Abstract Algebra Homework 7

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This problem set includes problems 10.3 numbers $4d$), 11.3 numbers 7, 16, 17 and 11.4 number 5.

4) Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$.

4d) Show that T/U is abelian.

<u>Proof</u>: Note that we have already showed that U is normal in T in part 4c). To show that T/U is abelian, we need to show that $(AU)(BU) = (BU)(AU)$, i.e. $ABU = BAU$ for all $A, B \in T$.

Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and let $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$. Then we have that

$$AB = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}$$

and

$$BA = \begin{pmatrix} a'a & a'b + b'c \\ 0 & c'c \end{pmatrix}.$$

This shows that $AB \neq BA$ in general. However, we want to show that $(AU)(BU) = (BU)(AU)$. Note that $(AU)(BU) = ABU$ and $(BU)(AU) = BAU$ since U is normal. Let $C = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in U$ where $z \in \mathbb{R}$. Then we have that

$$ABU = \begin{pmatrix} aa' & z(ab' + bc') \\ 0 & cc' \end{pmatrix}$$

and

$$BAU = \begin{pmatrix} a'a & z(a'b + b'c) \\ 0 & c'c \end{pmatrix}.$$

Notice that $aa' = a'a$ and $cc' = c'c$ since $a, a', c, c' \in \mathbb{R}$. So we see that ABU and BAU only differ in the upper right entry. This does not matter though since U is matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$. Notice that both $z(ab' + bc')$ and $z(a'b + b'c) \in \mathbb{R}$. Thus, AB and BA define the same coset in U, meaning that $ABU = BAU$. Thus T/U is abelian. $\square$

7) In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.

a) List the elements in $H+N$ and $H \cap N$.

Solution: We have that

$$\begin{aligned} H+N &= \{h+n \mid h \in H \,\text{and}\, n \in N\} \\ &= \{h+n \mid h \in \langle 4 \rangle \,\text{and}\, n \in \langle 6 \rangle\} \\ &= \{0,2,4,6,8,10,12,14,16,18,20,22\} \\ &= \langle 2 \rangle \end{aligned}$$

We also see that $H \cap N = \{0, 12\}$.
Remarks: Note that $H+N = \Big\langle \gcd(4,6) \Big\rangle$ in $\mathbb{Z}_{24}$ and also that $H \cap N = \Big\langle \operatorname{lcm}(4,6) \Big\rangle \mathbb{Z}$ in $\mathbb{Z}_{24}$. □

b) List the cosets in HN/N, showing the elements in each coset.

Solution: We know that

$$(H+N)/N := \{g+N \,|\, g \in H+N\}.$$

So we see that the cosets that partition $H+N$ are the following:

$$\begin{aligned} 0+N &= \{0,6,12,18\} \\ 2+N &= \{2,8,14,20\} \\ 4+N &= \{4,10,16,22\} \end{aligned}$$

□

c) List the costs in $H/(H \cap N)$, showing the elements in each coset.

Solution: We know that

$$H/(H \cap N) := \{aH \cap N \,|\, a \in H\}.$$

So we see that the cosets that partition the group H are the following:

$$\begin{aligned} 0+H \cap N &= \{0,12\} \\ 4+H \cap N &= \{4,16\} \\ 8+H \cap N &= \{8,20\} \end{aligned}$$

□

d) Give the correspondence between $(H+N)/N$ and $H/(H \cap N)$ described in the proof of the Second Isomorphism Theorem.

Solution: Recall that all subgroups of an abelian group are normal. Since $\mathbb{Z}_{24}$ is an abelian group, we have that H and N are normal in $\mathbb{Z}_{24}$. So we can apply the Second Isomorphism Theorem which tells us

$$H/(H \cap N) \cong (H+N)/N.$$

□

16) If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

Proof: To show that G is isomorphic to a subgroup of $G/H \times G/K$, we need to define a function ϕ and show it is a group homomorphism. Then we will show that $\ker \phi = \{e\}$ and use the First Isomorphism Theorem.

Let $\phi : G \mapsto G/H \times G/K$ be defined as $\phi(g) = (gH, gK)$. Clearly the function ϕ is well-defined. We now need to show this is indeed a group homomorphism. Let $a, b \in G$. Then

$$\begin{aligned}\phi(ab) &= (abH, abK)\\ &= (aH, aK)(bH, bK)\\ &= \phi(a)\phi(b).\end{aligned}$$

So ϕ is a homomorphism. From the First Isomorphism Theorem, we know that

$$G/\ker\phi \cong \phi(G).$$

Since $\phi(G)$ is a subgroup of $G/H \times G/K$ and we want to show $G \cong \phi(G)$, it suffices for us to show that $\ker\phi = \{e\}$, i.e. $\ker\phi = H \cap K$.

Let $g \in \ker\phi$. Then $\phi(g) = (gH, gK) = (H, K)$. That is, $gH = H$ and $gK = K$. So $g \in H \cap K$ and we have that $\ker\phi \subset H \cap K$. Contrarily, if $g \in H \cap K$ then we clearly see that $\phi(g) = (gH, gK) = (H, K)$ so that $H \cap K \subset \ker\phi$. Thus $\ker\phi = H \cap K$.

Since we have shown that $\ker\phi = H \cap K$, i.e. ϕ is one-to-one, we have proven that

$$G \cong \phi(G)$$

which is a subgroup of $G/H \times G/K$. □

17) Let $\phi : G_1 \mapsto G_2$ be a surjective group homomorphism. Let H_1 be a normal subgroup of G_1 and suppose that $\phi(H_1) = H_2$. Prove or disprove that $G_1/H_1 \cong G_2/H_2$.

Proof: We will disprove that $G_1/H_1 \cong G_2/H_2$ by giving a counterexample.

First note that since H_1 and H_2 are subgroups, they must contain at least the identity element. We consider the two trivial subgroups: $H_1 = \{e_{G_1}\}$ and $H_2 = \{e_{G_2}\}$. Then we have that $\phi(H_1) = \phi(e_{G_1}) = e_{G_2} = H_2$. So

$$G_1/H_1 \cong G_2/H_2 \iff G_1 \cong G_2. \tag{1}$$

In general, this is not true. So we need to specify G_1, G_2, and $\phi : G_1 \mapsto G_2$ so that equation (1) does not hold. Consider $G_1 = \mathbb{Z}_2$ and $G_2 = \{e_{G_2}\}$. Clearly $G_1 \ncong G_2$ since they have different orders: G_1 has order 2 while G_2 has order 1. For $\phi : G_1 \mapsto G_2$, we can just choose the identity homomorphism, i.e. $\phi(g) = e$ for all $g \in G_1$.

We need to verify that the function ϕ is a surjective homomorphism. To show it is surjective, let $y \in G_2$. Since $G_2 = \{e_{G_2}\}, y$ clearly must be e_{G_2}. We want to show there exist $x \in G_1$ such that $\phi(x) = y$, i.e. $\phi(x) = e_{G_2}$. As we showed earlier, $x = e_{G_1}$ works, i.e. $\phi(e_{G_1}) = e_{G_2}$. To show it is a homomorphism, let $a, b \in G_1$. We then have that

$$\begin{aligned}\phi(ab) &= e_{G_2}\\ &= e_{G_2}e_{G_2}\\ &= \phi(a)\phi(b).\end{aligned}$$

So ϕ is a homomorphism.

Since we have constructed a specific example showing that $G_1/H_1 \cong G_2/H_2$ does not always hold, we are done. □

5) Let G be a group and let i_g be an inner automorphism of G and define a map $G \mapsto Aut(G)$ by $g \mapsto i_g$. Prove that this map is a homomorphism with image $Inn(G)$ and kernel $Z(G)$. Use this result to conclude that

$$G/Z(G) \cong Inn(G)$$

Proof: Recall that $i_g(x) := gxg^{-1}$ for $g \in G$. We first show that the map is a homomorphism. Let $a, b \in G$. Then

$$\begin{aligned}i_{ab}(x) &= (ab)x(ab)^{-1}\\ &= abxb^{-1}a^{-1}\\ &= a(bxb^{-1})a^{-1}\\ &= i_a\big(i_b(x)\big)\\ &= (i_a i_b)(x)\end{aligned}$$

Since $i_{ab} = i_a i_b$ our map is a homomorphism.

By definition, we know that $Inn(G)$ is the set of inner automorphisms:

$$Inn(G) := \{i_g | g \in G\}.$$

So clearly the image for our homomorphism is $Inn(G)$ by definition. Next, we show that the kernel of this homomorphism is $Z(G)$, the center of G.

$$\begin{aligned}\{a \in G | i_a(x) = x \quad \forall x \in G\} &= \{a \in G | axa^{-1} = x \quad \forall x \in G\} \\ &= \{a \in G | ax = xa \quad \forall x \in G\} \\ &= Z(G)\end{aligned}$$

By the First Isomorphism Theorem, we conclude that $G/\ker\phi \cong \phi(G)$, i.e.

$$G/Z(G) \cong Inn(G).$$

$\square$