Abstract Algebra Homework 7

Joe Loser

March 18, 2016

This problem set includes problems 10.3 numbers $4d), 11.3$ numbers $7, 16, 17$ and 11.4 number 5.

4) Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$.

4d) Show that T/U is abelian.

Proof: Note that we have already showed that U is normal in T in part 4c). To show that T/U is abelian, we need to show that $(AU)(BU) = (BU)(AU)$ for all $A, B \in T$.

Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and let $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$. Then we have that

$$AB = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}$$

and

$$BA = \begin{pmatrix} a'a & a'b + b'c \\ 0 & c'c \end{pmatrix}.$$

This shows that $AB \neq BA$ in general. However, we want to show that $(AU)(BU) = (BU)(AU)$. Note that $(AU)(BU) = ABU$ and $(BU)(AU) = BAU$ since U is normal. Let $C = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in U$ where $z \in \mathbb{R}$. Then we have that

$$ABU = \begin{pmatrix} aa' & z(ab' + bc') \\ 0 & cc' \end{pmatrix}$$

and

$$BAU = \begin{pmatrix} a'a & z(a'b + b'c) \\ 0 & c'c \end{pmatrix}.$$

Notice that $aa' = a'a$ and $cc' = c'c$ since $a, a', c, c' \in \mathbb{R}$. So we see that ABU and BAU only differ in the upper right entry. This does not matter though since U is matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$. Notice that both $z(ab' + bc')$ and $z(a'b + b'c) \in \mathbb{R}$. Thus, AB and BA define the same coset in U, meaning that $ABU = BAU$. Thus T/U is abelian. $\square$

7) In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.

a) List the elements in $H + N$ and $H \cap N$.

b) List the cosets in HN/N, showing the elements in each coset.

c) List the costs in $H/(H \cap N)$, showing the elements in each coset.

d) Give the correspondence between HN/N and $H/(H \cap N)$ described in the proof of the Second Isomorphism Theorem.

16) If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

17) Let $\phi : G_1 \mapsto G_2$ be a surjective group homomorphism. Let H_1 be a normal subgroup of G_1 and suppose that $\phi(H_1) = H_2$. Prove or disprove that $G_1/H_1 \cong G_2/H_2$.

<u>Proof</u>: We will disprove that $G_1/H_1 \cong G_2/H_2$ by giving a counterexample. □

Give counter example.

5) Let G be a group and let i_g be an inner automorphism of G and define a map $G \mapsto Aut(G)$ by $g \mapsto i_g$. Prove that this map is a homomorphism with image $Inn(G)$ and kernel $Z(G)$. Use this result to conclude that

$$G/Z(G) \cong Inn(G)$$