# Abstract Algebra Homework 10

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This problem set includes problems $3b, 5b, 12$, and an extra problem from section 17.4.

3) Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$.

3b) $a(x) = 6x^4 - 2x^3 + x^2 - 3x + 1, b(x) = x^2 + x - 2$ in $\mathbb{Z}_7[x]$.

<u>Solution</u>: Performing long division, we have

$$
\begin{array}{r|r}
 & 6x^2 \quad - 8x + 21 \\ \hline
x^2 + x - 2 & 6x^4 - 2x^3 \quad + x^2 \quad - 3x \quad + 1 \\
 & - 6x^4 - 6x^3 + 12x^2 \qquad\qquad\qquad \\ \cline{2-2}
 & - 8x^3 + 13x^2 \quad - 3x \qquad\quad \\
 & 8x^3 \quad + 8x^2 - 16x \qquad\quad \\ \cline{2-2}
 & 21x^2 - 19x \quad + 1 \\
 & - 21x^2 - 21x + 42 \\ \cline{2-2}
 & - 40x + 43
\end{array}
$$

So we have that

$$
\begin{aligned}
a(x) &= (6x^2 - 8x + 21)(x^2 + x - 2) - 40x + 43 \\
&\equiv (6x^2 + 6x)(x^2 + x - 2) + 2x + 1.
\end{aligned}
$$

Thus $q(x) = 6x^2 + 6x$ and $r(x) = 2x + 1$. $\square$

5b) Find all of the zeros of $p(x) := 3x^3 - 4x^2 - x + 4$ in $\mathbb{Z}_5$.

<u>Solution</u>: We first find a root of $p(x)$ by looking at $p(0), p(1), p(2), p(3)$, and $p(4)$. We see that
$p(0) = 4 \neq 0$
$p(1) = 3 - 5 + 4 = 2 \neq 0$
$p(2) = 24 - 16 + 2 = 10 \equiv 0$
$p(3) = 81 - 36 + 1 \equiv 1 - 1 + 1 \equiv 1 \neq 0$ and
$p(4) = 3(64) - 64 = 2(64) = 128 \equiv 3 \neq 0$.

So $x = 2$ is a root of $p(x)$. By Corollary R21, 2 is a root of $p(x) \iff x - 2 | p(x)$. Performing the long division, we have that

$$
\begin{array}{r|r}
 & 3x^2 + 2x + 3 \\ \hline
x - 2 & 3x^3 - 4x^2 \quad - x + 4 \\
 & - 3x^3 + 6x^2 \qquad\qquad \\ \cline{2-2}
 & 2x^2 \quad - x \qquad \\
 & - 2x^2 + 4x \qquad \\ \cline{2-2}
 & 3x + 4 \\
 & - 3x + 6 \\ \cline{2-2}
 & 10
\end{array}
$$

Note that $10 \equiv 0$ so $r(x) = 0$. Thus

$$
p(x) = (3x^2 + 2x + 3)(x - 2) \tag{1}
$$

To further see that we cannot factorize this any more, we look to see if there are any roots of $q(x) := (3x^2+2x+3)$ in $\mathbb{Z}_5$. We have that

$q(0) = 3 \neq 0$
$q(1) = 8 \equiv 3 \neq 0$
$q(2) = 12 + 4 + 3 \equiv 19 \equiv 4 \neq 0$
$q(3) = 3(9) + 6 + 3 = 27 + 9 = 36 \equiv 1 \neq 0$
$q(4) = 3(16) + 8 + 3 \equiv 3(1) + 3 + 3 \equiv 4 \neq 0$

So $q(x)$ cannot be factorized any more. Thus we see that the only factorization of $p(x)$ in $\mathbb{Z}_5$ is as seen in equation (1). Hence the only root of $p(x)$ is $x = 2$. $\square$

12) If $F$ is a field, show that $F[x_1, \cdots, x_n]$ is an integral domain.

<u>Proof</u>: Note that since $F$ is a field, $F$ is also an integral domain. Recall by definition we have that

$$F[x_1, x_2] = (F[x_1])[x_2].$$

Without loss of generality, we will assume that $F[x_1, x_2]$ is the same as $F[x_2, x_1]$. We now proceed by induction.

i) Base Case: $n = 1$
By Corollary R19 in class, we immediately have that $F[x_1]$ is an integral domain.

ii) Induction Step
Assume that $F[x_1, \cdots, x_{n-1}]$ is an integral domain. Then we show that $F[x_1, \cdots, x_n]$ is an integral domain. We have that

$$F[x_1, \cdots, x_n] = \Big(F[x_1, \cdots, x_{n-1}]\Big)[x_n].$$

Note that $\Big(F[x_1, \cdots, x_{n-1}]\Big)$ is an integral domain and adjoining another variable, $x_n$, still makes $F[x_1, \cdots, x_n]$ an integral domain by the induction hypothesis. $\square$

E1) Write $\mathbb{Z}[\sqrt{7}]$ as a quotient ring of the polynomial ring $\mathbb{Z}[x]$ and then use this to find a familiar ring isomorphic to

$$R := \frac{\mathbb{Z}[\sqrt{7}]}{< 8 - \sqrt{7} >}.$$

<u>Solution</u>: First recall that for any integer $n \in \mathbb{Z}$ we have that

$$\mathbb{Z}[\sqrt{n}] := \{a + b\sqrt{n} | a, b, \in \mathbb{Z}\}.$$

Note that $x^2 - 7 = 0$ i.e. $x = \sqrt{7}$. Define $\phi : \mathbb{Z}[x] \to \mathbb{Z}[\sqrt{7}]$ by $\phi(f(x)) = f(\sqrt{7})$. We now show that $\phi$ is a ring homomorphism and is onto.

Let $a, b, \in \mathbb{Z}[x]$. Then we have that

$$\begin{aligned}\phi(a + b) &= (a + b)(\sqrt{7}) \\ &= a(\sqrt{7}) + b(\sqrt{7}) \\ &= \phi(a) + \phi(b).\end{aligned}$$

Also,

$$\begin{aligned}\phi(a \cdot b) &= (a \cdot b)(\sqrt{7}) \\ &= a(\sqrt{7}) \cdot b(\sqrt{7}) \\ &= \phi(a) \cdot \phi(b).\end{aligned}$$

So $\phi$ is a ring homomorphism.

Next we determine the kernel of $\phi$. We see that

$$\begin{aligned}\ker \phi &= \{f(x) \in \mathbb{Z}[x] | f(\sqrt{7}) = 0\} \\ &= \{g(x) \cdot (x^2 - 7) | g(x) \in \mathbb{Z}[x]\} \\ &= (x^2 - 7)\mathbb{Z}[x].\end{aligned}$$

To show $\phi$ is onto, note every element of $\mathbb{Z}[\sqrt{7}]$ is of the form $a+b\sqrt{7}$ where $a,b\in\mathbb{Z}$ and $a+b\sqrt{7}=\phi(a+bx)$. Thus $\phi$ is onto.

Applying the First Isomorphism Theorem, we see that

$$\mathbb{Z}[\sqrt{7}] \cong \frac{\mathbb{Z}[x]}{(x^2-7)\mathbb{Z}[x]}.$$

Using this expression, we now have that

$$R \cong \frac{\frac{\mathbb{Z}[x]}{(x^2-7)\mathbb{Z}[x]}}{(8-x)\mathbb{Z}[x]}.$$

Trying to get things into a form where we can use the Third Isomorphism Theorem, we get

$$R \cong \frac{\frac{\mathbb{Z}[x]}{(x^2-7)\mathbb{Z}[x]}}{\frac{(8-x,x^2-7)\mathbb{Z}[x]}{(x^2-7)\mathbb{Z}[x]}}.$$

Applying the Third Isomorphism now yields

$$R \cong \frac{\mathbb{Z}[x]}{(8-x,x^2-7)\mathbb{Z}[x]}.$$

Now we try to make $8-x=0$ i.e. $x=8$. Define a map $\psi:\mathbb{Z}[x]\to\mathbb{Z}[8]$ by $\psi(p(x))=p(8)$. First note that adjoining by 8 does not give us anything new. So $\mathbb{Z}[8]=\mathbb{Z}$. We now show that $\psi$ is indeed a ring homomorphism.

Let $a,b\in\mathbb{Z}[x]$. Then

$$\begin{aligned}\psi(a+b) &= (a+b)(8)\\ &= a(8)+b(8)\\ &= \psi(a)+\psi(b).\end{aligned}$$

Also,

$$\begin{aligned}\psi(a\cdot b) &= (a\cdot b)(8)\\ &= a(8)\cdot b(8)\\ &= \psi(a)\cdot\psi(b).\end{aligned}$$

So $\psi$ is a ring homomorphism.

We can find the kernel of $\psi$ by:

$$\begin{aligned}\ker\psi &= \{f(x)\in\mathbb{Z}[x]\,|\,f(8)=0\}\\ &= \{g(x)\cdot(8-x)\,|\,g(x)\in\mathbb{Z}[x]\}\\ &= (8-x)\mathbb{Z}[x].\end{aligned}$$

To show $\psi$ is onto, note every element of $\mathbb{Z}[8]$ is of the form $a+b(8)$ where $a,b\in\mathbb{Z}$ and $a+b(8)=\psi(a+bx)$. Thus $\psi$ is onto.

Note that

$$(8-x,x^2-7)\mathbb{Z}[x]=\{(8-x)a(x)+(x^2-7)b(x)\,|\,a(x),b(x)\in\mathbb{Z}[x]\}.$$

Applying our mapping $\psi$ now yields

$$\begin{aligned}R &\cong \frac{\mathbb{Z}}{(8-8,8^2-7)\mathbb{Z}}\\ &= \frac{\mathbb{Z}}{(0,57)\mathbb{Z}}\\ &\cong \frac{\mathbb{Z}}{57\mathbb{Z}}\\ &\cong Z_{57}.\end{aligned}$$

□