Abstract Algebra Homework 10

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April 24, 2016

This problem set includes problems $3b, 5b, 12$, and an extra problem from section 17.4.

3) Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$.

3b) $a(x) = 6x^4 - 2x^3 + x^2 - 3x + 1, b(x) = x^2 + x - 2$ in $\mathbb{Z}_7[x]$

<u>Solution</u>: Performing long division, we have

$$
\begin{array}{rl}
 & \;\; 6x^2 \;\; - 8x + 21 \\ \cline{2-2}
x^2 + x - 2 & \big) \; 6x^4 - 2x^3 \;\; + x^2 \;\; - 3x \;\; + 1 \\
 & \;\; - 6x^4 - 6x^3 + 12x^2 \\ \cline{2-2}
 & \qquad\quad - 8x^3 + 13x^2 \;\; - 3x \\
 & \qquad\qquad 8x^3 \;\; + 8x^2 - 16x \\ \cline{2-2}
 & \qquad\qquad\qquad 21x^2 - 19x \;\; + 1 \\
 & \qquad\qquad\quad - 21x^2 - 21x + 42 \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad - 40x + 43
\end{array}
$$

So we have that

$$
\begin{aligned}
a(x) &= (6x^2 - 8x + 21)(x^2 + x - 2) - 40x + 43 \\
&= (6x^2 + 6x)(x^2 + x - 2) + 2x + 1.
\end{aligned}
$$

Thus $q(x) = 6x^2 + 6x$ and $r(x) = 2x + 1$. □

5) Find all of the zeros of the given polynomial.

5b) $p(x) := 3x^3 - 4x^2 - x + 4$ in $\mathbb{Z}_5$

<u>Solution</u>: We first find a root of $p(x)$ by looking at $p(0), p(1), p(2), p(3)$, and $p(4)$. We see that
$p(0) = 4$
$p(1) = 3 - 5 + 4 = 2$
$p(2) = 24 - 16 + 2 = 10 \equiv 0$
$p(3) = 81 - 36 + 1 \equiv 1 - 1 + 1 \equiv 1$ and
$p(4) = 3(64) - 64 = 2(64) = 128 \equiv 3.$

So $x = 2$ is a root of $p(x)$. Performing the long division, we have that

$$
\begin{array}{rl}
 & \qquad 3x^2 + 2x + 3 \\ \cline{2-2}
x - 2 & \big) \; 3x^3 - 4x^2 \;\; - x + 4 \\
 & \;\; - 3x^3 + 6x^2 \\ \cline{2-2}
 & \qquad\quad 2x^2 \;\; - x \\
 & \qquad - 2x^2 + 4x \\ \cline{2-2}
 & \qquad\qquad\quad 3x + 4 \\
 & \qquad\qquad - 3x + 6 \\ \cline{2-2}
 & \qquad\qquad\qquad 10
\end{array}
$$

Note that $10 \equiv 0$ so $r(x) = 0$. Thus

$$p(x) = (3x^2 + 2x + 3)(x - 2) \tag{1}$$

To further see that we cannot factorize this any more, we look to see the roots of $q(x) := (3x^2 + 2x + 3)$ in $\mathbb{Z}_5$. We have that
$q(0) = 3$
$q(1) = 8 \equiv 3$
$q(2) = 12 + 4 + 3 \equiv 19 \equiv 4$
$q(3) = 3(9) + 6 + 3 = 27 + 9 = 36 \equiv 1$
$q(4) = 3(16) + 8 + 3 \equiv 3(1) + 3 + 3 \equiv 4$

So $q(x)$ cannot be factorized any much. Thus we see that the only factorization of $p(x)$ in $\mathbb{Z}_5$ is as seen in equation (1). Hence the only root of $p(x)$ is $x = 2$. $\square$

12) If F is a field, show that $F[x_1, \cdots, x_n]$ is an integral domain.

Proof: Note that since F is a field, F is also an integral domain. Recall by definition we have that

$$F[x_1, x_2] = (F[x_1])[x_2].$$

Without loss of generality, we will assume that $F[x_1, x_2]$ is the same as $F[x_2, x_1]$. We now proceed by induction.

i) Base Case: $n = 1$
By Corollary R19 in class, we immediately have that $F[x_1]$ is an integral domain.

ii) Induction Step
Assume that $F[x_1, \cdots, x_{n-1}]$ is an integral domain. Then we show that $F[x_1 \cdots x_n]$ is an integral domain. We have that

$$F[x_1, \cdots, x_n] = \Big(F[x_1, \cdots, x_{n-1}]\Big)[x_n].$$

Note that $\Big(F[x_1, \cdots, x_{n-1}]\Big)$ is an integral domain and adjoining another variable, x_n, still makes $F[x_1, \cdots, x_n]$ an integral domain by the induction hypothesis. $\square$

E1) Write $\mathbb{Z}[\sqrt{7}]$ as a quotient ring of the polynomial ring $\mathbb{Z}[x]$ and then use this to find a familiar ring isomorphic to

$$\frac{\mathbb{Z}[\sqrt{7}]}{< 8 - \sqrt{7} >}.$$

Solution: