A study of the stochastic Saltzman-Maasch model for climate dynamics in Pleistocene

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July 18, 2024

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Deterministic model

The Saltzman-Maasch model

The model explains central features of the glacial cycles observed in the climate record of the Pleistocene Epoch.

$$\begin{cases} \dot{X} = -X - Y - vZ \\ \dot{Y} = -pZ + rY - sY^2 - Y^3 \\ \dot{Z} = -q(X+Z) \end{cases}$$

The state variables X, Y, Z represent the anomalies (deviations from long-term averages) of

- X: the total continental ice mass
- Y: the atmospheric CO_2 concentration
- Z: the the mean temperature of the North Atlantic Deep Water

Deterministic Equilibria

The model possesses the equilibrium $E_0 = (0, 0, 0)$, for all the parameter values. Other equilibria points $E = (X, Y, Z)$ can be found imposing:

$$\begin{cases} -X - Y - vZ = 0 \\ -pZ + rY - sY^2 - Y^3 = 0 \\ -q(X + Z) = 0 \implies X = -Z \end{cases}$$

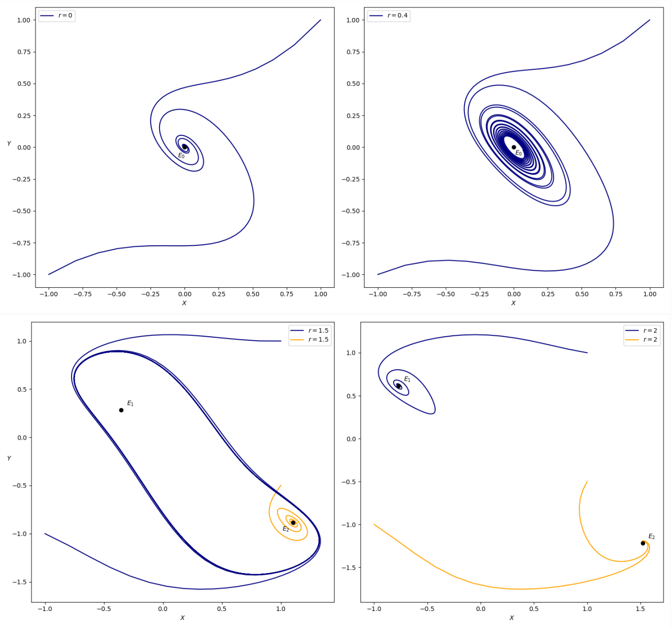
which leads to the non-trivial equilibrium points expression:

$$E_{1,2} = \left(-\frac{Y_{1,2}}{1-v},\ Y_{1,2},\ \frac{Y_{1,2}}{1-v} \right), \quad Y_{1,2} = -\frac{s}{2} \pm \sqrt{r + \frac{s^2}{4} - \frac{p}{1-v}}$$

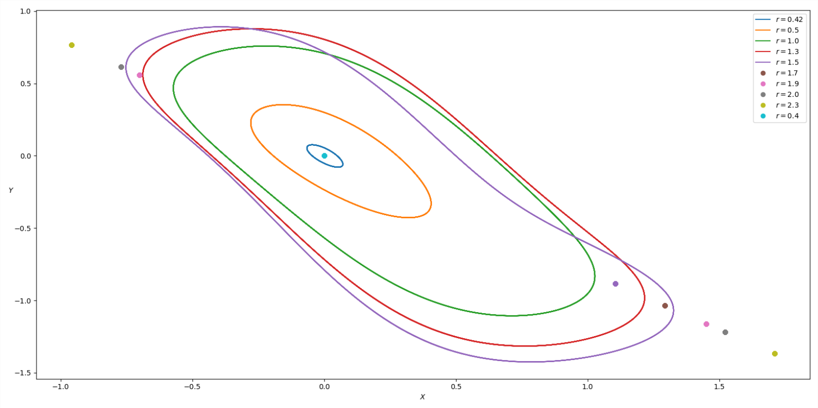
The parameter r is used as control and bifurcation parameter, fixing the others: $p = 1, q = 2.5, v = 0.2, s = 0.6$. Within this framework:

$$E_{1,2} = (-1.25Y_{1,2},\ Y_{1,2},\ 1.25Y_{1,2}), \quad Y_{1,2} = -0.3 \pm \sqrt{r - 1.16} \ .$$

Phase Portraits

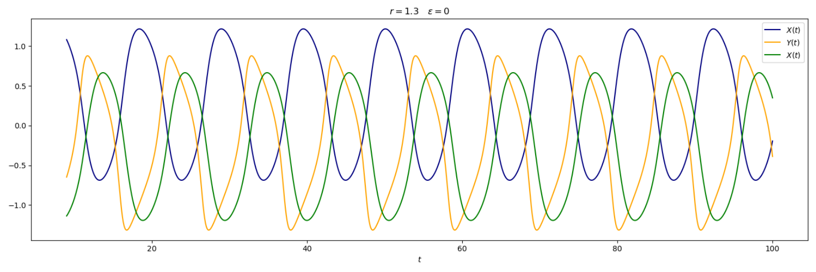


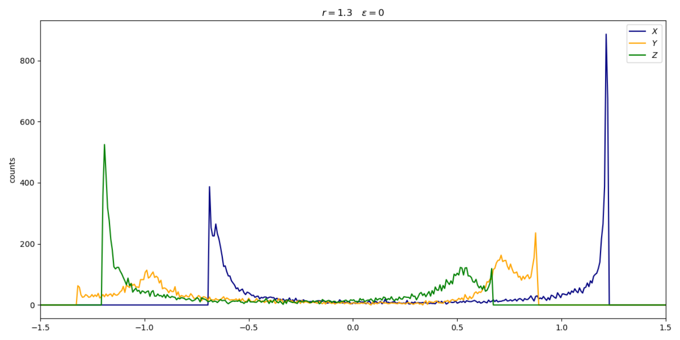
Attractors: monostability and bistability regions



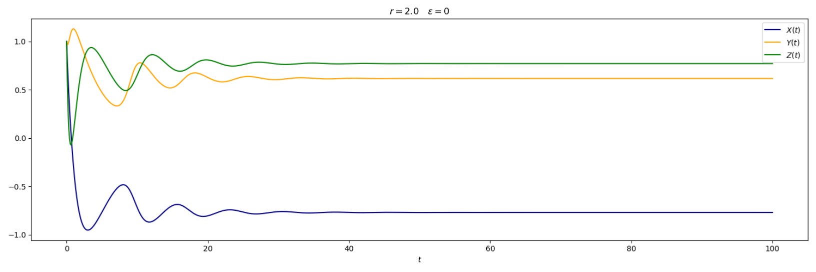
For $r < 0.42$: monostability in E_0. Around $r \approx 0.42$: bifurcation point and periodic cycles as attractors. Around $r \approx 1.5$, bistability with periodic cycles and E_2. Around $r \approx 1.7$ periodic cycles loss and monostability (brown point). For $r > 1.9$, bistability with E_1 and E_2.

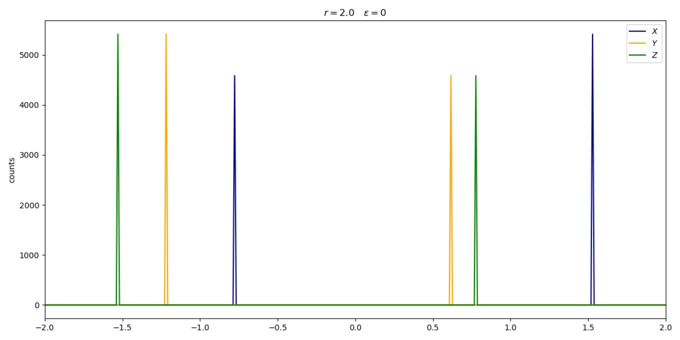
Time Behavior and Histograms



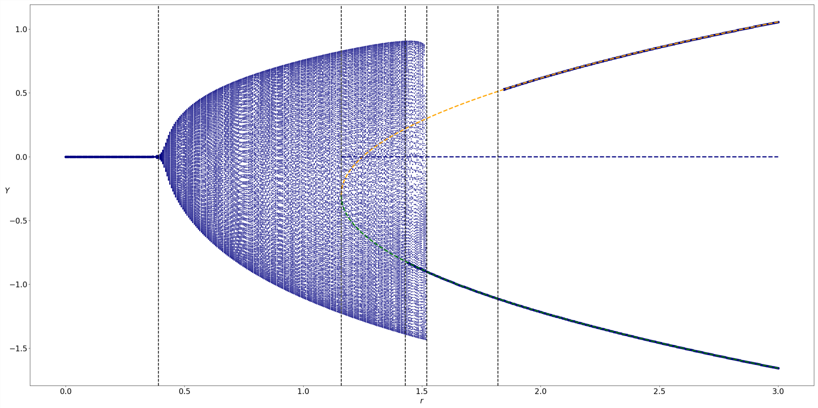


Time Behavior and Histograms





Deterministic Bifurcation Diagram



$E_0 = (0,0,0)$ is stable for $r < r_1 = 0.41704$. At $r = r_1$, bifurcation with the birth of a stable limit cycle. This cycle loses its stability at $r_4 = 1.517$. The equilibria E_1, E_2 appearing at $r = r_2 = 1.16$ are unstable at first. The equilibrium E_2 becomes stable at $r_3 = 1.42271$, E_1 at $r_5 = 1.82025$.

Deterministic Bifurcation Diagram

- In the parametric region $0 < r < r_1$, the system is monostable with the trivial equilibrium E_0 as a single attractor.
- In the zone $r_1 < r < r_3$, the climate system is monostable with the stable limit cycle as a single attractor.
- For $r_3 < r < r_4$, the system is bistable with two coexisting attractors, namely the equilibrium E_2 and the limit cycle.
- For $r_4 < r < r_5$, the deterministic system is monostable with the equilibrium E_2 representing a single attractor.
- If $r > r_5$, the system is bistable with two coexisting stable equilibria E_1 and E_2.

The Stochastic Model

The Stochastic Saltzman-Maasch model

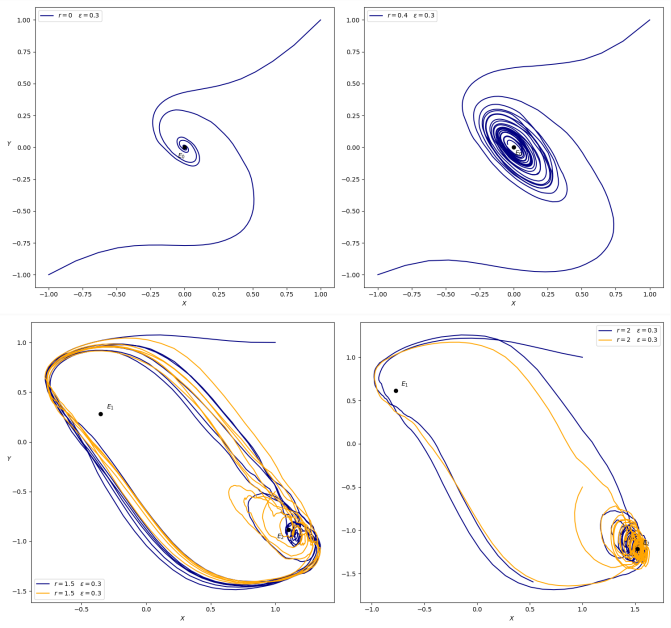
Now a stochastic dynamics is introduced, in order to reveal new features of noise-induced climate variability. Introduce in the model a **multiplicative white gaussian noise** term in the third equation.

$$\begin{cases} \dot{X} = -X - Y - vZ \\ \dot{Y} = -pZ + rY - sY^2 - Y^3 \\ \dot{Z} = -q(X + Z) + \epsilon Z\xi(t) \end{cases}$$

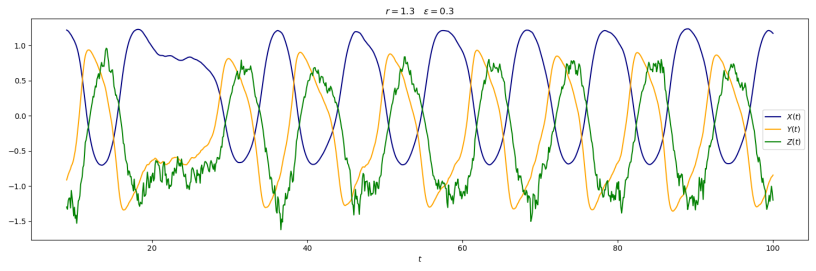
where $\xi(t)$ is the multiplicative white noise.

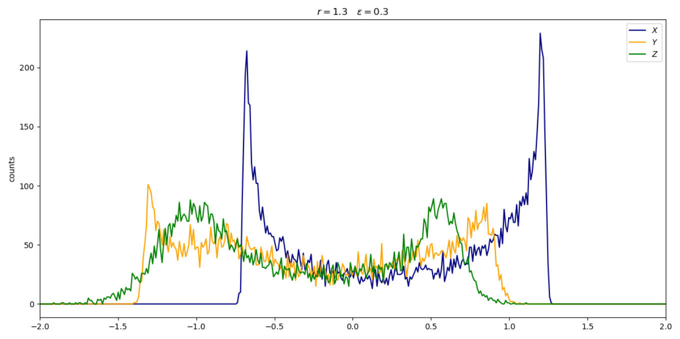
It will be shown that the climate system can be highly noise-excitable and it possesses large-amplitude fluctuations even in those regions where the deterministic model does not contain any self-sustained oscillations.

Noisy Trajectories: Noisy Phase Portraits

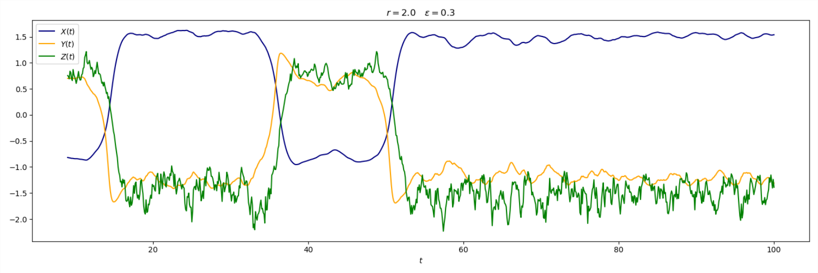


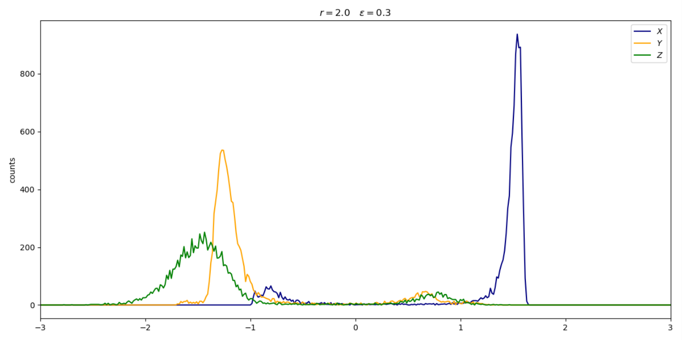
Time Behavior and Histograms





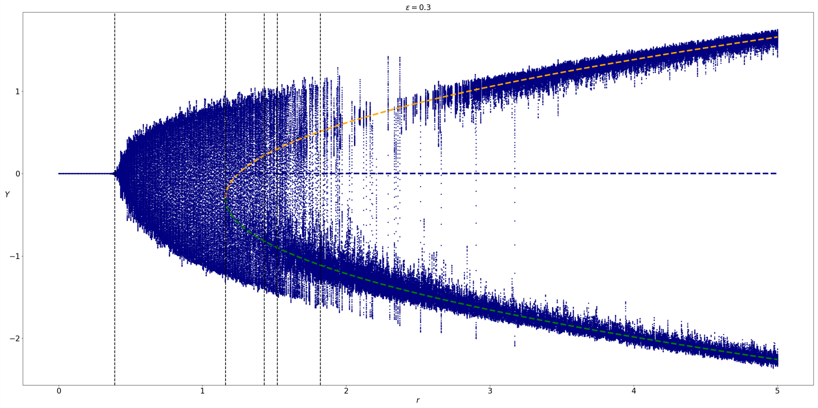
Time Behavior and Histograms



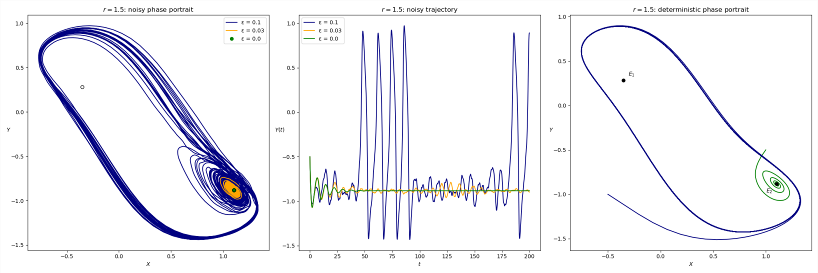


Bifurcation Diagram with Noise

How the Phase Portrait has changed under the effect of noise: presence of noise-induced phenomena of transition.

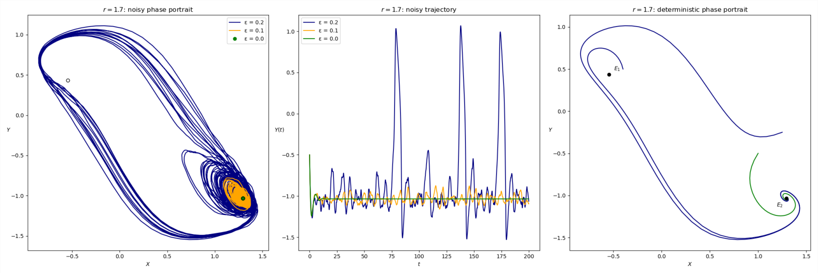


NITs



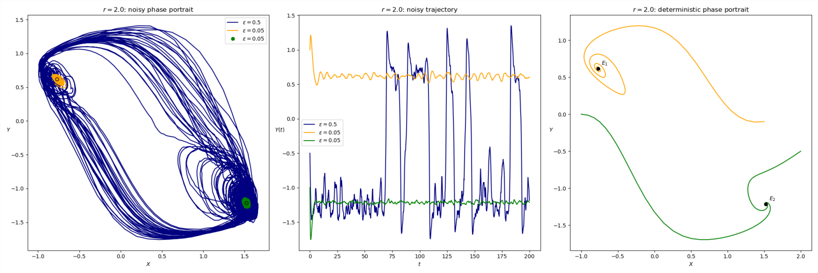
- $r = 1.5$: bistability region $r_3 < r < r_4$.
- For weak noise, $\epsilon = 0.03$, the solutions exhibit small amplitude stochastically-induced oscillations (SASIO) near E_2
- With increasing noise, the system starts to demonstrate large amplitude stochastically-induced oscillations (LASIO)
- This change is due to noise-induced transitions between the basins of attraction of the equilibrium E_2 and the periodic cycle

NITs



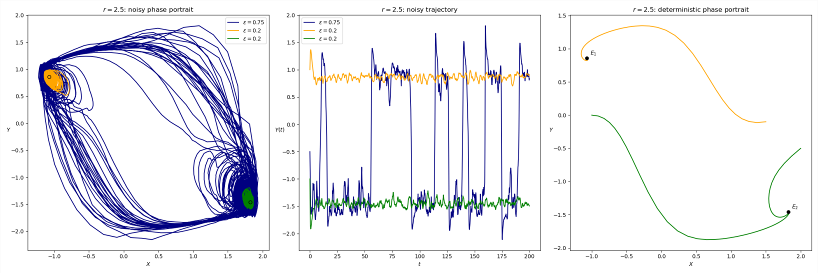
- Monostability region with $r = 1.7$: E_2 as single attractor and no stable cycles
- For small deviations of initial states (subthreshold zone), the trajectories tend to E_2. For larger deviations (superthreshold zone), the trajectory exhibits a large-amplitude loop
- For $\epsilon = 0.1$, SASIO is observed near E_2, while for $\epsilon = 0.2$, a complex regime with alternating SASIO and LASIO is evident

NITs



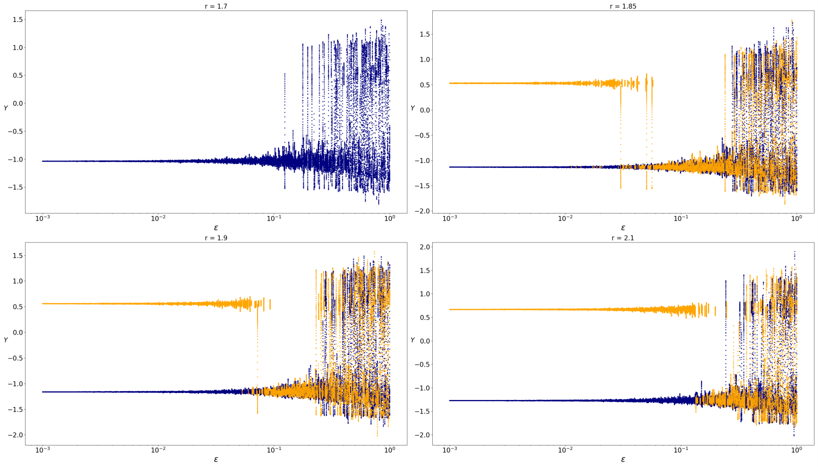
- For $r = 1.9$, the deterministic system has two stable equilibria, E_1 (warm climate state, in orange) and E_2 (cold state, in green)
- For $\epsilon = 0.05$, the solutions starting near E_1 and E_2 are concentrated near their corresponding equilibria (subthreshold)
- With increasing noise, mutual transitions between the basins of attraction of these equilibria are revealed with complex oscillatory regime with alternating SASIO and LASIO

NITs



- For $\epsilon = 0.2$, solutions starting near E_1, result in small-amplitude oscillations
- As noise increases, trajectories fall into the superthreshold zone, then to the subthreshold zone of equilibrium E_2 exhibiting SASIO
- With further noise increase, trajectories fall into the superthreshold zone of equilibrium E_2, generating large-amplitude oscillations around both E_1 and E_2

NITs



NITs

Interesting multiphase process of stochastic noise-induced transformations in the bistability zone $r > r_5 \approx 1.82$

- If a trajectory begins near E_2 (in blue), increasing noise reveals only two phases: small amplitude stochastic oscillations (SASIO) near E_1 and large amplitude stochastic oscillations (LASIO) between E_1 and E_2
- If the trajectory starts from E_1 (in orange), three phases are observed: SASIO near E_1, a transition to SASIO near E_2, and LASIO between E_1 and E_2. This reveals a sharp transition from warm to cold climate state

Improved Model

Improved Saltzman-Maasch model

1. Generalize the deterministic part of the model, adding nonlinearities and coupling terms
2. Generalize the multiplicative stochastic term: use a more sophisticated noise term, such as an Orstein-Uhlenbeck process $\eta(t)$, and incorporating a different dependency on the variables
3. Including external stochastic forcing terms

$$\begin{cases} \dot{X} = -a(X+Y) - vZ + \omega_X \xi(t) \\ \dot{Y} = -pZ + rY - sY^2 - Y^3 + bYZ + cZY^2 + \omega_Y \xi(t) \\ \dot{Z} = -q(X+Z) + \epsilon Z(1+dX)\eta(t) \end{cases}$$

where:

- $a, b, c, d \geq 0$
- $\xi(t)$ is a white noise term
- $\eta(t)$ can be a white noise term, or an Ornstein-Uhlenbeck process

Equilibrium Points

In addition to the equilibrium $E_0 = (0,0,0)$, the new model presents equilibrium points corresponding to the values of Y:

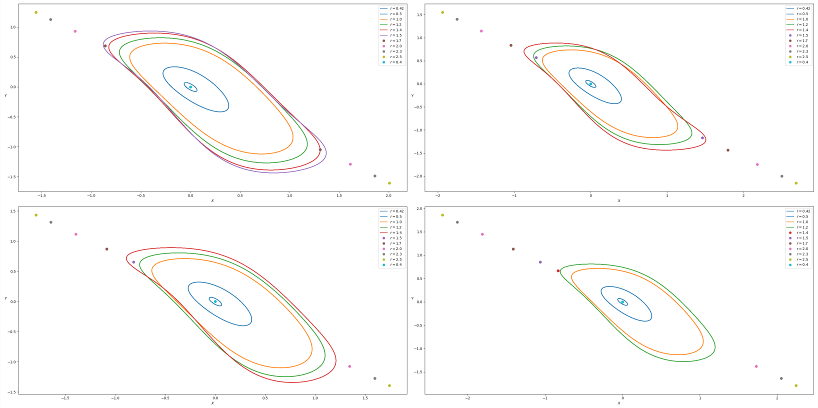
$$Y_{1,2} = \frac{\frac{ba}{a-v} - s \pm \sqrt{\left(\frac{ba}{a-v} - s\right)^2 - 4\left(\frac{ca}{a-v} - 1\right)\left(r - \frac{pa}{a-v}\right)}}{2\left(1 - \frac{ca}{a-v}\right)}$$

which leads to the non-trivial equilibrium points expression:

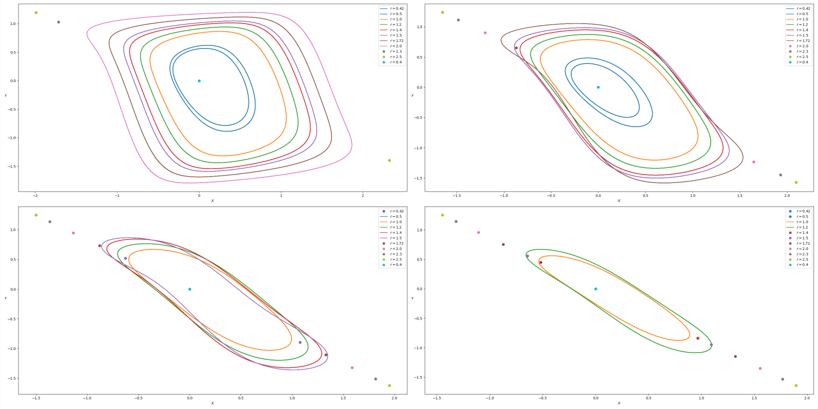
$$E_{1,2} = \left(-\frac{aY_{1,2}}{a-v},\ Y_{1,2}\ ,\ \frac{aY_{1,2}}{a-v}\right)$$

For $b = c = 0$ and $a = 1$, we obtain the previous deterministic model.

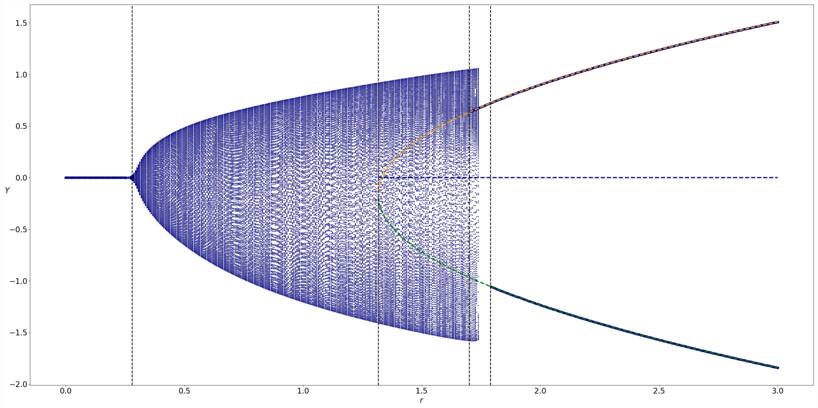
Effects of the parameters b and c



Effects of the parameter a

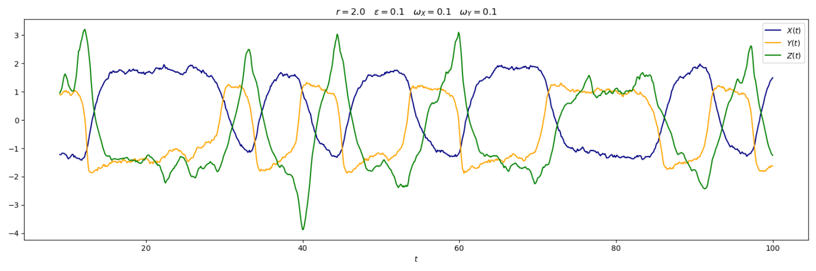


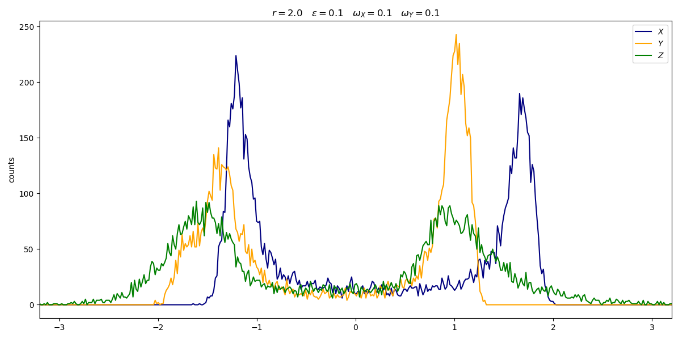
Deterministic Bifurcation Diagram



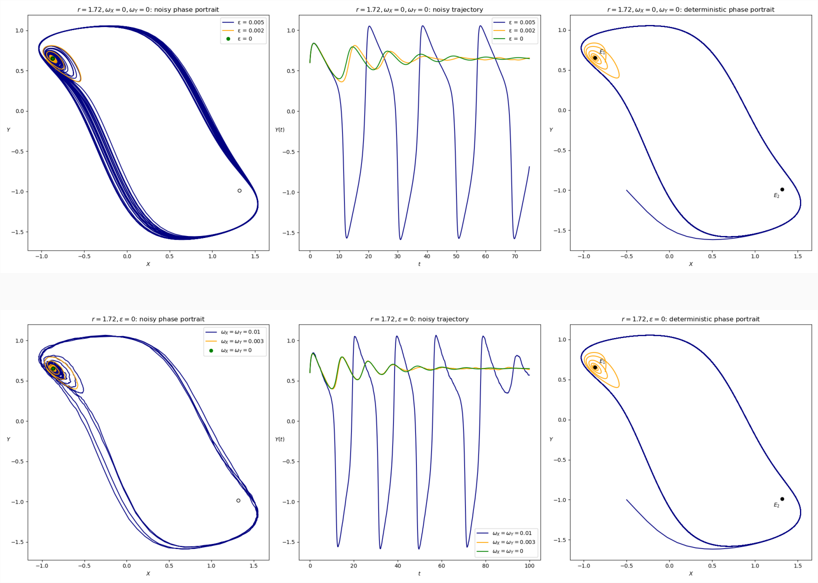
Fixed for the following analysis the values of the parameters:
$b = 0.3, c = 0.3, d = 0.2, a = 0.8$.
Interesting thing: with this combination of parameters, the equilibrium points E_1 and E_2 have the inverse stability.

Stochastic system: time Behavior and Histograms



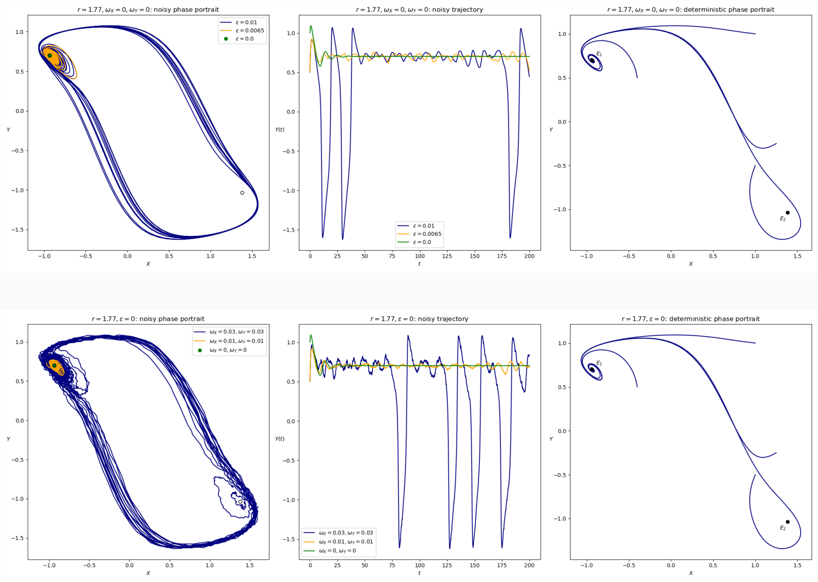


NITs

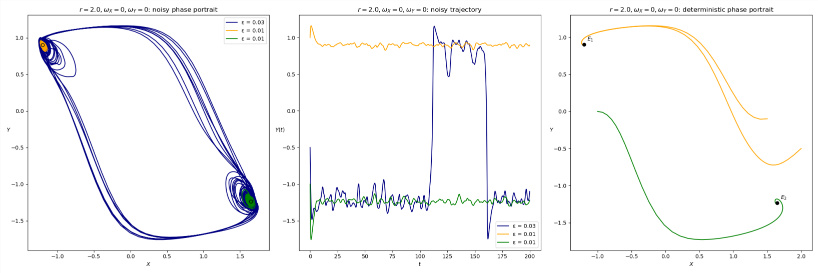


$r = 1.72, \omega_X = 0, \omega_Y = 0$: noisy phase portrait
ε = 0.005
ε = 0.002
ε = 0
$r = 1.72, \omega_X = 0, \omega_Y = 0$: noisy trajectory
ε = 0.005
ε = 0.002
ε = 0
$r = 1.72, \omega_X = 0, \omega_Y = 0$: deterministic phase portrait
E_1
E_2
$r = 1.72, \varepsilon = 0$: noisy phase portrait
$\omega_X = \omega_Y = 0.01$
$\omega_X = \omega_Y = 0.003$
$\omega_X = \omega_Y = 0$
$r = 1.72, \varepsilon = 0$: noisy trajectory
$\omega_X = \omega_Y = 0.01$
$\omega_X = \omega_Y = 0.003$
$\omega_X = \omega_Y = 0$
$r = 1.72, \varepsilon = 0$: deterministic phase portrait
E_1
E_2
X
Y
t
Y(t)

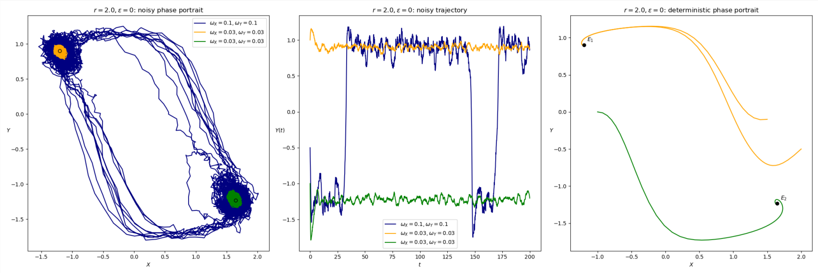
NITs


$r = 1.77, \omega_X = 0, \omega_Y = 0$: noisy phase portrait
$\varepsilon = 0.01$
$\varepsilon = 0.0065$
$\varepsilon = 0.0$
$r = 1.77, \omega_X = 0, \omega_Y = 0$: noisy trajectory
$r = 1.77, \omega_X = 0, \omega_Y = 0$: deterministic phase portrait
E_1
E_2
$r = 1.77, \varepsilon = 0$: noisy phase portrait
$\omega_X = 0.03, \omega_Y = 0.03$
$\omega_X = 0.01, \omega_Y = 0.01$
$\omega_X = 0, \omega_Y = 0$
$r = 1.77, \varepsilon = 0$: noisy trajectory
$r = 1.77, \varepsilon = 0$: deterministic phase portrait
X
Y
Y(t)
t

NITs



$r = 2.0, \omega_X = 0, \omega_Y = 0$: noisy phase portrait
ε = 0.03
ε = 0.01
ε = 0.01
$r = 2.0, \omega_X = 0, \omega_Y = 0$: noisy trajectory
$Y(t)$
t
$r = 2.0, \omega_X = 0, \omega_Y = 0$: deterministic phase portrait
E_1
E_2
X
Y

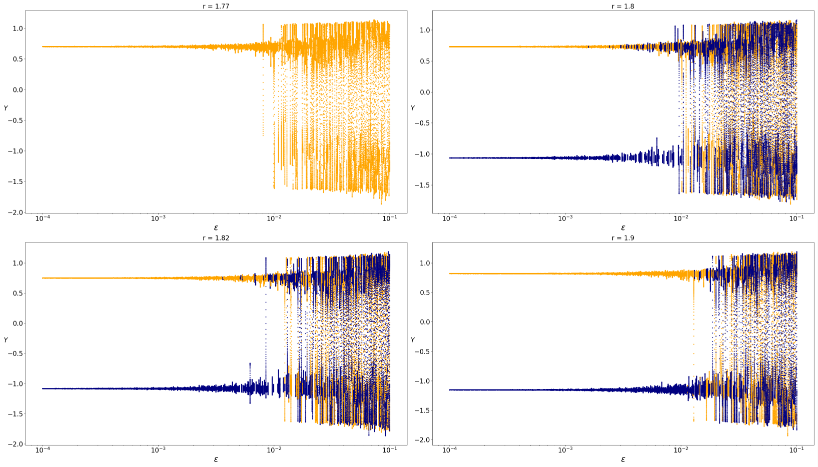


$r = 2.0, \varepsilon = 0$: noisy phase portrait
$\omega_X = 0.1, \omega_Y = 0.1$
$\omega_X = 0.03, \omega_Y = 0.03$
$\omega_X = 0.03, \omega_Y = 0.03$
$r = 2.0, \varepsilon = 0$: noisy trajectory
$Y(t)$
t
$r = 2.0, \varepsilon = 0$: deterministic phase portrait
E_1
E_2
X
Y

NITs

Some considerations on NITs in this case....

NITs



NITs