Class 1: Counting and Sets
18.05, Spring 2013

1 Learning Goals

1. Know the definitions and notations for sets, intersections, union, complement.
2. Be able to visualize set operations using Venn diagrams.
3. Understand the reason we need to find the size of sets in 18.05.
4. Be able to use the rule of product, permutations and combinations to count the elements in a set.

2 Counting

2.1 Motivating questions

Example 1. A coin is *fair* if it comes up heads or tails with equal probability.
You flip a fair coin three times. What is the probability that exactly one of the flips results in a head?

answer: With three flips, we can easily list the eight possible outcomes:

$$\{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

Three of these outcomes have exactly one head:

$$\{TTH, THT, HTT\}$$

Since all outcomes are equally likely, we have

$$P(\text{1 head in 3 flips}) = \frac{\text{number of outcomes with 1 head}}{\text{total number of outcomes}} = \frac{3}{8}.$$

Would listing the outcomes be practical with 10 flips?

A deck of 52 cards has 13 ranks (2, 3, ..., 9, 10, J, Q, K, A) and 4 suits ($\heartsuit, \spadesuit, \diamondsuit, \clubsuit$,).
A poker hand consists of 5 cards. A *One-pair* hand consists of two cards having one rank and the remaining three cards having three other ranks, e.g., $\{2\heartsuit, 2\spadesuit, 5\heartsuit, 8\clubsuit, \text{K}\diamondsuit\}$

Example 2. The probability of a one-pair hand is:
a) less than 5%

b) between 5% and 10%
c) between 10% and 20%
d) between 20% and 40%
e) greater than 40%

Since every set of five cards is equally likely, we can compute the probability of a one-pair hand as

$$P(\text{one-pair}) = \frac{\text{number of one-pair hands}}{\text{total number of hands}}$$

To find the exact probability, we need to *count* the number of elements in each of these sets. And we have to be clever about it, because there are too many elements to simply list them all. We will come back to this problem.

2.2 Sets and notation

Counting means counting the elements of a set, so we start with a brief review of sets. (If this is new to you, please come to office hours).

2.2.1 Definitions

A **set** S is a collection of elements. We use the following notation.
Element $x \in S$: the element x is in the set S.
Subset $A \subset S$: the set A is a subset of S if all of its elements are in S.
Complement A^c or $S - A$: The **complement** of A in S is the set of elements of S that are **not** in A.

Suppose A and B are subsets of S.
Union $A \cup B$: the *union* of A and B is the set of elements of S in A or B.
Intersection $A \cap B$: the *intersection* of A and B is the set of elements of S in A and B.
Empty set $\emptyset$: the *empty set* is the set with no elements.
Disjoint: A and B are **disjoint** if they have no common elements. That is, if $A \cap B = \emptyset$.
Difference $A - B$: the *difference* of A and B is the set of elements in A that are not in B.

Example 3. Start with the set of all months:

$$S = \{\text{Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec}\}.$$

Consider two subsets:

$$\begin{aligned} L &= \text{the month is long, i.e. 31 days} = \{\text{Jan, Mar, May, Jul, Aug, Oct, Dec}\} \\ R &= \text{the letter r is the name of the month} = \{\text{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec}\}. \end{aligned}$$

Our goal here is to look at different set operations.
Intersection: $L \cap R$ means both L and R occurred. $L \cap R = \{\text{Jan, Mar, Oct, Dec}\}$.
Union: $L \cup R$ means at least one of L and R occurred.
$L \cup R = \{\text{Jan, Feb, Mar, Apr, May, Jul, Aug, Sep, Nov, Oct, Dec}\}$.
Complement: L^c means everything that is *not* in L. $L^c = \{\text{Feb, Apr, Jun, Sep, Nov}\}$.
Difference: $L - R$ means everything that's in L and not in R.
So, $L - R = \{\text{May, Jul, Aug}\}$ and $P(L - R) = 3/12$.

There are often many ways to get the same set, e.g. $L^c = S - L$, $L - R = L \cap R^c$.

The relationship between union, intersection, and complement is given by **DeMorgan's laws**:

$$(A \cup B)^c = A^c \cap B^c$$

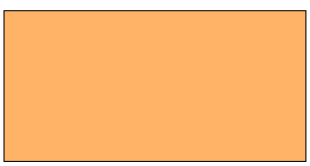
$$(A \cap B)^c = A^c \cup B^c$$

In words the first law says everything not in (A or B) is the same set as everything that's (not in A) and (not in B). The second law is similar.

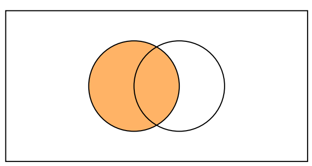
2.2.2 Venn Diagrams

Most people will have seen these. They offer an easy way to visualize set operations.

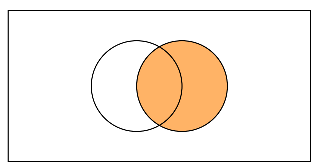
In all the figures the large rectangle stands for S, L is the region inside the circle on the left and R is the region inside the circle on the right. The shaded region is the set indicated.



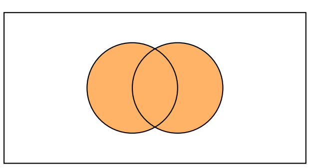
S



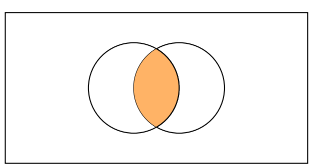
L



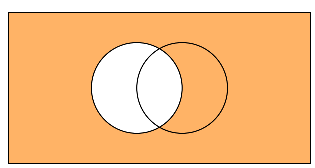
R



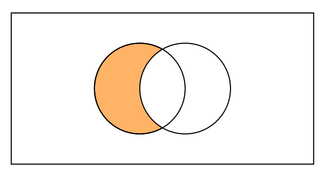
$L \cup R$



$L \cap R$

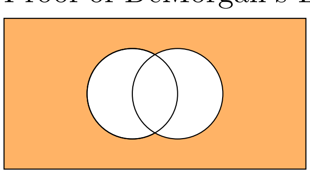


L^c

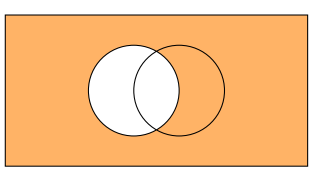


$L - R$

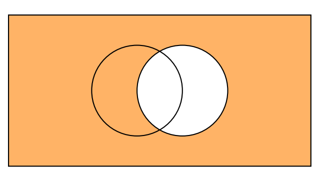
Proof of DeMorgan's Laws



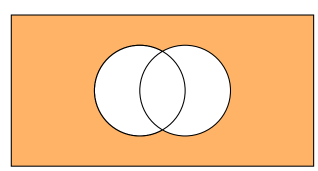
$(L \cup R)^c$



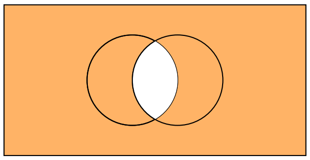
L^c



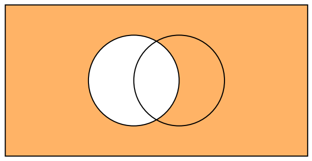
R^c



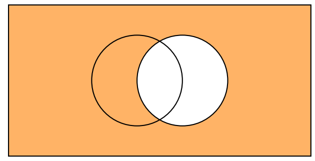
$L^c \cap R^c$



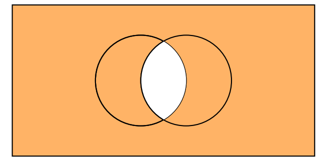
$(L \cap R)^c$



L^c



R^c



$L^c \cup R^c$

Example 4. i) Verify DeMorgan's laws for the subsets $A = \{1, 2, 3\}$ and $B = \{3, 4\}$ of the set $S = \{1, 2, 3, 4, 5\}$.
ii) Draw and label a Venn diagram with A the set of male students and B the set of sophomores. Shade the region illustrating the first law. Can you express the first law in this case as a non-technical English sentence?

answer: For each law we just work through both sides of the equation and show they are the same.
1. Law $(A \cup B)^c = A^c \cap B^c$:
Right hand side: $A \cup B = \{1, 2, 3, 4\} \Rightarrow (A \cup B)^c = \{5\}$.
Left hand side: $A^c = \{4, 5\}$, $B^c = \{1, 2, 5\} \Rightarrow A^c \cap B^c = \{5\}$.
The two sides are equal. QED

2. Law $(A \cap B)^c = A^c \cup B^c$:
Right hand side: $A \cap B = \{3\} \Rightarrow (A \cap B)^c = \{1, 2, 4, 5\}$.
Left hand side: $A^c = \{4, 5\}$, $B^c = \{1, 2, 5\} \Rightarrow A^c \cup B^c = \{1, 2, 4, 5\}$.
The two sides are equal. QED

2.2.3 Products of sets

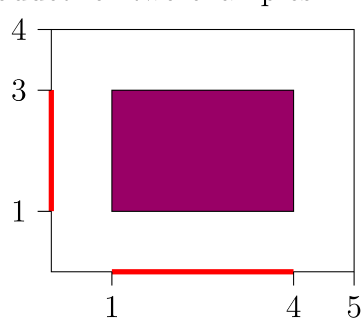
The product of sets S and T is the set of ordered pairs:

$$S \times T = \{(s, t) \,|\, s \in S, t \in T\}.$$

The following diagrams illustrate the set product for two examples.

$\times$	1	2	3	4
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2,2)	(2,3)	(2,4)
3	(3,1)	(3,2)	(3,3)	(3,4)

$\{1, 2, 3\} \times \{1, 2, 3, 4\}$



$[1, 4] \times [1, 3] \subset [0, 5] \times [0, 4]$

Note that if $A \subset S$ and $B \subset T$ then $A \times B \subset S \times T$.

2.3 Counting

If S is finite, then $|S|$ or $\#S$ denotes the number of elements.

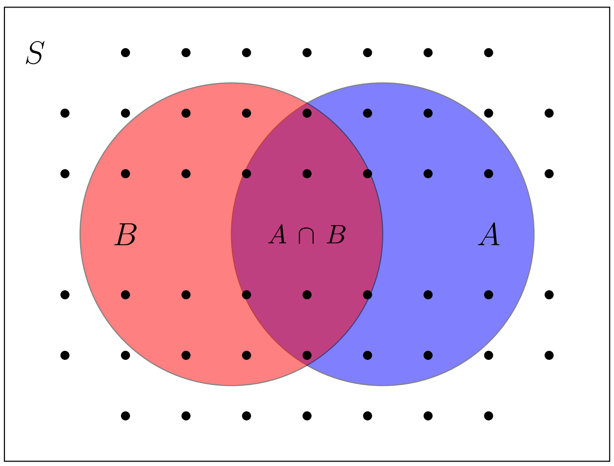
Two useful counting principles are the *inclusion-exclusion principle* and the *rule of product.*

2.3.1 Inclusion-exclusion principle

The *inclusion-exclusion principal* says

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We can illustrate this with a Venn diagram



In the figure, $|A|$ is the number of dots in $|A|$ and likewise for the other sets. The figure shows that $|A|+|B|$ *double*-counts $|A \cap B|$, which is why $|A \cap B|$ is subtracted off in the inclusion-exclusion formula.

Example 5. Suppose in a band of singers and guitarists, seven people sing, four play the guitar, and two do both. How big is the band?

<u>**answer:**</u> Let S be the set singers and G be the set guitar players. The inclusion-exclusion principle says

$$\text{size of band} = |S \cup G| = |S| + |G| - |S \cap G| = 7 + 4 - 2 = 9.$$

2.3.2 Rule of Product

The *Rule of Product* says:

> If there are n ways to perform action 1 followed by m ways to perform action 2, then there are $n \cdot m$ ways to perform action 1 followed by action 2.

We will also call this the **multiplication** rule.

Example 6. if you have 3 shirts and 4 pants then you can make $3 \cdot 4 = 12$ outfits.

The rule of product holds even if the ways to perform action 2 depend on action 1, as long as the *number* of ways to perform action 2 is independent of action 1. To illustrate this:

Example 7. There are 5 competitors in the 100m final at the Olympics. In how many ways can the gold, silver, and bronze medals be awarded?

<u>**answer:**</u> There are 5 ways to award the gold. Once that is awarded there are 4 ways to award the silver and then 3 ways for the bronze: answer $5 \cdot 4 \cdot 3 = 60$ ways.

Note how the choice of gold medalist affects the possibilities for the silver medalist, but not the number of possible silver medalists.

2.4 Permutations and combinations

2.4.1 Permutations

A permutation of a set is a particular ordering of its elements. For example, the set $\{a, b, c\}$ has six permutations: $abc, acb, bac, bca, cab, cba$. We could have found the number of permutations using the rule of product. That is, there are 3 ways to pick the first element, then 2 for the second and 1 for the first. This gives a total of $3 \cdot 2 \cdot 1 = 6$ permutations.

In general, the rule of product tells us that the number of permutations of a set of k elements is

$$k! = k(k-1) \cdots 3 \cdot 2 \cdot 1.$$

By extension we talk about the permutations of k things out of a set of n things. We show what this means with an example.

Example 8. List all the permutations of 3 elements out of the set $\{a, b, c, d\}$. <u>**answer:**</u> This is a long list,

$$\begin{array}{cccccc} abc & abd & acb & acd & adb & adc \\ bac & bad & bca & bcd & bda & bdc \\ cab & cad & cba & cbd & cda & cdb \\ dab & dac & dba & dbc & dca & dcb \end{array}$$

There are 24 permutations. Note that abc and acb count as distinct permutations. That is, for permutations the *order matters.*

Note also, the rule or product would have told us there are $4 \cdot 3 \cdot 2 = 24$ permutations without bothering to list them all.

2.4.2 Combinations

In contrast to permutations, in combinations order does not matter. We show what we mean with an example

Example 9. List all the combinations of 3 elements out of the set $\{a, b, c, d\}$.

answer: Such a combination is a collection of 3 elements without regard to order. So, *abc* and *cab* both represent the same combination. We can list all the combinations by listing all the subsets of exactly 3 elements.

$$\{a, b, c\} \quad \{a, b, d\} \quad \{a, c, d\} \quad \{b, c, d\}$$

There are only 4 combinations. Contrast this with the 24 permutations in the previous example. The factor of 6 comes because every combination of 3 things can be written in 6 different orders.

2.4.3 Formulas

We'll use the following notations.
${}_nP_k$ = number of permutations of k distinct elements from a set of size n
${}_nC_k = \binom{n}{k}$ = number of combinations of k elements from a set of size n
We emphasise that by the number of combinations of k elements we mean the number of subsets of size k.

These are given by the formulas:

$${}_nP_k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$$

$${}_nC_k = \frac{n!}{k!(n-k)!} = \frac{{}_nP_k}{k!}$$

The formula for ${}_nP_k$ follows from the rule of product. It also implies the formula for ${}_nC_k$ because a subset of size k can be ordered in $k!$ ways.

We can illustrate the relation between permutations and combinations by lining up the results of the previous two examples. (We rearrange the first table to make things more clear.)

abc	*acb*	*bac*	*bca*	*cab*	*cba*	$\{a, b, c\}$
abd	*adb*	*bad*	*bda*	*dab*	*dba*	$\{a, b, d\}$
acd	*adc*	*cad*	*cda*	*dac*	*dca*	$\{a, c, d\}$
bcd	*bdc*	*cbd*	*cdb*	*dbc*	*dcb*	$\{b, c, d\}$
Permutations: ${}_4P_3$						Combinations: ${}_4C_3$

Notice that each row in the permutations list consists of all permutations of the corresponding set in the combinations list.

2.4.4 Examples

Example 10. Find the following counts.
i) The number of ways to choose 2 out of 4 things.
ii) The number of ways to list 2 out of 4 things.
iii) The number of ways to choose 3 out of 10 things.

answer: (i) This is asking for combinations: $\binom{4}{2} = \frac{4!}{2!\,2!} = 6$.
(ii) This is asking for permuations: ${}_4P_2 = \frac{4!}{2!} = 12$.
(iii) This is asking for combinations: $\binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 120$.

Example 11. (i) Count the number of ways to get 3 heads in a sequence of 10 flips of a coin.
(ii) If the coin is fair, what is the probability of exactly 3 heads in 10 flips.

answer: (i) This asks for the subset of all sequences of heads and tails that have exactly 3 heads. That is, we have to choose 3 out of 10 sequence entries to be heads. This is the same question as in the previous example.

$$\binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 120.$$

(ii) Since the coin is fair there are 2^{10} equally likely ways to get a sequence of 10 heads or tails. The probability is $120/1024 = .117$.