

# Abstract Algebra Homework 9

Joe Loser

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This problem set includes problems 5b, 8, 18c, and 27 from section 16.6.

5b) For the given ring  $R$  with an ideal  $I$ , give an addition and multiplication table for  $R/I$ .

Solution: Recall that

$$R/I := \{r + I \mid r \in R\}.$$

We can easily see that the three elements of  $R/I$  are the following:

$$\begin{aligned} 0 + I &= \{0, 3, 6, 9\} \\ 1 + I &= \{1, 4, 7, 10\} \\ 2 + I &= \{2, 5, 8, 11\}. \end{aligned}$$

Below is the addition table for  $R/I$ . Note that it is implicit, but worth noting, that we are talking about the addition of the three cosets here in  $R/I$ .

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Similarly, here is the multiplication table for  $R/I$ . Again, we are talking about multiplication of the cosets in  $R/I$ .

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

□

8) Prove or disprove: The ring  $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$  is isomorphic to the ring  $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$ .

Proof: We will show that  $\mathbb{Q}(\sqrt{2}) \not\cong \mathbb{Q}(\sqrt{3})$ . To do this, we need to show that no homomorphism from  $\mathbb{Q}(\sqrt{2})$  to  $\mathbb{Q}(\sqrt{3})$  can be an isomorphism.

Suppose that  $\phi : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{3})$  is a homomorphism. We will begin by showing that  $\phi$  fixes  $\mathbb{Z}$  and  $\mathbb{Q}$ .

We first show that  $\phi(1) = 1$  since we do not get this for free by our definition of a ring homomorphism. Let  $x \in \mathbb{Z}$ . Suppose  $\phi(1) = a + b\sqrt{3}$  where  $a, b \in \mathbb{Q}$ . Then we have that

$$\begin{aligned} \phi(x) &= \phi(x \cdot 1) \\ &= \phi(x)\phi(1) \\ &= \phi(x\phi(1))\phi(1) \\ &= \phi(x)\phi(1)^2 \\ &\vdots \\ &= \phi(x)\phi(1)^n. \end{aligned}$$

So  $\phi(x) = \phi(x)\phi(1)^n$ . Thus  $1 = \phi(1)^n$ . Hence  $\phi(1) = 1$ .

We now use this result of  $\phi(1) = 1$  to show that  $\phi$  fixes  $\mathbb{Z}$ . That is, we extend it to show  $\phi(n) = n$  for all  $n \in \mathbb{Z}$ . To show this, let  $n \in \mathbb{Z}^+$ . Then

$$\begin{aligned}\phi(n) &= \phi(\underbrace{1 + \cdots + 1}_{n\text{-times}}) \\ &= \underbrace{\phi(1) + \cdots + \phi(1)}_{n\text{-times}} \\ &= \underbrace{1 + \cdots + 1}_{n\text{-times}} \\ &= n.\end{aligned}$$

The proof is similar for  $n \in \mathbb{Z}^-$  and we conclude that  $\phi(n) = n$  for all  $n \in \mathbb{Z}$ .

Next, we show that  $\phi$  fixes  $\mathbb{Q}$ . That is,  $\phi(y) = y$  for all  $y \in \mathbb{Q}$ . Let  $y = \frac{a}{b}$  where  $a, b \in \mathbb{Z}$ . Then

$$\begin{aligned}\phi(y) &= \phi\left(\frac{a}{b}\right) \\ &= \phi(ab^{-1}) \\ &= \phi(a)\phi(b^{-1}) \\ &= \frac{\phi(a)}{\phi(b)} \\ &= \frac{a}{b}\end{aligned}$$

since we just showed  $\phi(n) = n$  for all  $n \in \mathbb{Z}$ . Thus  $\phi(y) = y$  for all  $y \in \mathbb{Q}$ .

Therefore, if  $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$  we see that

$$\begin{aligned}\phi(a + b\sqrt{2}) &= \phi(a) + \phi(b\sqrt{2}) \\ &= \phi(a) + \phi(b)\phi(\sqrt{2}) \\ &= a + b\phi(\sqrt{2}).\end{aligned}$$

So we need to figure out what exactly  $\phi(\sqrt{2})$  is.

If  $\phi(\sqrt{2}) = c + d\sqrt{3}$  for some  $c, d \in \mathbb{Q}$  then what are the possible values of  $c$  and  $d$ ? First notice that  $\phi(2) = 2$  since  $2 \in \mathbb{Z}$ . We also know that  $2 = (\sqrt{2})^2$ . So then

$$\begin{aligned}\phi(2) &= \phi((\sqrt{2})^2) \\ &= \phi(\sqrt{2})\phi(\sqrt{2}) \\ &= (c + d\sqrt{3})^2 \\ &= c^2 + 3d^2 + 2cd\sqrt{3}.\end{aligned}$$

Thus  $2 = c^2 + 3d^2$  and  $0 = 2cd\sqrt{3}$ . Hence  $cd = 0$ . So either  $c = 0$  or  $d = 0$ .

If  $c = 0$  then we have  $2 = 3d^2$ . So  $d = \sqrt{\frac{2}{3}} \notin \mathbb{Q}$  which is a contradiction since  $d \in \mathbb{Q}$ . Similarly, if  $d = 0$  then we have that  $2 = c^2$ . So  $c = \sqrt{2} \notin \mathbb{Q}$  which is a contradiction since  $c \in \mathbb{Q}$ .

Thus  $\phi(\sqrt{2}) = c + d\sqrt{3}$  for  $c, d \in \mathbb{Q}$ . But we have shown that we cannot find a suitable  $c$  or  $d \in \mathbb{Q}$  such that this is satisfied. Hence we have shown that there is no isomorphism from  $\mathbb{Q}(\sqrt{2})$  to  $\mathbb{Q}(\sqrt{3})$ . Therefore, as rings,  $\mathbb{Q}(\sqrt{2}) \not\cong \mathbb{Q}(\sqrt{3})$ .  $\square$