Abstract Algebra Homework 9

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This problem set includes problems $5b, 8, 18c$, and 27 from section 16.6.

5b) For the given ring R with an ideal I, give an addition and multiplication table for R/I.

Solution: Recall that

$$R/I := \{r + I \mid r \in R\}.$$

We can easily see that the three elements of R/I are the following:

$$\begin{aligned} 0 + I &= \{0, 3, 6, 9\} \\ 1 + I &= \{1, 4, 7, 10\} \\ 2 + I &= \{2, 5, 8, 11\}. \end{aligned}$$

Below is the addition table for R/I. Note that it is implicit, but worth noting, that we are talking about the addition of the three cosets here in R/I.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Similarly, here is the multiplication table for R/I. Again, we are talking about multiplication of the cosets in R/I.

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\square$

8) Prove or disprove: The ring $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is isomorphic to the ring $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$.

Proof: We will show that $\mathbb{Q}(\sqrt{2}) \ncong \mathbb{Q}(\sqrt{3})$. To do this, we need to show that no homomorphism from $\mathbb{Q}(\sqrt{2})$ to $\mathbb{Q}(\sqrt{3})$ can be an isomorphism.

Suppose that $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$ is a homomorphism. We will begin by showing that ϕ fixes $\mathbb{Z}$ and $\mathbb{Q}$.

We first show that $\phi(1) = 1$ since we do not get this for free by our definition of a ring homomorphism. Let $x \in \mathbb{Z}$. Suppose $\phi(1) = a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$. Then we have that

$$\begin{aligned} \phi(x) &= \phi(x \cdot 1) \\ &= \phi(x)\phi(1) \\ &= \phi(x\dot{1})\phi(1) \\ &= \phi(x)\phi(1)^2 \\ &= \vdots \\ &= \phi(x)\phi(1)^n. \end{aligned}$$

So $\phi(x) = \phi(x)\phi(1)^n$. Thus $1 = \phi(1)^n$. Hence $\phi(1) = 1$.

We now use this result of $\phi(1) = 1$ to show that ϕ fixes $\mathbb{Z}$. That is, we extend it to show $\phi(n) = n$ for all $n \in \mathbb{Z}$. To show this, let $n \in \mathbb{Z}^+$. Then

$$\begin{aligned}
\phi(n) &= \phi\underbrace{(1 + \cdots + 1)}_{n\text{-times}} \\
&= \underbrace{\phi(1) + \cdots + \phi(1)}_{n\text{-times}} \\
&= \underbrace{1 + \cdots + 1}_{n\text{-times}} \\
&= n.
\end{aligned}$$

The proof is similar for $n \in \mathbb{Z}^-$ and we conclude that $\phi(n) = n$ for all $n \in \mathbb{Z}$.

Next, we show that ϕ fixes $\mathbb{Q}$. That is, $\phi(y) = y$ for all $y \in \mathbb{Q}$. Let $y = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. Then

$$\begin{aligned}
\phi(y) &= \phi\Big(\frac{a}{b}\Big) \\
&= \phi(ab^{-1}) \\
&= \phi(a)\phi(b^{-1}) \\
&= \phi(a)\phi(b)^{-1} \\
&= \frac{\phi(a)}{\phi(b)} \\
&= \frac{a}{b}
\end{aligned}$$

since we just showed $\phi(n) = n$ for all $n \in \mathbb{Z}$ and both $a, b \in \mathbb{Z}$. Thus $\phi(y) = y$ for all $y \in \mathbb{Q}$.

Therefore, if $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ we see that

$$\begin{aligned}
\phi(a + b\sqrt{2}) &= \phi(a) + \phi(b\sqrt{2}) \\
&= \phi(a) + \phi(b)\phi(\sqrt{2}) \\
&= a + b\phi(\sqrt{2}).
\end{aligned}$$

So we need to figure out what exactly $\phi(\sqrt{2})$ is.

If $\phi(\sqrt{2}) = c + d\sqrt{3}$ for some $c, d \in \mathbb{Q}$ then what are the possible values of c and d? First notice that $\phi(2) = 2$ since $2 \in \mathbb{Z}$. We also know that $2 = (\sqrt{2})^2$. So then

$$\begin{aligned}
\phi(2) &= \phi\big((\sqrt{2})^2\big) \\
&= \phi(\sqrt{2})\phi(\sqrt{2}) \\
&= (c + d\sqrt{3})^2 \\
&= c^2 + 3d^2 + 2cd\sqrt{3}.
\end{aligned}$$

Thus $2 = c^2 + 3d^2$ and $0 = 2cd\sqrt{3}$. Hence $cd = 0$. So either $c = 0$ or $d = 0$.

If $c = 0$ then we have $2 = 3d^2$. So $d = \sqrt{\frac{2}{3}} \notin \mathbb{Q}$ which is a contradiction since $d \in \mathbb{Q}$. Similarly, if $d = 0$ then we have that $2 = c^2$. So $c = \sqrt{2} \notin \mathbb{Q}$ which is a contradiction since $c \in \mathbb{Q}$.

Thus $\phi(\sqrt{2}) = c + d\sqrt{3}$ for $c, d \in \mathbb{Q}$. But we have shown that we cannot find a suitable c or $d \in \mathbb{Q}$ such that this is satisfied. Hence we have shown that there is no isomorphism from $\mathbb{Q}(\sqrt{2})$ to $\mathbb{Q}(\sqrt{3})$. Therefore, as rings, $\mathbb{Q}(\sqrt{2}) \ncong \mathbb{Q}(\sqrt{3})$. $\square$

18c) Let $\phi : R \to S$ be a ring homomorphism. Let 1_R and 1_S be the identities for R and S, respectively. If ϕ is onto, show that $\phi(1_R) = 1_S$.

Proof: Let R and S be rings with identities 1_R and 1_S. Suppose ϕ is a ring homomorphism from R to S.

Let $e = \phi(1_R)$. So $e \in S$. Also let $s \in S$. Since ϕ is onto, there exists an element $r \in R$ so that $\phi(r) = s$. Then we have that

$$\begin{aligned} es &= \phi(1_R)\phi(r) \\ &= \phi(1_R \cdot r) \\ &= \phi(r) \\ &= s. \end{aligned}$$

So $es = s$. We now show that that $se = s$ as well.

$$\begin{aligned} se &= \phi(r)\phi(1_R) \\ &= \phi(r \cdot 1_R) \\ &= \phi(r) \\ &= s. \end{aligned}$$

Therefore $e \in S$ and $es = s = se$ for all $s \in S$. Hence e is a multiplicative identity for the ring S. However, 1_S is an identity as well. We have shown in class that if a ring has a multiplicative identity, it must be unique. We did this by showing that the set of units in a ring is a group and we know that the identity in a group is unique, hence the 1 in a ring is unique. Thus, it must be the case that $e = 1_S$. Since we defined e as $e = \phi(1_R)$ we conclude that $\phi(1_R) = 1_S$. $\square$

27) Let R be a commutative ring. An element $a \in R$ is nilpotent if $a^n = 0$ for some positive integer n. Show that the set of all nilpotent elements forms an ideal in R.

Proof: Let N be the set of all nilpotent elements, i.e. the nilradical. To show N is an ideal, we need to show

(i) $(N,+) < (R,+)$
(ii) $rx \in N$ and $xr \in N$ for all $r \in R$ and for all $x \in N$.

Clearly N is nonempty as $0 = 0^1 \in N$. Let $x, y \in N$. Then $x^n = y^m = 0$ for some $n, m \in \mathbb{N}$. By the Binomial Theorem we have that

$$(x+y)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k y^{n+m-k}.$$

Note that if $k \geq n, x^k = 0$. So $x^k y^{n+m-k} = 0$. Similarly, if $k < n, y^{n+m-k} = 0$ so $x^k y^{n+m-k} = 0$. Thus $(x+y)^{n+m} = 0$ and hence $(x+y) \in N$. Notice that we needed the fact that R is commutative for this to hold.

To show that additive inverses exist in N, note that for $x \in N$ we have

$$\begin{aligned} (-x)^n &= (-1)^n x^n \\ &= (-1)^n \cdot 0 \\ &= 0. \end{aligned}$$

So $-x \in N$. Thus N is an additive subgroup of R.

Lastly, let $r \in R$ and $x \in N$. Since R is commutative, we should only need to show $(xr)^n = 0$ and then automatically would we have $(rx)^n = 0$. We show both calculations here nonetheless.

$$\begin{aligned} (xr)^n &= r^n x^n \\ &= r^n \cdot 0 \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} (rx)^n &= x^n r^n \\ &= 0 \cdot r^n \\ &= 0. \end{aligned}$$

So both $rx \in N$ and $xr \in N$.

Since N is an additive subgroup of R and N absorbs R on the left and right, N is an ideal of R. $\square$