Surface code quantum computing in a network with timing errors

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A networked approach to quantum computing made up of small cells of well controlled qubits, connected together by long range photonics connections has been shown to be a reasonable approach to achieinvg scalability with matter based qubit systems, such as trapped ions. This approach overcomes many of the challenges faced by matter based qubit systems allowing complex architectures, long range connections, and ... without any increased complexity in the cell design. Performing operations across the network - such as the four-body measurements required to implement the surface code - require the ability to keep time between remote cells. Here we identify the nature of errors that arise from inaccurate time keeping, and find thresholds for what level of inaccuracy can be tolerated by error correction with the surface code.

Error correcting codes, such as the surface code, are based on the repeated projective measurement of the stabilizers of the code. In order to understand how timing errors will affect the implementation the code, we must identify how these parity measurements will be affected. We consider the case where we aim to measure the Pauli Z operator on a single qubit, but instead measure in some slightly rotated basis $\cos\theta Z + \sin\theta X$. The projection into this rotated basis, can be viewed as a rotation of the state, followed by the ideal Z projection, and a finally a rotation back into the original basis. Thus we can write this rotated projection as,

$$P\left(\theta\right) = R^{\dagger}\left(\theta\right)\hat{P}_Z R\left(\theta\right)$$

Where P_Z and $P(\theta)$ are taken to be even projection operators into the bases Z and $\cos\theta Z + \sin\theta X$ respectively, and the rotation operator $R\left(\theta\right) = R_y\left(\theta\right) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y$. Defining $C = \cos\frac{\theta}{2}$ and $S = \sin\frac{\theta}{2}$, we can write this explicitly as,

$$P\left(\theta\right) = C^2 P_Z + S^2 Y P_Z Y + iSC\left[Y P_Z - P_Z Y\right]$$

Using the relationship $P_Z Y = Y P_{Z,odd}$ we can write,

$$P\left(\theta\right) = C^2 P_Z + S^2 P_{Z,odd} + iSCY\left[P_Z - P_{Z,odd}\right] \quad (1)$$

This operator exactly describes the effect of projecting into the rotated basis. However to understand the effect of errors, we must also include our uncertainty in the value of θ, which is drawn from some probability distribution that describes our noise model. If an observed even measurement outcome corresponds to the projector $P\left(\theta\right)$ being applied to the state with probability $q\left(\theta\right)$ then we obtain the density matrix describing the output state after this noisy measurement by integrating over this distribution.

$$\rho_{measured} = \int_\theta q\left(\theta\right) P\left(\theta\right)\rho P\left(\theta\right) d\theta$$

Using equation 1, we can then write:

$$\rho_{measured} = \int_\theta q(\theta)\left[C^4\rho_1 + iC^3 S\rho_2 + C^2 S^2\rho_3 + iCS^3\rho_4 + S^4\rho_5\right] d\theta$$

Where the five terms are given by,

$$\begin{aligned}
\rho_1 &= \rho_{even}\\
\rho_2 &= -\rho_{even}Y + Y\rho_{even} - YP_{Z,odd}\rho P_Z + P_Z\rho P_{Z,odd}Y\\
\rho_3 &= Y\rho_{odd}Y + Y\rho_{even}Y + P_Z\rho P_{Z,odd} + P_{Z,odd}\rho P_Z - YP_Z\rho P_{Z,odd}Y\\
\rho_4 &= -Y\rho_{odd} + \rho_{odd}Y - P_{Z,odd}\rho P_Z Y + YP_Z\rho P_{Z,odd}\\
\rho_5 &= \rho_{odd}
\end{aligned}$$

and we have used the definitions: $\rho_{even} = P_Z\rho P_Z$, and $\rho_{odd} = P_{Z,odd}\rho P_{Z,odd}$.

To understand how these erroneous terms manifest themselves in our final density matrix after projection, we must consider the nature of our distribution of errors, $q(\theta)$. We first find the form of the errors in the case that $q(\theta)$ is symmetric, and has no systematic component. Later, we will show how this result can be straightforwardly extended to the case of systematic errors.

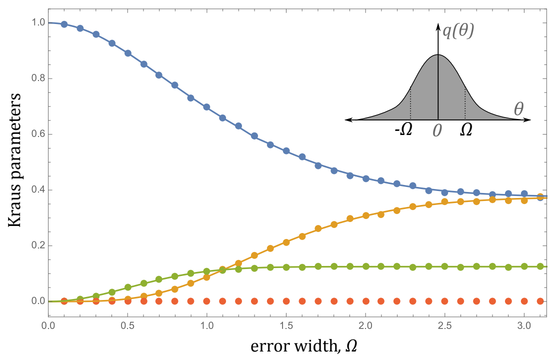
When $q(\theta)$ is symmetric about $\theta = 0$ the asymmetric terms in the integral, ρ_2 and ρ_4, vanish, and we are left with,

$$\rho_{measured} = \int_\theta q(\theta)\left[C^4\rho_1 + C^2 S^2\rho_3 + S^4\rho_5\right] d\theta$$

The remaining terms in the integral, can all be easily understood physically, with the exception of ρ_3, which contains terms such as $P_{Z,odd}\rho P_Z$. By applying a Z gate to the qubit with 50% probability, we split the density matrix into a sum of two parts, where these terms have opposite signs in each part. Overall then these problematic terms now cancel, leaving us with a form which can be more straightforwardly simulated.

Under these conditions, the noisy measurement can then be written as a superoperator,

$K_0 = P_{EVEN}$	$p_0 = \int q(\theta)\cos^4(\theta)d\theta$	
$K_1 = P_{ODD}$	$p_1 = \int q(\theta)\sin^4(\theta)d\theta$	
$K_2 = YP_{EVEN}$	$p_2 = \int q(\theta)\cos^2(\theta)\sin^2(\theta)d\theta$	
$K_3 = YP_{ODD}$	$p_3 = \int q(\theta)\cos^2(\theta)\sin^2(\theta)d\theta$	



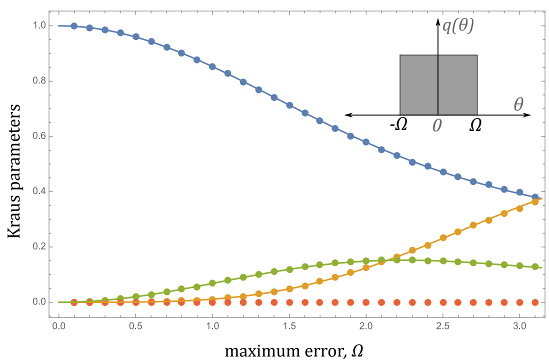


FIG. 1. Magnitude of the terms of the super operator under a) Normally distributed noise b) uniformly distributed noise. The solid lines show the value calculated directly from the integrals in the summary table. Data points show the results of simulation over 10^4 trials, where the value of θ was sampled from the distribution $q(\theta)$ in each trial. Red data points show the remaining amplitude of the simulated density matrix, not described by any of the four Kraus operators. This is vanishingly small, reinforcing that the model is a full description of the resulting errors.

$$\mathcal{S}(\rho) = \sum p_i K_i \rho K_i^\dagger$$

Where the Kraus operators, K_i, are given by the remaining terms of ρ_1, ρ_3 and ρ_5, and the probabilities, p_i, are found by integrating the constants over $q(\theta)$. These are summarized in table 1.

Case of systematic errors

If systematic error is present then the probability function, $q_{sys}(\theta)$, describing the distribution of the error does not average to zero, but rather $\int q_{sys}(\theta)\, d\theta = \bar{\theta}$. By changing variables to $\phi = \theta - \bar{\theta}$

$$\begin{aligned}
\rho_{measured} &= \int_\theta q_{sys}(\theta)\, P(\theta)\, \rho P(\theta)\, d\theta \\
&= \int_\phi q_{sys}\left(\phi+\bar{\theta}\right) P\left(\phi+\bar{\theta}\right) \rho P\left(\phi+\bar{\theta}\right) d\phi \\
&= \int_\phi q_{rand}(\phi)\, R^\dagger\left(\phi+\bar{\theta}\right) \hat{P}_Z R\left(\phi+\bar{\theta}\right) \rho R^\dagger\left(\phi+\bar{\theta}\right) \hat{P}_Z R\,(\phi \\
&= R\left(\bar{\theta}\right) \left\{ \int_\phi q_{rand}(\phi)\, R^\dagger(\phi)\, \hat{P}_Z R(\phi) \rho_{\bar{\theta}} R^\dagger(\phi)\, \hat{P}_Z R(\phi)\, d\phi \right\} \\
&= R\left(\bar{\theta}\right) \left\{ \int_\phi q_{rand}(\phi)\, P(\phi)\, \rho_{\bar{\theta}} P(\phi)\, d\phi \right\} R^\dagger\left(\bar{\theta}\right)
\end{aligned}$$

where $\rho_{\bar{\theta}} = R(\bar{\theta})\rho R^\dagger(\bar{\theta})$

So with systematic errors, we can also write the effect as a superoperators where a fixed rotation is followed by an application of random errors which average to zero.

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