

Abstract Algebra Homework 10

Joe Loser

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This problem set includes problems 3b, 5b, 12, and an extra problem from section 17.4.

3) Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$.

3b) $a(x) = 6x^4 - 2x^3 + x^2 - 3x + 1, b(x) = x^2 + x - 2$ in $\mathbb{Z}_7[x]$

Solution: Performing long division, we have

$$\begin{array}{r}
 \overline{6x^2 - 8x + 21} \\
 x^2 + x - 2 \overline{) } \\
 \underline{-6x^4 - 6x^3 + 12x^2} \\
 -8x^3 + 13x^2 - 3x \\
 \underline{8x^3 + 8x^2 - 16x} \\
 21x^2 - 19x + 1 \\
 \underline{-21x^2 - 21x + 42} \\
 -40x + 43
 \end{array}$$

So we have that

$$\begin{aligned}
 a(x) &= (6x^2 - 8x + 21)(x^2 + x - 2) - 40x + 43 \\
 &= (6x^2 + 6x)(x^2 + x - 2) + 2x + 1.
 \end{aligned}$$

Thus $q(x) = 6x^2 + 6x$ and $r(x) = 2x + 1$. □

5) Find all of the zeros of the given polynomial.

5b) $p(x) := 3x^3 - 4x^2 - x + 4$ in \mathbb{Z}_5

Solution: We first find a root of $p(x)$ by looking at $p(0), p(1), p(2), p(3)$, and $p(4)$. We see that

$$p(0) = 4$$

$$p(1) = 3 - 4 + 4 = 2$$

$$p(2) = 24 - 16 + 2 = 10 \equiv 0$$

$$p(3) = 81 - 36 + 1 \equiv 1 - 1 + 1 \equiv 1 \text{ and}$$

$$p(4) = 3(64) - 64 = 2(64) = 128 \equiv 3.$$

So $x = 2$ is a root of $p(x)$. Performing the long division, we have that

$$\begin{array}{r}
 \overline{3x^2 + 2x + 3} \\
 x - 2 \overline{) } \\
 \underline{-3x^3 + 6x^2} \\
 2x^2 - x \\
 \underline{-2x^2 + 4x} \\
 3x + 4 \\
 \underline{-3x + 6} \\
 10
 \end{array}$$

Note that $10 \equiv 0$ so $r(x) = 0$. Thus

$$p(x) = (3x^2 + 2x + 3)(x - 2) \tag{1}$$

To further see that we cannot factorize this any more, we look to see the roots of $q(x) := (3x^2 + 2x + 3)$ in \mathbb{Z}_5 . We have that

$$q(0) = 3$$

$$q(1) = 8 \equiv 3$$

$$q(2) = 12 + 4 + 3 \equiv 19 \equiv 4$$

$$q(3) = 3(9) + 6 + 3 = 27 + 9 = 36 \equiv 1$$

$$q(4) = 3(16) + 8 + 3 \equiv 3(1) + 3 + 3 \equiv 4$$

So $q(x)$ cannot be factorized any much. Thus we see that the only factorization of $p(x)$ in \mathbb{Z}_5 is as seen in equation (1). Hence the only root of $p(x)$ is $x = 2$. \square

12) If F is a field, show that $F[x_1, \dots, x_n]$ is an integral domain.

Proof: