MEASURING SKEWED OPERATORS

We consider the case where we aim to measure the pauli Z operator, but instead measure in some slightly rotated basis $\cos\theta Z + \sin\theta X$. The projection into this rotated basis, can be viewed as a rotation of the state, followed by the ideal Z projection, followed by an inverse rotation.

$$P\left(\theta\right) = R^{\dagger}\left(\theta\right)\hat{P}_{Z}R\left(\theta\right)$$

Where P_Z and $P(\theta)$ are taken to be even projection operators into the bases Z and $\cos\theta Z + \sin\theta X$, the rotation operator $R\left(\theta\right) = R_y\left(\theta\right) = \cos\frac{\theta}{2}\mathbb{I} - i\sin\frac{\theta}{2}Y$. writing $C = \cos\frac{\theta}{2}$ and $S = \sin\frac{\theta}{2}$,

$$\begin{aligned} P\left(\theta\right) &= C^2 P_Z + S^2 Y P_Z Y + iSC\left[Y P_Z - P_Z Y\right] \\ &= C^2 P_Z + S^2 Y^2 P_{Z,odd} + iSC\left[Y P_Z - Y P_{Z,odd}\right] \end{aligned}$$

By using the relationship $P_Z Y = Y P_{Z,odd}$ we can write this projector in two equivalent ways:

$$\begin{aligned} P\left(\theta\right) &= C^2 P_Z + S^2 P_{Z,odd} + iSCY\left[P_Z - P_{Z,odd}\right] \\ P\left(\theta\right) &= C^2 P_Z + S^2 P_{Z,odd} + iSC\left[P_{Z,odd} - P_Z\right]Y \end{aligned}$$

If an observed even measurement outcome corresponds to the projector $P\left(\theta\right)$being applied to the state with probability $q\left(\theta\right)$ then we can write the resultant state as:

$$\rho_{measured} = \int q\left(\theta\right)R^{\dagger}\left(\theta\right)P_Z R\left(\theta\right)\rho R^{\dagger}\left(\theta\right)P_Z R\left(\theta\right)d\theta$$

expanding this expression, and

$$\begin{aligned} \rho_{measured} &= \int q\left(\theta\right)P\left(\theta\right)\rho P\left(\theta\right)d\theta \\ &\left\{C^2 P_Z + S^2 P_{Z,odd} + iSCY\left[P_Z - P_{Z,odd}\right]\right\}\rho\left\{C^2 P_Z + S^2 P_{Z,odd} + iSC\left[P_{Z,odd} - P_Z\right]Y\right\} \\ &= C^4 P_Z\rho P_Z + S^4 P_{\hat{Z,odd}}\rho P_{\hat{Z,odd}} \\ &\quad + iC^3S\left[P_Z\rho P_{Z,odd}Y - P_Z\rho P_Z Y + Y P_Z\rho P_Z - Y P_{Z,odd}\rho P_Z\right] \\ &\quad + C^2S^2[P_Z\rho P_{Z,odd} + P_{Z,odd}\rho P_Z - Y P_Z\rho P_{Z,odd}Y + Y P_{Z,odd}\rho P_{Z,odd}Y \\ &\qquad + Y P_Z\rho P_Z Y - Y P_{Z,odd}\rho P_Z Y] \\ &\quad + iCS^3\left[P_{Z,odd}\rho P_{Z,odd}Y - P_{Z,odd}\rho P_Z Y + Y P_Z\rho P_{Z,odd} - Y P_{Z,odd}\rho P_{Z,odd}\right] \end{aligned}$$

Label $\rho_{even} = P_Z\rho P_Z$, and $\rho_{odd} = P_{Z,odd}\rho P_{Z,odd}$, then

$$\begin{aligned} \rho_{measured} &= \int q\left(\theta\right)P\left(\theta\right)\rho P\left(\theta\right)d\theta \\ &C^4\rho_{even} + S^4\rho_{odd} \\ &+ iC^3S\left[-\rho_{even}Y + Y\rho_{even} - Y P_{Z,odd}\rho P_Z + P_Z\rho P_{Z,odd}Y\right] \\ &+ C^2S^2\left[Y\rho_{odd}Y + Y\rho_{even}Y + P_Z\rho P_{Z,odd} + P_{Z,odd}\rho P_Z - Y P_Z\rho P_{Z,odd}Y - Y P_{Z,odd}\rho P_Z Y\right] \\ &+ iCS^3\left[-Y\rho_{odd} + \rho_{odd}Y - P_{Z,odd}\rho P_Z Y + Y P_Z\rho P_{Z,odd}\right] \end{aligned}$$

Assume $q\left(\theta\right)$ is symmetric about $\theta = 0$

Now all the assymetric terms in the integral vanish, and we are left with

$$\rho_{measured} = \int q(\theta)\{C^4\rho_{even} + S^4\rho_{odd} + C^2S^2[Y\rho_{odd}Y + Y\rho_{even}Y]$$
$$+C^2S^2[P_Z\rho P_{Z,odd} + P_{Z,odd}\rho P_Z - YP_Z\rho P_{Z,odd}Y - YP_{Z,odd}\rho P_ZY]\}d\theta$$

Apply Z with 50% probability - this introduces a sign in any $P_{Z,odd}$ operations, then the second line of terms in the equation above cancel to leave:

$$\rho_{measured} = \left[\int q(\theta)\,C^4 d\theta\right]\rho_{even} + \left[\int q(\theta)\,S^4 d\theta\right]\rho_{odd} + \left[\int q(\theta)\,C^2S^2 d\theta\right][Y\rho_{odd}Y + Y\rho_{even}Y]$$

So we can write the resultant channel as a superoperator

$$\mathcal{S}(\rho) = \sum p_i K_i \rho K_i^{\dagger}$$

With the following probabilities and Kraus operators:

$p_9 = \left[\int q(\theta)\,C^4 d\theta\right]$	$K_0 = P_Z$
$p_1 = \left[\int q(\theta)\,S^4 d\theta\right]$	$K_1 = P_{Z,odd}$
$p_2 = \left[\int q(\theta)\,C^2S^2 d\theta\right]$	$K_2 = YP_{Z,odd}$
$p_3 = \left[\int q(\theta)\,C^2S^2 d\theta\right]$	$K_3 = YP_Z$

Case of systematic errors

If systematic error is present then the probability function, $q_{sys}(\theta)$, describing the distribution of the error does not average to zero, but rather $\int q_{sys}(\theta)\,d\theta = \bar{\theta}$. By changing variables to $\phi = \theta - \bar{\theta}$

$$\begin{aligned}
\rho_{measured} &= \int_{\theta} q_{sys}(\theta)\,P(\theta)\,\rho P(\theta)\,d\theta \\
&= \int_{\phi} q_{sys}\left(\phi+\bar{\theta}\right)P\left(\phi+\bar{\theta}\right)\rho P\left(\phi+\bar{\theta}\right)d\phi \\
&= \int_{\phi} q_{rand}(\phi)\,R^{\dagger}\left(\phi+\bar{\theta}\right)\hat{P}_Z R\left(\phi+\bar{\theta}\right)\rho R^{\dagger}\left(\phi+\bar{\theta}\right)\hat{P}_Z R\left(\phi+\bar{\theta}\right)d\phi \\
&= R\left(\bar{\theta}\right)\left\{\int_{\phi} q_{rand}(\phi)\,R^{\dagger}(\phi)\,\hat{P}_Z R(\phi)\rho_{\bar{\theta}}R^{\dagger}(\phi)\,\hat{P}_Z R(\phi)\,d\phi\right\}R^{\dagger}\left(\bar{\theta}\right) \\
&= R\left(\bar{\theta}\right)\left\{\int_{\phi} q_{rand}(\phi)\,P(\phi)\,\rho_{\bar{\theta}}P(\phi)\,d\phi\right\}R^{\dagger}\left(\bar{\theta}\right)
\end{aligned}$$

where $\rho_{\bar{\theta}} = R(\bar{\theta})\rho R^{\dagger}(\bar{\theta})$

So with systematic errors, we can also write the effect as a superoperators where a fixed rotation is followed by an application of random errors which average to zero.