Abstract Algebra Homework 6

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This problem set includes problems from sections 11.3, 9.3, and 10.3.

8) If G is an abelian group and $n \in \mathbb{N}$, show that $\phi : G \to G$ defined by $g \mapsto g^n$ is a group homomorphism.

<u>Proof</u>: Let $a, b \in G$. It is easy to see that $\phi : G \to G$ is well-defined for all $g \in G$. Then we have that $\phi(a) = a^n$ and $\phi(b) = b^n$. Also we have that $\phi(ab) = (ab)^n = a^n b^n$. Similarly, $\phi(ba) = (ba)^n = b^n a^n$. Thus $\phi(ab) = \phi(ba)$ since G is an abelian group.

To show that $\phi : G \mapsto G$ is a group homomorphism, it must satisfy

$$\phi(ab) = \phi(a)\phi(b)$$

As we have calculated, we have shown that this holds as

$$\phi(ab) = \phi(ba) = a^n b^n = \phi(a)\phi(b)$$

Hence, $\phi : G \mapsto G$ is indeed a group homomorphism. $\square$

10) If $\phi : G \to H$ is a group homomorphism and G is cyclic, prove that $\phi(G)$ is also cyclic.

<u>Proof</u>: Let $G = \langle g \rangle$. We want to show there exist $x \in \phi(G)$ such that $\phi(G) = \langle x \rangle$. It is natural to think $\phi(g)$ generates $\phi(G)$. Then we have that

$$\phi(g^n) = \phi\underbrace{(g \cdot g \cdot g \cdots g)}_{\text{n times}} = \underbrace{\phi(g)\phi(g) \cdots \phi(g)}_{\text{n times}} = \big(\phi(g)\big)^n.$$

Since every element of G is of the form g^n, then this shows that every element $\phi(G)$ is of the form $\big(\phi(g)\big)^n$. Thus $\phi(G)$ is also cyclic and $\phi(G) = \langle \phi(g) \rangle$. $\square$

8) Prove that $\mathbb{Z}$ is not isomorphic to $\mathbb{Z}$.

Proof: Without loss of generality, assume the group binary operation is $+$ for both groups. We claim that $\mathbb{Q} \ncong \mathbb{Z}$ since $\mathbb{Z}$ is cyclic ($\mathbb{Z} = \langle 1 \rangle$) while $\mathbb{Q}$ is not. If $\mathbb{Q}$ were isomorphic to $\mathbb{Z}$ then $\mathbb{Q}$ must be cyclic. We will show that $\mathbb{Q}$ is not cyclic.

Clearly $\mathbb{Q}$ is not generated by 0 since $\langle 0 \rangle = \{0\}$. So we will consider any cyclic subgroup of $\mathbb{Q}$ generated by a non-zero element $a \in \mathbb{Q}$. Then, by definition of being a cyclic group, we have that

$$Q = \langle a \rangle = \{na \mid n \in \mathbb{Z}\}.$$

Notice that this subgroup in particular does not contain the element $\frac{a}{2}$ for example. If it did, then we would have that $\frac{a}{2} = \frac{1}{2}a = na$ for some $n \in \mathbb{Z}$. But, earlier we showed that $a \neq 0$ since $\mathbb{Q}$ is not generated by 0. Thus, n must be $\frac{1}{2}$. However, $\frac{1}{2} \notin \mathbb{Z}$. Therefore, $\mathbb{Q}$ is not cyclic.

Since we have shown that $\mathbb{Q}$ is not cyclic while $\mathbb{Z}$ is, it follows that $\mathbb{Q} \ncong \mathbb{Z}$ by the Structure Theorem for groups presented in recitation. $\square$

47) If $G \cong \overline{G}$ and $H \cong \overline{H}$, show that $G \times H \cong \overline{G} \times \overline{H}$.

Proof: To show two groups are isomorphic, we need to define a bijective group homomorphism between the two groups. Since $G \cong \overline{G}$, there exist a bijective group homomorphism $\phi : G \mapsto \overline{G}$. Similarly, since $H \cong \overline{H}$, there exist a bijective group homomorphism $\psi : H \mapsto \overline{H}$. So define

$$\phi \times \psi : G \times H \mapsto \overline{G} \times \overline{H}$$

by $(\phi \times \psi)(g, h) = (\phi(g), \psi(h))$ where $G \times H := \{(g, h) \mid g \in G, h \in H\}$. We now need to check that $\phi \times \psi$ is indeed a bijective group homomorphism.

i) To be explicit, we should verify that $\phi \times \psi$ is well-defined. It is well-defined if $(g, h) = (g', h') \implies (\phi \times \psi)(g, h) = (\phi \times \psi)(g', h')$. By definition of $\phi \times \psi$ to the left hand side, we have

$$(\phi(g), \psi(h)) = (\phi(g'), \psi(h'))$$

which is equivalent to the right hand side, so $\phi \times \psi$ is well-defined.

ii) We now show $\phi \times \psi$ is one-to-one. Let $(g_1, h_1), (g_2, h_2) \in G \times H$ such that

$$(\phi \times \psi)(g_1, h_1) = (\phi \times \psi)(g_2, h_2).$$

Then we have that $(\phi(g_1), \psi(h_1)) = (\phi(g_2), \psi(h_2))$ which is true if and only if $\phi(g_1) = \phi(g_2)$ and $\psi(h_1) = \psi(h_2)$. Since both ϕ and ψ are bijective, we have that $g_1 = g_2$ and $h_1 = h_2$ respectively. Thus $(g_1, h_1) = (g_2, h_2)$ and $\phi \times \psi$ is one-to-one.

iii) We now show that $\phi \times \psi$ is onto and thus we will have shown it is bijective. Let $\big(\phi(g), \psi(h)\big) \in \overline{G} \times \overline{H}$. We want to show there exist (g, h) such that

$$(\phi \times \psi)(g, h) = \big(\phi(g), \psi(h)\big).$$

By definition of $\phi \times \psi$, we have that this is immediately true. So $\phi \times \psi$ is onto and we then have that it is bijective.

iv) Lastly we need to show $\phi \times \psi$ is a homomorphism. Let $(g_1, h_1), (g_2, h_2) \in G \times H$. We want to show

$$(\phi \times \psi)\big((g_1, h_1)(g_2, h_2)\big) = (\phi \times \psi)(g_1, h_1)(\phi \times \psi)(g_2, h_2) \tag{1}$$

By definition, we have that $(g_1h_1)(g_2h_2) = (g_1g_2, h_1h_2)$. Working with the left hand side of equation (1), we have that

$$\begin{aligned}(\phi \times \psi)\big((g_1, h_1)(g_2, h_2)\big) &= (\phi \times \psi)(g_1g_2, h_1h_2)\\ &= \big(\phi(g_1g_2), \psi(h_1h_2)\big)\\ &= \big(\phi(g_1)\phi(g_2), \psi(h_1)\psi(h_2)\big) \because \phi, \psi\ bijective\end{aligned}$$

Working with the right hand side of equation (1), we also see that

$$\begin{aligned}(\phi \times \psi)(g_1, h_1)(\phi \times \psi)(g_2, h_2) &= \big(\phi(g_1), \psi(h_1)\big)\big(\phi(g_2), \psi(h_2)\big)\\ &= \big(\phi(g_1)\phi(g_2), \psi(h_1)\psi(h_2)\big)\end{aligned}$$

Thus, $\phi \times \psi$ is indeed a group homomorphism as it respects the group structure.

Since the function $\phi \times \psi$ is bijective and a group homomorphism, we have that $G \times H \cong \overline{G} \times \overline{H}$. □

4) Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}$.

4a) Show that U is a subgroup of T.

<u>Proof</u>: To show $U < T$ we will use the Subgroup Test.

i) We need to show the identity of T, I_2 is an element of U. Simply let $x = 0$ and we get the identity matrix. So the identity $I_2 \in U$.

ii) If $h_1, h_2 \in U$, we need to show $h_1h_2 \in U$. Let $h_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $h_2 = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ for $x, y \in \mathbb{R}$. Then we have that

$$h_1h_2 = \begin{pmatrix} 1 & y + x \\ 0 & 1 \end{pmatrix}$$

which is an element of U since $y + x \in \mathbb{R}$.

iii) Lastly, we need to show that if $h \in U$ then $h^{-1} \in U$. Let $h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then $h^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$. Since $x \in \mathbb{R}, -x \in \mathbb{R}$. Thus $h^{-1} \in U$. $\square$

4b) Prove that U is abelian.

<u>Proof</u>: To show U is abelian, we will simply calculate $h_1 h_2$ and compare it to $h_2 h_1$ for $h_1, h_2 \in U$. If $h_1 h_2 = h_2 h_1$ then U is abelian.

Let $h_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $h_2 = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$. Then we have that

$$h_1 h_2 = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix}$$

and also that

$$h_2 h_1 = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}.$$

Since $x, y \in \mathbb{R}, x + y = y + x$. Thus, $h_1 h_2 = h_2 h_1$ and so U is abelian. $\square$

4c) Prove that U is normal in T.

<u>Proof</u>: To show U is normal in T, we need to show that for all $t \in T$, $tUt^{-1} \subset U$.

Let $t = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for $a, b, c \in \mathbb{R}$. Then

$$\begin{aligned} t^{-1} &= \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \end{aligned}$$

So we then have that

$$\begin{aligned} tUt^{-1} &= \begin{pmatrix} a & c \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \\ &= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix} \begin{pmatrix} \frac{1}{1} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{b}{c} + \frac{ax+b}{c} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{ax}{c} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Notice that $\frac{ax}{c} \in \mathbb{R}$ since $a, x, c \in \mathbb{R}$ and note that since $ac \neq 0$, then $c \neq 0$ so we are not dividing by 0. Thus, $tUt^{-1} \subset U$. and hence U is normal in T. $\square$

4e) Is T normal in $GL_2(\mathbb{R})$?

<u>Proof</u>: No, T is *not* normal in $GL_2(\mathbb{R})$. Consider the following matrices:

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in T \text{ and } g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{R}).$$

Then we have that

$$\begin{aligned} gtg^{-1} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

Hence we see that $gtg^{-1} \notin T$. Then $gTg^{-1} \not\subset T$. Therefore, T is not a normal subgroup of $GL_2(\mathbb{R})$. $\square$