Supplementary Material

I. Overhead simulations

Here we present tables of the protocols used in producing the graphs of Fig. 6. Each table is a sample of the data points from these graphs. Throughout "RM" refers the Reed-Muller protocol (15-to-1), "Toff" refers to the Toffoli protocol (8 T to one 3-qubit Toffoli state) and otherwise protocols are labelled by the 'k" value of the Bravyi-Haah protocol involved. The initial round of distillation is the first number. Where a large jump in the "Number of magic states" column is observed the same protocol is being used between these values as that indicated at the start and end of the gap.

A. T states, $p_g = 10^{-3}$

1. Block Checking

Number of magic states	MSD protocol
100000.	{2,10}
200000.	{10,26}
600000.	{2,10}
1.00×10^{8}	{2,10}
5.00×10^{8}	{RM,10}
2.30×10^{13}	{RM,10}
3.00×10^{13}	{RM,2}
2.00×10^{14}	{RM,2}
3.00×10^{14}	{2,10,26}
6.00×10^{14}	{2,10,26}
7.00×10^{14}	{2,10,18}
9.00×10^{14}	{2,10,18}
3.00×10^{15}	{2,2,34}
4.00×10^{15}	{2,2,34}
7.00×10^{15}	{2,2,26}
1.00×10^{16}	{2,2,26}
3.00×10^{20}	{RM,18,26}
4.00×10^{20}	{RM,18,26}
2.00×10^{21}	{RM,10,34}
2.00×10^{21}	{RM,10,34}
5.00×10^{21}	{RM,34,18}
5.00×10^{21}	{RM,34,18}
1.00×10^{22}	{RM,26,18}
1.00×10^{22}	{RM,26,18}
3.00×10^{22}	{RM,18,18}
3.00×10^{22}	{RM,18,18}
4.00×10^{22}	{RM,34,10}
4.00×10^{22}	{RM,34,10}
8.00×10^{22}	{RM,26,10}
8.00×10^{22}	{RM,26,10}
2.00×10^{23}	{RM,10,18}
2.00×10^{23}	{RM,10,18}
3.00×10^{23}	{RM,18,10}
4.00×10^{23}	{RM,10,10}
6.00×10^{24}	{RM,10,10}
9.00×10^{24}	{RM,2,18}
9.00×10^{24}	{RM,2,18}
2.00×10^{25}	{RM,2,10}
5.00×10^{26}	{RM,2,10}
6.00×10^{26}	{RM,42,2}
7.00×10^{26}	{RM,42,2}
1.00×10^{27}	{RM,34,2}
3.00×10^{27}	{RM,26,2}
3.00×10^{27}	{RM,26,2}
6.00×10^{27}	{RM,18,2}
9.00×10^{27}	{RM,18,2}
5.00×10^{28}	{RM,10,2}
2.00×10^{30}	{RM,1,26}
8.00×10^{30}	{RM,1,26}

2. Module checking

Number of magic states	MSD protocol
600000.	{34,42}
700000.	{34,34}
800000.	{34,58}
2.00×10^{6}	{34,34}
3.00×10^{6}	{26,74}
4.00×10^{6}	{26,58}
6.00×10^{6}	{34,34}
7.00×10^{6}	{26,74}
8.00×10^{6}	{26,74}
2.00×10^{7}	{34,34}
3.00×10^{7}	{26,74}
4.00×10^{7}	{26,50}
6.00×10^{7}	{34,34}
7.00×10^{7}	{26,74}
2.00×10^{8}	{26,74}
3.00×10^{8}	{26,66}
4.00×10^{8}	{26,50}
6.00×10^{8}	{34,26}
7.00×10^{8}	{34,26}
8.00×10^{8}	{18,42}
2.00×10^{9}	{18,18}
3.00×10^{9}	{10,18}
4.00×10^{9}	{10,10}
7.00×10^{9}	{10,10}
8.00×10^{9}	{2,10}
2.00×10^{10}	{10,2}
3.00×10^{10}	{2,2}
4.00×10^{10}	{2,2}
6.00×10^{10}	{10,26,18}
8.00×10^{10}	{10,26,18}
1.20×10^{11}	{10,18,26}
2.00×10^{11}	{10,18,26}
2.80×10^{11}	{10,26,18}
3.20×10^{11}	{10,18,26}
6.00×10^{11}	{10,18,26}
6.40×10^{11}	{10,26,18}
8.00×10^{11}	{10,26,18}
1.08×10^{12}	{10,18,26}
1.28×10^{12}	{10,18,26}

1.60×10^{12}	{10,18,26}
2.00×10^{12}	{10,26,18}
2.56×10^{12}	{10,26,18}
3.24×10^{12}	{10,18,26}
4.00×10^{12}	{10,18,26}
5.12×10^{12}	{10,18,26}
6.48×10^{12}	{10,26,18}
8.00×10^{12}	{10,26,18}
1.02×10^{13}	{10,18,26}
1.79×10^{13}	{10,18,26}
2.00×10^{13}	{10,26,18}
3.00×10^{13}	{10,18,26}
4.00×10^{13}	{10,18,26}
6.00×10^{13}	{82,82,74}
7.00×10^{13}	{10,26,18}
8.00×10^{13}	{82,82,82}
2.00×10^{14}	{10,26,18}
3.00×10^{14}	{82,82,82}
4.00×10^{14}	{82,82,82}
6.00×10^{14}	{82,82,74}
7.00×10^{14}	{10,26,18}
8.00×10^{14}	{82,82,82}
2.00×10^{15}	{82,82,66}
3.00×10^{15}	{82,82,82}
4.00×10^{15}	{82,82,82}
6.00×10^{15}	{82,82,66}
7.00×10^{15}	{82,82,82}
8.00×10^{15}	{82,82,82}
2.00×10^{16}	{82,82,66}
3.00×10^{16}	{82,82,82}
4.00×10^{16}	{82,82,82}
6.00×10^{16}	{82,82,66}
7.00×10^{16}	{82,82,82}
8.00×10^{19}	{82,82,82}
2.00×10^{18}	{58,82,58}
3.00×10^{18}	{10,18,26}
4.00×10^{18}	{10,18,26}
6.00×10^{18}	{10,18,18}
7.00×10^{18}	{10,18,26}
8.00×10^{19}	{10,18,26}
2.00×10^{20}	{10,18,18}
3.00×10^{20}	{10,10,18}
4.00×10^{20}	{10,10,10}
7.00×10^{20}	{10,10,10}
8.00×10^{20}	{10,2,10}
2.00×10^{21}	{10,10,2}
1.00×10^{22}	{RM,30,30}
1.00×10^{23}	{RM,70,78}
1.00×10^{24}	{RM,70,78}
1.00×10^{25}	{RM,70,70}
1.00×10^{26}	{RM,70,70}
1.00×10^{27}	{RM,62,70}
1.00×10^{28}	{RM,70,62}
1.00×10^{29}	{RM,78,54}
1.00×10^{30}	{RM,14,30}

B. Toffoli states, $p_{\rm g} = 10^{-3}$

1. Block checking

Number of magic states	MSD protocol
33300.	{10,Toff}
100000.	{10,Toff}
133000.	{Toff,18}
200000.	{Toff,10}
333000.	{10,Toff}
1.00×10^{6}	{2,Toff}
3.33×10^{7}	{2,Toff}
7.14×10^{7}	{RM,10}
3.29×10^{12}	{RM,10}
4.27×10^{12}	{2,50,Toff}
5.12×10^{12}	{2,42,Toff}
6.83×10^{12}	{2,42,Toff}
1.00×10^{13}	{2,34,Toff}
1.67×10^{13}	{2,26,Toff}
2.00×10^{13}	{2,18,Toff}
3.33×10^{13}	{2,18,Toff}
1.00×10^{14}	{2,10,Toff}
3.00×10^{14}	{2,10,Toff}
3.33×10^{14}	{2,2,Toff}
3.33×10^{15}	{2,2,Toff}
4.29×10^{19}	{RM,18,26}
5.71×10^{19}	{RM,18,26}
2.86×10^{20}	{RM,10,34}
7.14×10^{20}	{RM,34,18}
1.43×10^{21}	{RM,26,18}
4.29×10^{21}	{RM,18,18}
5.71×10^{21}	{RM,34,10}
1.14×10^{22}	{RM,26,10}
1.43×10^{22}	{RM,26,10}
2.86×10^{22}	{RM,10,18}
4.29×10^{22}	{RM,18,10}
5.71×10^{22}	{RM,10,10}
8.57×10^{23}	{RM,10,10}
1.29×10^{24}	{RM,2,18}
2.86×10^{24}	{RM,2,10}
7.14×10^{25}	{RM,2,10}
8.57×10^{25}	{RM,42,2}
1.43×10^{26}	{RM,34,2}
4.29×10^{26}	{RM,26,2}
8.57×10^{26}	{RM,18,2}
1.29×10^{27}	{RM,18,2}
7.14×10^{27}	{RM,10,2}
2.86×10^{29}	{RM,1,26}
1.14×10^{30}	{RM,1,26}
1.29×10^{30}	{RM,1,34}

2. Module checking

Number of magic states	MSD protocol
33300.	{18,Toff}
1.00×10^{6}	{18,Toff}
1.33×10^{6}	{Toff,18}
1.67×10^{6}	{10,Toff}
3.33×10^{9}	{10,Toff}
6.67×10^{9}	{Toff,2}
1.00×10^{10}	{Toff,2}
1.67×10^{10}	{Toff,42,26}
2.00×10^{10}	{Toff,42,26}
3.33×10^{10}	{Toff,34,42}
4.00×10^{10}	{Toff,34,42}

5.33×10^{10}	{Toff,42,26}
6.67×10^{10}	{Toff,42,26}
8.00×10^{10}	{Toff,34,42}
1.07×10^{11}	{Toff,34,42}
1.33×10^{11}	{Toff,42,34}
1.67×10^{11}	{18,34,Toff}
2.00×10^{11}	{18,34,Toff}
2.67×10^{11}	{Toff,34,42}
4.27×10^{11}	{Toff,34,34}
5.33×10^{11}	{18,34,Toff}
6.40×10^{11}	{18,34,Toff}
8.53×10^{11}	{Toff,34,42}
1.33×10^{12}	{Toff,34,34}
1.67×10^{12}	{18,34,Toff}
3.33×10^{19}	{18,34,Toff}
6.67×10^{19}	{18,26,Toff}
1.00×10^{20}	{18,18,Toff}
1.67×10^{20}	{10,18,Toff}
2.00×10^{20}	{10,10,Toff}
3.33×10^{20}	{10,10,Toff}
3.33×10^{21}	{RM,98,Toff}
3.33×10^{22}	{RM,58,Toff}
3.33×10^{23}	{RM,98,Toff}
3.33×10^{24}	{RM,54,Toff}
3.33×10^{25}	{RM,98,Toff}
3.33×10^{26}	{RM,46,Toff}
3.33×10^{27}	{RM,98,Toff}
3.33×10^{28}	{RM,42,Toff}
3.33×10^{29}	{RM,6,Toff}

C. T states, $p_{\rm g} = 10^{-4}$

1. Block checking

Number of magic states	MSD protocol
100000.	{10}
200000.	{10}
300000.	{2}
800000.	{2}
6.00×10^{6}	{10,18}
1.00×10^{7}	{18,10}
6.00×10^{7}	{10,18}
9.00×10^{7}	{18,10}
4.00×10^{8}	{10,26}
9.00×10^{8}	{18,10}
3.00×10^{9}	{10,26}
5.00×10^{9}	{10,18}
8.00×10^{9}	{10,10}
2.00×10^{10}	{2,34}
3.00×10^{10}	{2,26}
4.00×10^{10}	{2,18}
1.60×10^{11}	{2,34}
2.40×10^{11}	{2,34}
3.20×10^{11}	{2,26}
4.00×10^{11}	{2,18}
9.00×10^{11}	{2,26}
1.08×10^{12}	{2,18}
2.24×10^{12}	{RM,34}
3.00×10^{12}	{RM,26}
4.32×10^{12}	{RM,18}
2.30×10^{13}	{RM,34}
4.00×10^{13}	{RM,18}
1.00×10^{14}	{RM,34}
2.00×10^{14}	{RM,34}
3.00×10^{14}	{RM,18}
2.00×10^{15}	{RM,34}
3.00×10^{15}	{RM,18}
1.00×10^{16}	{RM,34}
2.00×10^{16}	{RM,26}
3.00×10^{16}	{RM,18}
8.00×10^{16}	{RM,34}
2.00×10^{17}	{RM,26}
3.00×10^{17}	{RM,18}
6.00×10^{17}	{RM,34}
1.00×10^{18}	{RM,26}
3.00×10^{18}	{RM,18}
4.00×10^{18}	{RM,18}
2.00×10^{19}	{RM,10}
4.00×10^{19}	{2,74,58}
7.00×10^{19}	{2,58,58}
9.00×10^{19}	{2,50,58}
2.00×10^{20}	{2,34,58}
3.00×10^{20}	{2,26,58}
4.00×10^{20}	{2,18,66}
7.00×10^{20}	{2,18,58}
8.00×10^{20}	{2,10,66}
2.00×10^{21}	{2,10,58}
3.00×10^{21}	{2,42,26}
5.00×10^{21}	{2,34,26}
9.00×10^{21}	{2,26,26}
2.00×10^{22}	{2,18,18}
3.00×10^{22}	{2,18,18}
6.00×10^{22}	{2,10,26}
9.00×10^{22}	{2,10,18}
3.00×10^{29}	{RM,66,66}
4.00×10^{29}	{RM,58,66}
6.00×10^{29}	{RM,50,66}
7.00×10^{29}	{RM,50,58}
8.00×10^{29}	{RM,42,50}
2.00×10^{30}	{RM,34,66}
5.00×10^{30}	{RM,26,66}
6.00×10^{30}	{RM,18,66}
7.00×10^{30}	{RM,18,58}
8.00×10^{30}	{RM,18,50}

2. Module checking

Number of magic states	MSD protocol
5.00×10^{7}	{18,18}
6.00×10^{7}	{18,18}
7.00×10^{7}	{18,10}
8.00×10^{7}	{18,10}
2.00×10^{8}	{18,18}
6.00×10^{8}	{18,18}
7.00×10^{8}	{18,10}
8.00×10^{8}	{18,10}
2.00×10^{9}	{18,18}
4.00×10^{9}	{18,18}
6.00×10^{9}	{18,10}
8.00×10^{9}	{18,10}

2.00×10^{10}	{10,26}
3.00×10^{10}	{10,26}
4.00×10^{10}	{10,18}
6.00×10^{10}	{18,10}
8.00×10^{10}	{18,10}
1.20×10^{11}	{10,26}
3.20×10^{11}	{10,26}
4.00×10^{11}	{10,18}
6.00×10^{11}	{18,10}
8.00×10^{11}	{18,10}
1.08×10^{12}	{10,26}
3.24×10^{12}	{10,26}
4.00×10^{12}	{10,18}
5.12×10^{12}	{18,10}
6.48×10^{12}	{18,10}
8.00×10^{12}	{10,26}
2.00×10^{13}	{10,26}
3.00×10^{13}	{10,18}
4.00×10^{13}	{10,10}
7.00×10^{13}	{10,10}
8.00×10^{13}	{2,10}
2.00×10^{14}	{10,2}
3.00×10^{14}	{2,2}
4.00×10^{14}	{2,2}
1.00×10^{20}	{RM,38}
1.00×10^{21}	{RM,38}
1.00×10^{22}	{RM,38}
2.00×10^{22}	{18,58,82}
6.00×10^{22}	{18,58,82}
7.00×10^{22}	{18,58,74}
8.00×10^{22}	{18,66,66}
2.00×10^{23}	{18,58,82}
6.00×10^{23}	{18,58,82}
7.00×10^{23}	{18,58,74}
8.00×10^{23}	{18,66,66}
2.00×10^{24}	{18,58,82}
6.00×10^{24}	{18,58,82}
7.00×10^{24}	{18,58,66}
8.00×10^{24}	{18,66,58}
2.00×10^{25}	{18,58,82}
6.00×10^{25}	{18,58,82}
7.00×10^{25}	{18,58,66}
8.00×10^{25}	{18,66,58}
2.00×10^{26}	{18,50,82}
3.00×10^{26}	{18,50,82}
4.00×10^{26}	{18,50,82}
6.00×10^{26}	{18,58,74}
7.00×10^{26}	{18,58,66}
8.00×10^{26}	{18,66,58}
2.00×10^{27}	{18,34,50}
3.00×10^{27}	{18,26,42}
4.00×10^{27}	{18,26,34}
6.00×10^{27}	{10,18,58}
7.00×10^{27}	{10,26,34}
8.00×10^{27}	{10,18,42}
2.00×10^{28}	{10,18,18}
3.00×10^{28}	{10,18,10}
4.00×10^{28}	{10,10,10}
7.00×10^{28}	{10,10,10}
1.00×10^{29}	{RM,30,30}
1.00×10^{30}	{RM,78,78}

D. Toffoli states, $p_{\rm g} = 10^{-4}$

1. Block checking

Number of magic states	MSD protocol
33300.	{Toff}
66700.	{Toff}
100000.	{Toff,34}
200000.	{10,Toff}
267000.	{10,Toff}
333000.	{Toff,10}
1.00×10^{6}	{Toff,42}
1.33×10^{6}	{Toff,26}
2.00×10^{6}	{Toff,18}
3.33×10^{6}	{Toff,10}
6.67×10^{6}	{Toff,50}
2.00×10^{7}	{10,Toff}
3.00×10^{7}	{42,Toff}
3.33×10^{7}	{34,Toff}
1.00×10^{8}	{26,Toff}
1.67×10^{8}	{18,Toff}
3.33×10^{8}	{18,Toff}
6.67×10^{8}	{Toff,50}
1.67×10^{9}	{10,Toff}
3.33×10^{9}	{Toff,50}
1.33×10^{10}	{2,Toff}
3.60×10^{11}	{2,Toff}
4.29×10^{11}	{RM,26}
6.17×10^{11}	{RM,18}
3.29×10^{12}	{RM,34}
5.71×10^{12}	{RM,18}
1.43×10^{13}	{RM,34}
2.86×10^{13}	{RM,34}
4.29×10^{13}	{RM,18}
2.86×10^{14}	{RM,34}
4.29×10^{14}	{RM,18}
1.43×10^{15}	{RM,34}
2.86×10^{15}	{RM,26}
4.29×10^{15}	{RM,18}
1.14×10^{16}	{RM,34}
1.67×10^{16}	{Toff,26,42}
3.33×10^{16}	{Toff,18,50}
6.67×10^{16}	{Toff,26,26}
8.57×10^{16}	{RM,34}
1.43×10^{17}	{RM,26}
4.29×10^{17}	{RM,18}
5.71×10^{17}	{RM,18}
6.67×10^{17}	{10,18,Toff}
1.33×10^{18}	{Toff,26,74}
2.67×10^{18}	{Toff,18,74}
3.33×10^{18}	{Toff,26,58}
1.00×10^{19}	{Toff,18,82}
2.00×10^{19}	{Toff,10,82}
2.67×10^{19}	{Toff,10,74}
3.33×10^{19}	{2,10,Toff}
3.00×10^{22}	{2,10,Toff}
6.67×10^{22}	{2,2,Toff}
3.33×10^{23}	{2,2,Toff}
4.29×10^{28}	{RM,66,66}
5.71×10^{28}	{RM,58,66}

8.57×10^{28}	{RM,50,66}
1.14×10^{29}	{RM,42,50}
1.43×10^{29}	{RM,42,42}
2.86×10^{29}	{RM,34,66}
7.14×10^{29}	{RM,26,66}
8.57×10^{29}	{RM,18,66}
1.14×10^{30}	{RM,18,50}
1.29×10^{30}	{RM,18,42}

2. Module checking

Number of magic states	MSD protocol
1.33×10^{6}	{42,Toff}
1.33×10^{12}	{42,Toff}
1.67×10^{12}	{34,Toff}
6.67×10^{12}	{34,Toff}
1.00×10^{13}	{26,Toff}
1.33×10^{13}	{18,Toff}
1.67×10^{13}	{18,Toff}
2.00×10^{13}	{10,Toff}
3.33×10^{13}	{10,Toff}
6.67×10^{13}	{Toff,2}
1.00×10^{14}	{Toff,2}
3.33×10^{14}	{RM,10,Toff}
3.33×10^{15}	{RM,10,Toff}
3.33×10^{16}	{RM,10,Toff}
6.67×10^{16}	{74,82,Toff}
6.67×10^{26}	{74,58,Toff}
1.00×10^{27}	{82,34,Toff}
1.33×10^{27}	{82,26,Toff}
1.67×10^{27}	{26,74,Toff}
2.00×10^{27}	{26,58,Toff}
2.67×10^{27}	{26,42,Toff}
3.33×10^{27}	{26,34,Toff}
6.67×10^{27}	{18,26,Toff}
1.00×10^{28}	{18,18,Toff}
1.33×10^{28}	{10,18,Toff}
1.67×10^{28}	{10,18,Toff}
2.00×10^{28}	{10,10,Toff}
3.33×10^{28}	{10,10,Toff}
3.33×10^{29}	{RM,10,Toff}
3.33×10^{31}	{RM,10,Toff}
3.33×10^{32}	{RM,6,Toff}
3.33×10^{33}	{RM,10,Toff}
3.33×10^{34}	{RM,10,Toff}