Abstract Algebra Homework 3

Joe Loser

February 7, 2016

This problem set includes problems from sections 3.4: namely, problems $32, 33, 45, 46$ and 48.

32) Show that if G is a finite group of even order, then there is an $a \in G$ such that a is not the identity and $a^2 = e$.

<u>Proof</u>: Define $S \subseteq G$ by

$$S := \{a \in G : a \neq a^{-1}\}.$$

Notice that S is a *proper* subset of G since $e \notin S$. Since $(a^{-1})^{-1} = a$ for all $a \in G$, we see that $a \in S$ if and only if $a^{-1} \in S$. This is valid since a group is closed under inverses. So, we can pair up the elements of S each with their respective inverses:

$$S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, ..., a_n, a_n^{-1}\}. \tag{*}$$

Thus, we see that S has an even number of elements: $|S| = 2n$. Suppose $|G| = 2m$ by hypothesis. Then, $n < m$ and the number of elements $a \in G$ satisfying the condition that $a = a^{-1}$ is

$$2m - 2n = 2(m - n).$$

In particular, notice that an even number of elements in G are equal to their own inverses and are paired together in the ordering of $(*)$. Hence, since $e = e^{-1}$, there must be at least one other element $a \in G$ such that $a = a^{-1}$. $\square$

33) Let G be a group and suppose that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Prove that G is an abelian group.

<u>Proof</u>: We know that a group G is abelian if $\forall a, b \in G$ $a * b = b * a$. Let $a, b \in G$. Then we have that $(ab)^2 = (ab)(ab) = a^2b^2$. Since G is a group, both a and b have inverses which we denote a^{-1} and b^{-1}. Clearly multiplication in G is well-defined and it also contains inverses by definition of a group. So we can safely multiply both sides of the equation by a^{-1}. So we have that

$$
\begin{aligned}
a^{-1}\big((ab)(ab)\big) &= a^{-1}(a^2b^2)\\
\big(a^{-1}(ab)\big)(ab) &= (a^{-1}a^2)b^2\\
\big((a^{-1}a)b\big)(ab) &= \big((a^{-1}a)a\big)b^2\\
(eb)(ab) &= (ea)b^2\\
b(ab) &= ab^2 \qquad (1)
\end{aligned}
$$

Now, multiply (1) by b^{-1}. This yields

$$
\begin{aligned}
b(ab)b^{-1} &= (ab^2)b^{-1}\\
(ba)(bb^{-1}) &= (ab)(bb^{-1})\\
(ba)e &= (ab)e\\
ba &= ab
\end{aligned}
$$

Thus, we have shown that G is an abelian group. □

45) Prove that the intersection of two subgroups of a group G is also a subgroup of G.

Proof: Let G be a group and let $H_1 < G$ and $H_2 < G$ be subgroups of G. We want to show $H_1 \cap H_2 < G$. To show this, we want to satisfy the three conditions of the Subgroup Test (Proposition G4 in class).

We know that $e \in H_1$ and $e \in H_2$ since H_1 and H_2 are subgroups. So, $e \in H_1 \cap H_2$. In turn, this also shows that $H_1 \cap H_2$ is not empty.

Next, let $h \in H_1 \cap H_2$. Clearly $h \in H_1$ then. Since H_1 is a subgroup, then $h^{-1} \in H_1$. Similarly, $h \in H_2$ and since H_2 is a subgroup, $h^{-1} \in H_2$. Thus, $h^{-1} \in H_1 \cap H_2$.

Lastly, by definition, H_1 and H_2 are closed under the binary operation of G. Let $k, k' \in H_1 \cap H_2$. Then $k \in H_1$ and $k' \in H_1$. Since H_1 is a subgroup, $kk' \in H_1$. Similarly, $k \in H_2$ and $k' \in H_2$. Since H_2 is also a subgroup, $kk' \in H_2$. Hence, $kk' \in H_1 \cap H_2$.

We have shown that $H_1 \cap H_2$ contains the identity, inverses, and is closed under multiplication. Hence, it is a subgroup. □

46) Prove or disprove: If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

Proof: Let $H < G$ and $K < G$ be subgroups of a group G. The union $H \cup K$ need not be a subgroup of G. We will prove this by giving a simple counterexample.

Consider $\mathbb{Z}_6$, the cyclic group of order 6. We will look at two of its subgroups: namely those generated from 2 and 3 and show that the union of these two subgroups is not a subgroup. These two subgroups are $\{[0], [2], [4]\}$ and $\{[0], [3]\}$, i.e. $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Then,

$H \cup K = \{[0], [2], [3], [4]\}$ which is not a subgroup since it is not closed under addition. In particular, notice that $2 \in \mathbb{Z}_2$ and $3 \in \mathbb{Z}_3$ and hence are both in the union. However, their sum $5 = 2 + 3$ is not an element of $H \cup K$ because 5 is neither a multiple of 2 or 3.

Since we have shown a counterexample that the union of two subgroups of a group G need not yield a subgroup, we are done. $\square$

48) Let G be a group and $g \in G$. Show that

$$Z(G) = \{x \in G : gx = xg \; \forall g \in G\}.$$

is a subgroup of G. This subgroup is called the **center** of G.

Proof: For all $g \in G$ we have that $eg = ge = e$. Thus $e \in Z(G)$ which means $Z(G)$ is non-empty.

Let $a, b \in Z(G)$. Then for all $g \in G$ we have $ag = ga$ and $bg = gb$ so that

$$\begin{aligned}(ab)g &= a(bg)\\ &= a(gb)\\ &= (ag)b\\ &= (ga)b\\ &= g(ab)\end{aligned}$$

Therefore, $ab \in Z(G)$.

Lastly, let $c \in Z(G)$ and since $g \in G$, then $cg = gc$. We want to show that $Z(G)$ contains inverses. So, multiply both sides by c^{-1} twice. This is allowed since $Z(G)$ is a subgroup and hence contains inverses.

$$\begin{aligned}c^{-1}(cg)c^{-1} &= c^{-1}(gc)c^{-1}\\ (c^{-1}c)gc^{-1} &= c^{-1}g(cc^{-1})\\ egc^{-1} &= c^{-1}ge\\ gc^{-1} &= c^{-1}g\end{aligned}$$

Therefore, $c^{-1} \in Z(G)$ since we took c to be an arbitrary element of $Z(G)$.

Thus, $Z(G)$ is a subgroup. $\square$