both are in $S \cup T$. However $2 + 3 = 5 \notin S \cup T$. So $S \cup T$ is not a ring (and hence not a subring of R). □

□

28) A ring R is a Boolean ring if for every $a \in R, a^2 = a$. Show that every Boolean ring is a commutative ring.

Proof: We know that R is a commutative ring if $\forall a, b \in R, ab = ba$.

Let $a, b \in R$. Notice that since R is a Boolean ring and $a, b \in R$ we have that

$$\begin{aligned} a + b &= (a + b)^2 \\ &= (a + b)(a + b) \\ &= a(a + b) + b(a + b) \\ &= a^2 + ab + ba + b^2 \\ &= a + ab + ba + b^2 \quad \because a^2 = a \quad \text{and} \quad b^2 = b. \end{aligned}$$

By subtracting $a + b$ from both sides we have that $0 = ab + ba$. So $-ab = ba$. We are almost done since we want to show that $ab = ba$. To conclude, we will show that for all $c \in R, -c = c$. Let $c \in R$. Then

$$\begin{aligned} -c &= (-c)^2 \\ &= (-c)(-c) \\ &= -c(-c) \\ &= -(-c^2) \\ &= c^2 \\ &= c \quad \because R \text{ is boolean.} \end{aligned}$$

Thus $-ab = ba \implies ab = ba$ since $-c = c$ for all $c \in R$ and both a and b are arbitrary elements in R as well. □

34) Let p be prime. Prove that

$$Z_{(p)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } \gcd(b, p) = 1 \right\}$$

is a ring.

Proof: To show that $Z_{(p)}$ is a ring, we can verify directly by checking all of the properties of a ring using the definition of a ring. Or, better yet, we can show that $Z_{(p)}$ is a subring of a known ring and hence is a ring itself. We will show the latter.

Notice that as sets, $Z_{(p)} \subset \mathbb{Q}$ and \mathbb{Q} is a well-known ring. We will show that $Z_{(p)}$ is a subring of \mathbb{Q} by using the Subring Test.

i) To show that $Z_{(p)}$ is nonempty, simply take $a = 1, b = 1$ which is an element of $Z_{(p)}$ since $\gcd(b, p) = 1$ for any p prime.

ii) Next we show that for all $r, s \in Z_{(p)}, rs \in Z_{(p)}$. Let $r, s \in Z_{(p)}$ So $r = \frac{a}{b}, s = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$ and $\gcd(b, p) = \gcd(d, p) = 1$. Then we have that

$$\begin{aligned} rs &= \frac{a}{b} \cdot \frac{c}{d} \\ &= \frac{ac}{bd}. \end{aligned}$$

Notice that $ac \in \mathbb{Z}, bd \in \mathbb{Z}$. Also $\gcd(bd, p) = 1$ since $\gcd(b, p) = \gcd(d, p) = 1$. Thus $rs = \frac{ac}{bd} \in Z_{(p)}$.

iii) Lastly we show that for all $r, s \in Z_{(p)}, r - s \in Z_{(p)}$. Let $r, s \in Z_{(p)}$ as before in ii). Then

$$\begin{aligned} r - s &= \frac{a}{b} - \frac{c}{d} \\ &= \frac{ad - bc}{bd}. \end{aligned}$$

Notice that $ad - bc \in \mathbb{Z}, bd \in \mathbb{Z}$ and $\gcd(bd, p) = 1$ since $\gcd(b, p) = \gcd(d, p) = 1$. Thus $\frac{ad - bc}{bd} \in Z_{(p)}$.

By the Subring Test, we conclude that $Z_{(p)}$ is a subring of \mathbb{Q} and so $Z_{(p)}$ is a ring. □

38) An element x in a ring is called idempotent if $x^2 = x$. Prove that the only idempotent in an integral domain are 0 and 1. Find a ring with a idempotent x not equal to 0 or 1.

Proof: □